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Research article

Jensen, Ostrowski and Hermite-Hadamard type inequalities for h-convex stochastic processes by means of center-radius order relation

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Abstract: In optimization, convex and non-convex functions play an important role. Further, there is no doubt that convexity and stochastic processes are closely related. In this study, we introduce the notion of the h-convex stochastic process for center-radius order in the setting of interval-valued functions ($\mathcal{IVFS}$) which is novel in literature. By using these notions we establish Jensen, Ostrowski, and Hermite-Hadamard ($\mathcal{H.H}$) types inequalities for generalized interval-valued $\mathcal{CR}$-h-convex stochastic processes. Furthermore, the study provides useful examples to support its findings.

Keywords: Hermite-Hadamard; Ostrowski and Jensen inequalities; h-convexity; stochastic h-convex
Mathematics Subject Classification: 26A48, 26A51, 33B10, 39A12, 39B62

1. Introduction

In dealing with uncertain data, interval analysis provides a number of useful tools. This method may be used in models containing data that have inaccuracies as a result of measuring certain types of things in certain ways. As an example of a set-valued analysis, interval analysis is used in mathematical analysis and general topology. By using this technique, we can handle interval uncertainty in some deterministic real-world phenomena. In Moore's acclaimed book the mathematics of numerical analysis, interval analysis was introduced for the first time in numerical analysis, see Ref. [1]. Over the past fifty years, interval analysis has been widely applied to a variety of fields, such as the following: Computer graphics [2], interval differential equation [3], automatic error analysis [4] and neural network output optimization [5], etc.

It has long been recognized that convexity is a significant factor in areas such as probability theory, economics, optimal control theory, and fuzzy analysis, as well as a valuable source of inspiration in both the natural sciences and the applied sciences. Additionally, generalized convexity of mappings can be a powerful tool for solving a wide variety of nonlinear analysis, as well as applied analysis, problems in mathematics, and physics. A particularly exciting area is the study of convexity with integral problems. In recent years, integral inequalities have proven useful for qualitative and quantitative evaluations of convexity. In mathematics, the Hermite-Hadamard inequality is well known for being the first geometric interpretation of convex maps. A famous double inequality is defined as follow:

$$\zeta\left(\frac{f+g}{2}\right) \leq \frac{1}{g-f}\int_{f}^{g}\zeta(\varepsilon)d\varepsilon \leq \frac{\zeta(f)+\zeta(g)}{2}, \tag{1.1}$$

where $\zeta : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex function on interval I and $f, g \in I$ with $f < g$. Convexity classes of various types have been covered by this function, which has been refined, generalized, and extended in various ways by using h-convexity, see Refs. [6–16]. The following are some developments related to proposed inequalities using different integral operators for interval-valued functions, see Refs. [17–30]. Further, more it is of great importance in statistics and probability to understand stochastic convexity in order to calculate numerical estimates of existing probabilistic quantities. Initially an investigation of convex stochastic processes was conducted by Nikodem in 1980, see Ref. [31]. Several applications of stochastic convexity were given by Shaked et al. [32] in 1988. A further revision of results previously developed by authors was made by Skowronski in 1992, along with an introduction of some new notions associated with convex stochastic processes and some further results obtained, see Ref. [33]. A famous double inequality often called Hermite-Hadamard inequality was extended to convex stochastic processes by Kotrys in 2012, see Ref. [34]. In 2015, Nelson Merentes and his co-authors utilized Varoşanec [35], concept of h-convexity and revised previous results developed by different authors in context of h-convex stochastic processes this article develops Hermite-Hadamard, Schur and Jensen type inequalities by describing h-convex stochastic processes, see Ref. [36]. Some recent developments related to these inequalities for convex stochastic processes, see Refs. [37–45]. Moreover, Mevlüt Tunç and the following authors [46, 47] developed Ostrowski type inequalities for h-convexity as well as for h-convex stochastic process, respectively.

Based on the radius and midpoints of the interval, Bhunia [48], developed the center-radius order in 2014. Following authors developed these inequalities for harmonically $\mathcal{CR}$-h-convex and $\mathcal{CR}$-h-Godunova-Levin functions based on the notions of center-radius order in 2022, see Refs. [49, 50]. Center-radius order relations pertaining to h- convex functions offer the advantage of providing more precise inequality terms, and it is possible to demonstrate the validity of the argument by providing interesting illustrations. Therefore, understanding how convexity and inequality can be studied using a total order relation is essential. Compared to the different order relations used in interval analysis to develop inequalities, this order relation is quite different to calculate, we can use the midpoint and center of the interval to calculate it.

Inspired by Refs. [36, 46, 47, 49–52]. By combining center-radius order relation and stochastic h-convex process, we develop Hermite-Hadamard, Ostrowski, and Jensen type inequalities in the setting of interval-valued functions. In addition to the conclusions drawn, the study provides several examples.

2. Preliminaries

Concerning the notions that have been used but not defined, see Refs. [6, 49]. As you process the rest of the paper, it will be very useful if you are familiar with a few basic arithmetic concepts related to interval analysis.

$$[\sigma] = [\underline{\sigma}, \overline{\sigma}] \qquad (z \in \mathbf{R}, \underline{\sigma} \leqq z \leqq \overline{\sigma};\ \underline{\sigma}, \overline{\sigma} \in \mathbf{R}),$$
$$[\Omega] = [\underline{\Omega}, \overline{\Omega}] \qquad (z \in \mathbf{R}, \underline{\Omega} \leqq z \leqq \overline{\Omega};\ \underline{\Omega}, \overline{\Omega} \in \mathbf{R}),$$
$$[\sigma] + [\Omega] = [\underline{\sigma}, \overline{\sigma}] + [\underline{\Omega}, \overline{\Omega}] = [\underline{\sigma} + \underline{\Omega}, \overline{\sigma} + \overline{\Omega}]$$

and

$$\varepsilon\Omega = \varepsilon[\underline{\Omega}, \overline{\Omega}] = \begin{cases} \left[\varepsilon\underline{\Omega}, \varepsilon\overline{\Omega}\right], & (\varepsilon > 0); \\ \{0\}, & (\varepsilon = 0); \\ \left[\varepsilon\overline{\Omega}, \varepsilon\underline{\Omega}\right], & (\varepsilon < 0), \end{cases}$$

where $\varepsilon \in \mathbf{R}$.

Let $\mathbf{R_I}$ and $\mathbf{R_I^+}$ be the collection of all and positive intervals of $\mathbf{R}$, respectively. The following will discuss several algebraic properties of interval arithmetic.

Let $\Omega = [\underline{\Omega}, \bar{\Omega}] \in \mathbf{R_I}$, then $\Omega_C = \frac{\overline{\Omega}+\underline{\Omega}}{2}$ and $\Omega_{\mathcal{R}} = \frac{\overline{\Omega}-\underline{\Omega}}{2}$ are the center-radius of interval Ω. A center-radius form of interval Ω can be expressed as:

$$\Omega = \langle \Omega_C, \Omega_{\mathcal{R}} \rangle = \left\langle \frac{\overline{\Omega} + \underline{\Omega}}{2}, \frac{\overline{\Omega} - \underline{\Omega}}{2} \right\rangle.$$

Following are the relationships we use to determine the radius and center of an interval:

Definition 2.1. *(*see [49]) The $C\mathcal{R}$-order relation for $\Omega = [\underline{\Omega}, \overline{\Omega}] = \langle \Omega_C, \Omega_{\mathcal{R}} \rangle$, $\sigma = [\underline{\sigma}, \overline{\sigma}] = \langle \sigma_C, \sigma_{\mathcal{R}} \rangle \in \mathbf{R_I}$ represented as (see Figure 1).

$$\Omega \preceq_{C\mathcal{R}} \sigma \Longleftrightarrow \begin{cases} \Omega_C < \sigma_C, & \text{if } \Omega_C \neq \sigma_C; \\ \Omega_{\mathcal{R}} \leq \sigma_{\mathcal{R}}, & \text{if } \Omega_C = \sigma_C. \end{cases}$$

For any two intervals $\Omega, \sigma \in \mathbf{R_I}$, we have either $\Omega \preceq_{C\mathcal{R}} \sigma$ or $\sigma \preceq_{C\mathcal{R}} \Omega$. Riemann integral for $\mathcal{IVFS}$ are represented as:

Definition 2.2. (see [49]) Let $\eta : [f, g]$ be an $\mathcal{IVF}$ such that $\eta = [\underline{\eta}, \overline{\eta}]$. Then η is Riemann integrable (**IR**) on $[f, g]$ iff $\underline{\eta}$ and $\overline{\eta}$ are Riemann integrable on $[f, g]$, that is,

$$(\mathbf{IR}) \int_f^g \eta(\mathrm{s})d\mathrm{s} = \left[(\mathbf{R}) \int_f^g \underline{\eta}(\mathrm{s})d\mathrm{s}, (\mathbf{R}) \int_f^g \overline{\eta}(\mathrm{s})ds\right].$$

The pack of all (**IR**) $\mathcal{IVFS}$ on $[f, g]$ is represented by $\mathbf{IR}_{([f,g])}$. The collection of all center-radius order interval-valued functions are denoted by $\mathcal{CR}$-$\mathcal{IVFS}$.

Shi et al. [49] proved that the integral preserves order by using $\mathcal{CR}$-order relations.

Theorem 2.1. (see [49]) Let $\eta, \zeta : [f, g]$ be $\mathcal{IVFS}$ given by $\eta = [\underline{\eta}, \overline{\eta}]$ and $\zeta = [\underline{\zeta}, \overline{\zeta}]$. If $\eta(\mathrm{s}) \preceq_{\mathcal{CR}} \zeta(\mathrm{s})$, for all $s \in [f, g]$, then

$$\int_f^g \eta(\mathrm{s})d\mathrm{s} \preceq_{\mathcal{CR}} \int_f^g \zeta(\mathrm{s})d\mathrm{s}.$$

To support the above Theorem, we will now provide an illustration and some interesting example (see Figure 2).

Example 2.1. *Conider* $\eta = [z, 2z]$ *and* $\zeta = [z^2, z^2 + 2]$*, then,* $\forall\ z \in [0, 1]$.

$$\eta_{\mathcal{C}} = \frac{3z}{2}, \eta_{\mathcal{R}} = \frac{z}{2}, \zeta_{\mathcal{C}} = z^2 + 1 \quad and \quad \zeta_{\mathcal{R}} = 1.$$

From Definition 2.1, we have $\eta(z) \preceq_{\mathcal{CR}} \zeta(z)$*,* $\forall\ z \in [0, 1]$.

Since,

$$\int_0^1 [z, 2z]dz = \left[\frac{1}{2}, 1\right]$$

and

$$\int_0^1 [z^2, z^2 + 2]dz = \left[\frac{1}{3}, \frac{7}{3}\right].$$

From Theorem 2.1, we have

$$\int_0^1 \eta(z)dz \preceq_{\mathcal{CR}} \int_0^1 \zeta(z)dz.$$

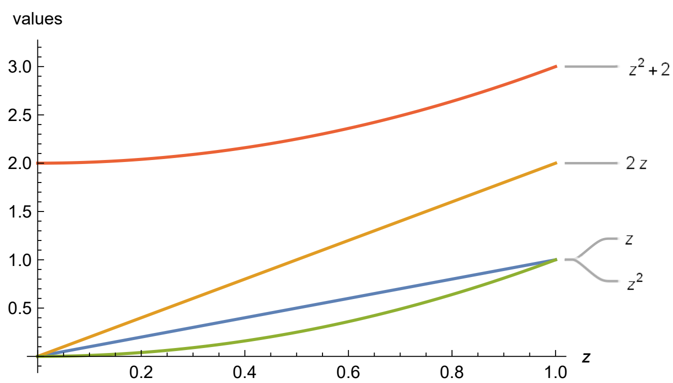

Figure 1. Graph shows that Definition 2.1 is valid.

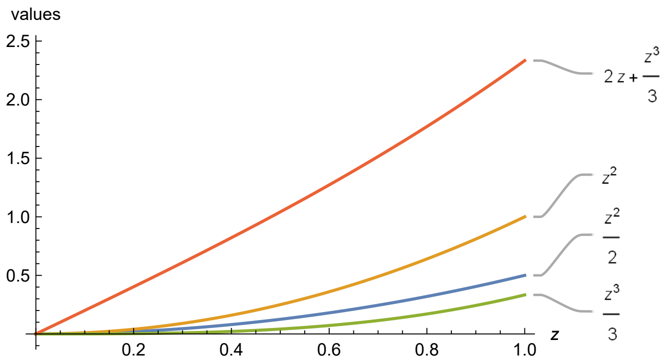

Figure 2. Graph shows that Theorem 2.1 holds.

2.1. New definitions and properties

Definition 2.3. *Consider* $(\Omega, \mathbb{A}, \mathbb{P})$ *be a probability space ($\mathcal{PBS}$). A function* $\zeta : \Omega \to \mathbf{R}$ *is said to be random variable if they satisfy the axioms of* $\mathbb{A}$*-measurable. A function* $\zeta : I \times \Omega \to \mathbf{R}$ *where* $I \subseteq \mathbf{R}$ *is called stochastic process if,* $\forall\, f \in I$ *the function* $\zeta(f, .)$ *is a random variable.*

2.1.1. Properties of stochastic process

A stochastic process $\zeta : I \times \Omega \to \mathbf{R}$ is

- Continuous in interval I, if $\forall\, f_o \in I$, we have
$$P - \lim_{f \to f_o} \zeta(f, .) = \zeta(f_o, .)$$
where $P - \lim$ represent the limit in probability space.
- Mean square continuous in interval I, if $\forall\, f_o \in I$, we have
$$\lim_{f \to f_o} \mathbf{E}\left[(\zeta(f, .) - \zeta(f_o, .))^2\right] = 0$$
where $\mathbf{E}\left[\zeta(f, .)\right]$ represent the expectation of random variable $\zeta(f, .)$.
- Mean-square differentiable at some point f, if one has random variable $\zeta' : I \times \Omega \to \mathbf{R}$, then this holds
$$\zeta'(f, .) = P - \lim_{f \to f_o} \frac{\zeta(f, .) - \zeta(f_o, .)}{f - f_o}.$$
- Mean square integral in interval I, if $\forall\, f \in I$, with $\mathbf{E}\left[\zeta(f, .)\right] < \infty$. Let $[f, g] \subseteq I$, $f = s_o < s_1 < s_2 ... < s_k$ is a partition of $[f, g]$. Consider $\zeta_n \in [s_{n-1}, s_n]$, $\forall\, n = 1, ..., k$. A random variable $S : \Omega \to \mathbf{R}$ is mean-square integral of the stochastic process ζ over interval $[f, g]$, if this holds
$$\lim_{k \to \infty} \mathbf{E}\left[\left(\sum_{n=1}^{k} \zeta(\zeta_n, .)(s_n - s_{n-1}) - S(.)\right)^2\right] = 0.$$
In that case, we write it as
$$S(.) = \int_f^g \zeta(s, .) ds \ (a.e). \tag{2.1}$$

Definition 2.4. *(See [49, 50]) Consider $h : [0,1] \to \mathbf{R}^+$. We say that $\zeta : [f, g] \to \mathbf{R}^+$ is called h-convex function, or that $\zeta \in SX(\mathcal{CR}\text{-}h, [f, g], \mathbf{R}^+)$, if $\forall\ f_1, g_1 \in [f, g]$ and $\varepsilon \in [0, 1]$, we have*

$$\zeta(\varepsilon f_1 + (1-\varepsilon)g_1) \leq\ h(\varepsilon)\zeta(f_1) + h(1-\varepsilon)\zeta(g_1). \tag{2.2}$$

In (2.2), if "$\leq$" is replaced with "$\geq$", then it is called h-concave function or $\zeta \in SV(\mathcal{CR}\text{-}h, [f, g], \mathbf{R}^+)$.

Definition 2.5. *(See [36]) Consider $h : [0,1] \to \mathbf{R}^+$. We say that $\zeta : I \times \Omega \to \mathbf{R}^+$ is called h-convex stochastic process, or that $\zeta \in SPX(\mathcal{CR}\text{-}h, I, \mathbf{R}^+)$, if $\forall\ f_1, g_1 \in I$ and $\varepsilon \in [0, 1]$, we have*

$$\zeta(\varepsilon f_1 + (1-\varepsilon)g_1, .) \leq\ h(\varepsilon)\zeta(f_1) + h(1-\varepsilon)\zeta(g_1, .). \tag{2.3}$$

In (2.3), if "$\leq$" is replaced with "$\geq$", then it is called h-concave stochastic process or $\zeta \in SPV(\mathcal{CR}\text{-}h, I, \mathbf{R}^+)$.

Definition 2.6. *(See [49, 50]) Consider $h : [0,1] \to \mathbf{R}^+$. We say that $\zeta = [\underline{\zeta}, \overline{\zeta}] : [f, g] \to \mathbf{R}^+_\mathbf{I}$ is called $\mathcal{CR}$-h-convex function, or that $\zeta \in SX(\mathcal{CR}\text{-}h, [f, g], \mathbf{R}^+_\mathbf{I})$, if $\forall\ f_1, g_1 \in [f, g]$ and $\varepsilon \in [0, 1]$, we have*

$$\zeta(\varepsilon f_1 + (1-\varepsilon)g_1) \preceq_{\mathcal{CR}}\ h(\varepsilon)\zeta(f_1) + h(1-\varepsilon)\zeta(g_1). \tag{2.4}$$

In (2.4), if "$\preceq_{\mathcal{CR}}$" is replaced with "$\succeq_{\mathcal{CR}}$", then it is called $\mathcal{CR}$-h-concave function or $\zeta \in SV(\mathcal{CR}\text{-}h, [f, g], \mathbf{R}^+_\mathbf{I})$.

Now let's introduce the concept for h-convex stochastic process for $\mathcal{CR}$-$\mathcal{IVFS}$

Definition 2.7. *(See [36, 50]) Consider $h : [0,1] \to \mathbf{R}^+$. We say that stochastic process $\zeta = [\underline{\zeta}, \overline{\zeta}] : I \times \Omega \to \mathbf{R}^+_\mathbf{I}$ where $[f, g] \subseteq I$ is called h-convex stochastic process for $\mathcal{CR}$-$\mathcal{IVFS}$ or that $\zeta \in SPX(\mathcal{CR}\text{-}h, [f, g], \mathbf{R}^+_\mathbf{I})$, if $\forall\ f_1, g_1 \in [f, g]$ and $\varepsilon \in [0, 1]$, we have*

$$\zeta(\varepsilon f_1 + (1-\varepsilon)g_1, .) \preceq_{\mathcal{CR}}\ h(\varepsilon)\zeta(f_1, .) + h(1-\varepsilon)\zeta(g_1, .). \tag{2.5}$$

In (2.5), if "$\preceq_{\mathcal{CR}}$" is replaced with "$\succeq_{\mathcal{CR}}$", then it is called h-concave stochastic process for $\mathcal{CR}$-$\mathcal{IVFS}$ or $\zeta \in SPV(\mathcal{CR}\text{-}h, [f, g], \mathbf{R}^+_\mathbf{I})$.

Remark 2.1. *(i) If $h = 1$, Definition 2.7 becomes a stochastic process for $\mathcal{CR}$-P-function.*
(ii) If $h(\varepsilon) = \frac{1}{\varepsilon}$, Definition 2.7 becomes a stochastic process for $\mathcal{CR}$-Godunova-Levin function.
(iii) If $h(\varepsilon) = \varepsilon$, Definition 2.7 becomes a stochastic process for $\mathcal{CR}$-convex function.
(iv) If $h = \varepsilon^s$, Definition 2.7 becomes a stochastic process for $\mathcal{CR}$-s-convex function.

3. Hermite-Hadamard inequality for $\mathcal{CR}$-h-convex stochastic process

Theorem 3.1. *Let $h : (0,1) \to \mathbf{R}^+$ and $h\left(\frac{1}{2}\right) \neq 0$. A function $\zeta : I \times \Omega \to \mathbf{R}^+_\mathbf{I}$ is h-convex stochastic process as well as mean square integrable for $\mathcal{CR}$-$\mathcal{IVFS}$. For every f,g $\in [f, g] \subseteq I$, if $\zeta \in SPX(\mathcal{CR}\text{-}h, [f, g], \mathbf{R}^+_\mathbf{I})$ and $\zeta \in\ \mathbf{R}^+_\mathbf{I}$. Almost everywhere, the following inequality is satisfied*

$$\frac{1}{2\left[h\left(\frac{1}{2}\right)\right]}\zeta\left(\frac{f+g}{2}, .\right) \preceq_{\mathcal{CR}} \frac{1}{g-f}\int_f^g \zeta(\varepsilon, .)d\varepsilon \preceq_{\mathcal{CR}} [\zeta(f, .) + \zeta(g, .)]\int_0^1 h(s)ds. \tag{3.1}$$

Proof. Since $\zeta \in SPX(\mathcal{CR}\text{-}h, [f,g], \mathbf{R}_{\mathbf{I}}^{+})$, and consequently integrate over $(0,1)$, we have

$$\begin{aligned}
&\frac{1}{\left[h\left(\frac{1}{2}\right)\right]}\zeta\left(\frac{f+g}{2},.\right) \preceq_{\mathcal{CR}} \zeta(sf+(1-s)g,.)+\zeta((1-s)f+sg,.)\\
&\frac{1}{\left[h\left(\frac{1}{2}\right)\right]}\zeta\left(\frac{f+g}{2},.\right) \preceq_{\mathcal{CR}} \left[\int_0^1 \zeta(sf+(1-s)g,.)ds+\int_0^1 \zeta((1-s)f+sg,.)ds\right]\\
&=\left[\int_0^1 \underline{\zeta}(sf+(1-s)g,.)ds+\int_0^1 \underline{\zeta}((1-s)f+sg,.)ds,\right.\\
&\quad\left.\int_0^1 \overline{\zeta}(sf+(1-s)g,.)ds+\int_0^1 \overline{\zeta}((1-s)f+sg,.)ds\right]\\
&=\left[\frac{2}{g-f}\int_f^g \underline{\zeta}(\varepsilon,.)d\varepsilon, \frac{2}{g-f}\int_f^g \overline{\zeta}(\varepsilon,.)d\varepsilon\right]\\
&=\frac{2}{g-f}\int_f^g \zeta(\varepsilon,.)d\varepsilon.
\end{aligned} \tag{3.2}$$

By Definition 2.7, we have

$$\zeta(sf+(1-s)g,.) \preceq_{\mathcal{CR}} h(s)\zeta(f,.)+h(1-s)\zeta(g,.).$$

Integration over $(0,1)$, we have

$$\int_0^1 \zeta(sf+(1-s)g,.)ds \preceq_{\mathcal{CR}} \zeta(f,.)\int_0^1 h(s)ds+\zeta(g,.)\int_0^1 h(1-s)ds.$$

Accordingly,

$$\frac{1}{g-f}\int_f^g \zeta(\varepsilon,.)d\varepsilon \preceq_{\mathcal{CR}} [\zeta(f,.)+\zeta(g,.)]\int_0^1 h(s)ds. \tag{3.3}$$

Now, combining (3.2) and (3.3), we get the required result

$$\frac{1}{2\left[h\left(\frac{1}{2}\right)\right]}\zeta\left(\frac{f+g}{2},.\right) \preceq_{\mathcal{CR}} \frac{1}{g-f}\int_f^g \zeta(\varepsilon,.)d\varepsilon \preceq_{\mathcal{CR}} [\zeta(f,.)+\zeta(g,.)]\int_0^1 h(s)ds.$$

□

Example 3.1. *Consider* $[f,g]=[0,1], h(s)=s, \forall\, s\in [0,1]$. *If* $\zeta:[f,g]\to \mathbf{R_I}^{+}$ *is defined as*

$$\zeta(\varepsilon,.)=[-2\varepsilon^2+3, 2\varepsilon^2+4], \quad \varepsilon\in[0,1].$$

Then,

$$\begin{aligned}
\frac{1}{2\left[h\left(\frac{1}{2}\right)\right]}\zeta\left(\frac{f+g}{2},.\right) &= \zeta\left(\frac{1}{2},.\right)=\left[\frac{5}{2},\frac{9}{2}\right],\\
\frac{1}{g-f}\int_f^g \zeta(\varepsilon,.)d\varepsilon &= \left[\int_0^1(-2\varepsilon^2+3)d\varepsilon, \int_0^1(2\varepsilon^2+4)d\varepsilon\right]=\left[\frac{7}{3},\frac{14}{3}\right],
\end{aligned}$$

$$[\zeta(f,.)+\zeta(g,.)]\int_0^1 h(s)ds=[2,5]\,.$$

As a result,

$$\left[\frac{5}{2},\frac{9}{2}\right]\preceq_{C\mathcal{R}}\left[\frac{7}{3},\frac{14}{3}\right]\preceq_{C\mathcal{R}}[2,5]\,.$$

This verify the above theorem.

Theorem 3.2. *Let $h:(0,1)\to\mathbf{R}^+$ and $h\left(\frac{1}{2}\right)\neq 0$. A function $\zeta: I\times\Omega\to\mathbf{R}_{\mathbf{I}}^+$ is h-convex stochastic process as well as mean square integrable for $C\mathcal{R}$-$\mathcal{IVFS}$. For every f,g $\in[f,g]\subseteq I$, if $\zeta\in SPX(C\mathcal{R}\text{-}h,[f,g],\mathbf{R}_{\mathbf{I}}^+)$ and $\zeta\in\ \mathbf{R}_{\mathbf{I}}^+$. Almost everywhere, the following inequality is satisfied*

$$\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\zeta\left(\frac{f+g}{2},.\right)\preceq_{C\mathcal{R}}\triangle_1\preceq_{C\mathcal{R}}\frac{1}{g-f}\int_f^g\zeta(\varepsilon,.)d\varepsilon\preceq_{C\mathcal{R}}\triangle_2$$

$$\preceq_{C\mathcal{R}}\left\{[\zeta(f,.)+\zeta(g,.)]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\right\}\int_0^1 h(s)ds,$$

where

$$\triangle_1=\frac{1}{4h\left(\frac{1}{2}\right)}\left[\zeta\left(\frac{3f+g}{4},.\right)+\zeta\left(\frac{3g+f}{4},.\right)\right],$$

$$\triangle_2=\left[\zeta\left(\frac{f+g}{2},.\right)+\frac{\zeta(f,.)+\zeta(g,.)}{2}\right]\int_0^1 h(s)ds.$$

Proof. Take $\left[f,\frac{f+g}{2}\right]$, we have

$$\zeta\left(\frac{3f+g}{4},.\right)\preceq_{C\mathcal{R}}h\left(\frac{1}{2}\right)\zeta\left(sf+(1-s)\frac{f+g}{2},.\right)+h\left(\frac{1}{2}\right)\zeta\left((1-s)f+s\frac{f+g}{2},.\right)$$

Integration over (0,1), we have

$$\begin{aligned}\zeta\left(\frac{3f+g}{2},.\right)&\preceq_{C\mathcal{R}}h\left(\frac{1}{2}\right)\left[\int_0^1\zeta\left(sf+(1-s)\frac{f+g}{2},.\right)ds+\int_0^1\zeta\left(s\frac{f+g}{2}+(1-s)g,.\right)ds\right]\\&=h\left(\frac{1}{2}\right)\left[\frac{2}{g-f}\int_f^{\frac{f+g}{2}}\zeta(\varepsilon,.)d\varepsilon+\frac{2}{g-f}\int_f^{\frac{f+g}{2}}\zeta(\varepsilon,.)d\varepsilon\right]\\&=h\left(\frac{1}{2}\right)\left[\frac{4}{g-f}\int_f^{\frac{f+g}{2}}\zeta(\varepsilon,.)d\varepsilon\right].\end{aligned}\tag{3.4}$$

Accordingly,

$$\frac{1}{4h\left(\frac{1}{2}\right)}\zeta\left(\frac{3f+g}{2},.\right)\preceq_{C\mathcal{R}}\frac{1}{g-f}\int_f^{\frac{f+g}{2}}\zeta(\varepsilon,.)d\varepsilon.\tag{3.5}$$

Similarly for interval $\left[\frac{f+g}{2}, g\right]$, we have

$$\frac{1}{4h\left(\frac{1}{2}\right)}\zeta\left(\frac{3g+f}{2},.\right) \preceq_{C\mathcal{R}} \frac{1}{g-f}\int_{\frac{f+g}{2}}^{g} \zeta(\varepsilon,.)d\varepsilon. \tag{3.6}$$

Adding inequalities (3.5) and (3.6), we get

$$\triangle_1 = \frac{1}{4h\left(\frac{1}{2}\right)}\left[\zeta\left(\frac{3f+g}{4},.\right)+\zeta\left(\frac{3g+f}{4},.\right)\right] \preceq_{C\mathcal{R}} \left[\frac{1}{g-f}\int_{f}^{g} \zeta(\varepsilon,.)d\varepsilon\right].$$

Now

$$\begin{aligned}
&\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\zeta\left(\frac{f+g}{2},.\right)\\
&= \frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\zeta\left(\frac{1}{2}\left(\frac{3f+g}{4},.\right)+\frac{1}{2}\left(\frac{3g+f}{4},.\right)\right)\\
&\preceq_{C\mathcal{R}} \frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\left[h\left(\frac{1}{2}\right)\zeta\left(\frac{3f+g}{4},.\right)+h\left(\frac{1}{2}\right)\zeta\left(\frac{3g+f}{4},.\right)\right]\\
&= \frac{1}{4h\left(\frac{1}{2}\right)}\left[\zeta\left(\frac{3f+g}{4},.\right)+\zeta\left(\frac{3g+f}{4},.\right)\right]\\
&= \triangle_1\\
&\preceq_{C\mathcal{R}} \frac{1}{4h\left(\frac{1}{2}\right)}\left\{h\left(\frac{1}{2}\right)\left[\zeta(f,.)+\zeta\left(\frac{f+g}{2},.\right)\right]+h\left(\frac{1}{2}\right)\left[\zeta(g,.)+\zeta\left(\frac{f+g}{2},.\right)\right]\right\}\\
&= \frac{1}{2}\left[\frac{\zeta(f,.)+\zeta(g,.)}{2}+\zeta\left(\frac{f+g}{2},.\right)\right]\\
&\preceq_{C\mathcal{R}} \left[\frac{\zeta(f,.)+\zeta(g,.)}{2}+\zeta\left(\frac{f+g}{2},.\right)\right]\int_0^1 h(s)ds\\
&= \triangle_2\\
&\preceq_{C\mathcal{R}} \left[\frac{\zeta(f,.)+\zeta(g,.)}{2}+h\left(\frac{1}{2}\right)\zeta(f,.)+h\left(\frac{1}{2}\right)\zeta(g,.)\right]\int_0^1 h(s)ds\\
&\preceq_{C\mathcal{R}} \left[\frac{\zeta(f,.)+\zeta(g,.)}{2}+h\left(\frac{1}{2}\right)[\zeta(f,.)+\zeta(g,.)]\right]\int_0^1 h(s)ds\\
&\preceq_{C\mathcal{R}} \left\{[\zeta(f,.)+\zeta(g,.)]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\right\}\int_0^1 h(s)ds.
\end{aligned}$$

□

Example 3.2. *Recall the Example 3.1, we have*

$$\frac{1}{4\left[h(\frac{1}{2})\right]^2}\zeta\left(\frac{f+g}{2},.\right)=\zeta\left(\frac{1}{2},.\right)=\left[\frac{5}{2},\frac{9}{2}\right],$$

$$\triangle_1 = \frac{1}{2}\left[\zeta\left(\frac{1}{4},.\right)+\zeta\left(\frac{3}{4},.\right)\right] = \left[\frac{19}{8},\frac{37}{8}\right],$$
$$\triangle_2 = \left[\frac{\zeta(0,.)+\zeta(1,.)}{2}+\zeta\left(\frac{1}{2},.\right)\right]\int_0^1 h(s)ds,$$
$$= \frac{1}{2}\left([2,5]+\left[\frac{5}{2},\frac{9}{2}\right]\right)$$
$$= \left[\frac{9}{4},\frac{19}{4}\right]$$

and

$$\left\{[\zeta(f,.)+\zeta(g,.)]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\right\}\int_0^1 h(s)ds = [2,5].$$

Thus, we obtain

$$\left[\frac{5}{2},\frac{9}{2}\right] \preceq_{CR} \left[\frac{19}{8},\frac{37}{8}\right] \preceq_{CR} \left[\frac{7}{3},\frac{14}{3}\right] \preceq_{CR} \left[\frac{9}{4},\frac{19}{4}\right] \preceq_{CR} [2,5].$$

This verify Theorem 3.2.

Theorem 3.3. *Let $h_1, h_2 : (0,1) \to \mathbf{R}^+$ and $h_1, h_2 \neq 0$. A functions $\zeta, \varphi : I \times \Omega \to \mathbf{R}_{\mathbf{I}}^+$ are h-convex stochastic process as well as mean square integrable for $\mathcal{CR}$-$\mathcal{IVFS}$. For every f,g $\in I$, if $\zeta \in SPX(\mathcal{CR}\text{-}h_1, [f,g], \mathbf{R}_{\mathbf{I}}^+)$, $\varphi \in SPX(\mathcal{CR}\text{-}h_2, [f,g], \mathbf{R}_{\mathbf{I}}^+)$ and $\zeta, \varphi \in \mathbf{IR}_I$. Almost everywhere, the following inequality is satisfied*

$$\frac{1}{g-f}\int_f^g \zeta(\varepsilon,.)\varphi(\varepsilon,.)d\varepsilon \preceq_{CR} M(f,g)\int_0^1 h_1(s)h_2(s)ds + N(f,g)\int_0^1 h_1(s)h_2(1-s)ds,$$

where

$$M(f,g) = \zeta(f,.)\varphi(f,.)+\zeta(g,.)\varphi(g,.), N(f,g) = \zeta(f,.)\varphi(g,.)+\zeta(g,.)\varphi(f,.).$$

Proof. Conider $\zeta \in SPX(\mathcal{CR}\text{-}h_1, [f,g], \mathbf{R}_{\mathbf{I}}^+)$, $\varphi \in SPX(\mathcal{CR}\text{-}h_2, [f,g], \mathbf{R}_{\mathbf{I}}^+)$ then, we have

$$\zeta(fs+(1-s)g,.) \preceq_{CR} h_1(s)\zeta(f,.)+h_1(1-s)\zeta(g,.),$$

$$\varphi(fs+(1-s)g,.) \preceq_{CR} h_2(s)\varphi(f,.)+h_2(1-s)\varphi(g,.).$$

Then,

$$\zeta(fs+(1-s)g,.)\,\varphi(fs+(1-s)g,.)$$
$$\preceq_{CR} (h(1-s)\zeta(f,.)+h(s)\zeta(g,.))\,(h(1-s)\varphi(f,.)+h(s)\varphi(g,.)).$$

Integration over (0,1), we have

$$\int_0^1 \zeta(fs+(1-s)g,.)\,\varphi(fs+(1-s)g,.)\,ds$$
$$= \left[\int_0^1 \underline{\zeta}(fs+(1-s)g,.)\,\underline{\varphi}(fs+(1-s)g,.)\,ds, \int_0^1 \overline{\zeta}(fs+(1-s)g,.)\,\overline{\varphi}(fs+(1-s)g,.)\,ds\right]$$

$$= \left[\frac{1}{g-f}\int_f^g \underline{\zeta}(\varepsilon,.)\underline{\varphi}(\varepsilon,.)d\varepsilon, \frac{1}{g-f}\int_f^g \overline{\zeta}(\varepsilon,.)\overline{\varphi}(\varepsilon,.d\varepsilon\right]$$
$$= \frac{1}{g-f}\int_f^g \zeta(\varepsilon,.)\varphi(\varepsilon,.)d\varepsilon$$
$$\preceq_{\mathcal{CR}} M(f,g)\int_0^1 h_1(s)h_2(s)ds + N(f,g)\int_0^1 h_1(s)h_2(1-s)ds.$$

It follows that

$$\frac{1}{g-f}\int_f^g \zeta(\varepsilon,.)\varphi(\varepsilon,.)d\varepsilon \preceq_{\mathcal{CR}} M(f,g)\int_0^1 h_1(s)h_2(s)ds + N(f,g)\int_0^1 h_1(s)h_2(1-s)ds.$$

Theorem is proved. □

Example 3.3. *Let* $[f,g]=[0,1], h_1(s)=s$, $h_2(s)=1$ *for all* $s\in(0,1)$. *If* $\zeta,\varphi:[f,g]\subseteq I\to \mathbf{R_I}^+$ *are defined as*

$$\zeta(\varepsilon,.)=[\varepsilon^2,\varepsilon^3+1] \ \ and \ \ \varphi(\varepsilon,.)=[\varepsilon^2,\varepsilon+2].$$

Then, we have

$$\frac{1}{g-f}\int_f^g \zeta(\varepsilon,.)\varphi(\varepsilon,.)d\varepsilon = \left[\frac{1}{5},\frac{16}{5}\right],$$
$$M(f,g)\int_0^1 h_1(s)h_2(s)ds = M(0,1)\int_0^1 sds = \left[\frac{1}{2},4\right]$$

and

$$N(f,g)\int_0^1 h_1(s)h_2(1-s)ds = N(0,1)\int_0^1 sds = \left[0,\frac{7}{2}\right].$$

Since

$$\left[\frac{1}{5},\frac{16}{5}\right] \preceq_{\mathcal{CR}} \left[\frac{1}{2},4\right]+\left[0,\frac{7}{2}\right]=\left[\frac{1}{2},\frac{15}{2}\right].$$

Consequently, Theorem 3.3 is verified.

Theorem 3.4. *Let* $h_1,h_2:(0,1)\to\mathbf{R}^+$ *and* $h_1,h_2\neq 0$. *A functions* $\zeta,\varphi: I\times\Omega\to\mathbf{R}_{\mathbf{I}}^+$ *are h-convex stochastic process as well as mean square integrable for* $\mathcal{CR}$-$\mathcal{IVFS}$. *For every f,g* $\in I$, *if* $\zeta\in SPX(\mathcal{CR}\text{-}h_1,[f,g],\mathbf{R}_{\mathbf{I}}^+)$, $\varphi\in SPX(\mathcal{CR}\text{-}h_2,[f,g],\mathbf{R}_{\mathbf{I}}^+)$ *and* $\zeta,\varphi\in$ $\mathbf{IR}_I$. *Almost everywhere, the following inequality is satisfied*

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\zeta\left(\frac{f+g}{2},.\right)\varphi\left(\frac{f+g}{2},.\right)$$
$$\preceq_{\mathcal{CR}} \frac{1}{g-f}\int_f^g \zeta(\varepsilon,.)\varphi(\varepsilon,.)d\varepsilon + M(f,g)\int_0^1 h_1(s)h_2(1-s)ds + N(f,g)\int_0^1 h_1(s)h_2(s)ds.$$

Proof. Since $\zeta\in SPX(\mathcal{CR}\text{-}h_1,[f,g],\mathbf{R}_{\mathbf{I}}^+)$, $\varphi\in SPX(\mathcal{CR}\text{-}h_2,[f,g],\mathbf{R}_{\mathbf{I}}^+)$, we have

$$\zeta\left(\frac{f+g}{2},.\right)\preceq_{\mathcal{CR}} h_1\left(\frac{1}{2}\right)\zeta\left(fs+(1-s)g,.\right)+h_1\left(\frac{1}{2}\right)\zeta\left(f(1-s)+sg,.\right),$$

$$\varphi\left(\frac{f+g}{2},.\right) \preceq_{CR} h_2\left(\frac{1}{2}\right)\varphi(fs+(1-s)g,.)+h_2\left(\frac{1}{2}\right)\varphi(f(1-s)+sg,.). \tag{3.7}$$

$$\begin{aligned}
&\zeta\left(\frac{f+g}{2},.\right)\varphi\left(\frac{f+g}{2},.\right)\\
&\preceq_{CR} h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[\zeta(fs+(1-s)g,.)\varphi(fs+(1-s)g,.)+\zeta(f(1-s)+sg,.)\varphi(f(1-s)+sg,.)]\\
&+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[\zeta(fs+(1-s)g,.)\varphi(f(1-s)+sg,.)+\zeta(f(1-s)+sg,.)\varphi(fs+(1-s)g,.)]\\
&\preceq_{CR} h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[\zeta(fs+(1-s)g,.)\varphi(fs+(1-s)g,.)+\zeta(f(1-s)+sg,.)\varphi(f(1-s)+sg,.)]\\
&+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[(h_1(s)\zeta(f,.)+h_1(1-s)\zeta(g,.))(h_2(1-s)\varphi(f,.)+h_2(s)\varphi(g,.))]\\
&+[(h_1(1-s)\zeta(f,.)+h_1(s)\zeta(g,.))(h_2(s)\varphi(f,.)+h_2(1-s)\varphi(g,.))]\\
&\preceq_{CR} h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[\zeta(fs+(1-s)g,.)\varphi(fs+(1-s)g,.)+\zeta(f(1-s)+sg,.)\varphi(f(1-s)+sg,.)]\\
&+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[(h_1(s)h_2(1-s)+h_1(1-s)h_2(s))M(f,g)+(h_1(s)h_2(s)+h_1(1-s)h_2(1-s))N(f,g)].
\end{aligned}$$

Integration over $(0,1)$, we have

$$\begin{aligned}
&\int_0^1 \zeta\left(\frac{f+g}{2},.\right)\varphi\left(\frac{f+g}{2},.\right)ds=\left[\int_0^1 \underline{\zeta}\left(\frac{f+g}{2},.\right)\underline{\varphi}\left(\frac{f+g}{2},.\right)ds, \int_0^1 \overline{\zeta}\left(\frac{f+g}{2},.\right)\overline{\varphi}\left(\frac{f+g}{2},.\right)ds\right]\\
&=\zeta\left(\frac{f+g}{2},.\right)\varphi\left(\frac{f+g}{2},.\right)ds \preceq_{CR} 2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\frac{1}{g-f}\int_f^g \zeta(\varepsilon,.)\varphi(\varepsilon,.)d\varepsilon\right]\\
&+2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[M(f,g)\int_0^1 h_1(s)h_2(1-s)ds+N(f,g)\int_0^1 h_1(s)h_2(s)ds\right].
\end{aligned}$$

Divide both sides by $\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}$ above in equation, we get the required result

$$\begin{aligned}
&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\zeta\left(\frac{f+g}{2},.\right)\varphi\left(\frac{f+g}{2},.\right)\\
&\preceq_{CR} \frac{1}{g-f}\int_f^g \zeta(\varepsilon,.)\varphi(\varepsilon,.)d\varepsilon+M(f,g)\int_0^1 h_1(s)h_2(1-s)ds+N(f,g)\int_0^1 h_1(s)h_2(s)ds.
\end{aligned}$$

As a result, the proof is completed.

□

Example 3.4. *Recall the Example 3.3, we have*

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\zeta\left(\frac{f+g}{2},.\right)\varphi\left(\frac{f+g}{2},.\right)=\zeta\left(\frac{3}{2},.\right)\varphi\left(\frac{3}{2},.\right)=\left[\frac{-21}{8},\frac{147}{8}\right],$$

$$\frac{1}{g-f}\int_f^g \zeta(\varepsilon,.)\varphi(\varepsilon,.)d\varepsilon = \left[\frac{5}{12},\frac{227}{12}\right],$$

$$M(f,g)\int_0^1 h_1(s)h_2(1-s)ds = M(1,2)\int_0^1 sds = [-4,20]$$

and

$$N(f,g)\int_0^1 h_1(s)h_2(s)ds = N(1,2)\int_0^1 sds = \left[-5,\frac{29}{2}\right].$$

It follows that

$$\left[\frac{-21}{8},\frac{147}{8}\right] \preceq_{CR} \left[\frac{5}{12},\frac{227}{12}\right] + [-4,20] + \left[-5,\frac{29}{2}\right] = \left[\frac{-103}{12},\frac{641}{12}\right].$$

This proves the above theorem.

3.1. Ostrowski type inequality

Here is a lemma to help us accomplish our objective [53].

Lemma 3.1. *Define $\zeta : I\times\Omega \subseteq \mathbf{R} \to \mathbf{R}$ be a stochastic process which is mean square differentiable on the interior of interval I. Also, if the derivative of ζ is mean square integrable on interval $[f,g]$, and $f,g\in I$, then this holds:*

$$\begin{aligned}&\zeta(z,.) - \frac{1}{g-f}\int_f^g \zeta(s,.)ds\\ &= \frac{(z-f)^2}{g-f}\int_0^1 s\zeta'(sz+(1-s)f,.)ds - \frac{(g-z)^2}{g-f}\int_0^1 s\zeta'(sz+(1-s)g,.)ds,\ \forall z\in[f,g].\end{aligned}$$

Theorem 3.5. *Define $h:(0,1)\to\mathbf{R}$ be a super-multiplicative as well as nonnegative function with having the property that $s\le h(s)$ for each $s\in(0,1)$. Let $\zeta : I\times\Omega\subseteq\mathbf{R}\to\mathbf{R}$ be a stochastic process which is mean square differentiable on the interior of interval I. Also, the derivative of η is mean square integrable on interval $[f,g]$, and $f,g\in I$. If $|\eta'|$ is h- convex stochastic process for $\mathcal{CR}$-$\mathcal{IVFS}$ on I, with holding this property $|\zeta'(z,.)|\le\beta$ for each z, then*

$$\left|\zeta(z,.) - \frac{1}{g-f}\int_f^g \zeta(s,.)ds\right| \preceq_{CR} \frac{\beta\left[(z-f)^2+(g-z)^2\right]}{g-f}\int_0^1 \left[h(s^2)+h(s-s^2)\right]ds$$

$\forall\ z\in[f,g]$.

Proof. From Lemma 3.1, we have $|\zeta'|$ is h- convex stochastic process for $\mathcal{CR}$-$\mathcal{IVFS}$, then

$$\begin{aligned}&\left|\zeta(z,.) - \frac{1}{g-f}\int_f^g \zeta(s,.)ds\right|\\ &\preceq_{CR} \frac{(z-f)^2}{g-f}\int_0^1 s\,|\zeta'(sz+(1-s)f,.)|\,ds + \frac{(g-z)^2}{g-f}\int_0^1 s\,|\zeta'(sz+(1-s)g,.)|\,ds\\ &\preceq_{CR} \frac{(z-f)^2}{g-f}\int_0^1 s\left[h(s)|\zeta'(z,.)| + h(1-s)|\zeta'(f,.),.|\right]ds\end{aligned}$$

$$+\frac{(g-z)^2}{g-f}\int_0^1 s[h(s)|\zeta'(z,.)|+h(1-s)|\zeta'(g,.),.|]ds$$

$$\preceq_{C\mathcal{R}} \frac{\beta(z-f)^2}{g-f}\int_0^1 \left[h^2(s)+h(s)h(1-s)\right]ds+\frac{(g-z)^2}{g-f}\int_0^1 \left[h^2(s)+h(s)h(1-s)\right]ds$$

$$\preceq_{C\mathcal{R}} \frac{\beta\left[(z-f)^2+(g-z)^2\right]}{g-f}\int_0^1 \left[h^2(s)+h(s)h(1-s)\right]ds.$$

The proof is completed. □

3.2. Jensen type inequality

Theorem 3.6. *Let $s_i \in \mathbf{R}^+$. If h is super multiplicative non-negative function and $\zeta : I \times \Omega \to \mathbf{R}$ is non-negative h-convex stochastic process for $\mathcal{CR}$-$\mathcal{IVFS}$ or we say that $\zeta \in SPX(h, I, \mathbf{R}_{\mathbf{I}}^+)$ with $z_i \in I$, then this holds*

$$\zeta\left(\frac{1}{S_k}\sum_{i=1}^{k} s_i z_i,.\right)\preceq_{C\mathcal{R}}\sum_{i=1}^{k}\left[h\left(\frac{s_i}{S_k}\right)\zeta(z_i,.)\right], \tag{3.8}$$

where $S_k = \sum_{i=1}^{k} s_i$.

By mathematical induction when $k = 2$, then Eq (3.8) is true. Suppose that Eq (3.8) holds for $k-1$, then,

$$\zeta\left(\frac{1}{S_k}\sum_{i=1}^{k} s_i z_i,.\right)=\zeta\left(\frac{s_k}{S_k}z_k+\sum_{i=1}^{k-1}\frac{s_i}{S_k}z_i,.\right)$$

$$=\zeta\left(\frac{s_k}{S_k}z_k+\frac{S_{k-1}}{S_k}\sum_{i=1}^{k-1}\frac{s_i}{S_{k-1}}z_i,.\right)$$

$$\preceq_{C\mathcal{R}} h\left(\frac{s_k}{S_k}\right)\zeta(z_k,.)+h\left(\frac{S_{k-1}}{S_k}\right)\zeta\left(\sum_{i=1}^{k-1}\frac{s_i}{S_{k-1}}z_i,.\right)$$

$$\preceq_{C\mathcal{R}} h\left(\frac{s_k}{S_k}\right)\zeta(z_k,.)+h\left(\frac{S_{k-1}}{S_k}\right)\sum_{i=1}^{k-1}\left[h\left(\frac{s_i}{S_{k-1}}\right)\zeta(z_i,.)\right]$$

$$\preceq_{C\mathcal{R}} h\left(\frac{s_k}{S_k}\right)\zeta(z_k,.)+\sum_{i=1}^{k-1}\left[h\left(\frac{s_i}{S_k}\right)\zeta(z_i,.)\right]$$

$$\preceq_{C\mathcal{R}} \sum_{i=1}^{k}\left[h\left(\frac{s_i}{S_k}\right)\zeta(z_i,.)\right].$$

Hence proved by mathematical induction

4. Conclusions

A center-radius order relation is introduced in this manuscript by considering h-convex stochastic processes for $\mathcal{IVFS}$. Using these notions, we developed inequalities of the Ostrwoski-type, Jensen-type, and $\mathcal{H}.\mathcal{H}$ types. A distinguishing feature of this order relation is that inequality terms derived from it produce precise results. Moreover, we generalize the findings of following authors [36, 46, 48, 50], in this article, which is a new approach for future study. Additionally, the study provides interesting examples to prove the validity of theorems. It is possible to use these ideas to take convex optimization to a new level. This concept should be useful to researchers working in a variety of scientific fields. In the future, researchers might look at determining equivalent inequalities using different integral operators for different types of convexity.

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Conflict of interest

The authors declare no conflicts of interest.

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