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Research article

On the rationality of generating functions of certain hypersurfaces over finite fields

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Abstract: Let a, n be positive integers and let p be a prime number. Let $\mathbb{F}_q$ be the finite field with $q = p^a$ elements. Let $\{a_i\}_{i=1}^{\infty}$ be an arbitrary given infinite sequence of elements in $\mathbb{F}_q$ and $a_1 \neq 0$. For each positive integer i, let $\{d_{i+j,i}\}_{j=0}^{\infty}$ be an arbitrary given sequence of positive integers with d_{ii} coprime to $q-1$. For each integer $n \geq 1$, let N_n, $\bar{N}_n$ and $\widetilde{N}_n$ denote the number of $\mathbb{F}_q$-rational points of the hypersurfaces defined by the following three equations:

$$a_1x_1 + \cdots + a_nx_n = b,$$

$$x_1^2 + \cdots + x_n^2 = b$$

and

$$a_1x_1^{d_{11}} + a_2x_1^{d_{21}}x_2^{d_{22}} + \cdots + a_nx_1^{d_{n1}}x_2^{d_{n2}} \cdots x_n^{d_{nn}} = b,$$

respectively. In this paper, we show that the generating function $\sum_{n=1}^{\infty} N_nt^n$ is a rational function in t. Moreover, we show that if p is an odd prime, then the generating functions $\sum_{n=1}^{\infty} \bar{N}_nt^n$ and $\sum_{n=1}^{\infty} \widetilde{N}_nt^n$ are both rational functions in t. Moreover, we present the explicit rational expressions of $\sum_{n=1}^{\infty} N_nt^n$, $\sum_{n=1}^{\infty} \bar{N}_nt^n$ and $\sum_{n=1}^{\infty} \widetilde{N}_nt^n$, respectively.

Keywords: generating function; rationality; hypersurface; finite field
Mathematics Subject Classification: 11T06, 11T24

1. Introduction

Let a and n be positive integers and let p be a prime number. Let $\mathbb{F}_q$ be the finite field with $q = p^a$ elements and let $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$ be its multiplicative group. Let $f(x_1, ..., x_n)$ be a polynomial over $\mathbb{F}_q$.

We set $N_n(f=0)$ to be the number of rational points $(x_1,...,x_n)\in\mathbb{F}_q^n$ on the hypersurface defined by the equation $f(x_1,...,x_n)=0$. That is, we have

$$N_n(f=0):=\sharp\{(x_1,...,x_n)\in\mathbb{F}_q^n|f(x_1,...,x_n)=0\}.$$

It is well known that there exists an exact formula (see, for example, pp. 275-289 of [13]) for the number $N_n(f=0)$ when $\deg(f)\le 2$. Moreover, finding the formula for $N_n(f=0)$ and related topic has attracted lots of authors for many years (see, for instance, [1, 2, 4, 6, 7, 8, 10, 22]). Generally speaking, it is difficult to present an explicit formula for $N_n(f=0)$. There are many authors investigating the nonnegative integer $N_n(f=0)$ by considering the rationality of the generating function of the sequence $\{N_n(f=0)\}_{n=1}^{\infty}$.

Let $\mathbb{Z}^+$ denote the set of positive integers. For the diagonal hypersurface

$$a_1{x_1}^{d_1}+\cdots+a_n{x_n}^{d_n}=b, a_i\in\mathbb{F}_q^*, b\in\mathbb{F}_q, d_i\in\mathbb{Z}^+,$$

much work has been done to find the number of its $\mathbb{F}_q$-rational points (see, for example, [3, 11, 12, 15, 16, 18, 19, 20, 23]). For any $z\in\mathbb{F}_q$ and any $e\in\mathbb{Z}^+$, we denote by

$$N_n^{(e)}(z):=N_n({x_1}^e+\cdots+{x_n}^e=z)$$

the number of $\mathbb{F}_q$-rational points of e-th diagonal hypersurface $x_1^e+\cdots+x_n^e=z$. In 1977, Chowla, Cowles and Cowles [5] proved that the generating function $\sum_{n=1}^{\infty}N_n^{(3)}(0)t^n$ is a rational function of t. In 1979, Myerson [14] found that the generating function $\sum_{n=1}^{\infty}N_n^{(4)}(0)t^n$ is a rational function of t. In 2021, Hong and Zhu [9] proved that the generating function $\sum_{n=1}^{\infty}N_n^{(3)}(z)t^n$ is rational and also gave its explicit rational expression. In 2022, Zhao, Feng, Hong and Zhu [21] studied the rationality of the generating function of the sequence $\{N_n^{(4)}(z)\}_{n=1}^{\infty}$, and proved that $\sum_{n=1}^{\infty}N_n^{(4)}(z)t^n$ is a rational function in t and gave its explicit expression.

Let $\{a_i\}_{i=1}^{\infty}$ be an arbitrary given infinite sequence of elements in $\mathbb{F}_q$ and $a_1\neq 0$. For each positive integer i, let $\{d_{i+j,i}\}_{j=0}^{\infty}$ be an arbitrary given sequence of positive integers with d_{ii} coprime to $q-1$. For each integer $n\ge 1$, let N_n, $\bar{N}_n$ and $\widetilde{N}_n$ denote the number of rational points $(x_1,...,x_n)\in\mathbb{F}_q^n$ of the hypersurfaces defined by the following three equations:

$$a_1x_1+\cdots+a_nx_n=b,$$

$$x_1^2+\cdots+x_n^2=b$$

and

$$a_1{x_1}^{d_{11}}+a_2x_1^{d_{21}}x_2^{d_{22}}+\cdots+a_nx_1^{d_{n1}}\cdots x_n^{d_{nn}}=b, \tag{1.1}$$

respectively. The exact formulas of N_n and $\bar{N}_n$ can be found in [11]. Furthermore, Wang and Sun [17] studied the equation (1.1), and gave the explicit formula of $\widetilde{N}_n$ under the restriction of p being odd. Although the explicit formulas for N_n, $\bar{N}_n$ and $\widetilde{N}_n$ are known, it is unclear whether or not the generating functions of the three sequences $\{N_n\}_{n=1}^{\infty}$, $\{\bar{N}_n\}_{n=1}^{\infty}$ and $\{\widetilde{N}_n\}_{n=1}^{\infty}$ are rational. If the answer is positive, can we give their explicit expressions?

In this paper, we mainly explore the above mentioned problem. In fact, we will show that the generating function $\sum_{n=1}^{\infty}N_nt^n$ is a rational function in t. Furthermore, if p is an odd prime, then we show that the generating functions $\sum_{n=1}^{\infty} \bar{N}_n t^n$ and $\sum_{n=1}^{\infty} \widetilde{N}_n t^n$ are rational functions in t. In other words, the main results of this paper can be stated as follows.

Theorem 1.1 *Let $\mathbb{F}_q$ be the finite field of q elements and $b \in \mathbb{F}_q$. Let $\{a_i\}_{i=1}^{\infty}$ be an arbitrary given infinite sequence of elements in $\mathbb{F}_q$ and $a_1 \in \mathbb{F}_q^*$. Let N_n denote the number of rational points $(x_1, ..., x_n) \in \mathbb{F}_q^n$ on the hyperplane $a_1x_1 + \cdots + a_nx_n = b$. Then the generating function $\sum_{n=1}^{\infty} N_n t^n$ is a rational function in t and*

$$\sum_{n=1}^{\infty} N_n t^n = \frac{t}{1 - qt}.$$

Theorem 1.2 *Let $\mathbb{F}_q$ be the finite field of q elements and odd characteristic. For any $b \in \mathbb{F}_q$, let $\bar{N}_n$ denote the number of rational points $(x_1, ..., x_n) \in \mathbb{F}_q^n$ on the hypersurface*

$$x_1^2 + \cdots + x_n^2 = b.$$

Then the generating function $\sum_{n=1}^{\infty} \bar{N}_n t^n$ is a rational function in t. Moreover, one has

$$\sum_{n=1}^{\infty} \bar{N}_n t^n = \frac{t + qt^2}{1 - (qt)^2} + \frac{t\eta(b) + v(b)t^2\eta(-1)}{1 - qt^2\eta(-1)},$$

where η is the quadratic character of $\mathbb{F}_q$ and the integer-valued function v on $\mathbb{F}_q$ is defined by

$$v(b) := \begin{cases} q - 1, & if\ b = 0, \\ -1, & if\ b \in \mathbb{F}_q^*. \end{cases} \tag{1.2}$$

Theorem 1.3 *Let $\mathbb{F}_q$ be the finite field of q elements and odd characteristic and $b \in \mathbb{F}_q$. Let $\{a_i\}_{i=1}^{\infty}$ be an arbitrary given infinite sequence of elements in $\mathbb{F}_q^*$. For each positive integer i, let $\{d_{i+j,i}\}_{j=0}^{\infty}$ be an arbitrary given sequence of positive integers with d_{ii} coprime to $q - 1$. Let $\widetilde{N}_n$ represent the number of rational points $(x_1, ..., x_n) \in \mathbb{F}_q^n$ on the hypersurface*

$$a_1x_1^{d_{11}} + a_2x_1^{d_{21}}x_2^{d_{22}} + \cdots + a_nx_1^{d_{n1}}x_2^{d_{n2}} \cdots x_n^{d_{nn}} = b.$$

Then the generating function $\sum_{n=1}^{\infty} \widetilde{N}_n t^n$ is a rational function in t. Furthermore, one has

$$\sum_{n=1}^{\infty} \widetilde{N}_n t^n = \begin{cases} \dfrac{t(1 + qt)}{(1 + t)(1 - qt)}, & if\ b = 0, \\ \dfrac{t}{(1 + t)(1 - qt)}, & if\ b \neq 0. \end{cases}$$

This paper is organized as follows. We present in Section 2 several preliminary lemmas that are needed in the proofs of Theorems 1.1 to 1.3. Finally, Section 3 is devoted to the proofs of Theorems 1.1 to 1.3.

2. Auxiliary lemmas

In this section, we present three lemmas which are needed in the proofs of Theorems 1.1 to 1.3.

Lemma 2.1 [13] *Let $\mathbb{F}_q$ be the finite field with q elements. Let N_n denote the number of rational points $(x_1, ..., x_n) \in \mathbb{F}_q^n$ on the hyperplane*

$$a_1x_1 + \cdots + a_nx_n = b$$

with $a_1, ..., a_n \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$. Then $N_n = q^{n-1}$.

Lemma 2.2 [13] *Let $\mathbb{F}_q$ be the finite field with q elements and q being odd. Let $\bar{N}_n(a_1, ..., a_n)$ denote the number of rational points $(x_1, ..., x_n) \in \mathbb{F}_q^n$ on the hypersurface*

$$a_1x_1^2 + \cdots + a_nx_n^2 = b$$

with $a_1, ..., a_n \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$. Then

$$\bar{N}_n(a_1, ..., a_n) = \begin{cases} q^{n-1} + v(b)q^{(n-2)/2}\eta((-1)^{n/2}a_1 \cdots a_n) & if\ 2|n, \\ q^{n-1} + q^{(n-1)/2}\eta((-1)^{(n-1)/2}ba_1 \cdots a_n) & if\ 2 \nmid n, \end{cases}$$

where $v(b)$ is defined as in (1.2) and η is the quadratic character of $\mathbb{F}_q$.

Lemma 2.3 [17] *Let $\mathbb{F}_q$ be the finite field of q elements and odd characteristic and $b \in \mathbb{F}_q$. Let $\{a_i\}_{i=1}^{\infty}$ be an arbitrary given infinite sequence of elements in $\mathbb{F}_q^*$. For any positive integer i, let*

$$\{d_{i+j,i}\}_{j=0}^{\infty}$$

be an arbitrary given sequence of positive integers with d_{ii} being coprime to $q-1$. Let $\widetilde{N}_n$ represent the number of rational points $(x_1, ..., x_n) \in \mathbb{F}_q^n$ on the hypersurface

$$a_1x_1^{d_{11}} + a_2x_1^{d_{21}}x_2^{d_{22}} + \cdots + a_nx_1^{d_{n1}}x_2^{d_{n2}} \cdots x_n^{d_{nn}} = b.$$

Then

$$\widetilde{N}_n = \begin{cases} (-1)^{n-1} + 2\sum\limits_{k=1}^{n-1}(-1)^{n-(k+1)}q^k & if\ b = 0, \\ \sum\limits_{k=0}^{n-1}(-1)^{n-(k+1)}q^k & if\ b \neq 0. \end{cases}$$

3. Proofs of Theorems 1.1 to 1.3

In this section, we present the proofs of Theorems 1.1 to 1.3. It is well known that the following identity on the formal power series is true:

$$\sum_{i=0}^{\infty} u^i = \frac{1}{1-u}.$$

This fact will be used freely in what follows.

First of all, we show Theorem 1.1.

Proof of Theorem 1.1. At first, by Lemma 2.1, one has

$$N_n = q^{n-1}.$$

Then we deduce that

$$\begin{aligned}&\sum_{n=1}^{\infty} N_n t^n\\ =&\sum_{n=1}^{\infty} q^{n-1} t^n\\ =&t\sum_{n=0}^{\infty}(qt)^n\\ =&\frac{t}{1-qt}\end{aligned}$$

as desired. That is, the generating function $\sum_{n=1}^{\infty} N_n t^n$ is a rational function in t.

This concludes the proof of Theorem 1.1. □

Consequently, we present the proof of Theorem 1.2.

Proof of Theorem 1.2. First of all, from Lemma 2.2 we can derive that for any positive integer n, one has

$$\bar{N}_n = q^{n-1} + v(b)q^{(n-2)/2}\eta((-1)^{n/2}) \tag{3.1}$$

if $2|n$, and

$$\bar{N}_n = q^{n-1} + q^{(n-1)/2}\eta((-1)^{(n-1)/2}b) \tag{3.2}$$

if $2 \nmid n$.

On the one hand, we have

$$\begin{aligned}&\sum_{n=1}^{\infty} \bar{N}_n t^n\\ =&\sum_{n=1}^{\infty}(\bar{N}_{2n}t^{2n} + \bar{N}_{2n-1}t^{2n-1})\\ =&\sum_{n=1}^{\infty} \bar{N}_{2n}t^{2n} + \sum_{n=1}^{\infty} \bar{N}_{2n-1}t^{2n-1}.\end{aligned} \tag{3.3}$$

On the other hand, using (3.1) and (3.2), one arrives at

$$\begin{aligned}
&\sum_{n=1}^{\infty} \bar{N}_{2n} t^{2n} \\
=&\sum_{n=1}^{\infty} (q^{2n-1} + v(b) q^{\frac{2n-2}{2}} \eta((-1)^{\frac{2n}{2}})) t^{2n} \\
=&\sum_{n=1}^{\infty} q^{2n-1} t^{2n} + v(b) \sum_{n=1}^{\infty} q^{n-1} \eta((-1)^n) t^{2n} \\
=&\sum_{n=1}^{\infty} q^{-1} q^{2n} t^{2n} + v(b) \sum_{n=1}^{\infty} q^{-1} q^n \eta((-1)^n)(t^2)^n \\
=&\frac{1}{q} \sum_{n=1}^{\infty} (qt)^{2n} + \frac{1}{q} v(b) \sum_{n=1}^{\infty} (qt^2 \eta(-1))^n \\
=&\frac{qt^2}{1-(qt)^2} + \frac{v(b) t^2 \eta(-1)}{1 - qt^2 \eta(-1)}
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} \bar{N}_{2n-1} t^{2n-1} \\
=&\sum_{n=1}^{\infty} \left(q^{2n-2} + q^{\frac{(2n-2)}{2}} \eta((-1)^{\frac{2n-2}{2}} b)\right) t^{2n-1} \\
=&\sum_{n=1}^{\infty} q^{2n-2} t^{2n-1} + \sum_{n=1}^{\infty} q^{n-1} \eta((-1)^{n-1} b) t^{2n-1} \\
=&t \sum_{n=1}^{\infty} (qt)^{2n-2} + \frac{\eta(b)}{qt\eta(-1)} \sum_{n=1}^{\infty} (qt^2 \eta(-1))^n \\
=&\frac{t}{1-(qt)^2} + \frac{t\eta(b)}{1 - qt^2 \eta(-1)}.
\end{aligned} \tag{3.5}$$

Finally, putting (3.4) and (3.5) into (3.3) gives us that

$$\sum_{n=1}^{\infty} \bar{N}_n t^n = \frac{t + qt^2}{1-(qt)^2} + \frac{t\eta(b) + v(b) t^2 \eta(-1)}{1 - qt^2 \eta(-1)}$$

as required. Thus the generating function $\sum_{n=1}^{\infty} \bar{N}_n t^n$ is a rational function in t.

This completes the proof of Theorem 1.2. □

In concluding this paper, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. At first, by using Lemma 2.3, we know that

$$\widetilde{N}_n = (-1)^{n-1} + 2 \sum_{k=1}^{n-1} (-1)^{n-(k+1)} q^k \tag{3.6}$$

if $b = 0$, and

$$\widetilde{N}_n = \sum_{k=0}^{n-1} (-1)^{n-(k+1)} q^k \tag{3.7}$$

if $b \neq 0$. Now let us divide the proof into the following two cases.

CASE 1. $b = 0$. Clearly, one has $\widetilde{N}_1 = 1$. Then by (3.6) one has

$$\begin{aligned}
&\sum_{n=1}^{\infty} \widetilde{N}_n t^n \\
=&t + \sum_{n=2}^{\infty} \Big((-1)^{n-1} + 2 \sum_{k=1}^{n-1} (-1)^{n-(k+1)} q^k \Big) t^n \\
=&\sum_{n=1}^{\infty} (-1)^{n-1} t^n + 2 \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} (-1)^{n-(k+1)} q^k t^n \\
=&-\sum_{n=1}^{\infty} (-t)^n + 2 \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} (-1)^{n-(k+1)} q^k t^n \\
=&\frac{t}{1+t} - 2 \sum_{k=1}^{\infty} (-q)^k \sum_{n=k+1}^{\infty} (-t)^n \\
=&\frac{t}{1+t} - 2 \sum_{k=1}^{\infty} (-q)^k \frac{(-t)^{k+1}}{1+t} \\
=&\frac{t}{1+t} + \frac{2t}{1+t} \sum_{k=1}^{\infty} (qt)^k \\
=&\frac{t}{1+t} + \frac{2qt^2}{(1+t)(1-qt)} \\
=&\frac{t(1+qt)}{(1+t)(1-qt)}.
\end{aligned}$$

That is, the generating function $\sum_{n=1}^{\infty} \widetilde{N}_n t^n$ is a rational function in t when $b = 0$.

CASE 2. $b \neq 0$. Then from (3.7), we derive that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \widetilde{N}_n t^n \\
=&\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (-1)^{n-(k+1)} q^k t^n \\
=&-\sum_{n=1}^{\infty} (-t)^n \sum_{k=0}^{n-1} (-q)^k \\
=&-\sum_{n=1}^{\infty} (-t)^n \frac{1-(-q)^n}{1-(-q)} \\
=&\frac{-1}{1+q} \sum_{n=1}^{\infty} (-t)^n (1-(-q)^n)
\end{aligned}$$

$$
\begin{aligned}
&=\frac{-1}{1+q}\sum_{n=1}^{\infty}(-t)^n+\frac{1}{1+q}\sum_{n=1}^{\infty}(tq)^n\\
&=\frac{-1}{1+q}\cdot\frac{-t}{1+t}+\frac{1}{1+q}\cdot\frac{tq}{1-tq}\\
&=\frac{1}{1+q}\Big(\frac{t}{1+t}+\frac{tq}{1-tq}\Big)\\
&=\frac{t-t^2q+tq+t^2q}{(1+q)(1+t)(1-tq)}\\
&=\frac{t(1+q)}{(1+q)(1+t)(1-tq)}\\
&=\frac{t}{(1+t)(1-qt)}
\end{aligned}
$$

as one expects. So the generating function $\sum_{n=1}^{\infty}\widetilde{N}_n t^n$ is a rational function in t when $b\neq 0$.

This finishes the proof of Theorem 1.3. □

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Conflict of interest

The authors declare that there is no conflict of interest.

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