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Research article

Controllability of a generalized multi-pantograph system of non-integer order with state delay

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Abstract: This paper presents the dynamical aspects of a nonlinear multi-term pantograph-type system of fractional order. Pantograph equations are special differential equations with proportional delays that are employed in many scientific disciplines. The pantograph mechanism, for instance, has been applied in numerous scientific disciplines like electrodynamics, engineering, and control theory. Because of its key rule in diverse fields, the current study establishes some necessary criteria for its controllability. The main idea of the proof is based on converting the system into a fixed point problem and introducing a suitable controllability Gramian matrix $\mathcal{G}_c$. The Gramian matrix $\mathcal{G}_c$ is used to demonstrate the linear system's controllability. Controllability criteria for the associated nonlinear system have been established in the sections that follow using the Schaefer fixed-point theorem and the Arzela-Ascoli theorem, as well as the controllability of the linear system and a few key assumptions. Finally, a computational example is listed.

Keywords: fractional order system; delayed Mittag-Leffler function; Grammian matrix; control function; fixed point theorem; multi-pantograph; controllability
Mathematics Subject Classification: 93Cxx

1. Introduction

In dynamical control systems, controllability is an essential tool and plays a vital role in diverse fields of sciences and engineering. In such systems, to achieve a specific goal, an input control function is acquired to drive the system state from some known state to a desirable state. The dynamics of control systems are usually modeled using ordinary differential equations, partial differential equations, or even more precisely fractional order differential equations. The research work done in [1–3], demonstrates some modern and classical work on control theory.

In recent years, an increasing interest has been seen in fractional mathematical models in order to increase the quality of modeling real-world phenomena and enhance system stability. Fractional order derivatives have gained a nominal rule in modern research in diverse fields of science such as physics, chemistry, mathematical biology, and engineering. Being an accurate and precise method of modeling dynamical systems, it has attracted many great researchers and mathematicians in various applied circumstances; see [4–10] for details. In contrast, integer order controllers have been generalized to fractional order controllers [11], whereas Manabe has explored fractional order systems in the area of automatic control. Similar work on fractional order controllers and discretization techniques has also been carried out in [12]. Stability analysis of a noninteger order PID controller, optimization, and design have been explored in [13]. For further study on some new ideas in dynamic systems and control in the framework of fractional calculus, we suggest the research work done in [14–20].

Several systems in our surroundings have a great dependence on their entire past states besides their reliance on their recent states. Such systems include chemical processes, transmission lines, rolling mill systems and our industrial systems. Laplace and Condorcet introduced delay differential equations and delay integro-differential equations in the eighteenth century to model such systems mathematically. Many techniques have been used in the literature to solve such systems with state or control delays [21–23]. Several research studies have also been carried out towards applications of noninteger order systems in diverse fields, and some useful results have been obtained. In the controllability analysis of nonlinear systems, the main difficulty one has to face is the solution of such systems. The most commonly used techniques for finding solutions are the numerical technique, the spectral method, etc. Due to its high accuracy and precision, the spectral method of solutions is advantageous over the other methods of solutions. A similar method of solutions has been utilized in the solutions of linear fractional differential equations by the authors in [24–30].

Among other qualitative aspects like stability, existence, uniqueness of solutions, etc., controllability is a key concept and has a tremendous role in mathematical control theory. It is used to control an object's behavior to get the intended result. In recent approaches toward the controllability of nonlinear systems, the most powerful and appropriate method is the fixed-point technique. Some fixed-point techniques have particularly been utilized to establish controllability results, depending on the nonlinear function being used in the systems. In [31] Balachandran considered a neutral fractional integro-differential system with distributed delays and explored its controllability results. Balachandran and Krishnan [32], established controllability conditions for a nonlinear fractional order system with multiple delays. In [33] Muslim and George have investigated the controllability of a fractional dynamical system in a Banach space. Controllability analysis of fractional order neutral-type systems with impulsive effects and state delay has been explored in [34]. The relative controllability of a dynamical system governed by a fractional order system with a pure delay has been studied in [35]. The existence results and controllability conditions of a nonlinear system with damping in Hilbert space have been considered in [36]. Kumar in [37] has recently explored fractional order damped delay systems with multiple delays for relative controllability. Yapeng et al. in [38] have investigated the controllability results of a dynamical system with input delay, governed by a fractional order integro-differential system. Their inclusion is given by

$$\begin{cases} {}^c\mathbb{D}^r v(t) = \mathbb{M}v(t) + \mathbb{N}u(t) + \mathbb{Q}u(t-\rho) + h(t, v(t)) \\ \qquad\qquad + g(t, v(t), \int_0^t f(t, s, v(s))ds), \; v(t) \in \mathbb{R}^n, \; t \in J = [0, \ell], \\ v(0) = v_0, \; u(t) = \psi(t), \; -\rho \le t \le 0, \end{cases}$$

where $r \in (0,1)$, $\mathbb{M}$ is an $n \times n$ matrix, $\mathbb{N}$ and $\mathbb{Q}$ are matrices of order $n \times m$, and the functions $h : J \times \mathbb{R}^n \to \mathbb{R}^n$, $g : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $f : J \times J \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and nonlinear.

Nawaz et al. [39], explored the controllability of a dynamical system modeled by a noninteger order differential system with control and state delay. Very recently, in another paper, Nawaz et al. [40] utilized the delayed Mittag-Leffler matrix functions and Schauder's fixed point techniques for controllability results of a nonlinear system with pure delay in the framework of fractional calculus. Their inclusion in the linear case is given by

$$\begin{cases} {}^c\mathbb{D}^r v(t) = \mathbb{M}v(t-\rho) + \mathbb{N}u(t), \; v(t) \in \mathbb{R}^n, \; \rho > 0, \; t \in J = [0, \ell], \\ v(t) = \psi(t), -\rho \le t \le 0, \end{cases}$$

and the corresponding nonlinear system is described by

$$\begin{cases} {}^c\mathbb{D}^r v(t) = \mathbb{M}v(t-\rho) + \mathbb{N}u(t) + h(t, v(t-\rho), u(t)), \; v(t) \in \mathbb{R}^n, \; \rho > 0, \; t \in J = [0, \ell], \\ v(t) = \psi(t), -\rho \le t \le 0, \end{cases}$$

where ${}^c\mathbb{D}^r v(t)$ represents the Caputo derivative of $v(t)$ with $0 < r \le 1$. $\mathbb{M}$ and $\mathbb{N}$ are the matrices of order $n \times n$ and $n \times m$, respectively. $v : J \to \mathbb{R}^n$ is continuously differentiable on $[0, \ell]$ with $\ell > (k-1)\rho, k \in \mathcal{N} = \{1, 2, \ldots\}$. $u(t) \in \mathbb{R}^m$ is the input control function, and $h : J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous nonlinear function.

The pantograph equation is a special delay differential equation that plays a nominal role in describing numerous phenomena [41]. The equation was initially introduced by Ockendon and Taylor [42]. It has a tremendous rule in dynamical systems, electrodynamics, control systems, etc [43]. The equation has been generalized by different researchers in diverse forms for establishing existence and stability results [44–48]. However, to the best of our knowledge, no work has been carried out on the controllability of a dynamical system governed by a fractional order generalized multi-pantograph system with state delay. Motivated by the above work, especially [38, 40] and [49, 50], in this paper we present the controllability of a generalized multi-pantograph system in the Caputo sense described by the equation

$$\begin{cases} {}^c\mathbb{D}^r v(t) = \mathbb{M}v(t-\rho) + \mathbb{N}u(t) \\ \qquad\qquad + g(t, v(t-\rho), v(\eta_1 t), \ldots, v(\eta_n t)), \; v(t) \in \mathbb{R}^n, \; t \in J = [0, \ell], \\ v(t) = \psi(t), -\rho \le t \le 0, \end{cases} \tag{1.1}$$

where $0 < \mu_i < 1, i = 1, 2, \ldots, n$, $0 < r < 1$, $\mathbb{M} \in \mathbb{R}^{n \times n}$, $\mathbb{N} \in \mathbb{R}^{n \times m}$, $u(t) \in \mathbb{R}^m$ is the input control function, $v : [-\tau, \ell] \to \mathbb{R}^n$ is differentiable and continuous on $[0, \ell]$ with $\ell > (k-1)\rho, k \in \mathcal{N} = \{1, 2, \ldots\}$, $\rho > 0$ is a state delay, $\psi \in C^1_\rho = C^1([-\rho, 0], \mathbb{R}^n)$, and $g : I \times \mathbb{R}^{(i+1)n} \to \mathbb{R}^n$ is a nonlinear continuous function.

2. Background materials

Definition 2.1. *[48] The fractional integral of a suitable function $f : [0,\infty) \to \mathbb{R}$, of order $\nu > 0$ is defined as*

$$I_0^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s) ds.$$

$\Gamma(\nu)$, represents the gamma function of ν.

Definition 2.2. *[48] The Caputo fractional derivative of a suitable function $f : [0,\infty) \to \mathbb{R}$, of order $\nu > 0$ is defined as*

$${}^cD^\nu f(t) = \frac{1}{\Gamma(q-\nu)} \int_0^t (t-s)^{q-\nu-1} f^{(q)}(s) ds, \; q-1 < \nu \leq q.$$

Here $q = [\nu] + 1$. In particular for $q = 1$, we have $0 < \nu \leq 1$. Consequently, one may arrive at

$${}^cD^\nu f(t) = \frac{1}{\Gamma(1-\nu)} \int_0^t (t-s)^{-\nu} f'(s)\, ds.$$

Definition 2.3. *[49] Given a state matrix $\mathbb{M}$, a state variable $v \in \mathbb{R}^{n\times 1}$ and delay ρ, the delayed Mittag-Leffler type matrix function $\mathscr{E}_\rho^{\mathbb{M}t^r}$, in a single parameter r is defined as*

$$\mathscr{E}_\rho^{\mathbb{M}t^r} = \begin{cases} \Theta, & -\infty < t < -\rho, \\ I, & -\rho \leq t \leq 0, \\ I + \sum_{k=1}^\infty \frac{\mathbb{M}^k (t-(k-1)\rho)^{kr}}{\Gamma(kr+1)}, & (k-1)\rho < t \leq k\rho, \end{cases} \tag{2.1}$$

where Θ is a null matrix, $\mathbb{M} \in \mathbb{R}^{n\times n}$, and I represents an identity matrix.

Definition 2.4. *[50] Given a state matrix $\mathbb{M}$, a state variable $v \in \mathbb{R}^{n\times 1}$ and a delay $\rho > 0$, the delayed Mittag-Leffler type matrix function $\mathscr{E}_{\rho,\bar{r}}^{\mathbb{M}t^r}$, in two parameters r and $\bar{r}$ is defined as*

$$\mathscr{E}_{\rho,\bar{r}}^{\mathbb{M}t^r} = \begin{cases} \Theta, & -\infty < t < -\rho, \\ I \frac{(\rho+t)^{r-1}}{\Gamma(\bar{r})}, & -\rho \leq t \leq 0, \\ \frac{(\rho+t)^{r-1}}{\Gamma(\bar{r})} I + \sum_{k=1}^\infty \frac{\mathbb{M}^k (t-(k-1)\tau)^{(k+1)r-1}}{\Gamma(kr+\bar{r})}, & (k-1)\rho < t \leq k\rho, \end{cases} \tag{2.2}$$

where Θ is a null matrix, $\mathbb{M} \in \mathbb{R}^{n\times n}$, and I represents an identity matrix.

Lemma 2.1. *The qth order derivatives of each of the single and double parameter delayed Mittag-Leffler functions have the following forms:*

$$\mathscr{E}_{\rho,1-q}^{\mathbb{M}t^{r-q}} = \begin{cases} \Theta, & -\infty < t < -\rho, \\ \Theta, & -\rho \leq t \leq 0, \\ \displaystyle\sum_{k=1}^\infty \frac{\mathbb{M}^k (t-(k-1)\rho)^{kr-q}}{\Gamma(kr+1-q)}, & (k-1)\rho < t \leq k\rho, \end{cases} \tag{2.3}$$

and

$$\mathscr{E}_{\rho,\bar{r}-q}^{\mathbb{M}t^{r-q}}=\begin{cases}\Theta, & -\infty<t<-\rho,\\ I\dfrac{\Gamma(r)(t+\rho)^{r-1-q}}{\Gamma(\bar{r})\Gamma(r-q)}, & -\rho\leq t\leq 0,\\ I\dfrac{\Gamma(r)(t+\rho)^{r-1-q}}{\Gamma(\bar{r})\Gamma(r-q)}+\sum\limits_{k=1}^{\infty}\dfrac{\mathbb{M}^k\Gamma((k+1)r)(t-(k-1)\rho)^{kr+r-q-1}}{\Gamma(kr+\bar{r})\Gamma(kr+r-q)}, & (k-1)\rho<t\leq k\rho,\end{cases}\tag{2.4}$$

where q is a positive integer.

Proof. Differentiating Eqs (2.1) and (2.2) q times in a row makes it simple to determine the outcome. □

Lemma 2.2. *For a square matrix $\mathbb{M}\in\mathbb{R}^{n\times n}$ with constant entries, the Inequality*

$$\|\mathscr{E}_{\rho}^{\mathbb{M}t^r}\|\leq\mathscr{E}_r(\|\mathbb{M}\|t^r),\quad (k-1)\rho\leq t\leq k\rho,\quad k=\{1,2,\ldots\},$$

hold, where $\mathscr{E}_r(\|\mathbb{M}\|t^r)=\sum_{k=0}^{\infty}\frac{\mathbb{M}t^r}{\Gamma(kr+1)}, r>0, t\in\mathbb{R}$ denotes the Mittag-Leffler matrix function.

Proof. By the results given in Eq (2.1), we have

$$\begin{aligned}\|\mathscr{E}_{\rho}^{\mathbb{M}t^r}\|&=\|I+\sum_{k=1}^{\infty}\frac{\mathbb{M}^k(t-(k-1)\rho)^{kr}}{\Gamma(kr+1)}\|,\quad (k-1)\rho\leq t\leq k\rho,\quad k=\{1,2,\ldots\},\\ &\leq\|I\|+\sum_{k=1}^{\infty}\frac{\|\mathbb{M}^k\|\|(t-(k-1)\rho)^{kr}\|}{\Gamma(kr+1)},\\ &\leq\|I\|+\sum_{k=1}^{\infty}\frac{(\|\mathbb{M}\|t)^{kr}}{\Gamma(kr+1)},\\ &=\mathscr{E}_r(\|\mathbb{M}\|t^r),\end{aligned}$$

which we needed to prove. □

Lemma 2.3. *[50] A solution $v\in C([-\rho,\ell],\mathbb{R}^n)$, of the system*

$$\begin{cases}{}^c\mathbb{D}^r v(t)=\mathbb{M}v(t-\rho)+f(t,v(t)), & v(t)\in\mathbb{R}^n,\ t\in J=[0,\ell],\ \rho>0,\\ v(t)=\psi(t), & -\rho\leq t\leq 0,\end{cases}$$

where $f:J\to\mathbb{R}^n$ is a continuous function, is characterized by

$$v(t)=\mathscr{E}_{\rho}^{\mathbb{M}t^r}\psi(-\rho)+\int_{-\rho}^{0}\mathscr{E}_{\rho}^{\mathbb{M}(t-\rho-s)^r}\psi'(s)ds+\int_{0}^{t}\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}f(s,v(s))ds.$$

In addition, we define

$$\begin{aligned}&\|v\|=\sum_{i=1}^{n}|v_i|, v_i\in v, \|\mathbb{M}\|=\max_{1\leq j\leq n}\sum_{i=1}^{n}|m_{ij}|, m_{ij}\in\mathbb{M},\\ &\|\mathbb{N}\|=\max_{1\leq j\leq m}\sum_{i=1}^{n}|n_{ij}|, n_{ij}\in\mathbb{N},\\ &\|\psi\|_C=\max_{t\in[-\rho,0]}|\psi(t)|.\end{aligned}$$

Table 1. Description of the notations and symbols used

${}^{c}D^{\alpha}$	Caputo derivative of fractional order alpha
$\mathcal{G}_c$	Controllability Gramian matrix
$C\left(J, \mathbb{R}^n\right)$	Banach space of vector valued continuous functions
$\mathscr{E}_{\rho,\bar{r}}^{\mathbb{M}t^r}$	Two parameter delayed Mittag-Leffler function
ρ	A fixed delay
$\psi(t)$	An arbitrary continuously differentiable function, i.e., $\psi \in C^1 = C^1_\rho\left([-\rho, 0], \mathbb{R}^n\right)$
$0 < \eta_i < 1$	Proportional delays
FOS	Fractional order system

3. Controllability analysis

In the following, we look into the dynamical system's controllability. We have divided the system into linear and nonlinear components, and controllability results were established for each instance. The Caputo derivative, the delayed Mittag-Leffler function, and some fixed-point approaches are the major tools we use in this work.

3.1. *Linear fractional-order system*

This section explores the controllability results of the linear system Eq (4.2), which is given by

$$\begin{cases} {}^{c}\mathbb{D}^{r}v(t) = \mathbb{M}v(t-\rho) + \mathbb{N}u(t), \; v(t) \in \mathbb{R}^n, \; t \in J = [0, \ell], \\ v(t) = \psi(t), -\rho \leq t \leq 0. \end{cases} \tag{3.1}$$

Utilizing Lemma 2.2, solution $v(t)$ to the system Eq (3.1) can be expressed as given by

$$v(t) = \mathscr{E}_{\rho}^{\mathbb{M}t^r}\psi(-\rho) + \int_{-\rho}^{0} \mathscr{E}_{\rho}^{\mathbb{M}(t-\rho-s)^r}\psi'(s)ds + \int_{0}^{t} \mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\mathbb{N}u(s)ds. \tag{3.2}$$

The Gramian controllability matrix $\mathcal{G}_c(0, \ell)$, for $t, \in J$ and $v(t) = \psi(t), t \in [-\rho, 0]$, is defined as

$$\mathcal{G}_c(0, \ell) = \int_{0}^{\ell} \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r}\mathbb{N}\mathbb{N}^{*}\mathscr{E}_{\rho,r}^{\mathbb{M}^{*}(\ell-\rho-s)^r}ds, \tag{3.3}$$

where $*$ represents the matrix transpose.

Definition 3.1. *The linear fractional-order system Eq* (3.1) *is said to be controllable on an interval* $[0, \ell]$*, if there exists an admissible control function* $u(t)$ *such that the solution Eq* (3.2) *to the system Eq* (3.1) *fulfills the conditions* $v(0) = \psi(0)$ *and* $v(\ell) = v_{sd}$.

Theorem 3.1. *The linear fractional-order system Eq* (3.1) *is controllable on* $[0, \ell]$*, if and only if the Gramian matrix* $\mathcal{G}_c(0, \ell)$ *Eq* (3.3) *is invertible.*

Proof. **Sufficiency:** Assume that $\mathcal{G}_c(0, \ell)$ is invertible on $[0, \ell]$. Then, the control function $u(t)$ that steers the system Eq (3.1) from an initial state $\psi(0)$ to any desirable state v_{sd} is given by

$$u(s) = \mathbb{N}^{*}\mathscr{E}_{\rho,r}^{\mathbb{M}^{*}(\ell-\rho-s)^r}\mathcal{G}_c^{-1}(0, \ell)[v_{sd} - \mathscr{E}_{\rho}^{\mathbb{M}\ell^r}\psi(-\rho) + \int_{-\rho}^{0} \mathscr{E}_{\rho}^{\mathbb{M}(\ell-\rho-s)^r}\psi'(s)ds]. \tag{3.4}$$

Substituting $t = \ell$ in Eq (3.2) and plugging Eq (3.4) in the resultant equation, we have

$$\begin{aligned}
v(\ell) &= \mathscr{E}_{\rho}^{\mathbb{M}\ell^r}\psi(-\rho) + \int_{-\rho}^{0} \mathscr{E}_{\rho}^{\mathbb{M}(\ell-\rho-s)^r}\psi'(s)ds \\
&+ \int_{0}^{\ell} \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r}\mathbb{N}\mathbb{N}^* \mathscr{E}_{\rho,r}^{\mathbb{M}^*(\ell-\rho-s)^r}\mathcal{G}_c^{-1}(0,\ell) \\
&\times [v_{sd} - \mathscr{E}_{\rho}^{\mathbb{M}\ell^r}\psi(-\rho) - \int_{-\rho}^{0} \mathscr{E}_{\rho}^{\mathbb{M}(\ell-\rho-s)^r}\psi'(s)ds]ds \\
&= \mathscr{E}_{\rho}^{\mathbb{M}\ell^r}\psi(-\rho) + \int_{-\rho}^{0} \mathscr{E}_{\rho}^{\mathbb{M}(\ell-\rho-s)^r}\psi'(s)ds + \mathcal{G}_c(0,\ell)\mathcal{G}_c^{-1}(0,\ell) \\
&\times [v_{sd} - \mathscr{E}_{\rho}^{\mathbb{M}\ell^r}\psi(-\rho) - \int_{-\rho}^{0} \mathscr{E}_{\rho}^{\mathbb{M}(\ell-\rho-s)^r}\psi'(s)ds].
\end{aligned} \tag{3.5}$$

Simplification of the last equation yields

$$v(\ell) = v_{sd},$$

which implies that the system Eq (3.1) is controllable.
Necessity: Let $\det[\mathcal{G}_c(0,\ell)] = 0$, i.e., $\mathcal{G}_c^{-1}(0,\ell)$ is not well defined. Then, there exists a nonzero state $\underline{v}$ that satisfies the following condition:

$$\begin{aligned}
&\underline{v}^*\mathcal{G}_c(0,\ell)\underline{v} = 0, \\
&\Rightarrow \int_0^{\ell} \underline{v}^* \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r}\mathbb{N}\mathbb{N}^* \mathscr{E}_{\rho,r}^{\mathbb{M}^*(\ell-\rho-s)^r}\underline{v}ds = 0, \\
&\Rightarrow \int_0^{\ell} \|\underline{v}^* \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r}\mathbb{N}\|^2 ds = 0,
\end{aligned}$$

and following the above set of implications, one arrive at

$$\underline{v}^* \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r}\mathbb{N} = 0, \tag{3.6}$$

for all $\rho, s \in [0,\ell]$. Let the system Eq (3.1) be controllable on $[0,\ell]$, and there exist two control input functions $\hat{u}(t)$ and $\tilde{u}(t)$ such that

$$v(\ell) = \mathscr{E}_{\rho}^{\mathbb{M}\ell^r}\psi(-\rho) + \int_{-\rho}^{0} \mathscr{E}_{\rho}^{\mathbb{M}(\ell-\rho-s)^r}\psi'(s)ds + \int_0^t \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r}\mathbb{N}\hat{u}(s)ds = 0, \tag{3.7}$$

and

$$v(\ell) = \mathscr{E}_{\rho}^{\mathbb{M}\ell^r}\psi(-\rho) + \int_{-\rho}^{0} \mathscr{E}_{\rho}^{\mathbb{M}(\ell-\rho-s)^r}\psi'(s)ds + \int_0^t \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r}\mathbb{N}\tilde{u}(s)ds = \underline{v}. \tag{3.8}$$

From Eqs (3.7) and (3.8), we obtain

$$\begin{aligned}
&\underline{v} = \int_0^t \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r}\mathbb{N}(\tilde{u}(s) - \hat{u}(s)ds, \\
&\Rightarrow \underline{v}^*\underline{v} = \int_0^t \underline{v}^* \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r}\mathbb{N}(\tilde{u}(s) - \hat{u}(s)ds.
\end{aligned} \tag{3.9}$$

By utilizing Eq (3.6), one may write

$$\underline{v}^*\underline{v} = 0,$$

which is a contradiction to the fact that $\underline{v} \neq 0$. Hence, our supposition that $\mathcal{G}_c^{-1}(0, \ell)$ is not well defined is wrong, and the theorem statement that $\mathcal{G}_c(0, \ell)$ is invertible is true. □

Remark 3.1. *The behavior of physical systems is modeled mathematically using linear fractional-order systems (FOS). They can be identified by a differential equation with fractional-order derivatives, which are non-integer exponents that provide greater modeling freedom for complex processes. A system's ability to be controlled by outside inputs is referred to as its controllability. The finding mentioned above has multiple uses in various fields. Aerospace engineers have created control systems for aerospace vehicles like spacecraft, satellites, and missiles using linear FOS controllability. Analysis of controllability aids in identifying the bare minimum of control inputs necessary to direct the system to a desired state. Moreover, linear FOS controllability has been employed in robotics to construct robot control systems. Designing effective control systems with the fewest possible control inputs is made possible by the consideration of controllability. Power electronic systems like inverters and converters can be designed using the equations. Its dynamical characteristic aids in the development of effective control schemes capable of controlling output voltage and current.*

3.2. Nonlinear fractional-order system

The aim of this section is to establish controllability conditions for the nonlinear system utilizing some fixed point techniques. The nonlinear system is described by the inclusion:

$$\begin{cases} {}^c\mathbb{D}^r v(t) = \mathbb{M}v(t-\rho) + \mathbb{N}u(t) \\ \qquad\qquad + g(t, v(t-\rho), v(\eta_1 t), \ldots, v(\eta_n t)),\ v(t) \in \mathbb{R}^n,\ t \in J = [0, \ell], \\ v(t) = \psi(t), -\rho \leq t \leq 0. \end{cases} \tag{3.10}$$

Utilizing Lemma (2.2), the solution $v(t)$ of the system Eq (3.10) is given by

$$\begin{aligned} v(t) = \mathscr{E}_\rho^{\mathbb{M}t^r}\psi(-\rho) + \int_{-\rho}^0 \mathscr{E}_\rho^{\mathbb{M}(t-\rho-s)^r}\psi'(s)ds + \int_0^t \mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\mathbb{N}u(s)ds \\ + \int_0^t \mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r} g(s, v(s-\rho), v(\eta_1 s), \ldots, v(\eta_n s))ds. \end{aligned} \tag{3.11}$$

To establish our results, we consider the following assumptions:

A_1 : There exists a nonzero constant K_1 such that the nonlinear continuous function $g : I\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ satisfies $g(t, v(t-\rho), v(\eta_1 t), \ldots, v(\eta_n t)) \leq K_1,\ \forall\ t \in [0, \ell] = J$.

A_2 : To avoid tedious calculations, it is assumed that

$$\begin{cases} n_1 = \|\psi(t)\|, \quad n_2 = \|\psi'(t)\|, \quad t \in [-\rho, 0], \\ m_1 = \|\mathscr{E}_{\rho}^{\mathbb{M}t^r}\| = \sup_{0\le s\le t\le \ell} |\mathscr{E}_{\rho}^{\mathbb{M}t^r}|, \\ m_2 = \|\mathscr{E}_{\rho}^{\mathbb{M}(t-\rho-s)^r}\| = \sup_{0\le s\le t\le \ell} |\mathscr{E}_{\rho}^{\mathbb{M}(t-\rho-s)^r}|, \\ m_3 = \|\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\| = \sup_{0\le s\le t\le \ell} |\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}|, \\ m_4 = \|\mathscr{E}_{\rho,1-q}^{\mathbb{M}t^{r-q}}\| = \sup_{0\le s\le t\le \ell} |\mathscr{E}_{\rho,1-q}^{\mathbb{M}t^{r-q}}|, \\ m_5 = \|\mathscr{E}_{\rho,1-q}^{\mathbb{M}(t-\rho-s)^{r-q}}\| = \sup_{0\le s\le t\le \ell} |\mathscr{E}_{\rho,1-q}^{\mathbb{M}(t-\rho-s)^{r-q}}|, \\ m_6 = \|\mathscr{E}_{\rho,r-q}^{\mathbb{M}(t-\rho-s)^{r-q}}\| = \sup_{0\le s\le t\le \ell} |\mathscr{E}_{\rho,r-q}^{\mathbb{M}(t-\rho-s)^{r-q}}|, \\ m_7 = \sup_{0\le s\le t\le \ell} |(t-s)^{n-r-1}|. \end{cases}$$

Theorem 3.2. *If the hypotheses A_1, A_2 hold, and the linear system Eq* (3.1) *is controllable on* $[0, \ell]$*, then the nonlinear Eq* (3.10) *is also controllable on* $[0, \ell]$.

Proof. To establish the desired results of controllability, we transform the Eq (3.11) into operator form. So, define an operator $T : C_n \to C_n$ by

$$\begin{aligned} (Tv)(t) = {} & \mathscr{E}_{\rho}^{\mathbb{M}t^r}\psi(-\rho) + \int_{-\rho}^{0} \mathscr{E}_{\rho}^{\mathbb{M}(t-\rho-s)^r}\psi'(s)ds + \int_0^t \mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\mathbb{N}u(s)ds \\ & + \int_0^t \mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r} g(s, v(s-\rho), v(\eta_1 s), \ldots, v(\eta_n s))ds, \end{aligned} \tag{3.12}$$

and taking norms, we have

$$\begin{aligned} \|(Tv)(t)\| \le {} & \|\mathscr{E}_{\rho}^{\mathbb{M}t^r}\| \, \|\psi(-\rho)\| + \int_{-\rho}^{0} \|\mathscr{E}_{\rho}^{\mathbb{M}(t-\rho-s)^r}\| \, \|\psi'(s)\| ds \\ & + \int_0^t \|\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\| \, \|\mathbb{N}\| \, \|u(s)\| ds \\ & + \int_0^t \|\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\| \, \|g(s, v(s-\rho), v(\eta_1 s), \ldots, v(\eta_n s))\| ds \\ \le {} & m_1 n_1 + \rho m_2 n_2 + \ell m_3 \|\mathbb{N}\| K_2 + \ell m_3 K_1 \\ = {} & m_1 n_1 + \rho m_2 n_2 + \ell m_3 \left(\|\mathbb{N}\| K_2 + K_1\right) = \lambda_1, \end{aligned} \tag{3.13}$$

where

$$u(s) = \mathbb{N}^* \mathscr{E}_{\rho,r}^{\mathbb{M}^*(\ell-\rho-s)^r} \mathcal{G}_c^{-1}(0, \ell)\Psi, \tag{3.14}$$

and

$$\begin{aligned} \Psi = {} & v_{sd} - \mathscr{E}_{\rho}^{\mathbb{M}\ell^r}\psi(-\rho) - \int_{-\rho}^{0} \mathscr{E}_{\rho}^{\mathbb{M}(\ell-\rho-s)^r}\psi'(s)ds \\ & - \int_0^{\ell} \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r} g(s, v(s-\rho), v(\eta_1 s), \ldots, v(\eta_n s))ds. \end{aligned}$$

Taking norms, one has

$$
\begin{aligned}
\|u(s)\| &\leq \|\mathbb{N}^*\| \, \|\mathscr{E}_{\rho,r}^{\mathbb{M}^*(\ell-\rho-s)^r}\| \, \|\mathcal{G}_c^{-1}(0,\ell)\| \, [\|v_{sd}\| + \|\mathscr{E}_{\rho}^{\mathbb{M}t^r}\| \, \|\psi(-\rho)\| + \int_{-\rho}^{0} \|\mathscr{E}_{\rho}^{\mathbb{M}(\ell-\rho-s)^r}\| \, \|\psi'(s)\| ds \\
&\quad + \int_0^{\ell} \|\mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r}\| \, \|g(s, v(s-\rho), v(\eta_1 s), \ldots, v(\eta_n s))\| ds] \\
&\leq \|\mathbb{N}^*\| m_3 \|\mathcal{G}_c^{-1}(0,\ell)\| \, [\|v_{sd}\| + m_1 n_1 + \rho m_2 n_2 + \ell m_3 K_1] = K_2.
\end{aligned}
\tag{3.15}
$$

Define a closed convex subset $\mathbb{S}_{\lambda_1}$, by

$$
\mathbb{S}_{\lambda_1} = \{v \in C_n(J) : \|v\| \leq \lambda_1\}.
$$

From this, we see that the operator T maps $\mathbb{S}_r$ into itself. What remains to be proved is that the operator T has a fixed point. The continuity of T follows from the continuity of g. Then, by the Arzela-Ascoli theorem, it follows that T is completely continuous as well. This property in turn shows that there exists a fixed point $v \in \mathbb{S}_{\lambda_1}$ by Schauder's fixed point theorem and assumes that $Tv = v$. Moreover, by substituting $t = \ell$ in Eq (3.12) and then plugging Eq (3.14) in the resultant equation, we have

$$
\begin{aligned}
(Tv)(\ell) &= \mathscr{E}_{\rho}^{\mathbb{M}\ell^r} \psi(-\rho) + \int_{-\rho}^{0} \mathscr{E}_{\rho}^{\mathbb{M}(\ell-\rho-s)^r} \psi'(s) ds \\
&\quad + \int_0^{\ell} \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r} \mathbb{N}\mathbb{N}^* \mathscr{E}_{\rho,r}^{\mathbb{M}^*(\ell-\rho-s)^r} \mathcal{G}_c^{-1}(0,\ell) ds \\
&\quad \times [v_{sd} - \mathscr{E}_{\rho}^{\mathbb{M}t^r} \psi(-\rho) - \int_{-\rho}^{0} \mathscr{E}_{\rho}^{\mathbb{M}(\ell-\rho-s)^r} \psi'(s) ds \\
&\quad - \int_0^{\ell} \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r} g(s, v(s-\rho), v(\eta_1 s), \ldots, v(\eta_n s)) ds] \\
&\quad + \int_0^{\ell} \mathscr{E}_{\rho,r}^{\mathbb{M}(\ell-\rho-s)^r} g(s, v(s-\rho), v(\eta_1 s), \ldots, v(\eta_n s)) ds \\
&= v_{sd},
\end{aligned}
\tag{3.16}
$$

which shows that the input function $u(t)$ defined by Eq (3.14) transfers the system state from $\psi(-\rho)$ to a desired state v_{sd} in time $t = \ell$. Hence, the system Eq (3.10) is controllable. □

Remark 3.2. *The findings examined in this section have numerous applications in physics, engineering, biology, economics, and finance, among other disciplines. The outcomes can be applied to the design of control systems. They can aid in system stabilization, oscillation reduction, and improved control performance. Chemical reaction modeling can also be useful. The reaction's non-integer order dynamics can be captured by the fractional order, and the reaction's temporal delay can be explained by the state delay. Nonlinear fractional order systems with state delay of the pantograph type can be used to mimic financial markets. They can assist with stock price forecasting and market behavior analysis.*

Our next controllability result is based on Schaefer's fixed point theorem.

Lemma 3.1. *(Schaefer's theorem): Let* $\mathbb{V}$ *be a Banach space and* $h : \mathbb{V} \to \mathbb{V}$ *be continuous and compact. Moreover, assume the set* $S = \{v \in \mathbb{V} : v = \xi h(v)\}$*,* $\xi \in [0, 1]\}$*, has a solution for* $\xi = 1$*, and all other solutions for* $0 < \xi < 1$ *are unbounded.*

Theorem 3.3. *The nonlinear system Eq* (3.14) *is controllable on* $[0, \ell]$*, if the assumptions (*A_1*&*A_2*) hold and the linear system Eq* (3.1) *is controllable on* $[0, \ell]$.

Proof. Define a Banach space $V = \{v : v, v^{(q)}, {}^cD^\nu(v) \in (I, R^n)\}$, endowed with the norm $\|v\| = \max\{\|v\|, \|v^{(q)}(t)\|, \|{}^cD^\nu v(t)\|, \|u\|\}$. Also, define an operator $T : C_n \to C_n$, by

$$\begin{aligned}(Tv)(t) = \mathscr{E}_\rho^{\mathbb{M}t^r}\psi(-\rho) + \int_{-\rho}^0 \mathscr{E}_\rho^{\mathbb{M}(t-\rho-s)^r}\psi'(s)ds + \int_0^t \mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\mathbb{N}u(s)ds \\ + \int_0^t \mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r} g(s, v(s-\rho), v(\eta_1 s), \ldots, v(\eta_n s))ds,\end{aligned} \tag{3.17}$$

where $u(t)$ is as defined by Eq (3.14).

To show that the operator T satisfies Schaefer's fixed point theorem, we will go through several steps:

Step I. As a first step, we show that the set $\zeta(T) = \{v \in V : v = \sigma Tv, 0 \le \sigma \le 1\}$ is bounded in $[0, \ell]$. For $v \in \zeta(T)$ and $t \in [0, \ell]$, we have

$$\begin{aligned}v(t) = \sigma\mathscr{E}_\rho^{\mathbb{M}t^r}\psi(-\rho) + \sigma\int_{-\rho}^0 \mathscr{E}_\rho^{\mathbb{M}(t-\rho-s)^r}\psi'(s)ds + \sigma\int_0^t \mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\mathbb{N}u(s)ds \\ + \sigma\int_0^t \mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r} g(s, v(s-\rho), v(\eta_1 s), \ldots, v(\eta_n s))ds.\end{aligned} \tag{3.18}$$

Taking norms and utilizing the assumptions (A_1&A_2), we have

$$\begin{aligned}\|v(t)\| &\le \|\mathscr{E}_\rho^{\mathbb{M}t^r}\|\,\|\psi(-\rho)\| + \int_{-\rho}^0 \|\mathscr{E}_\rho^{\mathbb{M}(t-\rho-s)^r}\|\,\|\psi'(s)\|ds \\ &+ \int_0^t \|\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\|\,\|\mathbb{N}\|\,\|u(s)\|ds \\ &+ \int_0^t \|\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\|\|g(s, v(s-\rho), v(\eta_1 s), \ldots, v(\eta_n s))\|ds \\ &\le m_1n_1 + \rho m_2n_2 + \ell m_3\|\mathbb{N}\|K_2 + \ell m_3K_1 \\ &= m_1n_1 + \rho m_2n_2 + \ell m_3\left(\|\mathbb{N}\|K_2 + K_1\right) = \lambda_1.\end{aligned} \tag{3.19}$$

Also using Lemma (2.1) and following the same steps as above, we obtain

$$\begin{aligned} v^{(q)}(t) &= \sigma\mathscr{E}_{\rho,1-q}^{\mathbb{M}t^{r-q}}\psi(-\rho) + \sigma\int_{-\rho}^{0}\mathscr{E}_{\rho,1-r}^{\mathbb{M}(t-\rho-s)^{r-q}}\psi'(s)ds + \sigma\int_{0}^{t}\mathscr{E}_{\rho,r-q}^{\mathbb{M}(t-\rho-s)^{r-q}}\mathbb{N}u(s)ds \\ &+ \sigma\int_{0}^{t}\mathscr{E}_{\rho,r-q}^{\mathbb{M}(t-\rho-s)^{r-q}}g(s,v(s-\rho),v(\eta_1 s),\ldots,v(\eta_n s))ds \\ &= \sigma\mathscr{E}_{\rho,1-q}^{\mathbb{M}t^{r-q}}\psi(-\rho) + \sigma\int_{0}^{t}\mathscr{E}_{\rho,r-q}^{\mathbb{M}(t-\rho-s)^{r-q}}\mathbb{N}u(s)ds \\ &+ \sigma\int_{0}^{t}\mathscr{E}_{\rho,r-q}^{\mathbb{M}(t-\rho-s)^{r-q}}g(s,v(s-\rho),v(\eta_1 s),\ldots,v(\eta_n s))ds, \end{aligned} \tag{3.20}$$

and taking norms, it would yield

$$\begin{aligned} \|v^{(q)}(t)\| &\le \|\mathscr{E}_{\rho,1-q}^{\mathbb{M}t^{r-q}}\|\,\|\psi(-\rho)\| + \int_{0}^{t}\|\mathscr{E}_{\rho,r-q}^{\mathbb{M}(t-\rho-s)^{r-q}}\|\,\|\mathbb{N}\|\,\|u(s)\|ds \\ &+ \int_{0}^{t}\|\mathscr{E}_{\rho,r-q}^{\mathbb{M}(t-\rho-s)^{r-q}}\|\,\|g(s,v(s-\rho),v(\eta_1 s),\ldots,v(\eta_n s))\|ds \\ &= m_4 n_1 + \ell m_6\|\mathbb{N}\|K_2 + \ell m_6 K_1 = \lambda_2. \end{aligned} \tag{3.21}$$

Now, by the definition of a Caputo derivative, we have

$$\begin{aligned} \|^{c}D^{r}v(t)\| &\le \|\frac{1}{\Gamma(q-r)}\|\,\|\int_{0}^{t}(t-s)^{q-r-1}ds\|v^{(q)}(s)\| \\ &\le \|\frac{1}{\Gamma(q-r)}\|\frac{\ell^{q-r}}{q-r}\lambda_2. \end{aligned} \tag{3.22}$$

The last inequality demonstrates that ${}^{c}D^{r}v(t)$ is bounded. This implies that $\zeta(T)$ is bounded as well, since $\|v\| = \{\|v\|, \|v^{(q)}\|, \|^{c}D^{r}v(t)\|, \|u\|\}$.

Step II. We show that the operator T is completely continuous, i.e.,

(a) $T\mathbb{B}_{\lambda_1}$ is uniformly bounded.

Let $\mathbb{B}_{\lambda_1} = \{v \in V : \|v\| \le \lambda_1\}$. The bounded set $\mathbb{B}_{\lambda_1}$ is mapped into the equicontinuous family by the operator T. Then, for $t_1, t_2 \in J, 0 < t_1 < t_2 < \ell$ and $v \in \mathbb{B}_{\lambda_1}$, we have

$$\begin{aligned} &\|(Tv)(t_2) - Tv)(t_1)\| \\ &\le \|(\mathscr{E}_{\rho}^{\mathbb{M}t_2^r} - \mathscr{E}_{\rho}^{\mathbb{M}t_1^r})\psi(-\rho) + \int_{-\rho}^{0}(\mathscr{E}_{\rho}^{\mathbb{M}(t_2-\rho-s)^r} - \mathscr{E}_{\rho}^{\mathbb{M}(t_1-\rho-s)^r})\psi'(s)ds\| \\ &+ \|\int_{0}^{t_1}(\mathscr{E}_{\rho,r}^{\mathbb{M}(t_2-\rho-s)^r} - \mathscr{E}_{\rho,r}^{\mathbb{M}(t_1-\rho-s)^r})\mathbb{N}\mathbb{N}^*\mathscr{E}_{\rho,r}^{\mathbb{M}^*(\ell-\rho-s)^r}\mathcal{G}_c^{-1}(0,\ell)\Psi ds\| \\ &+ \|\int_{t_1}^{t_2}\mathscr{E}_{\rho,r}^{\mathbb{M}(t_2-\rho-s)^r}\mathbb{N}\mathbb{N}^*\mathscr{E}_{\rho,r}^{\mathbb{M}^*(\ell-\rho-s)^r}\mathcal{G}_c^{-1}(0,\ell)\Psi ds\| \\ &+ \|\int_{0}^{t_1}(\mathscr{E}_{\rho,r}^{\mathbb{M}(t_2-\rho-s)^r} - \mathscr{E}_{\rho,r}^{\mathbb{M}(t_1-\rho-s)^r})g(s,v(s-\rho),v(\eta_1 s),\ldots,v(\eta_n s))ds\| \\ &+ \|\int_{t_1}^{t_2}\mathscr{E}_{\rho,r}^{\mathbb{M}(t_2-\rho-s)^r}g(s,v(s-\rho),v(\eta_1 s),\ldots,v(\eta_n s))ds\|. \end{aligned} \tag{3.23}$$

As above, from Eq (3.14) we have

$$\|(Tu)(t_2)-(Tu)(t_1)\| \leq \|\mathbb{N}^*\|\,\|(\mathscr{E}_{\rho,r}^{\mathbb{M}^*(\ell-\rho-t_2)^r}-\mathscr{E}_{\rho,r}^{\mathbb{M}^*(\ell-\rho-t_1)^r})\|\,\|\mathcal{G}_c^{-1}(0,\ell)\|\,\|\Psi\|. \tag{3.24}$$

It yields

$$\begin{aligned}
&\|{}^cD^r(Tv)(t_2)-{}^cD^r(Tv)(t_1)\| \\
&\leq \|\frac{1}{\Gamma(q-r)}\int_{t_1}^{t_2}(t_2-s)^{q-r-1}(Tv)^{(q)}(s)ds\| \\
&+\frac{1}{\Gamma(q-r)}\int_0^{t_1}[(t_2-s)^{q-r-1}-(t_1-s)^{q-r-1}](Tv)^{(q)}(s)ds.
\end{aligned} \tag{3.25}$$

Evidently,

$$\begin{aligned}
&\lim_{t_2\to t_1}\|(Tv)(t_2)-(Tv)(t_1)\|\to 0,\\
&\lim_{t_2\to t_1}\|(Tv)^{(q)}(t_2)-(Tv)^{(q)}(t_1)\|\to 0,\\
&\lim_{t_2\to t_1}\|{}^cD^r(Tv)(t_2)-{}^cD^r(Tv)(t_1)\|\to 0.
\end{aligned}$$

Hence, $\{(Tv): v\in\mathbb{B}_{\lambda_1}\}$ is an equicontinuous family of functions that satisfies the uniform boundedness condition.

(b) The operator T is compact.

To prove the compactness of the operator T, let $\epsilon\in(0,1)$ be a real number and $[0,\ell]$ be fixed, then for every $v\in B_{\lambda_1}$, we have

$$\begin{aligned}
(T_\epsilon v)(t) &= \mathscr{E}_\rho^{\mathbb{M}t^r}\psi(-\rho)+\int_{-\rho}^0\mathscr{E}_\rho^{\mathbb{M}(t-\rho-s)^r}\psi'(s)ds+\int_0^{t-\epsilon}\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\mathbb{N}u(s)ds\\
&+\int_0^{t-\epsilon}\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}g(s,v(s-\rho),v(\eta_1 s),\ldots,v(\eta_n s))ds.
\end{aligned} \tag{3.26}$$

As above, we acquire that $\{(T_\epsilon v): v\in\mathbb{B}_{\lambda_1}\}$ is an equicontinuous family of functions that satisfies the uniform boundedness condition. Then, we have

$$\begin{aligned}
\|(Tv)(t)-(T_\epsilon v)(t)\| &\leq \int_{t-\epsilon}^t\|\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\|\,\|\mathbb{N}\|\,\|u(s)\|ds\\
&+\int_{t-\epsilon}^t\|\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\|\,\|g(s,v(s-\rho),v(\eta_1 s),\ldots,v(\eta_n s))\|ds\\
&\leq \epsilon\, m_3\|\mathbb{N}\|K_2+\epsilon\, m_3K_1\\
&=\epsilon\, m_3\left(\|\mathbb{N}\|K_2+K_1\right).
\end{aligned} \tag{3.27}$$

In the same way, we also have

$$\begin{aligned}
\|(Tv)^{(q)}(t)-(T_\epsilon v)^{(q)}(t)\| &\int_{t-\epsilon}^t\|\mathscr{E}_{\rho,r-q}^{\mathbb{M}(t-\rho-s)^{r-q}}\|\,\|\mathbb{N}\|\,\|u(s)\|ds\\
&+\int_{t-\epsilon}^t\|\mathscr{E}_{\rho,r-q}^{\mathbb{M}(t-\rho-s)^{r-q}}\|\,\|g(s,v(s-\rho),v(\eta_1 s),\ldots,v(\eta_n s))\|ds\\
&\leq \epsilon m_6\|\mathbb{N}\|K_2+\epsilon m_6K_1\\
&=\epsilon m_6\left(\|\mathbb{N}\|K_2+K_1\right).
\end{aligned} \tag{3.28}$$

Now, according to the definition of a Caputo derivative, we have

$$\begin{aligned}
&\|{}^{c}D^{r}(Tv)^{(q)}(t) - {}^{c}D^{r}(T_{\epsilon}v)^{(q)}(t)\| \\
&\leq \|\frac{1}{\Gamma(q-r)}\| \, \| \int_{0}^{t} (t-s)^{q-r-1}[(Tv)^{(q)}(s) - (T_{\epsilon}v)^{(q)}(s)]ds\|.
\end{aligned} \tag{3.29}$$

Evidently,

$$\begin{aligned}
&\lim_{\epsilon\to 0} \|(Tv)(t) - (T_{\epsilon}v)(t)\| \to 0, \\
&\lim_{\epsilon\to 0} \|(Tv)^{(q)}(t) - (T_{\epsilon}v)^{(q)}(t)\| \to 0, \\
&\lim_{\epsilon\to 0} \|{}^{c}D^{r}(Tv)(t) - {}^{c}D^{r}(T_{\epsilon}v)(t)\| \to 0.
\end{aligned}$$

Hence, $\{(Tv) : v \in \mathbb{B}_{\lambda_1}\}$ is compact in V by the Arzola-Ascoli theorem.

Step III. To demonstrate the continuity of T, we assume two more hypotheses.

A_3 : $\lim_{k\to\infty}\|v_k - v(t)\| \to 0$, where $V = \{v_1, v_2, \dots, v_k\}$.

A_4 : There exists a positive constant $\tilde{\omega} = \max\{\|v_k\|, \|u_k\|, \|{}^{c}D^{r}(v_k)\|\}$, $\forall\ k$ and $t, \in [0, \ell]$.

In the light of the hypothesis (A_3 & A_4), we have

$$g(t, v_k(t-\rho), v_k(\eta_1 t), \dots, v_k(\eta_n t)) \leq g(t, v(t-\rho), v(\eta_1 t), \dots, v(\eta_n t)).$$

By the Fatou-Lebesgue theorem, we have

$$\begin{aligned}
\|(Tv_k)(t) - (Tv)(t)\| \leq & \int_0^t \|\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\| \, \|\mathbb{N}\| \, \|u_k(s) - u(s)\| ds \\
& + \int_0^t \|\mathscr{E}_{\rho,r}^{\mathbb{M}(t-\rho-s)^r}\| \, \|g(s, v_k(s-\rho), v_k(\eta_1 s), \dots, v_k(\eta_n s)) \\
& - g(s, v(s-\rho), v(\eta_1 s), \dots, v(\eta_n s))\| ds \\
\leq & \int_0^t m_3 \|\mathbb{N}\| \, \|u_k(s) - u(s)\| ds \\
& + \int_0^t m_3 \|g(s, v_k(s-\rho), v_k(\eta_1 s), \dots, v_k(\eta_n s)) \\
& - g(s, v(s-\rho), v(\eta_1 s), \dots, v(\eta_n s))\| ds,
\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
&\|u_k(s) - u(s)\| \leq m_3^2 \|\mathbb{N}^*\| \, \|\mathcal{G}_c^{-1}(0, \ell)\| \\
&\qquad \int_0^{\ell} \|g(s, v_k(s-\rho), v_k(\eta_1 s), \dots, v_k(\eta_n s)) - g(s, v(s-\rho), v(\eta_1 s), \dots, v(\eta_n s))\| ds.
\end{aligned} \tag{3.31}$$

In a similar way, one can write

$$\begin{aligned}
\|(Tv_k)^{(q)}(t) - (Tv)^{(q)}(t)\| \leq & \int_0^t m_6 \|\mathbb{N}\| \, \|u_k(s) - u(s)\| ds \\
& + \int_0^t m_6 \|g(s, v_k(s-\rho), v_k(\eta_1 s), \dots, v_k(\eta_n s)) \\
& - g(s, v(s-\rho), v(\eta_1 s), \dots, v(\eta_n s))\| ds,
\end{aligned} \tag{3.32}$$

where $\|u_k(s) - u(s)\|$ is as given by Eq (3.31). Then, by the definition of Caputo derivative, we have

$$\begin{aligned} &\|{}^cD^r(Tv_k)^{(q)}(t) - {}^cD^r(Tv)^{(q)}(t)\| \\ &\le \|\frac{1}{\Gamma(q-r)}\| \, \| \int_0^t (t-s)^{q-r-1}[(Tv_k)^{(q)}(s) - (Tv)^{(q)}(s)]ds\|. \end{aligned} \tag{3.33}$$

Evidently, Eqs (3.30), (3.32) and (3.33) diminish as k approaches infinity. So, T is continuous and has a fixed point $V \in B_{\lambda_1}$, which is the solution of (3.10), by Schaefer's fixed point theorem and the Arzola-Ascoli theorem. Overall, the system (3.10) is controllable in $[0, \ell]$. □

Remark 3.3. *The modeling of chemical reactions can be done using fractional-order systems with state delay. The reaction's non-integer order dynamics can be captured by the fractional order, and the reaction's temporal delay can be explained by the state delay. Population dynamics in ecology can be modeled using the model with state delay. The state delay can serve as a representation of the lag in how quickly one population reacts to changes in a different population. They can aid in the analysis and design of more stable and oscillation-free power systems.*

4. Experimental & computational section

Consider the following fractional-order system with constant delay:

$$\begin{cases} {}^c\mathbb{D}^{0.6}v(t) = \mathbb{M}v(t-0.25) + \mathbb{N}u(t) \\ \qquad\qquad + g(t, v(t-\rho), v(\eta_1 t), \dots, v(\eta_n t)), \ v(t) \in \mathbb{R}^3, \ t \in J = [0,1], \\ v(t) = \psi(t), -0.25 \le t \le 0, \end{cases} \tag{4.1}$$

where

$$M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, N = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix}, g = \begin{bmatrix} 0 \\ 0 \\ \frac{\cos(t)e^{-v_1(t-0.25)}}{1+v_1^2(t/2)+v_2^2(2t/3)+v_3^2(3t/4)} \end{bmatrix}.$$

Then, by definition (2.4), the Gramian matrix (3.3), is given by

$$\begin{aligned} \mathcal{G}_c(0,1) &= \int_0^1 \mathscr{E}_{0.25,\,0.6}^{\mathbb{M}(1-0.25-s)^{0.6}} \mathbb{N}\mathbb{N}^* \mathscr{E}_{0.25,\,0.6}^{\mathbb{M}^*(1-0.25-s)^{0.6}} ds \\ &= \mathcal{G}_{c_1}(0,0.25) + \mathcal{G}_{c_2}(0.25,0.50) + \mathcal{G}_{c_3}(0.50,0.75) + \mathcal{G}_{c_4}(0.75,1), \end{aligned}$$

where

$$\begin{aligned} \mathcal{G}_{c_1}(0,0.25) &= \int_0^{0.25} [I\frac{(1-s)^{-2/5}}{\Gamma(0.6)} + M\frac{(0.75-s)^{1/5}}{\Gamma(1.2)} + M^2\frac{(0.50-s)^{4/5}}{\Gamma(1.8)} + M^3\frac{(0.25-s)^{7/5}}{\Gamma(2.4)}] \\ &\quad \times \mathbb{B}\mathbb{B}^*[I\frac{(1-s)^{-2/5}}{\Gamma(0.6)} + M\frac{(0.75-s)^{1/5}}{\Gamma(1.2)} + M^2\frac{(0.50-s)^{4/5}}{\Gamma(1.8)} + M^3\frac{(0.25-s)^{7/5}}{\Gamma(2.4)}]^* ds \\ &= \begin{bmatrix} 0.006378 & 0.199877 & 0.089102 \\ 0.199877 & 6.413748 & 2.828639 \\ 0.089102 & 2.828639 & 1.253624 \end{bmatrix}. \end{aligned}$$

For $\mathcal{G}_{c_2}(0.25, 0.50)$, we have

$$\begin{aligned}\mathcal{G}_{c_2}(0.25, 0.50) &= \int_{0.25}^{0.50} [I\frac{(1-s)^{-2/5}}{\Gamma(0.6)} + M\frac{(0.75-s)^{1/5}}{\Gamma(1.2)} + M^2\frac{(0.50-s)^{4/5}}{\Gamma(1.8)}] \\ &\times \mathbb{B}\mathbb{B}^*[I\frac{(1-s)^{-2/5}}{\Gamma(0.6)} + M\frac{(0.75-s)^{1/5}}{\Gamma(1.2)} + M^2\frac{(0.50-s)^{4/5}}{\Gamma(1.8)}]^* ds \\ &= \begin{bmatrix} 0.003621 & 0.102320 & 0.056632 \\ 0.102320 & 2.909774 & 1.620265 \\ 0.056632 & 1.620265 & 0.907244 \end{bmatrix}.\end{aligned}$$

For $\mathcal{G}_{c_3}(0.50, 0.75)$, we have

$$\begin{aligned}\mathcal{G}_{c_3}(0.50, 0.75) &= \int_{0.20}^{0.75} [I\frac{(1-s)^{-2/5}}{\Gamma(0.6)} + M\frac{(0.75-s)^{1/5}}{\Gamma(1.2)}] \\ &\times \mathbb{B}\mathbb{B}^*[I\frac{(1-s)^{-2/5}}{\Gamma(0.6)} + M\frac{(0.75-s)^{1/5}}{\Gamma(1.2)}]^* ds \\ &= \begin{bmatrix} 0.034617 & 0.181316 & 0.132416 \\ 0.181316 & 1.422942 & 1.009067 \\ 0.132416 & 1.009067 & 0.716850 \end{bmatrix}.\end{aligned}$$

For $\mathcal{G}_{c_4}(0.75, 1)$, we have

$$\begin{aligned}\mathcal{G}_{c_4}(0.75, 1) &= \int_{0.75}^{1} [I\frac{(1-s)^{-2/5}}{\Gamma(0.6)}] \times \mathbb{B}\mathbb{B}^*[I\frac{(1-s)^{-2/5}}{\Gamma(0.6)}]^* ds \\ &= \begin{bmatrix} 1.708663 & 1.708663 & 1.708663 \\ 1.708663 & 1.708663 & 1.708663 \\ 1.708663 & 1.708663 & 1.708663 \end{bmatrix}.\end{aligned}$$

Adding $(\mathcal{G}_{c_1} - \mathcal{G}_{c_4})$, we have

$$\mathcal{G}_c(0, 1) = \begin{bmatrix} 1.753281 & 2.192176 & 1.986815 \\ 2.192176 & 12.45512 & 7.166634 \\ 1.986815 & 7.166634 & 4.586382 \end{bmatrix},$$

which is invertible, as $\det(\mathcal{G}_c(0, 1)) = 1.32627801423548$. Also, the nonlinear function g satisfies the assumptions $(A_1 - A_4)$. Hence, the nonlinear system (4.1) is controllable on $[0, 1]$.

Remark 4.1. *Comparing the existing literature with our proposed model, we provide the advantages of the proposed model over the existing models and results. The work carried out in the neighborhood of the existing results focused on the study of other various models, but our proposed model has not been given attention by researchers. The authors in [31] proposed a neutral fractional integro-differential system incorporating distributed delays and studied results related to controllability. Also, the researchers in [32] introduced a nonlinear fractional order system with multiple delays and studied its dynamics. Subsequently, the work carried out in [33] focused on investigation and formulation of a dynamical system in Banach spaces. The manuscript by [34]* *examines the controllability analysis of fractional order neutral-type systems with impulsive effects and state delay. In the paper [35], the authors explored the relative controllability of a dynamical system regulated by a fractional order system with a pure delay. In [36], the existence findings and controllability requirements of a nonlinear system with damping in Hilbert space were taken into consideration. Very recently, the controllability outcomes of a dynamical system with input delay, controlled by a fractional order integro-differential system, have been studied by authors in [38]. The controllability of a dynamical system modelled by a noninteger order differential system with control and state delay was investigated by [39]. Recently, in another study [40], controllability results of a nonlinear system with pure delay were obtained using the delayed Mittag-Leffler matrix functions and Schauder's fixed point procedures.*

One new factor in our model is incorporation of a delay term, called a pantograph equation. Pantograph equations are a class of functional differential equations that have applications in mathematical modeling, such as population dynamics, control theory, and fluid dynamics [41]. The first attempt was made by the researchers [42]. Pantograph equations are used to design and analyze mechanical linkages, such as suspension systems, steering systems, and robotics. These linkages can be used in a wide range of applications, such as automotive, aerospace, and industrial machinery. Also, these types of equations are used in the design and analysis of electric circuits and systems. They are used to model and predict the behavior of complex circuits, such as power transmission lines, filters, and amplifiers [43].

Researchers have generalized the equation in a variety of ways to show its existence and stability [44–46]. The formulation of the underlying model and controllability of this dynamical system driven by a fractional order generalized multi-pantograph system with state delay have not, as far as we are aware, been studied. In this paper, we formulate as well as demonstrate the controllability of a generalized multi-pantograph system in the Caputo sense defined by the equation. The motivation was provoked by the above and more precisely [38, 40] and [49, 50]. After a comprehensive literature review our novel model obtained for

$$\begin{cases} {}^{c}\mathbb{D}^{r} v(t) = \mathbb{M}v(t-\rho) + \mathbb{N}u(t) \\ \qquad\qquad + g(t, v(t-\rho), v(\eta_1 t), \ldots, v(\eta_n t)), \; v(t) \in \mathbb{R}^n, \; t \in J = [0, \ell], \\ v(t) = \psi(t), -\rho \leq t \leq 0. \end{cases} \tag{4.2}$$

The descriptions of all symbols and mathematical notions have already been mentioned in the beginning of this research paper.

5. Conclusions

We established the controllability criteria for a nonlinear multi-pantograph system of fractional order utilizing the combined techniques of Schaefer's fixed point theorem and the Arzela-Ascoli theorem in this article. We transformed the suggested system into a fixed-point problem, defined the controllability Gramian matrix $\mathcal{G}_c$ and the control function $u(t)$ and proved that $\mathcal{G}_c$ must be invertible for the linear system to be controllable. With the aid of the linear part controllability and some assumptions on the nonlinear function, we established controllability criteria for the nonlinear system utilizing Schaefer's fixed-point theorem and the Arzela-Ascoli theorem. For the authenticity of the established results, an example has been added in the last section of the article.

Our proposed model contains three new features: The first is the insertion of a state delay. The second is the use of a multi-term pantograph nature function. And the third is the use of a fractional derivative for freedom in the order of the derivative. After the formulation of the main model, we explore results related to qualitative aspects of the model. These followed by the controllability of the linear and non-linear cases. Pantograph equations are used to describe the motion of charged particles in a magnetic field. They are also used to model the behavior of quantum systems, such as quantum dots and quantum wells. Also, the equations can be applied to model economic systems such as stock prices, interest rates, and inflation rates. They can be used to predict the behavior of these systems over time and identify the factors that influence them. Overall, pantograph equations have a wide range of applications in various fields and are an important tool for modeling and analyzing complex systems.

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Data availability

No data were used to support this study.

Conflict of interest

The authors declare that there is no conflict of interest.

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