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Research article

Energy decay of solution for nonlinear delayed transmission problem

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Abstract: In this work, we consider a nonlinear transmission problem in the bounded domain with a delay term in the first equation. Under conditions on the weight of the damping and the weight of the delay, we prove general stability estimates by introducing a suitable Lyapunov functional and using the properties of convex functions.

Keywords: nonlinear transmission problem; delay term; decay rate; Lyapunov functions
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1. Introduction

One of the specific features of most control systems for real technological processes is the presence of a time lag. The effect of a time delay complicates stabilization and reduces the quality indicators of automatic systems. Equations of this type are called differential-difference equations. An important class of objects with a delay is formed by objects in which the delay is contained only in control signals. These are the so-called objects with a delay in control. This class includes numerous production processes in metallurgy, thermal power engineering, chemical, oil refining, paper, food and many other industries.

Controllability and observability of systems with delay are studied in the works of Gabasov [10]. Researches have paid more attention to the problem of the synthesis of control systems for stationary processes with delay. Methods for the parametric synthesis of continuous systems for the automatic control of objects with a delay for a given controller structure are given. It is noted that continuous systems with typical controllers can provide a satisfactory quality of regulation with a small amount of time delay. A certain improvement in the quality of control of objects with delay can be obtained by using impulse controllers. Ya Tsypkin [19] showed that the introduction of an impulse element into systems to control objects with a delay can significantly increase the stability margin. For many processes with a delay, the use of systems with pulse-width modulation is the most appropriate. At the same time, few studies addressed the issues of building pulse-width systems to control objects with a delay. To control processes with a supremal delay, a controller called the Rezvik controller was proposed. The controller of this type contains a block that implements the inverse operator of the part of the object without delay, and an auxiliary positive feedback that contains a delay element. Although the Resvik regulator allows some degree of lag compensation, it has serious drawbacks. The first disadvantage is associated with the presence of positive feedback, which reduces stability, and the second disadvantage is associated with the difficulty of its implementation. In this regard, the Resvik controller does not have many applications. Among the various methods for controlling processes with delay, the Smith method and optimal control methods have the widest range of application.

To solve control problems, it is important to know whether a given object has the property of being controllable in terms of transferring from one given state to another. This is called the controllability property. Another important property of an object that needs to be known when constructing optimal control systems is observability. Due to the mass nature and importance of objects of this class, they are the focus in the present work. Note that, as for systems without delay, an important tool for studying the stability of systems with delay is the second method of Lyapunov. Here a significant contribution was made by Krasovskii, who proposed, instead of the Lyapunov function, to consider functionals with better properties. Absolute stability conditions for nonlinear systems with delay were studied. In systems with significant delay, the possibility of occurrence of periodic regimes is greater than in systems without delay [2].

The study of nonlinear dynamics in delayed systems of radiophysical and electronic nature is of interest both for solving many traditional practical problems of radio electronics (e. g. transient theory, excitation of parasitic oscillations, amplification and generation of short pulses, amplification of signals with a complex spectral composition), and in connection with new prospective applications of chaotic signals in communication systems, information processing, radar and electronic countermeasures. The study of complex irregular behavior (spatio-temporal chaos) in such systems can serve as a key to solve the problem of turbulence. The paper considers self-oscillatory systems with delay, models of devices for vacuum microwave electronics and systems of parametrically interacting waves, and it discusses the suppression of instabilities in such systems using chaos control methods.

Router queue management algorithms are modern and relevant problems that arise when building information systems. Models are constructed in the form of a system of nonlinear differential equations with variable delay. The general properties of these models and the corresponding solutions of differential equations are investigated. The choice of buffering parameters can significantly affect not only the speed of the communication channel, but also other transmission parameters, among which is speed stability. In distributed automatic control systems, in particular, this leads to a loss of stability of the control loop and/or a significant deterioration in the performance of control regulation. In practical cases, periodic fluctuations in the data transfer rate are often observed, which are not well understood, but significantly affect the operation of a distributed system. Some issues have not been studied sufficiently, especially in relation to distributed systems of automation and control, where the task is not only to deliver information, but also hard time synchronization. In this context, of particular interest are the cases when there are modes of periodic variation over a wide range of delay in data transmission. From a mathematical point of view, there is a system of nonlinear differential equations with a variable delay of a particular form that changes periodically. In the general case, this theory is not sufficiently developed; some special cases are modeled in the article. In our article, we consider this topic whereby we show that the regime of periodically varying delay naturally arises in such systems.

In this paper, we consider a nonlinear transmission problem with a delay term, for $(x,t) \in \Omega \times \mathbb{R}^+$

$$\begin{cases} \partial_{tt}u - au_{xx} + \mu_1 f_1(\partial_t u) + \mu_2 f_2(\partial_t u(t-\tau)) = 0, \\ \\ \partial_{tt}v - bv_{xx} = 0 \text{ in } (L_1, L_2) \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

where $0 < L_1 < L_2 < L_3, \Omega =]0, L_1[\cup]L_2, L_3[, a, b, \mu_1$ and μ_2 are positive constants and $\tau > 0$ is a delay. System (1.1) is subjected to the following boundary and transmission conditions

$$\begin{aligned} u(0,t) &= u(L_3,t) = 0, \\ u(L_i,t) &= v(L_i,t), \quad i = 1,2, \\ au_x(L_i,t) &= bv_x(L_i,t), \quad i = 1,2, \end{aligned} \tag{1.2}$$

and the initial conditions

$$\begin{aligned} &u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x) && \text{in } \Omega, \\ &v(x,0) = v_0(x), \quad \partial_t v(x,0) = v_1(x) && \text{in }]L_1, L_2[, \\ &u(t-\tau) = f_0(t-\tau) && \text{in } \Omega \times [0,\tau]. \end{aligned} \tag{1.3}$$

For $f_1(s) = s$, and in the absence of delay ($\mu_2 = 0$), Systems (1.1)–(1.3) were investigated in [5], for $\Omega =]0, L_1[$, where the authors showed the exponential stability of the total energy. Zennir and Feng [21] investigated a transmission problem in thermoelasticity and showed that the energy is exponentially stable (see also [16, 17]). On the contrary, if $\mu_1 = 0$, that is, there exists only the delay part in the first equation, the problems of (1.1)–(1.3) become unstable (see [6, 8]).

In [18], Nicaise and Pignotti examined the wave equation with a delay in the linear internal feedback

$$\begin{aligned} &\partial_{tt}u - \Delta_x u + \mu_1 \partial_t u + \mu_2 \partial_t u(t-\tau) = 0 && \text{in } \Omega \times (0,\infty), \\ &u = 0 && \text{on } \Gamma_D \times (0,\infty), \\ &\tfrac{\partial u}{\partial \nu} = 0 && \text{on } \Gamma_N \times (0,\infty). \end{aligned} \tag{1.4}$$

They proved, under the assumption that $\mu_2 < \mu_1$, that the energy is exponentially stable. However, for the opposite case ($\mu_2 \geq \mu_1$), they were able to construct a sequence of delays for which the corresponding solution is unstable. The same results were obtained for the case when both the damping and the delay are acting at the boundary.

In [2], Benaissa and Louhbi examined the problem (1.4) in the nonlinear situation. They proved the global existence of solutions in Sobolev spaces.

In [3], Benaissa and Bahlil extended the results of [2] for the Timoshenko beam system for the case when the waves propagation speeds are equal.

Benseghir [4] considered the problems (1.1)–(1.3) when f_1 and f_2 are linear. He proved, under the assumption that $\mu_2 < \mu_1$, that the energy is exponentially stable.

In this article, we aim to investigate (1.1)–(1.3) and establish a general decay result under the condition of a suitable relation between the weight of the delay term in the feedback, the weight of the term without delay and the waves propagation speeds. The proof is based on the Lyapunov functional method and makes use of some properties of convex functions, the generalized Young's inequality and Jensen's inequality. The convexity arguments were introduced and developed by [7, 9, 11–14, 20] and used, with appropriate modifications, in [1, 15, 22].

The paper is organized as follows. In Section 2 we prepare some needed results and lemmas. In Section 3, we present and prove our generalized stability result.

2. Assumptions

First, we recall and use the following assumptions on the functions f_1 and f_2:

A1: We assume that the function $f_1 \in C(\mathbb{R}, \mathbb{R})$ is a non-decreasing function such that there exist $\varepsilon_1, c_1, c_2 > 0$ and a convex and increasing function $H : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ of the class $C^1(\mathbb{R}^+) \cap C^2(]0, +\infty[)$ satisfying that $H(0) = 0$ and that H is linear on $[0, \varepsilon']$ or

$$(H'(0) = 0 \ \text{ and } H'' > 0 \text{ on }]0, \varepsilon']),$$

such that

$$c_1|s| \le |f_1(s)| \le c_2|s| \quad if \ |s| \ge \varepsilon', \tag{2.1}$$

$$s^2 + f_1^2(s) \le H^{-1}(s f_1(s)) \quad if \ |s| \le \varepsilon'. \tag{2.2}$$

$f_2 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$

$$|f_2'(s)| \le c_3, \tag{2.3}$$

$$\alpha_1 s f_2(s) \le F_2(s) \le \alpha_2 s f_1(s), \tag{2.4}$$

where

$$F_2(s) = \int_0^s f_2(r)dr,$$

$$\alpha_2\mu_2 \le \alpha_1\mu_1. \tag{2.5}$$

As in [18] the new variable

$$z(x, \rho, t) = \partial_t u(x, t - \tau\rho), \qquad \text{in } \Omega \times (0, 1) \times \mathbb{R}^+. \tag{2.6}$$

Now, note that

$$\partial_t z(x, \rho, t) = \partial_{tt} u(x, t - \tau\rho)$$

and

$$z_\rho(x,\rho,t) = -\tau\partial_{tt}u(x,t-\tau\rho).$$

Thus,

$$\tau\partial_t z(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad in\ \Omega\times(0,1)\times\mathbb{R}^+. \tag{2.7}$$

Then, Problem (1.1) is equivalent to

$$\begin{cases} \partial_{tt}u - au_{xx} + \mu_1 f_1(\partial_t u) + \mu_2 f_2(z(x,1,t)) = 0, \\ \partial_{tt}v - bv_{xx} = 0, \text{in } (L_1,L_2)\times\mathbb{R}^+, \\ \tau\partial_t z(x,\rho,t) + z_\rho(x,\rho,t) = 0, \text{in } \Omega\times(0,1)\times\mathbb{R}^+, \end{cases} \tag{2.8}$$

with the initial conditions

$$\begin{cases} u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x) & \text{in } \Omega, \\ v(x,0) = v_0(x), \quad \partial_t v(x,0) = v_1(x) & \text{in } (L_1,L_2), \\ z(x,0,t) = \partial_t u & \text{in } \Omega\times\mathbb{R}^+, \\ z(x,\rho,0) = f_0(x,-\rho\tau) & \text{in } \Omega\times(0,1), \end{cases} \tag{2.9}$$

and the transmission conditions

$$\begin{aligned} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i=1,2, \\ au_x(L_i,t) = bv_x(L_i,t), \quad i=1,2. \end{aligned} \tag{2.10}$$

3. Stability

Here, we consider the question of asymptotic behavior for (1.1)–(1.3). To this end, let ξ be a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1} < \xi < \tau\frac{\mu_1-\alpha_2\mu_2}{\alpha_2}. \tag{3.1}$$

We define the total energy associated with the solution of the problems (2.8)–(2.10) by

$$E(t) = \frac{1}{2}\int_\Omega \partial_t u^2 dx + \frac{a}{2}\int_\Omega u_x^2 dx + \frac{1}{2}\int_{L_1}^{L_2}\partial_t v^2 dx + \frac{b}{2}\int_{L_1}^{L_2} v_x^2 dx + \xi\int_\Omega\int_0^1 F_2(z(x,\rho,t))d\rho dx. \tag{3.2}$$

The main result reads as follows.

Theorem 3.1. *Let (u,v,z) be the solution of* (2.8)–(2.10). *Assume that the hypothesis (A1) holds and*

$$\frac{a}{b} < \frac{L_3+L_1-L_2}{2(L_2-L_1)}. \tag{3.3}$$

Then there exist positive constants C_1–C_3 and ε_0, such that

$$E(t) \leq C_1 H_1^{-1}(C_2 t + C_3), \quad \forall t \geq 0, \tag{3.4}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)} ds, \tag{3.5}$$

and

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0, \varepsilon'], \\ tH'(\varepsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \varepsilon']. \end{cases} \tag{3.6}$$

The proof of Theorem 3.1 will be carried out through the following Lemmas.

Lemma 3.1. *Let (u, v, z) be the solution of (2.8)–(2.10). Assume that $\mu_1 \geq \mu_2$. Then, the energy functional defined by (3.2) satisfies*

$$\partial_t E(t) - \left(\mu_1 - \tfrac{\xi \alpha_2}{\tau} - \mu_2 \alpha_2\right) \int_\Omega \partial_t u f_1(\partial_t u) dx - \left(\tfrac{\xi}{\tau}\alpha_1 - \mu_2(1 - \alpha_1)\right) \int_\Omega z(x, 1, t) f_2(z(x, 1, t)) dx.$$

Proof. Multiplying the first equation in (2.8) by $\partial_t u$, the second equation by $\partial_t v$ and using integration by parts, we get

$$\tfrac{1}{2}\partial_t \left(\int_\Omega \partial_t u^2 dx + a \int_\Omega u_x^2 dx + \int_{L_1}^{L_2} \partial_t v^2 dx + b \int_{L_1}^{L_2} v_x^2 dx \right) = -\mu_1 \int_\Omega \partial_t u f_1(\partial_t u) dx - \mu_2 \int_\Omega \partial_t u f_2(z(x, 1, t)) dx. \tag{3.7}$$

On the other hand, multiplying $(2.8)_3$ by $\xi f_2((z(x, \rho, t))$, we can then integrate the result over $\Omega \times (0, 1)$, to obtain

$$\begin{aligned} \xi \int_\Omega \int_0^1 z'(x, \rho, t) f_2(z(x, \rho, t)) d\rho dx &= -\tfrac{\xi}{\tau} \int_\Omega \int_0^1 \tfrac{\partial}{\partial \rho} F_2(z(x, \rho, t)) d\rho dx, \\ &= -\tfrac{\xi}{\tau} \int_\Omega (F_2(z(x, 1, t)) - F_2(z(x, 0, t))) dx. \end{aligned}$$

Then

$$\xi \partial_t \int_\Omega \int_0^1 F_2(z(x, \rho, t)) d\rho dx = -\frac{\xi}{\tau} \int_\Omega F_2(z(x, 1, t)) dx + \frac{\xi}{\tau} \int_\Omega F_2(\partial_t u) dx. \tag{3.8}$$

By (3.7) and (3.8) and using (2.4) yields

$$\partial_t E(t) \leq -\left(\mu_1 - \tfrac{\xi \alpha_2}{\tau}\right) \int_\Omega \partial_t u(x, t) f_1(\partial_t u) dx - \tfrac{\xi}{\tau} \int_\Omega F_2(z(x, 1, t)) dx - \mu_2 \int_\Omega \partial_t u f_2(z(x, 1, t)) dx.$$

By F_2^*, we denote the conjugate of the convex function F_2 i.e., $F_2^* = \sup_{t \in \mathbb{R}^+} (st - F_2(t))$. Then F_2^* is the Legendre transform of F_2, which is given by Daoulatli et al. [7]

$$F_2^*(s) = s(F_2')^{-1}(s) - F_2\left[(F_2')^{-1}(s)\right], \quad \forall s \geq 0, \tag{3.9}$$

and satisfies the following inequality

$$st \leq F_2^*(s) + F_2(t), \quad \forall s, t \geq 0. \tag{3.10}$$

Then, from the definition of F_2, we get

$$F_2^*(s) = sf_2^{-1}(s) - F_2\left[f_2^{-1}(s)\right], \quad \forall s \geq 0. \tag{3.11}$$

Hence

$$F_2^*(f_2(z(x,1,t)) = z(x,1,t)f_2(z(x,1,t) - F_2(z(x,1,t)) \leq (1-\alpha_1)z(x,1,t)f_2(z(x,1,t). \tag{3.12}$$

Using (3.9), (3.10) and (3.12), we have

$$\begin{aligned}
\partial_t E(t) &= -\left(\mu_1 - \tfrac{\xi\alpha_2}{\tau}\right)\int_\Omega \partial_t u(x,t) f_1(\partial_t u)dx - \tfrac{\xi}{\tau}\int_\Omega F_2(z(x,1,t))dx \\
&\quad +\mu_2 \int_\Omega (F_2(\partial_t u) + f_2^*(f_2(z(x,1,t)))dx \\
&\leq -\left(\mu_1 - \tfrac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_\Omega \partial_t u(x,t) f_1(\partial_t u)dx \\
&\quad -\tfrac{\xi}{\tau}\int_\Omega F_2(z(x,1,t))dx + \mu_2\int_\Omega f_2^*(f_2(z(x,1,t)))dx.
\end{aligned}$$

Using (2.4) and (3.1), we obtain

$$\partial_t E(t) \leq -\left(\mu_1 - \tfrac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_\Omega \partial_t u f_1(\partial_t u)dx - \left(\tfrac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)f_2(z((x,1,t)))dx.$$

This completes the proof.

Lemma 3.2. *Let* (u, v, z) *be the solution of* (2.8). *Then the functional* F_1 *defined by*

$$F(t) = \int_\Omega u\partial_t u dx + \int_{L_1}^{L_2} v\partial_t v dx \tag{3.13}$$

satisfies, along the solution, the estimate

$$\begin{aligned}
\partial_t F(t) \leq &\int_\Omega \partial_t u^2 dx + \int_{L_1}^{L_2}\partial_t v^2 dx - (a - \varepsilon c_0)\int_\Omega u_x^2 dx - b\int_{L_1}^{L_2} v_x^2 dx \\
&+c\mu_1\int_\Omega f_1^2(\partial_t u)dx + c\mu_2\int_\Omega f_2^2(z(x,1,t))dx.
\end{aligned}$$

Proof. By taking the time derivative of (3.13) and using (2.8), we get

$$\begin{aligned}
\partial_t F(t) = &\int_\Omega \partial_t u^2 dx + \int_{L_1}^{L_2}\partial_t v^2 dx - a\int_\Omega u_x^2 dx - b\int_{L_1}^{L_2} v_x^2 dx \\
&-\mu_1\int_\Omega u f_1(\partial_t u)dx - \mu_2\int_\Omega u f_2(z(x,1,t))dx.
\end{aligned} \tag{3.14}$$

By applying young's and Poincaré's inequalities, we have

$$\begin{aligned}
\partial_t F(t) \leq &\int_\Omega \partial_t u^2 dx + \int_{L_1}^{L_2}\partial_t v^2 dx - (a - \varepsilon c_0)\int_\Omega u_x^2 dx - b\int_{L_1}^{L_2} v_x^2 dx \\
&+c\mu_1\int_\Omega f_1^2(\partial_t u)dx + c\mu_2\int_\Omega f_2^2(z(x,1,t))dx,
\end{aligned}$$

where c_0 is Poincaré's constant. Then (3.14) is established.

Now, inspired by [16], we introduce the functional

$$q(x)=\begin{cases} x-\frac{L_1}{2}, & 0\le x\le L_1,\\ x-\frac{L_2+L_3}{2}, & L_1\le x\le L_2,\\ \frac{L_2-L_3-L_1}{2(L_2-L_1)}(x-L_1)+\frac{L_1}{2}, & L_2\le x\le L_3.\end{cases} \tag{3.15}$$

Next, we define the functionals

$$D_1(t)=-\int_\Omega q(x)u_x\partial_t u dx, \text{ and } D_2(t)=-\int_{L_1}^{L_2} q(x)v_x\partial_t v dx.$$

Lemma 3.3. *For any $\varepsilon_2>0$, we have the following estimates*

$$\begin{aligned}\partial_t D_1(t) &\le C(\varepsilon_2)\int_\Omega \partial_t u^2 dx+\left(\frac{a}{2}+2\varepsilon_2\right)\int_\Omega u_x^2 dx+C(\varepsilon_2)\int_\Omega f_1^2(\partial_t u)dx\\ &-\frac{a}{4}[(L_3-L_2)u_x^2(L_2,t)+L_1u_x^2(L_1,t)]+C(\varepsilon_2)\int_\Omega f_2^2(z(x,1,t))dx,\end{aligned} \tag{3.16}$$

and

$$\partial_t D_2(t) \le \frac{L_2-L_3-L_1}{4(L_2-L_1)}\left(\int_{L_1}^{L_2}\partial_t v^2 dx+b\int_{L_1}^{L_2} v_x^2 dx\right)+\frac{b}{4}\left[(L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)\right]. \tag{3.17}$$

Proof. Differentiating $D_1(t)$ with respect to t, we obtain

$$\partial_t D_1(t) = -\int_\Omega q(x)u_{tx}\partial_t u dx-\int_\Omega q(x)au_x(u_{xx}(x,t)-\mu_1 f_1(\partial_t u)-\mu_2 f_2(z(x,1,t))dx. \tag{3.18}$$

Integrating by parts, we have

$$\int_\Omega q(x)u_{tx}\partial_t u\, dx=-\frac{1}{2}\int_\Omega q'(x)u_t^2 dx+\frac{1}{2}\left[q(x)u_t^2\right]_{\partial\Omega}. \tag{3.19}$$

On the other hand,

$$\int_\Omega q(x)au_xu_{xx}\, dx=-\frac{1}{2}\int_\Omega aq'(x)u_x^2 dx+\frac{1}{2}\left[aq(x)u_x^2\right]_{\partial\Omega}. \tag{3.20}$$

Then by Young's and Poincaré's inequalities, and by using the boundary conditions in (1.2), we have

$$\begin{aligned}\partial_t D_1(t) &\le C(\varepsilon_2)\int_\Omega \partial_t u^2 dx+\left(\frac{a}{2}+2\varepsilon_2\right)\int_\Omega u_x^2 dx+C(\varepsilon_2)\int_\Omega f_1^2(\partial_t u)dx-\frac{a}{4}[(L_3-L_2)u_x^2(L_2,t)\\ &+L_1u_x^2(L_1,t)]+C(\varepsilon_2)\int_\Omega f_2^2(z(x,1,t))dx\end{aligned} \tag{3.21}$$

for any $\varepsilon_2>0$.

In the same way, we take the derivative of $D_2(t)$ with respect to t, to obtain

$$\begin{aligned}\partial_t D_2(t) &= -\int_{L_1}^{L_2} q(x)v_{tx}\partial_t v dx-\int_{L_1}^{L_2} q(x)v_x\partial_{tt}v dx\le \frac{L_2-L_3-L_1}{4(L_2-L_1)}\left(\int_{L_1}^{L_2}\partial_t v^2 dx+b\int_{L_1}^{L_2} v_x^2 dx\right)\\ &+\frac{b}{4}\left[(L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)\right].\end{aligned} \tag{3.22}$$

This gives (3.17).

Let us now, introduce the following functional related to the delayed term

$$I(t) = \int_\Omega \int_0^1 e^{-2\tau\rho} F_2(z(x,\rho,t))d\rho\, dx. \tag{3.23}$$

Then the following result in the next lemma holds.

Lemma 3.4. *Let* (u, v, z) *be a solution of* (2.8). *Then*

$$\partial_t I(t) \leq -2I(t) - \frac{e^{-2\tau}}{\tau} \int_\Omega F_2(z(x,1,t))dx + \frac{1}{\tau} \int_\Omega F_2(\partial_t u)dx. \tag{3.24}$$

Proof. Taking the differentiation of (3.23) with respect to t and using $(2.8)_3$, we have

$$\begin{aligned}
\partial_t I(t) &= \int_\Omega \int_0^1 e^{-2\tau\rho} \partial_t z(x,\rho,t) f_2(z(x,\rho,t))d\rho dx \\
&= -\tfrac{1}{\tau} \int_\Omega \int_0^1 e^{-2\tau\rho} z_\rho(x,\rho,t) f_2(z(x,\rho,t))d\rho dx \\
&= -\tfrac{1}{\tau} \int_\Omega \int_0^1 e^{-2\tau\rho} \tfrac{\partial}{\partial\rho} F_2(z(x,\rho,t))d\rho dx \\
&= -\tfrac{1}{\tau} \int_\Omega \int_0^1 \Big[\tfrac{\partial}{\partial\rho}\Big(e^{-2\tau\rho} F_2(z(x,\rho,t))\Big) d\rho dx + 2\tau e^{-2\tau\rho} F_2(z(x,\rho,t))\Big] d\rho dx \\
&= -\tfrac{1}{\tau} \int_\Omega \Big(e^{-2\tau} F_2(z(x,1,t)) - F_2(\partial_t u)\Big) dx - 2\int_\Omega \int_0^1 e^{-2\tau\rho} F_2(z(x,\rho,t))d\rho dx \\
&\leq -2I(t) - \tfrac{e^{-2\tau}}{\tau} \int_\Omega F_2(z(x,1,t))dx + \tfrac{1}{\tau} \int_\Omega F_2(\partial_t u)dx.
\end{aligned}$$

We define the Lyapunov functional

$$L(t) = NE(t) + I(t) + \delta_1 F(t) + \delta_2 D_1(t) + \delta_3 D_2(t). \tag{3.25}$$

where N, δ_2, δ_3 and δ_4 are positive constants.
Proof of Theorem 3.1.

By the boundary conditions in (1.2), it is not hard to see that

$$a^2 u_x^2(L_i,t) = b^2 v_x^2(L_i,t) \quad i = 1,2. \tag{3.26}$$

By combining (3.7), (3.14), (3.16), (3.17) and (3.24) and taking into account (3.26),

$$\begin{aligned}
\partial_t L(t) \leq\ & (\delta_2 c(\varepsilon_2) + \delta_1) \int_\Omega \partial_t u^2 dx + \left(\left(\frac{a}{2} + 2\varepsilon_2\right)\delta_2 - (a - \varepsilon c_0)\delta_1\right) \int_\Omega u_x^2 dx \\
& + \left(\frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}\delta_3 + \delta_1\right) \int_{L_1}^{L_2} \partial_t v^2 dx + \left(\frac{L_2 - L_3 - L_1}{4(L_2 - L_1)} b\delta_3 - b\delta_1\right) \int_{L_1}^{L_2} v_x^2 dx \\
& - \left(N\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha 2\right) - \frac{\alpha_2}{\tau}\right) \int_\Omega \partial_t u f_1(\partial_t u)dx
\end{aligned}$$

$$-\left(N(\frac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)) + \alpha_1\frac{e^{-2\tau}}{\tau} - C(\delta_1\mu_2 + \delta_2 c(\varepsilon_2))\right)$$
$$\times \int_\Omega z(x,1,t) f_2(z(x,1,t))dx$$
$$-(\delta_2 - \frac{a}{b}\delta_3)\frac{a(L_3-L_2)}{4}u_x^2(L_2,t) - (\delta_2 - \frac{a}{b}\delta_3)\frac{aL_1}{4}u_x^2(L_1,t)$$
$$+ (C(\varepsilon_2)\delta_2 + c\mu_1\delta_1)\int_\Omega f_1^2(\partial_t u)dx.$$

At this point, we have to choose our constants very carefully, such that all this coefficients in (3.27) will be negative.

Indeed, under the assumption of (3.3), we can always find δ_1, δ_2 and δ_3 such that

$$\frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}\delta_3 + \delta_1 < 0, \quad \delta_2 > \frac{a}{b}\delta_3 \quad \delta_1 > \frac{\delta_2}{2}. \tag{3.27}$$

Finally, we keep in mind (3.1) and choose N large enough such that the coefficients in (3.27) are negatives.

Consequently, from the above, we deduce that there exist positive constants d_1 and d_2 such that (3.27) becomes

$$\partial_t L(t) \;\; \le \;\; -d_1 \int_\Omega u_x^2 dx - d_2 \int_{L_1}^{L_2}\left(\partial_t v^2 + v_x^2\right)dx + c\int_\Omega \left(\partial_t u^2 + f_1^2(\partial_t u)\right)dx. \tag{3.28}$$

Then, (3.28) can became

$$\partial_t L(t) \le -dE(t) + c\int_\Omega \left(\partial_t u^2 + f_1^2(\partial_t u)\right)dx. \tag{3.29}$$

The next lemma will be very useful, as it means that there is equivalence between the energy functional and the appropriate Lyapunov function.

Lemma 3.5. *For $N, N > 1$, we have that*

$$\beta_1 L(t) \le E(t) \le \beta_2 L(t) \tag{3.30}$$

holds for two positive constants β_1 and β_2.

The last term on the right hand side of (3.29) should be estimated. To this end, we define

$$\Omega^+ = \{x \in \Omega : |\partial_t u| \ge \varepsilon'\}, \; \Omega^- = \{x \in \Omega : |\partial_t u| \le \varepsilon'\}.$$

From (2.1) and (2.2), it follows that

$$\int_{\Omega^+}(|\partial_t u|^2 + |f_1(\partial_t u)|^2)dx \;\; \le \;\; \mu_1\int_{\Omega^+}\partial_t u.f_1(\partial_t u)dx \le -\mu_1 E'(t). \tag{3.31}$$

Case1. H is linear on $[0, \varepsilon']$. In this case one can easily check that there exists $\mu_1' > 0$ such that $|f_1(s)| \leq \mu_1 |s|$ for all $|s| \leq \varepsilon'$, thus

$$\int_{\Omega^-} (|\partial_t u|^2 + |f_1(\partial_t u)|^2) dx \leq \mu_1' \int_{\Omega^-} \partial_t u . f_1(\partial_t u) dx \leq -\mu_1' E'(t). \tag{3.32}$$

Substitution of (3.31) and (3.32) into (3.29) gives

$$(L(t) + \mu E(t))' \leq C H_2(E(t)), \tag{3.33}$$

where $\mu = c(\mu_1 + \mu_1')$.

Case2. $H'(0) = 0$ and $H'' > 0$ on $]0, \varepsilon']$. Since H is convex and increasing, H^{-1} is concave and increasing. By virtue of (2.1), the reversed Jensen's inequality for a concave function and (3.7), it follows that

$$\begin{aligned} \int_{\Omega^-} (|\partial_t u|^2 + |f_1(\partial_t u)|^2) dx &\leq \int_{\Omega^-} H^{-1}(\partial_t u f_1(\partial_t u)) dx \\ &\leq |\Omega| H^{-1}\left(\frac{1}{|\Omega|} \int_{\Omega^-} \partial_t u f_1(\partial_t u) dx\right) \\ &\leq C H^{-1}(-C' E'(t)). \end{aligned} \tag{3.34}$$

A combination of (3.29), (3.31) and (3.34) yields

$$(L(t) + C\mu_1 E(t))' \leq -C_3 E(t) + C_5 H^{-1}(-C' E'(t)), t \in \mathbb{R}^+. \tag{3.35}$$

By H^*, we denote the conjugate of the convex function H, i.e.,

$$H^*(s) = \sup_{t \in \mathbb{R}^+} (st - H(t)).$$

Then H^* is the Legendre transform of H, which is given by (see [1, 7])

$$H^*(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)], \quad \forall s \geq 0, \tag{3.36}$$

and which satisfies the following inequality

$$st \leq H^*(s) + H(t), \quad \forall s, t \geq 0. \tag{3.37}$$

The relation (3.36) and the facts that $H'(0) = 0$ and $(H')^{-1}$ and H are increasing functions yield

$$H^*(s) \leq s(H')^{-1}(s), \quad \forall s \geq 0. \tag{3.38}$$

Making use of $E'(t) \leq 0$, $H''(t) \geq 0$, (3.35) and (3.38) we derive the following for $\varepsilon_0 > 0$ small enough

$$
\begin{aligned}
&[H'(\varepsilon_0 E(t))(L(t) + C\mu_1 E(t)) + C_5 C' E(t)]' \\
&\quad = \varepsilon_0 E'(t) H''(\varepsilon_0 E(t))(L(t) + C\mu_1 E(t)) + H'(\varepsilon_0 E(t))(L'(t) + C\mu_1 E'(t)) + C_5 C' E'(t) \\
&\quad \leq -C_3 H'(\varepsilon_0 E(t)) E(t) + \widetilde{C}_5 H'(\varepsilon_0 E(t)) H^{-1}(-C' E'(t)) + \widetilde{C}_5 C' E'(t) \\
&\quad \leq -C_3 H'(\varepsilon_0 E(t)) E(t) + \widetilde{C}_5 H^*(H'(\varepsilon_0 E(t))) \qquad (3.39) \\
&\quad \leq -C_3 H'(\varepsilon_0 E(t)) E(t) + \widetilde{C}_5 H'(\varepsilon_0 E(t)) \varepsilon_0 E(t) \\
&\quad \leq -\widetilde{C}_3 H'(\varepsilon_0 E(t)) E(t) \\
&\quad = -\widetilde{C}_3 H_2(E(t)).
\end{aligned}
$$

We note that in the second inequality, we used (3.37) and $0 \leq H'(\varepsilon_0 E(t)) \leq H'(\varepsilon_0 E(0))$. Let

$$
\widetilde{L}(t) = \begin{cases} L(t) + \mu E(t), \text{if } H \text{ is linear on}[0, \varepsilon'], \\ H'(\varepsilon_0 E(t))(L(t) + C\mu_1 E(t)) + C_5 C' E(t), \\ \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \varepsilon'], \end{cases} \tag{3.40}
$$

from (3.33) and (3.39), it follows that

$$
\widetilde{L}'(t) \leq -c_4 H_2(E(t)), \quad \forall s \geq 0. \tag{3.41}
$$

On the other hand, we can choose $M > 0$ larger as needed, and we find from Lemma (3.5) that $E(t)$ is equivalent to $L(t)$. So, $\widetilde{L}'(t)$ is also equivalent to $E(t)$. By the fact that H_2 is increasing, we obtain

$$
\widetilde{L}'(t) \leq -c_4 H_2(\widetilde{L}'(t)), \quad \forall t \geq 0. \tag{3.42}
$$

Noting that $H_1' = -1/H_2$, (see [11]) we have the following from (3.42)

$$
\widetilde{L}'(t) H_1'(\widetilde{L}'(t)) \geq \widetilde{c}_4 \quad \forall t \geq 0. \tag{3.43}
$$

A simple integration over $(0, t)$ yields

$$
H_1'(\widetilde{L}'(t)) \geq H_1(\widetilde{L}'(0)) + \widetilde{c}_4 t.
$$

Then, using the fact that H_1^{-1} is decreasing, we have

$$
\widetilde{L}'(t) \leq H_1^{-1}\left(\widetilde{L}'(0))\widetilde{c}_4 t\right). \tag{3.44}
$$

Consequently, the equivalence of L, $\widetilde{L}$ and E, yields the estimate

$$
E(t) \leq \omega_1 H_1^{-1}(\omega_2 t + \omega_3). \tag{3.45}
$$

4. Conclusions

In the study of many physical phenomena and processes, a small or large parameter is often singled out, therefore mathematical models of these phenomena or processes can be singularly perturbed by dynamical systems. The study of the dynamics of equations of this type is of great interest.

Several types of singularly perturbed systems with an infinite-dimensional phase space are studied in this work including equations with a large delay and equations of parabolic type with small diffusion. For such systems, the problem is solved by investigating the local dynamics, i.e., the behavior of the solutions in some small fixed neighborhood of the equilibrium state, and finding an asymptotic approximation of the steady regimes.

The equations with delay considered in this paper arise naturally as mathematical models in many applications, especially in biology, medicine, neurodynamics, radiophysics, electronics, laser physics, information processing and transmission systems. Among them, an important place is occupied by systems in which the delay time is relatively large.

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Conflict of interest

The authors declare that there is no conflicts of interest.

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