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Research article

Regularity of weak solutions to a class of fourth order parabolic variational inequality problems arising from swap option pricing

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Abstract: This article concerns the regularity of weak solutions for a variational inequality problem constructed by a fourth-order parabolic operator which has received much attention recently. We first consider the internal regular estimate of weak solutions using the difference type test function. Then, the near edge regularity and global regularity of weak solutions are analyzed by using the finite cover principle. Since the quadratic gradient of the weak solution does not satisfy the conditions for a test function, we have constructed a test function using a spatial difference operator to complete the proof of regularity. The results show that the weak solution has a second order regularity and an $L^\infty(0,T;H^2(\Omega))$ estimation independent of the lower order norm of the weak one.

Keywords: variational inequality problems; fourth order parabolic operator; regularity
Mathematics Subject Classification: 35K99, 97M30

1. Introduction

A swap option is a kind of exotic option involving two stocks. Assume that the stock price $\{S_i(t), t \in [0,T]\}$ follows the following stochastic process:

$$\mathrm{d}S_i(t) = \mu_i S_i(t)\mathrm{d}t + \sigma_i S_i(t)\mathrm{d}S_i(t), S_i(0) = s_i,$$

where s_i is known, μ_i is the yield rate of stock $\{S_i(t), t \in [0,T]\}$, and σ_i is the volatility, $i = 1, 2$. In the financial contract, the swap option allows investors to convert stock $\{S_1(t), t \in [0,T]\}$ to stock $\{S_2(t), t \in [0,T]\}$ within the time interval $[0,T]$. (It is just a right, not an obligation, to make the conversion.) If the yield and turnover of stock $\{S_1(t), t \in [0,T]\}$ are better than those of $\{S_2(t), t \in [0,T]\}$, investors need to compensate the agent for a certain amount of cash K, so the value of the swap option on the maturity date T is [1–3]

$$V(S_1(T), S_2(T), T) = \max\{S_2(T) - S_1(T) - K, 0\}.$$

Since investors can execute at any time within the interval $[0, T]$,

$$V(S_1(t), S_2(t), t) \geq \max\{S_2(t) - S_1(t) - K, 0\}.$$

From [4], the value of the swap option with maturity time T at time 0 satisfies the following parabolic variational inequality:

$$\begin{cases} L_0 V \geq 0, & (s_1, s_2, t) \in \mathrm{R}_+ \times \mathrm{R}_+ \times [0, T], \\ V - \max\{s_2 - s_1 - K, 0\} \geq 0, & (s_1, s_2, t) \in \mathrm{R}_+ \times \mathrm{R}_+ \times [0, T], \\ L_0 V \times (V - \max\{s_2 - s_1 - K, 0\}) = 0, & (s_1, s_2, t) \in \mathrm{R}_+ \times \mathrm{R}_+ \times [0, T], \\ V(s_1, s_2, T) = \max\{s_2 - s_1 - K, 0\}, & (s_1, s_2) \in \mathrm{R}_+ \times \mathrm{R}_+. \end{cases}$$

Let r be the risk-free interest rate in the securities market, and define q_i as the dividend rate of the stock $\{S_i(t), t \in [0, T]\}$, $i = 1, 2$. Parabolic operator $L_0 V$ can be written as

$$L_0 V = \partial_t V + \frac{1}{2}\sigma_1^2 s_1^2 \partial_{s_1 s_1} V + \frac{1}{2}\sigma_2^2 s_2^2 \partial_{s_2 s_2} V + (r - q_1) s_1 \partial_{s_1} V + (r - q_2) s_2 \partial_{s_2} V - rV.$$

If the stock has transaction costs, the operator Lv has a more complex structure in which

$$\sigma_i^2 = \sigma_i^2(\partial_{s_i s_i} V, \partial_{s_i} V, V), \ i = 1, 2.$$

Readers can read about the Leland model, the Barles and Soner's model and the Davis model in [5].

The author of this study focuses on more complex models, considering a certain kind of variational inequality problem

$$\begin{cases} Lu \geq 0, & (x, t) \in \Omega_T, \\ u - u_0 \geq 0, & (x, t) \in \Omega_T, \\ Lu(u - u_0) = 0, & (x, t) \in \Omega_T, \\ u(0, x) = u_0(x), & x \in \Omega, \\ u(t, x) = 0, & (x, t) \in \partial\Omega \times (0, T) \end{cases} \tag{1}$$

with the non-Newtonian polytropic operator

$$Lu = \partial_t u - \Delta u + \gamma |u|^{p-2} u, \ p > 0. \tag{2}$$

Recently, there are many studies about the theoretical research of variational inequality problems. Tao Wu in [6] used a fourth-order p-Laplacian Kirchhoff operator and considered the following variation-inequality initial-boundary value problem

$$\begin{cases} \min\{L\phi, \phi - \phi_0\} = 0, & (x, t) \in Q_T, \\ \phi(0, x) = \phi_0(x), & x \in \Omega, \\ \phi(t, x) = 0, & (x, t) \in \partial\Omega \times (0, T). \end{cases} \tag{3}$$

Based on the Leray-Schauder principle, the existence of solutions to the auxiliary problem is proved. The existence and uniqueness of solution to (3) is then studied in which the parabolic operator $L\phi$ is extended. The 2-D value variational inequality problem is also considered in [7] using the limit method. For the existence of weak solutions to variational inequalities, readers can refer to the literature [8–10]. The stability and uniqueness of weak solutions have also been a hot topic in recent years. Literature [7,11] analyzes the upper bound estimate of the difference between two weak solutions and proves the stability and uniqueness of the weak solution about the initial value. However, there is currently no literature on the regularity of weak solutions of such variational inequalities. Literature [12] analyzes parametric boundary value problems using optimal variational iteration methods and convergence control techniques, showing that the optimal variational iteration method is an effective method for solving such problems. The authors of [13] developed a scheme to examine fractional-order shock wave equations and wave equations occurring in the motion of gases in the Caputo sense with their main finding being the handling of the recurrence relation that produces the series solutions after only a few iterations. Finally, the authors of [14] suggested a He-Laplace variational iteration method for the study of some partial differential equations arising in physical phenomena such as chemical kinetics and population dynamics.

In summary, we investigate the regularity of weak solutions for the variational inequality problem (1). We use a variety of techniques, including integral inequalities, partial derivatives, flattening operators, and the finite cover principle, to obtain internal, near edge, and global regularity estimates. Additionally, we provided an $L^\infty(0,T;H^2(\Omega))$ estimation using the quadratic gradient of the weak solution.

2. Statement of the problem and its background

We will recall several well-known aspects about the solution to problem (1) and provide a set of maximal monotone maps defined in [1–3]

$$G = \{u|u(x) = 0, x > 0; u(x) \in [0, -M_0], \quad x = 0\}, \tag{4}$$

where M_0 is a positive constant. With a similar method to that used in [6,7], variational inequality problem (1) admits a generalized solution (u,ξ) that satisfies
(a) $u \in L^\infty(0,T,H^1(\Omega)) \cap L^\infty(0,T,L^p(\Omega))$, $\partial_t u \in L^\infty(0,T,L^2(\Omega))$ and $\xi \in G$,
(b) $u(x,t) \geq u_0(x)$, $u(x,0) = u_0(x)$ for any $(x,t) \in \Omega_T$,
(c) for every test-function $\varphi \in C^1(\bar{\Omega}_T)$,

$$\int\int_{\Omega_T} \partial_t u\cdot\varphi + \nabla u\nabla\varphi + \gamma|u|^{p-2}u\varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \xi\cdot\varphi \mathrm{d}x\mathrm{d}t. \tag{5}$$

Indeed, for any generalized solution u, (5) can be rewritten as

$$\int\int_{\Omega_T} \partial_t u\cdot\varphi + \Delta u\cdot\varphi + \gamma|u|^{p-2}u\varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \xi\cdot\varphi \mathrm{d}x\mathrm{d}t. \tag{6}$$

Following a similar way of [6,7], applying the comparison principle shows that

$$u_0 \leq u \leq |u_0|_\infty + 1 \text{ for any } (x,t) \in \Omega_T. \tag{7}$$

This paper focuses on the internal regularity of weak solutions within a subarea $\Omega' \subset\subset \Omega$. In doing so, we introduce the difference operator

$$\Delta_h^i u(x,t) = \frac{u(x+he_i,t) - u(x,t)}{h},$$

where e_i is the unit vector in the direction x_i. As stated in the literature [12], the difference operator has the following results.

Lemma 2.1. (1) Let $\Delta_h^{i*} = -\Delta_{-h}^i$ be the conjugate operator of Δ_h^i. Then we have

$$\int_{\mathrm{R}_n} f(x)\Delta_h^i g(x)\mathrm{d}x = -\int_{\mathrm{R}_n} g(x)\Delta_{-h}^i f(x)\mathrm{d}x$$

in other words, $\int_{\mathrm{R}_n} f(x)\Delta_h^i g(x)\mathrm{d}x = \int_{\mathrm{R}_n} g(x)\Delta_h^{i*} f(x)\mathrm{d}x$.

(2) Operator Δ_h^i has the following commutative results

$$D_j\Delta_h^i f(x) = \Delta_h^i D_j f(x),\ j = 1, 2, \cdots, n.$$

(3) If $u \in W^{1,p}(\Omega)$, for any $\Omega' \subset\subset \Omega$,

$$\|\Delta_h^i u\|_{L^p(\Omega')} \leq \|D_i u\|_{L^p(\Omega')},\ \|\Delta_h^{i*} u\|_{L^p(\Omega')} \leq \|D_i u\|_{L^p(\Omega')}.$$

(4) Let h be small enough. If $\|\Delta_h^i u\|_{L^p(\Omega)} \leq C$, then

$$\|D_i u\|_{L^p(\Omega)} \leq C,$$

where C is independent of h.

3. Internal regularity of solution

In this section, we will investigate the internal regularity of weak solutions. To do so, we require the following auxiliary result.

Lemma 3.1. Let $\eta \in C_0^\infty(\Omega)$ be the cutoff factor on $\Omega' \subset\subset \Omega$ that satisfies

$$0 \leq \eta \leq 1,\ \eta = 1 \text{ in } \Omega',\ \mathrm{dist}(\mathrm{supp}\eta, \Omega) \geq 2d$$

where $d = \mathrm{dist}(\Omega', \Omega)$; then,

$$\int_\Omega |\Delta_h^{i*}(\eta^2\Delta_h^i u)|^2\mathrm{d}x \leq 2\int_\Omega \eta^2|\nabla\Delta_h^i u|^2\mathrm{d}x + 8\int_\Omega |\nabla\eta|^2|\nabla\Delta_h^i u|^2\mathrm{d}x. \tag{8}$$

Proof. It follows from Lemma 2.1 (3) that

$$\int_\Omega |\Delta_h^{i*}(\eta^2\Delta_h^i u)|^2\mathrm{d}x = \int_\Omega |\Delta_{-h}^i(\eta^2\Delta_h^i u)|^2\mathrm{d}x \leq \int_\Omega |\nabla(\eta^2\Delta_h^i u)|^2\mathrm{d}x,$$

such that

$$\int_\Omega |\Delta_h^{i*}(\eta^2\Delta_h^i u)|^2\mathrm{d}x \leq \int_\Omega |\eta^2\nabla\Delta_h^i u + 2\eta\Delta_h^i u\nabla\eta|^2\mathrm{d}x.$$

Using $(a+b)^2 \leq 2a^2 + 2b^2$ gives

$$\int_\Omega |\Delta_h^{i*}(\eta^2\Delta_h^i u)|^2\mathrm{d}x \leq 2\int_\Omega |\eta^2\nabla\Delta_h^i u|^2\mathrm{d}x + 8\int_\Omega |\eta\Delta_h^i u\nabla\eta|^2\mathrm{d}x.$$

Therefore, the proof of Lemma 2.2 is finished (note that $0 \leq \eta \leq 1$). □

It is important to note that the quadratic gradient $\triangle u$ does not satisfy the condition for weak solutions. Therefore, we plan to construct a test function using the spatial difference operator Δ_h^i. Let $h < D$ and choose $\varphi = \Delta_h^{i*}(\eta^2\Delta_h^i u)$ as a test function in (5). Since $u \in H_0^1(\Omega)$, we have $\varphi \in H_0^1(\Omega)$, so that

$$\begin{aligned}&\int\int_{\Omega_T} \partial_t u \cdot \Delta_h^{i*}(\eta^2\Delta_h^i u) + \nabla u \nabla \Delta_h^{i*}(\eta^2\Delta_h^i u) + \gamma|u|^{p-2}u\Delta_h^{i*}(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t \\ &= \int\int_{\Omega_T} \xi \cdot \Delta_h^{i*}(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{9}$$

Now we prove that

$$\int\int_{\Omega_T} \partial_t u\Delta_h^{i*}(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t = \int_\Omega (\Delta_h^i u(x,T))^2\eta^2\mathrm{d}x - \int_\Omega (\Delta_h^i u_0)^2\eta^2\mathrm{d}x. \tag{10}$$

It follows from Lemma 2.1 (2) that

$$\int\int_{\Omega_T} \partial_t u\Delta_h^{i*}(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \partial_t(\Delta_h^i u)\eta^2\Delta_h^i u\mathrm{d}x\mathrm{d}t,$$

such that

$$\int\int_{\Omega_T} \partial_t u\Delta_h^{i*}(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int\int_{\Omega_T} \partial_t(\Delta_h^i u)^2\eta^2\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int\int_{\Omega_T} \partial_t((\Delta_h^i u)^2\eta^2)\mathrm{d}x\mathrm{d}t$$

Thus (10) follows. Combining (9) and (10) and using Lemma 2.1 (2) to $\int\int_{\Omega_T} \nabla u\nabla\Delta_h^{i*}(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t$ gives

$$\begin{aligned}&\int_0^T\int_\Omega \nabla\Delta_h^i u \cdot \nabla(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t + \gamma\int_0^T\int_{\Omega_T} |u|^{p-2}u\Delta_h^{i*}(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t \\ &\le \int\int_{\Omega_T} \xi \cdot \Delta_h^{i*}(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t + \int_\Omega (\Delta_h^i u_0)^2\eta^2\mathrm{d}x.\end{aligned} \tag{11}$$

Because

$$\begin{aligned}&\int_0^T\int_\Omega \nabla\Delta_h^i u \cdot \nabla(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t \\ &= 2\int_0^T\int_\Omega \eta\nabla\eta \cdot (\nabla\Delta_h^i u)\Delta_h^i u\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \eta^2(\nabla\Delta_h^i u)^2\mathrm{d}x\mathrm{d}t,\end{aligned}$$

(11) can be written as

$$\begin{aligned}&\int_0^T\int_\Omega \eta^2(\nabla\Delta_h^i u)^2\mathrm{d}x\mathrm{d}t \\ &\le \int\int_{\Omega_T} \xi \cdot \Delta_h^{i*}(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t + \int_\Omega |\Delta_h^i u_0|^2\eta^2\mathrm{d}x \\ &-2\int_0^T\int_\Omega \eta\nabla\eta \cdot (\nabla\Delta_h^i u)\Delta_h^i u\mathrm{d}x\mathrm{d}t - \gamma\int_0^T\int_{\Omega_T} |u|^{p-2}u\Delta_h^{i*}(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{12}$$

Applying Holder and Young inequalities as well as Lemma 3.1,

$$\begin{aligned}&\int\int_{\Omega_T} \xi \cdot \Delta_h^{i*}(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t \le 2M_0^2T|\Omega| + \tfrac{1}{8}\int\int_{\Omega_T} |\Delta_h^{i*}(\eta^2\Delta_h^i u)|^2\mathrm{d}x\mathrm{d}t \\ &\le 2M_0^2T|\Omega| + \tfrac{1}{4}\int_\Omega \eta^2|\nabla\Delta_h^i u|^2\mathrm{d}x + \int_\Omega |\nabla\eta|^2|\Delta_h^i u|^2\mathrm{d}x,\end{aligned} \tag{13}$$

$$\begin{aligned}&\gamma\int_0^T\int_{\Omega_T} |u|^{p-2}u\Delta_h^{i*}(\eta^2\Delta_h^i u)\mathrm{d}x\mathrm{d}t \\ &\le 2\gamma^2(|u_0|_\infty + 1)^{2p-2}T|\Omega| + \tfrac{1}{4}\int_\Omega \eta^2|\nabla\Delta_h^i u|^2\mathrm{d}x + \int_\Omega |\nabla\eta|^2|\Delta_h^i u|^2\mathrm{d}x,\end{aligned} \tag{14}$$

$$\begin{aligned}&2\int_0^T\int_\Omega \eta\nabla\eta \cdot (\nabla\Delta_h^i u)\Delta_h^i u\mathrm{d}x\mathrm{d}t \\ &\le \tfrac{1}{8}\int_0^T\int_\Omega \eta^2(\nabla\Delta_h^i u)^2\mathrm{d}x\mathrm{d}t + 8\int_0^T\int_\Omega |\nabla\eta|^2(\Delta_h^i u)^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{15}$$

Substituting (13), (14) and (15) (note that $0 \le \eta \le 1$), it is clear to verify

$$\begin{aligned}&\tfrac{3}{8}\int_0^T \int_\Omega \eta^2(\nabla\Delta_h^i u)^2 \mathrm{d}x\mathrm{d}t \\ &\le 2M_0^2 T|\Omega| + 8\int_\Omega |\nabla\eta|^2|\Delta_h^i u|^2 \mathrm{d}x + \int_\Omega |\Delta_h^i u_0|^2\eta^2 \mathrm{d}x + 2\gamma^2(|u_0|_\infty + 1)^{2p-2}T|\Omega|.\end{aligned}$$

It follows from Lemma 2.1 (3) that

$$\int_0^T \int_\Omega |\nabla\eta|^2(\Delta_h^i u)^2 \mathrm{d}x\mathrm{d}t \le C\int_0^T \int_\Omega (\Delta_h^i u)^2 \mathrm{d}x\mathrm{d}t \le C\int_0^T \int_\Omega (\nabla u)^2 \mathrm{d}x\mathrm{d}t,$$

$$\int_\Omega (\Delta_h^i u_0)^2\eta^2 \mathrm{d}x\mathrm{d}t \le \int_\Omega (\nabla u_0)^2 \mathrm{d}x\mathrm{d}t,$$

such that

$$\int_0^T \int_\Omega (\nabla\Delta_h^i u)^2 \mathrm{d}x\mathrm{d}t \le C\left(\int_0^T \int_\Omega (\nabla u)^2 \mathrm{d}x\mathrm{d}t + \int_\Omega |\nabla u_0|^2 \mathrm{d}x\mathrm{d}t\right).$$

This, from Lemma 2.1 (4), implies that

$$\int_0^T \int_\Omega |\Delta u|^2 \mathrm{d}x\mathrm{d}t \le C\left(\int_0^T \int_\Omega (\nabla u)^2 \mathrm{d}x\mathrm{d}t + \int_\Omega |\nabla u_0|^2 \mathrm{d}x\mathrm{d}t\right).$$

Theorem 3.1. If $u \in L^\infty(0,T;H^1(\Omega))$, $u_0 \in H^1(\Omega)$, then for any $\Omega' \subset\subset \Omega$,

$$\|u\|_{L^\infty(0,T;H^2(\Omega'))} \le C\left(\|u\|_{L^\infty(0,T;H^1(\Omega))} + \|u_0\|_{H^1(\Omega)}\right). \tag{16}$$

4. Global regularity of solution

In this section, we examine the near-edge regularity of weak solutions. Suppose U is a neighborhood containing $x_0 \in \partial\Omega$. Drawing inspiration from reference [12], we introduce a flattening operator Ψ, which belongs to C^2 to transform the proof process into an internal regularity problem in Section 3.

Theorem 4.1. Assume $u \in L^\infty(0,T;H^1(\Omega))$ and $u_0 \in H^1(\Omega)$. For any $x_0 \in \partial\Omega$, if x_0 belongs to U, then

$$\|u\|_{L^\infty(0,T;H^2(\Omega\cap U))} \le C\left(\|u\|_{L^\infty(0,T;H^1(\Omega))} + \|u_0\|_{H^1(\Omega)}\right).$$

According to the finite-covers theorem, there are finite neighborhoods $U_1,\ U_2,\cdots,U_N$, satisfies

$$\bigcup_{i=1}^N U_i = \partial\Omega,$$

such that

$$\|u\|_{L^\infty(0,T;H^2(\Omega\cap U_i))} \le C\left(\|u\|_{L^\infty(0,T;H^1(\Omega))} + \|u_0\|_{H^1(\Omega)}\right), i = 1,2,\cdots,N. \tag{17}$$

Combining Theorem 4.1 and (17), we give the following global regular estimation.

Theorem 4.2. If $u \in L^\infty(0,T;H^1(\Omega))$ and $u_0 \in H^1(\Omega)$, then

$$\|u\|_{L^\infty(0,T;H^2(\Omega))} \le C\left(\|u\|_{L^\infty(0,T;H^1(\Omega))} + \|u_0\|_{H^1(\Omega)}\right). \tag{18}$$

Indeed, from the perspective of $L^\infty(0,T;H^2(\Omega))$ norm estimation, we have a better result. Choose Δu as a test function in (6), such that

$$\frac{1}{2}\int_0^T \int_\Omega |\partial_t \nabla u|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |\Delta u|^2 + \gamma|u|^{p-2}u\cdot\Delta u \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \xi\cdot\Delta u \mathrm{d}x\mathrm{d}t.$$

Note that $\int_0^T \int_\Omega |\partial_t \nabla u|^2 \mathrm{d}x\mathrm{d}t = \int_\Omega |\nabla u(,T)|^2 \mathrm{d}x - \int_\Omega |\nabla u_0|^2 \mathrm{d}x$. Thus,

$$\int_0^T \int_\Omega |\Delta u|^2 + \gamma |u|^{p-2} u \cdot \Delta u \mathrm{d}x\mathrm{d}t \leq \int_0^T \int_\Omega \xi \cdot \Delta u \mathrm{d}x\mathrm{d}t + \int_\Omega |\nabla u_0|^2 \mathrm{d}x. \tag{19}$$

Applying Holder and Young inequalities and combining with (8), we have

$$\gamma \int_0^T \int_{\Omega_T} |u|^{p-2} u \cdot \Delta u \mathrm{d}x\mathrm{d}t \leq 2\gamma^2 (|u_0|_\infty + 1)^{2p-2} T|\Omega| + \frac{1}{4} \int_\Omega |\Delta u|^2 \mathrm{d}x. \tag{20}$$

Combining with (4) also gives

$$\int_0^T \int_\Omega \xi \cdot \Delta u \mathrm{d}x\mathrm{d}t \leq 2M_0^2 T|\Omega| + \frac{1}{4} \int \int_{\Omega_T} |\Delta u|^2 \mathrm{d}x\mathrm{d}t. \tag{21}$$

Substituting (20) and (21) in (19), one can get

$$\frac{1}{2} \int_0^T \int_\Omega |\Delta u|^2 \mathrm{d}x\mathrm{d}t \leq \int_\Omega |\nabla u_0|^2 \mathrm{d}x + 2\gamma^2 (|u_0|_\infty + 1)^{2p-2} T|\Omega| + 2M_0^2 T|\Omega|.$$

Rearranging the above inequality, we prepare the following theorem which is better than the global regular estimation in (18).

Theorem 4.3. If $u_0 \in H^1(\Omega)$, then

$$\|u\|_{L^\infty(0,T;H^2(\Omega))} \leq C,$$

where C depends on $p, \gamma, \int_\Omega |\nabla u_0|^2 \mathrm{d}x, |u_0|_\infty, T$ and $|\Omega|$.

5. Conclusions

This paper discusses the regularity of weak solutions for a class of variational inequality problems involving the fourth order parabolic operator. The existence of weak solutions is typically established through first order energy estimates, which can be proven by selecting an appropriate test function in the weak solution equation. As weak solutions exhibit higher-order norm estimates, scholars have paid much attention to their regularity. However, constructing test functions using second-order partial derivatives, which do not satisfy the conditions for weak solutions, is a challenge when proving the regularity of these solutions. Nonetheless, the spatial difference operator retains the differential order of the weak solution u, leading us to employ spatial difference operators and cutoff factors in test function construction. In this paper, we have constructed a test function

$$\varphi = \Delta_h^{i*}(\eta^2 \Delta_h^i u)$$

using these operators, which satisfies the condition for weak solutions. Subsequently, we have used the test function φ in (5) and the cutoff factor η to obtain inequality (11) which is a cornerstone for general energy estimates. As variational inequalities are more complex than equal parabolic equations, we have introduced a maximum monotone operator in weak solutions (5) and (11) based on [6,7]. We have combined the Holder and Young inequalities to obtain an estimate of the internal regularity of the weak solution without imposing any further existence conditions on the second partial derivative of the weak solution u.

At present, this paper gives the limiting condition $\gamma \geq 0$. Although this condition is not directly used in this paper, according to literature [6,7], this condition ensures that $u \in L^\infty(0, T, H^1(\Omega)) \cap L^\infty(0, T, L^p(\Omega))$, $\partial_t u \in L^\infty(0, T, L^2(\Omega))$. So this paper continues to use this restriction and we will try to weaken it in the future.

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Conflict of interest

The author declares no conflict of interest.

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