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Research article

Existence and blowup of solutions for non-divergence polytropic variation-inequality in corn option trading

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Abstract: This paper focuses on a class of variation-inequality problems involving non-divergence polytropic parabolic operators. The penalty method is employed, along with the Leray Schauder fixed point theory and limit progress, to determine the existence of solutions. The study also delves into the blow-up phenomena of the solution, revealing that under certain conditions, the solution will blow up in finite time.

Keywords: variation-inequality problem; generalized solution; non-divergence polytropic operator; existence; blow up
Mathematics Subject Classification: 35K99, 97M30

1. Introduction

First, consider a kind of variation-inequality problem

$$\begin{cases} Lu \geq 0, & (x,t) \in \Omega_T, \\ u \geq u_0, & (x,t) \in \Omega_T, \\ Lu \cdot (u - u_0) = 0, & (x,t) \in \Omega_T, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{1}$$

with the non-Newtonian polytropic operator,

$$Lu = \partial_t u - u^m \nabla(|\nabla u^m|^{p-2} \nabla u^m) - \gamma |\nabla u^m|^p. \tag{2}$$

Here, $\Omega \subset R^N (N \geq 2)$ is a bounded domain with an appropriately smooth boundary $\partial\Omega$, $p \geq 2$, $m > 0$ and u_0 satisfies

$$u_0 > 0, u_0^m \in W^{1,p}(\Omega) \cap L^\infty(\Omega).$$

The theory of variation-inequality problems has gained significant attention due to its applications in option pricing. These applications are discussed in references [1–3], where more details on the financial background can be found. In recent years, there has been a growing interest in the study of variation-inequality problems, with a particular emphasis on investigating the existence and uniqueness of solutions.

In 2022, Li and Bi considered a two dimension variation-inequality system [4],

$$\begin{cases} \min\{L_i u_i - f_i(x,t,u_1,u_2), u_i - u_{i,0}\} = 0, & (x,t) \in \Omega_T, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases}$$

involving a degenerate parabolic operator

$$L_i u_i = \partial_t u_i - \mathrm{div}(|\nabla u_i|^{p_i-2}\nabla u_i), i = 1,2.$$

Using the comparison principle of $L_i u_i$ and norm estimation techniques, the sequence of upper and lower solutions for the auxiliary problem is obtained. The existence and uniqueness of weak solutions are then analyzed. While reference [5] considers the initial boundary value problem under a single variational inequality, the author explores more complex non-divergence parabolic operators

$$Lu = \partial_t u - u\mathrm{div}(a(u)|\nabla u|^{p(x)-2}\nabla u) - \gamma|u|^{p(x)} - f(x,t).$$

Reference [5] constructs a more intricate auxiliary problem and proves that the weak solutions are both unique and existent by using progressive integration and various inequality amplification techniques. Readers can refer to references [6–8] for further information on these interesting results.

In the field of differential equations, there are various literature available on initial boundary value problems that involve the non-Newtonian polytropic operator. In [9,10], the authors focused on a specific class of initial boundary value problems that feature the non-Newtonian polytropic operator

$$\begin{cases} \partial_t u - \nabla(|\nabla u^m|^{p-2}\nabla u^m) + h(x,t)u^\alpha = 0, & (x,t) \in \Omega_T, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = 0, & (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

To investigate the existence of a weak solution, they made use of topologic degree theory.

Currently, there is no literature on the study of variational inequalities under non-Newtonian polytropic operators (2). Therefore, we attempt to use the results of partial differential equations from literature [5,9,10] to investigate the existence and blow-up properties of weak solutions for variational inequalities (1). Additionally, considering the degeneracy of the operator Lu at $u = 0$ and $\nabla u^m = 0$, some traditional methods for existence proofs are no longer applicable. Here, we attempt to use the fixed point theorem to solve this issue, and obtain the existence and blow up of generalized solutions.

2. Statement of the problem and its background

We first consider the case of variation-inequality in corn options. During the harvest season, farmers face the problem of corn storage, while flour manufacturers are concerned about the downtime caused by a lack of raw materials.

In exchange for the farmer storing the raw materials in the warehouse, the flour manufacturer promises the farmer the following contract:

Farmers at any time within a year have the right to sell corn at the agreed price K.

Assuming that the current time is 0, the corn price S_t follows the time interval $[0,\ T]$, and is given by:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t,$$

where S_0 is known, μ represents the annual growth rate of corn price, and σ represents the volatility rate. $\{W_t, t \geq 0\}$ stands for a winner process, representing market noise.

In addition, to avoid significant economic losses for flour manufacturers due to rapid increases in raw material prices, obstacle clauses are often included in the following form: if the price of corn rises more than B, the option contract becomes null and void. According to literature [1–3], the value V of the option contract at any time $t \in [0, T]$ satisfies

$$\begin{cases} \min\{L_0 V, V - \max\{S - K, 0\}\} = 0, & (S, t) \in (0, B) \times (0, T), \\ u(0, x) = \max\{S - K, 0\}, & S \in (0, B), \\ u(t, B) = 0, & t \in (0, T), \\ u(t, 0) = 0, & t \in (0, T), \end{cases} \tag{3}$$

where $L_0 V = \partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V + rS\partial_S V - rV$, r is the risk-free interest rate of the agricultural product market; B is the upper bound of corn prices, which prevents flour manufacturers from incurring significant losses due to rising corn prices. On the one hand,if $x = lnS$, then (3) can be written as

$$\begin{cases} \min\{L_1 V, V - \max\{e^x - K, 0\}\} = 0, & (x, t) \in (-\infty, \ln B) \times (0, T), \\ u(0, x) = \max\{e^x - K, 0\}, & x \in (-\infty, \ln B), \\ u(t, \ln B) = 0, & t \in (0, T), \\ u(t, 0) = 0, & t \in (0, T), \end{cases}$$

where $L_1 V = \partial_t V + \frac{1}{2}\sigma^2 \partial_{xx} V + r\partial_x V - rV$. It can be seen that problem (4) is a constant coefficient parabolic variational inequality problem, which has long been studied by scholars (see [1–3] for details). On the other hand, when there are transaction costs involved in agricultural product trading, the constant σ in the operator LV is no longer valid and is often related to $\partial_S V$, as well as V itself. For instance, the well-known Leland model [5] adjusts volatility σ into a non-divergence structure represented by

$$\sigma^2 = \sigma_0^2(1 + Le \cdot \mathrm{sign}(V\partial_x(|\partial_x V|^{p-2}\partial_x V))), p \geq 2, \tag{4}$$

where σ denotes the original volatility and Le corresponds to the Leland number.

Inspired by these findings, we aim to explore more intricate variation-inequality models in (1). When $m = 1$, the non-divergence polytropic structure $u^m\nabla(|\nabla u^m|^{p-2}\nabla u^m)$ in model (1) degenerates into a similar n-dimensional structure as model (4). It's worth noting that while model (4) only considers one type of risky asset and is defined in a 1-dimensional space, model (1) studies the problem in an n-dimensional space.

Variation-inequality (1) degenerates when either $u = 0$ or $\nabla u^m = 0$. Classically, there would be no traditional solution. Following a similar way in [1,3], we consider generalized solutions and first give a class of maximal monotone maps $G : [0, +\infty) \to [0, +\infty)$ satisfies

$$G(x) = 0 \text{ if } x > 0,\ G(x) > 0 \text{ if } x = 0. \tag{5}$$

Definition 2.1 A pair (u, ξ) is called a generalized solution for variation-inequality, if for any fixed $T > 0$,
(a) $u^m \in L^\infty(0, T, W_0^{1,p}(\Omega))$, $\partial_t u \in L^2(\Omega_T)$,
(b) $\xi \in G(u - u_0)$ for any $(x, t) \in \Omega_T$,
(c) $u(x, t) \geq u_0(x)$, $u(x, 0) = u_0(x)$ for any $(x, t) \in \Omega_T$,
(d) for every test-function $\varphi \in C^1(\bar{\Omega}_T)$, there holds

$$\int\int_{\Omega_T} \partial_t u \cdot \varphi + u^m |\nabla u^m|^{p-2} \nabla u^m \nabla \varphi \mathrm{d}x\mathrm{d}t + (1-\gamma) \int\int_{\Omega_T} |\nabla u^m|^p \varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \xi \cdot \varphi \mathrm{d}x\mathrm{d}t.$$

As far as what was mentioned above, $u^m \nabla(|\nabla u^m|^{p-2} \nabla u^m)$ degenerates when $u^m = 0$ or $\nabla u^m = 0$. We set and use a parameter $\varepsilon \in [0, 1]$ to regularize $u^m \nabla(|\nabla u^m|^{p-2} \nabla u^m)$ in operator Lu and the initial boundary condition. Meanwhile, we use ε to construct a penalty function $\beta_\varepsilon(\cdot)$ and use it to control the inequalities in (1) that the penalty map $\beta_\varepsilon : \mathrm{R}_+ \to \mathrm{R}_-$ satisfies

$$\beta_\varepsilon(x) = 0 \text{ if } x > \varepsilon,\ \beta_\varepsilon(x) \in [-M_0, 0) \text{ if } x \in [0, \varepsilon]. \tag{6}$$

In other words, we consider the following regular problem

$$\begin{cases} L_\varepsilon u_\varepsilon = -\beta_\varepsilon(u_\varepsilon - u_0), & (x, t) \in Q_T, \\ u_\varepsilon(x, 0) = u_{0\varepsilon}(x), & x \in \Omega, \\ u_\varepsilon(x, t) = \varepsilon, & (x, t) \in \partial Q_T, \end{cases} \tag{7}$$

where

$$L_\varepsilon u_\varepsilon = \partial_t u_\varepsilon - u_\varepsilon^m \nabla((|\nabla u_\varepsilon^m|^2 + \varepsilon)^{p-2} \nabla u_\varepsilon^m) - \gamma(|\nabla u_\varepsilon^m|^2 + \varepsilon)^{p-2} |\nabla u_\varepsilon^m|^2.$$

Similar to [4,5], problem (8) admits a solution u_ε satisfies $u_\varepsilon^m \in L^\infty(0, T; W^{1,p}(\Omega))$, $\partial_t u_\varepsilon^m \in L^\infty(0, T; L^2(\Omega))$, and the identity

$$\int_\Omega (\partial_t u_\varepsilon \cdot \varphi + u_\varepsilon^m (|\nabla u_\varepsilon^m|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon^m \nabla \varphi + (1-\gamma)(|\nabla u_\varepsilon^m|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla u_\varepsilon^m|^2 \varphi) \mathrm{d}x = -\int_\Omega \beta_\varepsilon(u_\varepsilon - u_0) \varphi \mathrm{d}x, \tag{8}$$

with $\varphi \in C^1(\bar{\Omega}_T)$. Meanwhile, for any $\varepsilon \in (0,\ 1)$,

$$u_{0\varepsilon} \leq u_\varepsilon \leq |u_0|_\infty + \varepsilon,\ u_{\varepsilon_1} \leq u_{\varepsilon_2} \text{ for } \varepsilon_1 \leq \varepsilon_2. \tag{9}$$

Indeed, define $A_\theta(u_\varepsilon) = \theta u_\varepsilon^m + (1-\theta) u_\varepsilon$,

$$L_\varepsilon^{\theta,\omega} u_\varepsilon = \partial_t u_\varepsilon - A_\theta \mathrm{div}\left(\left(|\nabla A_\theta(u_\varepsilon)|^2 + \varepsilon\right)^{\frac{p-2}{2}} \nabla A_\theta(u_\varepsilon)\right) - \gamma(|\nabla A_\theta(u_\varepsilon)|^2 + \varepsilon)^{p-2} |\nabla A_\theta(u_\varepsilon)|^2.$$

One can use a map based on Leray-Schauder fixed point theory

$$M : L^\infty(0, T; W_0^{1,p}(\Omega)) \times [0, 1] \to L^\infty(0, T; W_0^{1,p}(\Omega)), \tag{10}$$

that is,

$$\begin{cases} L_\varepsilon^{\theta,\omega} u_\varepsilon = -\theta \beta_\varepsilon(u_\varepsilon - u_0), & (x, t) \in \Omega_T, \\ u_\varepsilon(x, 0) = u_{0\varepsilon}(x) = u_0 + \varepsilon, & x \in \Omega, \\ u_\varepsilon(x, t) = \varepsilon, & (x, t) \in \partial\Omega_T, \end{cases} \tag{11}$$

so that by proving the boundedness, continuity and compactness of operator M, the existence result of (6) can be established. For details, see literature [11], omitted here.

3. Existence of solution

In this section, we consider the existence of a generalized solution to variation-inequality (1). Since $u_\varepsilon^m \in L^\infty(0,T;W^{1,p}(\Omega))$, $\partial_t u_\varepsilon^m \in L^\infty(0,T;L^2(\Omega))$, by combining with (9), we may infer that the sequence $\{u_\varepsilon, \varepsilon \geq 0\}$ contains a subsequence $\{u_{\varepsilon_k}, k = 1, 2, \cdots\}$ and a function u, $\varepsilon_k \to 0$ as $k \to \infty$,

$$u_{\varepsilon_k} \to u \text{ a.e. in } \Omega_T \text{ as } k \to \infty, \tag{12}$$

$$u_{\varepsilon_k}^m \overset{\text{weak}}{\to} u^m \text{ in } L^\infty(0,T;W_0^{1,p}(\Omega)) \text{ as } k \to \infty, \tag{13}$$

$$\partial_t u_{\varepsilon_k}^m \overset{\text{weak}}{\to} \partial_t u^m \text{ in } L^2(\Omega_T) \text{ as } k \to \infty. \tag{14}$$

From (9), one can easily show that $u_{\varepsilon_k} \leq u$, $\forall (x,t) \leq \Omega_T$, $k = 1, 2, 3, \cdots$. So, one can infer that for all $(x,t) \in \Omega_T$,

$$-\beta_{\varepsilon_k}(u_{\varepsilon_k} - u_0) \to \xi \text{ as } k \to \infty. \tag{15}$$

Next, we pass the limit $k \to \infty$. It follows from (13), that for any $(x,t) \in \Omega_T$, $k = 1, 2, 3, \cdots$,

$$u_{\varepsilon_k}^m (|\nabla u_{\varepsilon_k}^m|^2 + \varepsilon_k)^{\frac{p-2}{2}} \nabla u_{\varepsilon_k}^m \overset{weak}{\to} \chi_1 \text{ in } L^1(\Omega) \text{ as } k \to \infty, \tag{16}$$

$$(|\nabla u_{\varepsilon_k}^m|^2 + \varepsilon_k)^{\frac{p-2}{2}} |\nabla u_{\varepsilon_k}^m|^2 \overset{weak}{\to} \chi_2 \text{ in } L^1(\Omega) \text{ as } k \to \infty. \tag{17}$$

so that pass the limit $k \to \infty$,

$$\int_\Omega \partial_t u \cdot \varphi + \chi_1 \nabla\varphi \mathrm{d}x + (1-\gamma)\int_\Omega \chi_2 \varphi \mathrm{d}x = \int_\Omega \xi \cdot \varphi \mathrm{d}x. \tag{18}$$

Choosing $\varphi = u_{\varepsilon_k} - u$ in (8) and turning ε into ε_k, one can infer that

$$\int_\Omega \partial_t u_{\varepsilon_k} \cdot \varphi + u_{\varepsilon_k}^m (|\nabla u_{\varepsilon_k}^m|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_{\varepsilon_k}^m \nabla\varphi + (1-\gamma)(|\nabla u_{\varepsilon_k}^m|^2 + \varepsilon_k)^{\frac{p-2}{2}} |\nabla u_{\varepsilon_k}^m|^2 \varphi \mathrm{d}x = -\int_\Omega \beta_{\varepsilon_k}(u_{\varepsilon_k} - u_0)\varphi \mathrm{d}x. \tag{19}$$

Subtracting (18) from (19) and integrating it from 0 to T,

$$\begin{aligned} &\int\int_{\Omega_T} (\partial_t u_{\varepsilon_k} - \partial_t u) \cdot \varphi + \left[u_{\varepsilon_k}^m (|\nabla u_{\varepsilon_k}^m|^2 + \varepsilon_k)^{\frac{p-2}{2}} \nabla u_{\varepsilon_k}^m - \chi_1 \right] \nabla\varphi \mathrm{d}x\mathrm{d}t \\ &+(1-\gamma)\int\int_{\Omega_T} \left[(|\nabla u_{\varepsilon_k}^m|^2 + \varepsilon_k)^{\frac{p-2}{2}} |\nabla u_{\varepsilon_k}^m|^2 - \chi_2 \right] \varphi \mathrm{d}x\mathrm{d}t \\ &= -\int_0^t \int_\Omega [\beta_{\varepsilon_k}(u_{\varepsilon_k} - u_0) + \xi] \cdot \varphi \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{20}$$

From (32), we infer that

$$\lim_{k\to\infty} \int_0^t \int_\Omega [\beta_{\varepsilon_k}(u_{\varepsilon_k} - u_0) + \xi] \cdot \varphi \mathrm{d}x\mathrm{d}t = 0, \tag{21}$$

$$\int\int_{\Omega_T} \left[(|\nabla u_{\varepsilon_k}^m|^2 + \varepsilon_k)^{\frac{p-2}{2}} |\nabla u_{\varepsilon_k}^m|^2 - \chi_2 \right] \varphi \mathrm{d}x\mathrm{d}t = 0. \tag{22}$$

Recall that $u_{\varepsilon_k}(x,0) = u_0(x) + \varepsilon_k$ for any $x \in \Omega$. Then

$$\int\int_{\Omega_T} (\partial_t u_{\varepsilon_k} - \partial_t u) \cdot \varphi \mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_\Omega (u_{\varepsilon_k} - u)^2 \mathrm{d}x - \frac{1}{2}\varepsilon_k^2 \geq 0.$$

Note that $\varepsilon_k \to 0$ as $k \to \infty$. So we may infer that $\int\int_{\Omega_T} (\partial_t u_{\varepsilon_k} - \partial_t u) \cdot \varphi \mathrm{d}x\mathrm{d}t \geq 0$ if k is large enough. Then, removing the non negative term on the left hand-side in (20) and passing the limit $k \to \infty$, it is clear to verify that

$$\lim_{k\to\infty} \int\int_{\Omega_T} \left[u_{\varepsilon_k}^m (|\nabla u_{\varepsilon_k}^m|^2 + \varepsilon_k)^{\frac{p-2}{2}} \nabla u_{\varepsilon_k}^m - \chi_1 \right] \nabla\varphi \mathrm{d}x\mathrm{d}t \leq 0, \tag{23}$$

Note that $u^m |\nabla u^m|^{p-2} \nabla u^m = |\nabla u^{\mu m}|^{p-2} \nabla u^{\mu m}$, $\mu = \frac{p}{p-1}$. As mentioned in [12], it follows from (9) that

$$\begin{aligned} &\left[u_{\varepsilon_k}^m (|\nabla u_{\varepsilon_k}^m|^2 + \varepsilon_k)^{\frac{p-2}{2}} \nabla u_{\varepsilon_k}^m - u^m |\nabla u^m|^{p-2} \nabla u^m \right] (\nabla u_{\varepsilon_k}^{\mu m} - \nabla u^{\mu m}) \\ &\geq \left[|\nabla u_{\varepsilon_k}^{\mu m}|^{p-2} \nabla u_{\varepsilon_k}^{\mu m} - |\nabla u^{\mu m}|^{p-2} \nabla u^{\mu m} \right] \left(\nabla u_{\varepsilon_k}^m - \nabla u^m \right) \\ &\geq C |\nabla u_{\varepsilon_k}^{\mu m} - \nabla u^{\mu m}|^p \geq 0. \end{aligned} \tag{24}$$

Thus, by using $\operatorname{sgn}(\nabla u_{\varepsilon_k}^{\mu m} - \nabla u^{\mu m}) = \operatorname{sgn}(\nabla\varphi)$, leads to

$$\left[(|\nabla u_{\varepsilon_k}^m|^2 + \varepsilon_k)^{\frac{p-2}{2}} \nabla u_{\varepsilon_k}^m - |\nabla u^m|^{p-2} \nabla u^m \right] \nabla\varphi \geq 0. \tag{25}$$

Subtracting (24) from (25), one can see that

$$\int\int_{\Omega_T} \left[u^m |\nabla u^m|^{p-2} \nabla u^m - \chi_1 \right] \nabla\varphi \mathrm{d}x\mathrm{d}t \leq 0. \tag{26}$$

Obviously, if we swap u_{ε_k} and u , one can get another inequality

$$\int\int_{\Omega_T} \left[\chi_1 - u^m |\nabla u^m|^{p-2} \nabla u^m \right] \nabla\varphi \mathrm{d}x\mathrm{d}t \leq 0. \tag{27}$$

Combining (26) and (27), we obtain (28) below and give the following Lemma.
Lemma 3.1 For any $t \in (0, T]$ and $x \in \Omega$,

$$\chi_1 = u^m |\nabla u^m|^{p-2} \nabla u^m \text{ a.e. in } \Omega_T, \tag{28}$$

$$\left\| \nabla u_{\varepsilon_k}^{\mu m} - \nabla u^{\mu m} \right\|_{L^p(\Omega)} \to 0 \text{ as } k \to \infty. \tag{29}$$

Proof. One can deduce that (29) is an immediate result of combining (23), (24) and (29).

Following a similar proof showed in (16)–(28), one can infer that

$$\chi_2 = |\nabla u^m|^p \text{ a.e. in } \Omega_T. \tag{30}$$

Further, we prove $\xi \in G(u - u_0)$. When $u_{\varepsilon_k} \geq u_0 + \varepsilon$,we have $\beta_{\varepsilon_k}(u_{\varepsilon_k} - u_0) = 0$, so

$$\xi(x,t) = 0 \Leftrightarrow u > u_0. \tag{31}$$

If $u_0 \leq u_{\varepsilon_k} < u_0 + \varepsilon_k, \beta_\varepsilon(u_\varepsilon - u_0) \leq 0$ and $\beta_\varepsilon(\cdot) \in C^2(\mathrm{R})$ imply that

$$\xi(x,t) \leq 0 \Leftrightarrow u = u_0. \tag{32}$$

Combining (31) and (32), it can be easily verified that $\xi \in G(u - u_0)$.

Further, passing the limit $k \to \infty$ in the second line of (44) and the third line of (6),

$$u(x,0) = u_0(x) \text{ in } \Omega,\ u(x,t) \geq u_0(x) \text{ in } \Omega_T.$$

Combining the equations above, we infer that (u, ξ) satisfies the conditions of Definition 2.1, such that (u, ξ) is a generalized solution of (1).
Theorem 3.1 Assume that $u_0^m \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\gamma \leq 1$. Then variation-inequality (1) admits at least one solution (u, ξ) within the class of Definition 2.1.

4. Blowup of solution

In this section, we consider the blowup of the generalized solution when $\gamma > 2$ and try to prove it by contradiction. Assume (u, ξ) is a generalized solution of (1). Taking $\varphi = u^m$ in Definition 1, it is easy to see that

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(\cdot,t)^{m+1}\mathrm{d}x + (2-\gamma)\int_\Omega |\nabla u^m|^p u^m \mathrm{d}x = \int_\Omega \xi \cdot u^m \mathrm{d}x. \tag{33}$$

It follows from (5), (9), and $\xi \in G(u-u_0)$ that $\int_\Omega \xi \cdot u^m \mathrm{d}x \geq 0$. Let

$$E(t) = \int_\Omega u(\cdot,t)^{\omega m+1}\mathrm{d}x.$$

It follows from (c) in Definition 2.1 that $E(t) \geq 0$ for any $t \in (0, T]$, so one can infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \geq (m+1)(\gamma-2)\int_\Omega |\nabla u^m|^p u^m \mathrm{d}x. \tag{34}$$

Using the Poincare inequality gives

$$\int_\Omega |\nabla u^m|^p u^m \mathrm{d}x = \frac{p}{(\omega+p)m}\int_\Omega |\nabla u^{(\frac{1}{p}+1)m}|^p \mathrm{d}x \geq \frac{p}{(p+1)m}\int_\Omega |u|^{(p+1)m}\mathrm{d}x. \tag{35}$$

Here, $\int_\Omega |u|^{(p+1)m}\mathrm{d}x$ need to keep shrinking. By the Holder inequality

$$E(t) \leq C(|\Omega|)\left(\int_\Omega |u|^{(p+1)m}\mathrm{d}x\right)^{\frac{m+1}{(p+1)m}},$$

so that

$$\int_\Omega |u|^{(p+1)m}\mathrm{d}x \geq C(|\Omega|)E(t)^{\frac{(p+1)m}{m+1}}. \tag{36}$$

Combining (34)–(36), one can find that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \geq C(|\Omega|)\frac{p(m+1)(\gamma-2)}{(p+1)m}E(t)^{\frac{(p+1)m}{m+1}}. \tag{37}$$

Note that $mp > 1$. Using variable separation method, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)^{\frac{1-mp}{m+1}} \leq C(|\Omega|)\frac{p(\gamma-2)(1-mp)}{(p+1)m}, \tag{38}$$

such that

$$E(t) \geq \frac{1}{\left(E(0)^{\frac{1-mp}{m+1}} - C(|\Omega|)\frac{p(\gamma-2)(mp-1)}{(p+1)m}t\right)^{\frac{m+1}{mp-1}}}. \tag{39}$$

This means that the generalized solution blows up at $T^* = \frac{E(0)^{\frac{1-mp}{m+1}}(p+1)m}{p(\gamma-2)(mp-1)C(|\Omega|)}$.

Theorem 4.1 Assume $mp > 1$. if $\gamma > 2$, the generalized solution (u, ξ) of variation-inequality (1) blows up in finite time.

5. Conclusions

In this study, the existence and blowup of a generalized solution to a class of variation-inequality problems with non-divergence polytropic parabolic operators

$$Lu = \partial_t u - u^m \nabla(|\nabla u^m|^{p-2}\nabla u^m) - \gamma|\nabla u^m|^p.$$

We first consider the existence of generalized solution. Due to the use of integration by parts, $-\gamma|\nabla u^m|^p$ becomes $(1-\gamma)|\nabla u^m|^p$. In the process of proving $u_\varepsilon^m \in L^\infty(0,T;W^{1,p}(\Omega))$ and $\partial_t u_\varepsilon^m \in L^\infty(0,T;L^2(\Omega))$ in [4,5], $(1-\gamma)|\nabla u^m|^p$ is required to be greater than 0, therefore eliciting the restriction $\gamma \le 1$. Regarding the restriction of p, the condition $p \ge 1$ is required in (24) and the above formula. As what mentioned, we have used the results $u_\varepsilon^m \in L^\infty(0,T;W^{1,p}(\Omega))$ and $\partial_t u_\varepsilon^m \in L^\infty(0,T;L^2(\Omega))$ in literature [4,5] where $p \ge 2$ is required, therefore we require the restriction that $p \ge 2$. The results show that variation-inequality (1) admits at least one solution (u,ξ) when $\gamma \le 1$.

Second, we analyzed the blowup phenomenon of a generalized solution. In (38), mp must be big than 1, otherwise (39) is invalid. The results show that the generalized solution (u,ξ) of the variation-inequality (1) blows up in finite time when $\gamma \ge 2$.

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Conflict of interest

The author declares no conflict of interest.

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