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Research article

On the rate of convergence of Euler–Maruyama approximate solutions of stochastic differential equations with multiple delays and their confidence interval estimations

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Abstract: In this paper, we investigate Euler–Maruyama approximate solutions of stochastic differential equations (SDEs) with multiple delay functions. Stochastic differential delay equations (SDDEs) are generalizations of SDEs. Solutions of SDDEs are influenced by both the present and past states. Because these solutions may include past information, they are not necessarily Markov processes. This makes representations of solutions complicated; therefore, approximate solutions are practical. We estimate the rate of convergence of approximate solutions of SDDEs to the exact solutions in the L^p-mean for $p \geq 2$ and apply the result to obtain confidence interval estimations for the approximate solutions.

Keywords: stochastic differential delay equation; Euler–Maruyama approximation; rate of convergence; confidence interval
Mathematics Subject Classification: 60H10, 65U05

1. Introduction

Stochastic differential equations (SDEs) are used to describe phenomena in the context of natural science and engineering. To consider models of phenomena whose future states depend not only on the present states but also on the past states, stochastic differential delay equations (SDDEs) are used. For example, we refer to [1], in which the delay market model is studied. The study of SDDEs was started in a previous paper [2], in which Ito and Nisio considered SDEs that depend on infinite past processes. Following the paper, many properties of SDDEs have been discovered. Refer to the paper by Ivanov et al. [3] for a survey.

Because solutions of SDDEs are influenced by past events, they do not have Markov properties. This makes representations of solutions complicated; therefore, approximate solutions of SDDEs have been studied.

In [4], the strong discrete time approximation of an SDDE with a single constant time delay is studied. Under the global Lipschitz condition, the explicit solutions for linear stochastic delay equations are given. The global Lipschitz condition has been relaxed to the local Lipschitz condition in [5] and [6]. However, studies of SDDEs with a single constant time delay are still ongoing under the global Lipschitz condition (cf. [7] and [8]).

As mentioned in [9], when we consider approximate solutions of SDEs, it is important to know the error rate of convergence of approximate solutions to the exact solutions. In [10] and [11], Kanagawa obtained the rate of convergence of Euler–Maruyama approximate solutions of SDEs in the L^2-mean and L^p-mean for some $p \geq 2$, respectively. In [12], we stated the rate of convergence in the L^2-mean for SDDEs with a single delay function. In [13], we studied the cases of SDDEs with multiple delay functions and stated the rate of convergence in the L^2-mean. However, in [12] and [13], the results were given without detail of proof.

The Euler–Maruyama approximation scheme for SDEs is implemented by a step function, which is a discretization of Brownian motion $B(t)$. The random increments are given by $\varDelta B_n = B(t_n) - B(t_{n-1}), n = 1, 2, \ldots$, and the values provided by pseudo-random variables. Brownian motion, however, moves the time interval $[t_{n-1}, t_n]$. With the awareness of such issues, Kanagawa proposed a method of confidence interval estimations to predict the exact solutions in [14]. His method is supported by stochastic analysis ([10] and [11]) and Chebyshev's inequality. Through confidence interval estimations, we can expect sample paths of solutions of SDEs in each time interval (see Figure 6 in [9] and [15]). We remark that it is not possible to obtain confidence intervals by generating many trajectories using simulation studies and measuring the results, because we cannot obtain information on the behavior of $\{B(t), t \in (t_{n-1}, t_n)\}$. Refer to Chapter 11 in [16], [9], [17] and [18] for details on simulation studies for SDEs. In the case of SDDEs, the confidence interval estimations for the Euler–Maruyama approximate solutions were studied in [12] and [13] only for the special case of SDDEs, which have no drift terms following [14] and [15].

This paper is a continuation of our previous works [12] and [13]. In this paper, we consider SDDEs under the global Lipschitz condition containing multiple delay functions. In this sense, the model considered in this paper is an extension of those in [4], [7] and [8]. The purpose of this paper is two-fold. The first is to state a convergence theorem in the L^p-mean for some $p \geq 2$ following the method by Kanagawa ([11]). We remark that the convergence theorem showed in this paper includes the results in [12] and [13]. The second is to present the confidence interval estimations for the general case of SDDEs, which have diffusion terms and drift terms.

This paper is organized as follows. In the following section, we provide a setting of the SDDE and its Euler–Maruyama approximation scheme. After that we state our main theorem, i.e., the rate of convergence in the L^p-mean for some $p \geq 2$. In Section 3, we provide confidence interval estimations for the Euler–Maruyama approximate solutions of SDDEs for the general case and the special case. In Section 4, we present numerical examples of confidence interval estimations provided in Section 3. In Section 5, we provide proofs for the general case of SDDEs. We also provide the proofs for the special case of SDDEs.

2. Setting of SDDE and its solution

2.1. Setting of SDDE

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. On the space, we provide a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ that satisfies the usual conditions; that is, it is right continuous and $\mathcal{F}_0$ contains all P-null sets. We denote such a space by $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$. Let $B(t) = {}^t(B_1(t), \ldots, B_m(t))$ be an m-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^n)$ denotes the space of continuous functions $\xi : [-\tau, 0] \to \mathbb{R}^n$ with norms generated by $\sup_{-\tau\leq t\leq 0} |\xi(t)|$. Additionally, $C_{\mathcal{F}_t}([a, b]; \mathbb{R}^n)$ denotes the family of $\mathcal{F}_t$-measurable $C([a, b]; \mathbb{R}^n)$-valued random variables.

We next provide a setting of an SDDE. We first introduce delay functions as follows:

(**D**) (i) Let $\delta_i(t)$ be a Borel measurable function such that

$$-\tau \leq \delta_i(t) \leq t \text{ for } i = 1, \ldots, \ell. \tag{2.1}$$

(ii) For each δ_i, we assume that

$$|\delta_i(t) - \delta_i(s)| \leq \rho|t - s|, \;\; s, t \geq 0 \text{ for some positive constant } \rho. \tag{2.2}$$

Initial data are given by information for $t \leq 0$, which is denoted by $\{\xi(t), -\tau \leq t \leq 0\} \in C_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$. For ξ, we assume the following:

(**P**) (i) For some $p \geq 2$, there exists $K_0 < \infty$ such that

$$\sup_{-\tau\leq t\leq 0} E[|\xi(t)|^p] = K_0. \tag{2.3}$$

(ii) For the same p in (2.3), there exist $K_1 > 0$ and $\gamma \in (0, 1]$ such that

$$E[|\xi(t) - \xi(s)|^p] \leq K_1(t - s)^\gamma, \; -\tau \leq s < t \leq 0. \tag{2.4}$$

Let $f(x_0, \ldots, x_\ell)$ and $g(x_0, \ldots, x_\ell)$ be $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^{n\times\ell})$-measurable functions with values in $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively. Let T be a positive constant. We consider the following n-dimensional SDDE:

$$\begin{aligned} dX(t) &= f(X(t), \ldots, X(\delta_\ell(t)))dt + g(X(t), \ldots, X(\delta_\ell(t)))dB(t), \quad 0 \leq t \leq T, \\ X(t) &= \xi(t), \quad -\tau \leq t \leq 0. \end{aligned} \tag{2.5}$$

For the functions f and g, we assume the following:

(**H**) For any $x_0, \ldots, x_\ell, \overline{x}_0, \ldots, \overline{x}_\ell \in \mathbb{R}^n$, there exists $K_2 > 0$ such that

$$\begin{aligned} &|f(x_0, \ldots, x_\ell) - f(\overline{x}_0, \ldots, \overline{x}_\ell)|^2 \vee |g(x_0, \ldots, x_\ell) - g(\overline{x}_0, \ldots, \overline{x}_\ell)|^2 \\ \leq &K_2(|x_0 - \overline{x}_0|^2 + \cdots + |x_\ell - \overline{x}_\ell|^2), \end{aligned} \tag{2.6}$$

where $a \vee b := \max\{a, b\}$.

We remark that (**H**) implies that

$$|f(x_0, \ldots, x_\ell)|^2 \vee |g(x_0, \ldots, x_\ell)|^2 \leq K(1 + |x_0|^2 + \cdots + |x_\ell|^2), \tag{2.7}$$

where

$$K := 2(K_2 \vee |f(0, \ldots, 0)|^2 \vee |g(0, \ldots, 0)|^2). \tag{2.8}$$

2.2. Euler–Maruyama approximation

We next explain the Euler–Maruyama approximation scheme for the SDDE (2.5). We set a time step Δ with an integer N such that

$$\Delta := \tau/N \leq \frac{1}{\rho+1}. \tag{2.9}$$

Using the time step, we provide a discrete approximate solution of the SDDE (2.5) by

$$\begin{aligned} \overline{y}((k+1)\Delta) =& \overline{y}(k\Delta) + f\left(\overline{y}(k\Delta),\ldots,\overline{y}(I^{(\ell)}_{k\Delta}\Delta)\right)\Delta + g\left(\overline{y}(k\Delta),\ldots,\overline{y}(I^{(\ell)}_{k\Delta}\Delta)\right)\Delta B_k, \\ & \qquad k=0,1,\ldots,N-1, \\ \overline{y}(t) =& \xi(t), \quad -\tau \leq t \leq 0, \end{aligned} \tag{2.10}$$

where $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$ and $I^{(i)}_{k\Delta}$ is the integer part of $\delta_i(k\Delta)/\Delta$ for $i = 1, 2, \ldots, \ell$.

We consider a continuous approximate solution. Let $1_S(t)$ denote the indicator function of a subset of time interval S. We set

$$\begin{aligned} z_0(t) &:= \sum_{k=0}^{\infty} 1_{[k\Delta,(k+1)\Delta)}(t)\overline{y}(k\Delta), \\ z_i(t) &:= \sum_{k=0}^{\infty} 1_{[k\Delta,(k+1)\Delta)}(t)\overline{y}(I^{(i)}_{k\Delta}\Delta), \quad i=1,\ldots,\ell, \end{aligned} \tag{2.11}$$

and define a continuous Euler–Maruyama approximate solution:

$$y(t) = \begin{cases} \xi(t), \quad -\tau \leq t \leq 0, \\ \xi(0) + \int_0^t f(z_0(s),\ldots,z_\ell(s))ds + \int_0^t g(z_0(s),\ldots,z_\ell(s))dB(s), \ 0 \leq t \leq T. \end{cases}$$

We remark that for each k, the discrete solution $\bar{y}(k\Delta)$ and the continuous solution $y(k\Delta)$ are the same. Then, we obtain the rate of convergence in the L^p-mean as follows.

Theorem 2.1. *For the SDE with multiple delays* (2.5)*, we assume* (**D**)*,* (**P**) *and* (**H**)*. Then, for $p \geq 2$ in* (**P**) *there exist constants C_1 and C_2 such that*

$$\begin{aligned} & E\left[\sup_{0\leq t\leq T}|X(t)-y(t)|^p\right] \\ \leq & 4^{p-1}K_2^{p/2}T^p(1+c_p)(\ell+2)^{p/2-1}(C_1+\ell C_2)\,\Delta^\gamma \cdot \exp\left[4^{p-1}K_2^{p/2}T^p(1+c_p)(\ell+2)^{p/2-1}(\ell+1)\right], \end{aligned} \tag{2.12}$$

where $c_p = \frac{p^{p(p+1)/2}}{2^{p/2}(p-1)^{p(p-1)/2}}$.

Remark 2.1. *We remark that rates of convergence of approximate solutions of SDEs in the L^p-mean are given by the strong order to the power $p/2$ (e.g. Theorem 1 in* [*11*]*). For SDDEs, the rates of convergence are influenced by not only p but also the Hölder continuous of initial data γ.*

We provide the proof of Theorem 2.1 in the final section.

3. Confidence interval estimations for approximate solutions

In this section, we consider confidence interval estimations for approximate solutions. Theorem 2.1 implies an error estimation for Euler-Maruyama approximate solutions of diffusion processes governed by SDEs with multiple delays. The inequality (2.12) tells us

$$\lim_{\Delta\to 0} E\left[\sup_{0\le t\le T} |X(t)-y(t)|^2\right] = 0.$$

The definition of time step (2.9) and Chebyshev's inequality imply that for any $\epsilon > 0$,

$$\begin{aligned} P\left\{\sup_{0\le t\le T} |X(t)-y(t)| \le \epsilon\right\} \ge& 1 - E\left[\sup_{0\le t\le T} |X(t)-y(t)|^2\right]/\epsilon^2 \\ \ge& 1 - O(N^{-1})/\epsilon^2 \quad \text{as } N \to \infty. \end{aligned}$$

This inequality provides only the order of the hazard rate; hence, more information is required to obtain a confidence interval estimation. In this section, we present refined estimations for a general case and a special case.

3.1. General case

Using the inequality (2.12) in Theorem 2.1 with $p = 2$, we obtain the following confidence interval estimation.

Theorem 3.1. *For the SDE with multiple delays* (2.5)*, we assume* (**D**)*,* (**P**) *and* (**H**)*. Then, for any* $\epsilon > 0$

$$P\left[\sup_{0\le t\le T} |X(t)-y(t)| \le \epsilon\right] \ge 1 - \frac{20}{\epsilon^2} K_2 T^2 (C_1 + \ell C_2) e^{20K_2T^2(\ell+1)} \Delta^\gamma. \tag{3.1}$$

Proof. Theorem 2.1 and Chebyshev's inequality imply that

$$\begin{aligned} P\left\{\sup_{0\le t\le T} |X(t)-y(t)| > \epsilon\right\} <& \frac{1}{\epsilon^2} E\left[\sup_{0\le t\le T} |X(t)-y(t)|^2\right] \\ <& \frac{20}{\epsilon^2} K_2 T^2 (C_1 + \ell C_2) e^{20K_2T^2(\ell+1)} \Delta^\gamma, \end{aligned}$$

which proves Theorem 3.1. □

3.2. Special case

The expressions of constants C_1 and C_2 in Theorems 2.1 are complicated, and they are given in (5.5) and (5.9), respectively (see also (5.7) in Lemma 5.3). For SDDEs which have no drift terms, the expressions become simpler. We next consider the special case and provide confidence interval estimations.

For the case $f \equiv 0$ in (2.5), the SDDE is given by

$$d\widetilde{X}(t) = g\left(\widetilde{X}(t), \ldots, \widetilde{X}(\delta_\ell(t))\right) dB(t), \quad 0 \le t \le T,$$

$$\widetilde{X}(t) = \xi(t), \quad -\tau \le t \le 0. \tag{3.2}$$

$\widetilde{y}(t)$ denotes a discreate Euler–Maruyama approximate solution of $\widetilde{X}(t)$. Then, we obtain the following confidence interval estimation.

Corollary 3.1. *For the SDE with multiple delays* (3.2)*, we assume* (**D**)*,* (**P**) *and* (**H**)*. Then, there exist constants* $\widetilde{C}_1$ *and* $\widetilde{C}_2$ *such that, for any* $\epsilon > 0$

$$P\left[\sup_{0\le t\le T}\left|\widetilde{X}(t) - \widetilde{y}(t)\right| \le \epsilon\right] \ge 1 - \frac{8}{\epsilon^2}K_2T(\widetilde{C}_1 + \ell\widetilde{C}_2)e^{8K_2T(\ell+1)}\varDelta^\gamma, \tag{3.3}$$

where

$$\widetilde{C}_1 = K\{1 + \widetilde{C}_3(\ell + 1)\},\ \widetilde{C}_3 = 2(K_0 + KT)e^{2KT(\ell+1)},$$

and $\widetilde{C}_2$ *is a constant which is given as follows:*
for delay functions $\delta_i(t), i = 1, \ldots, \ell,$

$$\widetilde{C}_2^i = \begin{cases} \widetilde{C}_1(\rho + 1), & 0 \le I_{k\varDelta}^{(i)}\varDelta \le \delta_i(t), \\ \widetilde{C}_1\rho, & 0 \le \delta_i(t) \le I_{k\varDelta}^{(i)}\varDelta, \\ 2\widetilde{C}_1(\rho + 1) + 2K_1(\rho + 1)^\gamma, & I_{k\varDelta}^{(i)}\varDelta \le 0 \le \delta_i(t), \\ 2\widetilde{C}_1\rho + 2K_1\rho^\gamma, & \delta_i(t) \le 0 \le I_{k\varDelta}^{(i)}\varDelta, \\ K_1(\rho + 1)^\gamma, & I_{k\varDelta}^{(i)}\varDelta \le \delta_i(t) \le 0 \text{ or } \delta_i(t) \le I_{k\varDelta}^{(i)}\varDelta \le 0, \end{cases} \tag{3.4}$$

and we set

$$\widetilde{C}_2 := \max_{i=1,\ldots,\ell} \widetilde{C}_2^i. \tag{3.5}$$

4. Numerical examples

4.1. Multi-dimensional SDDE

Example 4.1. *Recall that* $B(t) = {}^t(B_1(t), \ldots, B_m(t))$ *is an m-dimensional standard Brownian motion. Given an n-dimensional vector function* $f = (f_1, \ldots, f_n)$ *and an* $n \times m$*-matrix function* $g = (g_{ij})$*, we consider the following n-dimensional SDDE:*

$$dX_l(t) = f_l(X(t), \ldots, X(\delta_\ell(t)))dt + \sum_{j=1}^m g_{lj}(X(t), \ldots, X(\delta_\ell(t)))dB_j(t),\ 0 \le t \le T, \tag{4.1}$$

which is the l-th component of the n-dimensional SDDE. We assume that Lipschitz's constant in (2.2) *is given by* $\rho = 10^{-1}$*. To apply Theorem 3.1, we give the setting as follows:*
Finish time: $T = 1$;
Time step: $\varDelta = 10^{-4}$;
Uniform boundedness of initial data: $K_0 = 10^{-1}$;
Hölder continuity of initial data: $\gamma = 1, K_1 = 10^{-1}$;
The constant K_2 *in* (2.6) : $K_2 = 10^{-2(n-1)}$.

In the case of $\ell = n = 2$, we set C_2 as given in (5.9) (*see also* (5.7) *in Lemma 5.3*). *Then, Theorem 3.1 implies that*

$$P\left\{\sup_{0\le t\le 1}|X(t)-y(t)|\le 1.94\times 10^{-2}\right\}\ge 0.9.$$

Under the same setting as listed above, we present ϵ's in (3.1) *for other cases of ℓ and n with the confidence levels 0.9 and 0.95 in Tables 1 and 2, respectively.*

Table 1. Numerical results for n and ℓ on Example 4.1 with the confidence level 0.9.

$n\backslash\ell$	1	2	3	4	5	10
2	1.068×10^{-2}	1.944×10^{-2}	3.028×10^{-2}	4.404×10^{-2}	6.156×10^{-2}	2.470×10^{-1}
3	8.760×10^{-4}	1.444×10^{-3}	2.038×10^{-3}	2.684×10^{-3}	3.399×10^{-3}	8.314×10^{-3}
4	8.742×10^{-5}	1.440×10^{-4}	2.030×10^{-4}	2.671×10^{-4}	3.379×10^{-4}	8.224×10^{-4}

Table 2. Numerical results for n and ℓ on Example 4.1 with the confidence level 0.95.

$n\backslash\ell$	1	2	3	4	5	10
2	1.151×10^{-2}	2.749×10^{-2}	4.282×10^{-2}	6.228×10^{-2}	8.706×10^{-2}	3.494×10^{-1}
3	1.239×10^{-3}	2.043×10^{-3}	2.882×10^{-3}	3.797×10^{-3}	4.807×10^{-3}	1.176×10^{-2}
4	1.236×10^{-4}	2.037×10^{-4}	2.870×10^{-4}	3.778×10^{-4}	4.778×10^{-4}	1.163×10^{-3}

4.2. One-dimensional SDDE

We consider the following SDE with piecewise constant arguments (Example 1 in [19]):

$$\begin{aligned} dX(t) &= \{X(t)+X([t])+X([t-1])\}dt+\{X(t)+X([t-1])\}dB(t),\ 0\le t\le 2,\\ X(t) &= 1,\ -1\le t\le 0. \end{aligned}$$

Since the example above has piecewise constant time delays, the explicit solution is given by

$$X(t)=\begin{cases} \exp\left\{\dfrac{t}{2}+B(t)\right\}\left(\xi(0)+\displaystyle\int_0^t e^{-\frac{s}{2}-B(s)}ds+\int_0^t e^{-\frac{s}{2}-B(s)}dB(s)\right), & t\in[0,1],\\ \exp\left\{\dfrac{t-1}{2}+B(t)-B(1)\right\}\left(X(1)+X(1)\displaystyle\int_1^t e^{-\frac{s-1}{2}-\{B(s)-B(1)\}}\}ds+\int_0^t e^{-\frac{s-1}{2}-\{B(s)-B(1)\}}dB(s)\right), & t\in[1,2]. \end{cases}$$

In the case of delay functions, the representations of explicit solutions of SDDEs by the stochastic integral are complicated. In such a case, we consider approximate solutions for SDDEs.

Example 4.2. *We consider the following one-dimensional SDDE with two-time delay functions.*

$$dX(t)=\{X(t)+X(\delta_1(t))+X(\delta_2(t))\}dt+\{X(t)+X(\delta_1(t))+X(\delta_2(t))\}dB(t), \tag{4.2}$$

where Lipschitz's constant in (2.2) *is given by* $\rho = 10^{-1}$. *To apply Theorem 3.1, we give the setting as follows:*

Finish time: $T = 2$;

Time step: $\varDelta = 10^{-4}$;

Uniform boundedness of initial data: $K_0 = 10^{-1}$;

Hölder continuity of initial data: $\gamma = 1, K_1 = 10^{-1}$;

The constant K_2 *in* (2.6) : $K_2 = 10^{-2}$.

We set C_2 *in the same manner as that in Example 4.1. Then, Theorem 3.1 implies that*

$$P\left\{\sup_{0\le t\le 1} |X(t) - y(t)| \le 0.15\right\} \ge 0.9.$$

4.3. Special case

Example 4.3. *We consider the SDDE* (3.2). *To apply Corollary 3.1, we give the setting as follows:*

Finish time: $T = 1$;

Time step: $\varDelta = 10^{-4}$;

Lipschitz constant of the time delay: $\rho = 10^{-1}$;

Uniform boundedness of initial data: $K_0 = 10^{-1}$;

Hölder continuity of initial data: $\gamma = 1, K_1 = 10^{-1}$;

The constant K_2 *in* (2.6) : $K_2 = 10^{-2(n-1)}$.

Under the conditions above, $\widetilde{C}_2$ *in* (3.5) *is determined. In the case of* $\ell = n = 2$, *we obtain that*

$$P\left\{\sup_{0\le t\le 1} \left|\widetilde{X}(t) - \widetilde{y}(t)\right| \le 8.04 \times 10^{-3}\right\} \ge 0.9.$$

Under the same setting as listed above, we present ϵ*'s in* (3.3) *for other cases of* ℓ *and* n *with the confidence levels 0.9 and 0.95 in Tables 3 and 4, respectively.*

Table 3. Numerical results for n and ℓ on Example 4.3 with the confidence level 0.9.

$n\backslash\ell$	1	2	3	4	5	10
2	5.458×10^{-3}	8.040×10^{-3}	1.041×10^{-2}	1.276×10^{-2}	1.512×10^{-2}	2.961×10^{-2}
3	4.797×10^{-4}	6.935×10^{-4}	8.691×10^{-4}	1.027×10^{-3}	1.177×10^{-3}	1.893×10^{-3}
4	4.791×10^{-5}	6.925×10^{-5}	8.675×10^{-5}	1.025×10^{-4}	1.174×10^{-4}	1.886×10^{-4}

Table 4. Numerical results for n and ℓ on Example 4.3 with the confidence level 0.95.

$n\backslash\ell$	1	2	3	4	5	10
2	7.718×10^{-3}	1.137×10^{-2}	1.472×10^{-2}	1.804×10^{-2}	2.145×10^{-2}	4.187×10^{-2}
3	6.784×10^{-4}	9.808×10^{-4}	1.229×10^{-3}	1.453×10^{-3}	1.664×10^{-3}	2.678×10^{-3}
4	6.775×10^{-5}	9.794×10^{-5}	1.227×10^{-4}	1.450×10^{-4}	1.660×10^{-4}	2.667×10^{-4}

5. Proofs

5.1. Proof of Theorem 2.1.

Lemma 5.1. *Under the assumptions* (**P**) (i) and (**H**),

$$\sup_{t\in[-\tau,T]} E[|y(t)|^p] \le 3^{p-1}\{K_0 + C_3T\}\cdot \exp\{3^{p-1}C_3(\ell+1)T\}, \tag{5.1}$$

where

$$C_3 = K^{p/2}(\ell+2)^{p/2-1}\left(T^{p-1} + T^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right). \tag{5.2}$$

Proof. (i) For the case $-\tau \le t \le 0$,

$$E[|y(t)|^p] \le \sup_{-\tau\le t\le 0} E[|\xi(t)|^p] = K_0.$$

(ii) For the case $0 \le t \le T$,

$$\begin{aligned}
&E[|y(t)|^p]\\
=&E\left[\left|\xi(0) + \int_0^t f(z_0(s),\dots,z_\ell(s))ds + \int_0^t g(z_0(s),\dots,z_\ell(s))dB(s)\right|^p\right]\\
\le&3^{p-1}\left\{E[|\xi(0)|^p] + E\left[\left|\int_0^t f(z_0(s),\dots,z_\ell(s))ds\right|^p\right] + E\left[\left|\int_0^t g(z_0(s),\dots,z_\ell(s))dB(s)\right|^p\right]\right\}\\
=:&3^{p-1}(I_1 + I_2 + I_3).
\end{aligned}$$

Using Hölder's inequality and the inequality (2.7), we obtain

$$\begin{aligned}
I_2 \le& t^{p-1}\cdot E\left[\int_0^t |f(z_0(s),\dots,z_\ell(s))|^p ds\right]\\
\le& K^{p/2}t^{p-1}E\left[\int_0^t \{1+|z_0(s)|^2+\cdots+|z_\ell(s)|^2\}^{p/2}ds\right]\\
\le& K^{p/2}(\ell+2)^{p/2-1}t^{p-1}E\left[\int_0^t \{1+|z_0(s)|^p+\cdots+|z_\ell(s)|^p\}ds\right].
\end{aligned}$$

Using Burkholder–Davis–Gundy's inequality, we obtain

$$\begin{aligned}
I_3 \le& t^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2} E\left[\int_0^t |g(z_0(s),\dots,z_\ell(s))|^p ds\right]\\
\le& K^{p/2}t^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2} E\left[\int_0^t \{1+|z_0(s)|^2+\cdots+|z_\ell(s)|^2\}^{p/2}ds\right]\\
\le& K^{p/2}(\ell+2)^{p/2-1}t^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2} E\left[\int_0^t \{1+|z_0(s)|^p+\cdots+|z_\ell(s)|^p\}ds\right].
\end{aligned}$$

Thus,

$$I_2 + I_3 \le K^{p/2}(\ell+2)^{p/2-1}\left(t^{p-1} + t^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)\left\{t + (\ell+1)\int_0^t \sup_{-\tau\le u\le s} E[|y(u)|^p]ds\right\}$$
$$\le C_3\left\{t + (\ell+1)\int_0^t \sup_{-\tau\le u\le s} E[|y(u)|^p]ds\right\}.$$

Using Gronwall's lemma, we obtain

$$\sup_{-\tau\le t\le T} E[|y(t)|^p] \le 3^{p-1}\{K_0 + C_3 T\}\cdot \exp\{3^{p-1}C_3(\ell+1)T\}. \square$$

Following Lemma 5.1, we set

$$C_4 := 3^{p-1}\{K_0 + C_3 T\}\cdot \exp\{3^{p-1}C_3(\ell+1)T\}, \tag{5.3}$$

which does not depend on Δ.

Lemma 5.2. *Under the assumptions* (**D**)*,* (**P**) (i) *and* (**H**)*, for any* $t \in [0, T]$

$$E[|y(t) - z_0(t)|^p] \le 2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left(\Delta^{p-1} + \Delta^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)\{1 + C_4(\ell+1)\}\Delta. \tag{5.4}$$

Proof. For a fixed t, there exists k such that $t \in [k\Delta, (k+1)\Delta)$. We remark that $z_0(t) = \bar{y}(k\Delta)$. In the same manner as those for showing Lemma 5.1, we obtain

$$E[|y(t) - z_0(t)|^p] = E[|y(t) - \bar{y}(k\Delta)|^p]$$
$$\le E\left[\left|\int_{k\Delta}^t f(z_0(s), \dots, z_\ell(s))ds + \int_{k\Delta}^t g(z_0(s), \dots, z_\ell(s))dB(s)\right|^p\right]$$
$$\le 2^{p-1}\left\{E\left[\left|\int_{k\Delta}^t f(z_0(s), \dots, z_\ell(s))ds\right|^p\right] + E\left[\left|\int_{k\Delta}^t g(z_0(s), \dots, z_\ell(s))dB(s)\right|^p\right]\right\}$$
$$\le 2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left(\Delta^{p-1} + \Delta^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)E\left[\int_{k\Delta}^t \{1 + |z_0(s)|^p + \cdots + |z_\ell(s)|^p\}ds\right]$$
$$\le 2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left(\Delta^{p-1} + \Delta^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)\{1 + C_4(\ell+1)\}\Delta. \square$$

Following Lemma 5.2, we set

$$C_1 := 2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left(1 + \left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)\{1 + C_4(\ell+1)\}, \tag{5.5}$$

which does not depend on Δ either.

Lemma 5.3. *Under the assumptions* (**D**)*,* (**P**) *and* (**H**)*, for any* $t \in [0, T]$

$$E[|y(\delta_i(t)) - z_i(t)|^p] \le C_2^i \Delta^\gamma,\ i = 1, \dots, \ell, \tag{5.6}$$

where C_2^i is given as follows:

$$C_2^i = \begin{cases} 2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left((\rho+1)^{p-1}+(\rho+1)^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)\{1+C_4(\ell+1)\}(\rho+1), \\ \hfill 0 \le I_{k\varDelta}^{(i)}\varDelta \le \delta_i(t), \\ 2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left(\rho^{p-1}+\rho^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)\{1+C_4(\ell+1)\}\rho, \\ \hfill 0 \le \delta_i(t) \le I_{k\varDelta}^{(i)}\varDelta, \\ 2^{p-1}\Big\{2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left((\rho+1)^{p-1}+(\rho+1)^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)\{1+C_4(\ell+1)\}(\rho+1) \\ \hfill +K_1(\rho+1)^{\gamma}\Big\}, \\ \hfill I_{k\varDelta}^{(i)}\varDelta \le 0 \le \delta_i(t), \\ 2^{p-1}\left\{K_1\rho^{\gamma}+2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left((\rho+1)^{p-1}+(\rho+1)^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)\{1+C_4(\ell+1)\}\rho\right\}, \\ \hfill \delta_i(t) \le 0 \le I_{k\varDelta}^{(i)}\varDelta, \\ K_1(\rho+1)^{\gamma}, \hfill I_{k\varDelta}^{(i)}\varDelta \le \delta_i(t) \le 0 \text{ or } \delta_i(t) \le I_{k\varDelta}^{(i)}\varDelta \le 0. \end{cases} \tag{5.7}$$

Proof. (I) Case in which $0 \le I_{k\varDelta}^{(i)}\varDelta \le \delta_i(t)$:

For a fixed $t \in [0,T]$, there exists k such that $t \in [k\varDelta, (k+1)\varDelta)$. Then,

$$y(\delta_i(t)) - z_i(t) = y(\delta_i(t)) - y(I_{k\varDelta}^{(i)}\varDelta).$$

As $\delta_i(t) - I_{k\varDelta}^{(i)}\varDelta \le (\rho+1)\varDelta$,

$$\begin{aligned} &E[|y(\delta_i(t)) - z_i(t)|^p] \\ \le& 2^{p-1}\left\{E\left[\left|\int_{I_{k\varDelta}^{(i)}\varDelta}^{\delta_i(t)} f(z_0(s),\dots,z_\ell(s))ds\right|^p\right] + E\left[\left|\int_{I_{k\varDelta}^{(i)}\varDelta}^{\delta_i(t)} g(z_0(s),\dots,z_\ell(s))dB(s)\right|^p\right]\right\} \\ \le& 2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left(\{(\rho+1)\varDelta\}^{p-1}+\{(\rho+1)\varDelta\}^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)\{1+C_4(\ell+1)\}(\rho+1)\varDelta. \end{aligned}$$

This inequality and $\varDelta < 1$ imply a constant C_2^i for this case.

(II) Case in which $0 \le \delta_i(t) \le I_{k\varDelta}^{(i)}\varDelta$:

As $I_{k\varDelta}^{(i)}\varDelta - \delta_i(t) \le \delta_i(k\varDelta) - \delta_i(t) \le \rho\varDelta$,

$$\begin{aligned} &E[|y(\delta_i(t)) - z_i(t)|^p] \\ \le& 2^{p-1}\left\{E\left[\left|\int_{\delta_i(t)}^{I_{k\varDelta}^{(i)}\varDelta} f(z_0(s),\dots,z_\ell(s))ds\right|^p\right] + E\left[\left|\int_{\delta_i(t)}^{I_{k\varDelta}^{(i)}\varDelta} g(z_0(s),\dots,z_\ell(s))dB(s)\right|^p\right]\right\} \\ \le& 2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left((\rho\varDelta)^{p-1}+(\rho\varDelta)^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)\{1+C_4(\ell+1)\}\rho\varDelta. \end{aligned}$$

(III) Case in which $I_{k\varDelta}^{(i)}\varDelta \le 0 \le \delta_i(t)$:

As $\delta_i(t) \le \delta_i(t) - I^{(i)}_{k\Delta}\Delta \le (\rho+1)\Delta$, Lemma 5.2, and (**P**) (ii) imply that

$$
\begin{aligned}
&E[|y(\delta_i(t)) - z_i(t)|^p] \\
\le & 2^{p-1}\left\{E\left[|y(\delta_i(t)) - \xi(0)|^p\right] + E[|\xi(0) - \xi(I^{(i)}_{k\Delta}\Delta)|^p]\right\} \\
\le & 2^{p-1}\Bigg\{2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left(\delta_i(t)^{p-1} + \delta_i(t)^{p/2-1}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)\{1 + C_4(\ell+1)\}\delta_i(t) \\
&\qquad + K_1(-I^{(i)}_{k\Delta}\Delta)^\gamma\Bigg\} \\
\le & 2^{p-1}\Bigg\{2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left((\rho+1)^{p-1}\Delta^{p-\gamma} + (\rho+1)^{p/2-1}\Delta^{p/2-\gamma}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right)\{1 + C_4(\ell+1)\}(\rho+1) \\
&\qquad + K_1(\rho+1)^\gamma\Bigg\}\Delta^\gamma.
\end{aligned}
$$

(IV) Case in which $\delta_i(t) \le 0 \le I^{(i)}_{k\Delta}\Delta$:
As $I^{(i)}_{k\Delta}\Delta \le I^{(i)}_{k\Delta}\Delta - \delta_i(t) \le \delta_i(k\Delta) - \delta_i(t) \le \rho\Delta$, in the same manner as that in case (III) we obtain

$$
\begin{aligned}
&E[|y(\delta_i(t)) - z_i(t)|^p] \\
\le & 2^{p-1}\left\{E\left[|y(\delta_i(t)) - \xi(0)|^p\right] + E[|\xi(0) - \xi(I^{(i)}_{k\Delta}\Delta)|^p]\right\} \\
\le & 2^{p-1}\Bigg\{K_1\rho^\gamma + 2^{p-1}K^{p/2}(\ell+2)^{p/2-1}\left((\rho+1)^{p-1}\Delta^{p-\gamma} + (\rho+1)^{p/2-1}\Delta^{p/2-\gamma}\left\{\frac{p(p-1)}{2}\right\}^{p/2}\right) \\
&\qquad \{1 + C_4(\ell+1)\}\rho\Bigg\}\Delta^\gamma.
\end{aligned}
$$

(V) Cases in which $I^{(i)}_{k\Delta}\Delta \le \delta_i(t) \le 0$ or $\delta_i(t) \le I^{(i)}_{k\Delta}\Delta \le 0$:
As $|\delta_i(t) - I^{(i)}_{k\Delta}\Delta| \le (\rho+1)\Delta$,

$$
\begin{aligned}
E[|y(\delta_i(t)) - z_i(t)|^p] \le & E\left[\left|\xi(\delta_i(t)) - \xi(I^{(i)}_{k\Delta}\Delta)\right|^p\right] \\
\le & K_1\left|\delta_i(t) - I^{(i)}_{k\Delta}\Delta\right|^\gamma \\
\le & K_1(\rho+1)^\gamma\Delta^\gamma. \ \square
\end{aligned}
\tag{5.8}
$$

Following Lemma 5.3, we set

$$
C_2 := \max_{i=1,\dots,\ell} C_2^i. \tag{5.9}
$$

Proof of Theorem 2.1. For any $t_1 \le T$,

$$
\begin{aligned}
&E[\sup_{0\le t\le t_1} |X(t) - y(t)|^p] \\
=& E\left[\sup_{0\le t\le t_1}\left|\int_0^t \{f(X(s),\dots,X(\delta_\ell(s))) - f(z_0(s),\dots,z_\ell(s))\}\,ds\right.\right. \\
&\qquad \left.\left. + \int_0^t \{g(X(s),\dots,X(\delta_\ell(s))) - g(z_0(s),\dots,z_\ell(s))\}\,dB(s)\right|^p\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le 2^{p-1}E\left[\sup_{0\le t\le t_1}\left|\int_0^t \{f(X(s),\dots,X(\delta_l(s)))-f(z_0(s),\dots,z_\ell(s))\}\,ds\right|^p\right]\\
&\qquad\qquad + 2^{p-1}E\left[\sup_{0\le t\le t_1}\left|\int_0^t \{g(X(s),\dots,X(\delta_\ell(s)))-g(z_0(s),\dots,z_\ell(s))\}\,dB(s)\right|^p\right]\\
&=: 2^{p-1}(I_4+I_5). \qquad (5.10)
\end{aligned}
$$

Using Hölder's inequality, for any $t_1 \le T$ we have

$$
I_4 \le T^{p-1}E\left[\left|\int_0^{t_1}\{f(X(s),\dots,X(\delta_\ell(s)))-f(z_0(s),\dots,z_\ell(s))\}^2\,ds\right|^{p/2}\right]. \qquad (5.11)
$$

Using Burkholder–Davis–Gundy's inequality, for any $t_i \le T$ we have

$$
I_5 \le T^{p-1}c_pE\left[\left|\int_0^{t_1}\{g(X(s),\dots,X(\delta_\ell(s)))-g(z_0(s),\dots,z_\ell(s))\}^2\,ds\right|^{p/2}\right], \qquad (5.12)
$$

where $c_p = \frac{p^{p(p+1)/2}}{2^{p/2}(p-1)^{p(p-1)/2}}$. The inequalities (5.11) and (5.12), and the assumption (**H**) imply that

$$
\begin{aligned}
&\text{(the right-hand side of (5.10))}\\
&\le 2^{p-1}K_2^{p/2}T^{p-1}(1+c_p)(\ell+2)^{p/2-1}\cdot E\left[\int_0^{t_1}\left\{|X(s)-z_0(s)|^p+\sum_{i=1}^{\ell}|X(\delta_i(s))-z_i(s)|^p\right\}ds\right]\\
&\le 2^{2(p-1)}K_2^{p/2}T^{p-1}(1+c_p)(\ell+2)^{p/2-1}\\
&\qquad\cdot\left\{E\left[\int_0^{t_1}\left\{|X(s)-y(s)|^p+\sum_{i=1}^{\ell}|X(\delta_i(s))-y(\delta_i(s))|^p\right\}ds\right]\right.\\
&\qquad\qquad\left.+E\left[\int_0^{t_1}\left\{|y(s)-z_0(s)|^p+\sum_{i=1}^{\ell}|y(\delta_i(s))-z_i(s)|^p\right\}ds\right]\right\}\\
&\le 4^{p-1}K_2^{p/2}T^{p-1}(1+c_p)(\ell+2)^{p/2-1}\left\{(\ell+1)\int_0^{t_1}E\left[\sup_{0\le r\le s}|X(r)-y(r)|^p\right]ds\right.\\
&\qquad\left.+T\left(E\left[\sup_{0\le s\le T}|y(s)-z_0(s)|^p\right]+\sum_{i=1}^{\ell}E\left[\sup_{0\le s\le T}|y(\delta_i(s))-z_i(s)|^p\right]\right)\right\}. \qquad (5.13)
\end{aligned}
$$

Lemma 5.2, Lemma 5.3 and Gronwall's lemma imply that

$$
\begin{aligned}
&\text{(the right-hand side of (5.13))}\\
&\le 4^{p-1}K_2^{p/2}T^{p-1}(1+c_p)(\ell+2)^{p/2-1}\left\{(\ell+1)\int_0^{t_1}E\left[\sup_{0\le r\le s}|X(r)-y(r)|^2\right]ds+T(C_1\Delta+\ell C_2\Delta^\gamma)\right\}\\
&\le 4^{p-1}K_2^{p/2}T^{p}(1+c_p)(\ell+2)^{p/2-1}(C_1\Delta+\ell C_2\Delta^\gamma)\cdot\exp\left[4^{p-1}K_2^{p/2}T^p(1+c_p)(\ell+2)^{p/2-1}(\ell+1)\right]\\
&\le 4^{p-1}K_2^{p/2}T^{p}(1+c_p)(\ell+2)^{p/2-1}(C_1+\ell C_2)\cdot\exp\left[4^{p-1}K_2^{p/2}T^p(1+c_p)(\ell+2)^{p/2-1}(\ell+1)\right]\Delta^\gamma. \ \square
\end{aligned}
$$

5.2. Proof of Corollary 3.1.

We prove Corollary 3.1 in the following steps:

1st step (corresponding to Lemma 5.1)

For the case $-\tau \le t \le 0$, we obtain that $E[|\widetilde{y}(t)|^2] \le K_0$.

For the case $0 \le t \le T$, we obtain that

$$\begin{aligned} E[|\widetilde{y}(t)|^2] =& E\left[\left|\xi(0) + \int_0^t g(z_0(s), z_1(s), \dots, z_l(s))dB(s)\right|^2\right] \\ \le& 2\left\{E[|\xi(0)|^2] + E\left[\int_0^t |g(z_0(s), z_1(s), \dots, z_l(s)|^2 dB(s)\right]\right\}. \end{aligned}$$

This inequality and Gronwall's lemma imply that

$$\sup_{-\tau\le t\le T} E[|\widetilde{y}(t)|^2] \le 2(K_0 + KT)e^{2KT(\ell+1)} = \widetilde{C}_3. \tag{5.14}$$

2nd step (corresponding to Lemma 5.2)

We consider the approximate solution of SDDE (2.10) with $f \equiv 0$ and use the same notation $z_i(t), i = 0, 1, \dots, \ell$ in (2.11) in the following step. We obtain

$$\begin{aligned} E[|\widetilde{y}(t) - z_0(t)|^2] \le& E\left[\left|\int_{k\varDelta}^t g(z_0(s), z_1(s), \dots, z_l(s))dB(s)\right|^2\right] \\ \le& KE\left[\int_{k\varDelta}^t \{1 + |z_0(s)|^p + \cdots + |z_\ell(s)|^p\}\, ds\right] \\ \le& K\left\{1 + \widetilde{C}_3(\ell + 1)\right\}\varDelta = \widetilde{C}_1\varDelta. \end{aligned} \tag{5.15}$$

3rd step (corresponding to Lemma 5.3)

In the fifth case in (3.4), we can show that the constant $\widetilde{C}_2^i$ coincides with C_2^i by using the inequality (5.8) with $p = 2$.

We next provide the proof for the third case in (3.4). As $\delta_i(t) \le \delta_i(t) - I_{k\varDelta}^{(i)}\varDelta \le (\rho + 1)\varDelta$, we have

$$\begin{aligned} E\left[|\widetilde{y}(\delta_i(t)) - z_i(t)|^2\right] \le& 2\left\{E\left[|\widetilde{y}(\delta_i(t)) - \xi(0)|^2\right] + E\left[\left|\xi(0) - \xi\left(I_{k\varDelta}^{(i)}\varDelta\right)\right|^2\right]\right\} \\ =:& 2\{I_6 + I_7\}. \end{aligned} \tag{5.16}$$

Using the estimation (5.15) with $t = 0$, we have

$$I_6 \le \widetilde{C}_1(\rho + 1)\varDelta. \tag{5.17}$$

Using the estimate (5.8), we have

$$I_7 \le K_1(\rho + 1)^\gamma \varDelta^\gamma. \tag{5.18}$$

Then, (5.17) and (5.18) imply that

$$\text{(Right-hand side of (5.16))} \le 2\left\{\widetilde{C}_1(\rho + 1) + K_1(\rho + 1)^\gamma\right\}\varDelta^\gamma.$$

For the other three cases in (3.4), proofs are given in the same manner as those for showing the cases (I), (II) and (IV) in the proof of Lemma 5.3.

We set $\widetilde{C}_2$ as (3.5). Then, for any $t \in [0, T]$ we have

$$E[|\overline{y}(\delta_i(t)) - z_i(t)|^2] \leq \widetilde{C}_2 \Delta^\gamma. \tag{5.19}$$

4th step

In the case of $f \equiv 0$, we obtain that

$$\begin{aligned}
&E\left[\sup_{0\leq t\leq t_1}\left|\widetilde{X}(t) - \overline{y}(t)\right|^2\right] \\
\leq &4E\left[\int_0^{t_1}\left|g\left(\widetilde{X}(s), \ldots, \widetilde{X}(\delta_\ell(s))\right) - g\left(z_0(s), \ldots, z_\ell(s)\right)\right|^2 ds\right] \\
\leq &4K_2 E\left[\int_0^{t_1}\left(\left|\widetilde{X}(s) - z_0(s)\right|^2 + \sum_{i=1}^{\ell}\left|\widetilde{X}(\delta_i(s)) - z_i(s)\right|^2\right) ds\right] \\
\leq &8K_2 E\left[\int_0^{t_1}\left(\left|\widetilde{X}(s) - \overline{y}(s)\right|^2 + \sum_{i=1}^{\ell}\left|\widetilde{X}(\delta_i(s)) - \overline{y}(\delta_i(t))\right|^2\right) ds\right] \\
&\qquad + 8K_2 E\left[\int_0^{t_1}\left(|\overline{y}(s) - z_0(s)|^2 + \sum_{i=1}^{\ell}|\overline{y}(\delta_i(t)) - z_i(s)|^2\right) ds\right] \\
=: &8K_2(I_8 + I_9).
\end{aligned} \tag{5.20}$$

For I_8, we have

$$I_8 \leq (\ell + 1)\int_0^{t_1} E\left[\sup_{0\leq r\leq s}\left|\widetilde{X}(r) - \overline{y}(s)\right|^2\right] ds. \tag{5.21}$$

Using (5.14) and (5.19), we have

$$\begin{aligned}
I_9 \leq &T\left\{E\left[\sup_{0\leq s\leq T}|\overline{y}(s) - z_0(s)|^2\right] + \sum_{i=1}^{\ell} E\left[\sup_{0\leq s\leq T}|\overline{y}(\delta_i(t)) - z_i(s)|^2\right]\right\} \\
\leq &T\left(\widetilde{C}_1\Delta + \ell\widetilde{C}_2\Delta^\gamma\right).
\end{aligned} \tag{5.22}$$

(5.21), (5.22) and Gronwall's lemma imply that

$$E\left[\sup_{0\leq t\leq T}|\widetilde{X}(t) - \overline{y}(t)|^2\right] \leq 8K_2 T\left(\widetilde{C}_1 + \ell\widetilde{C}_2\right)e^{8K_2T(\ell+1)}\Delta^\gamma. \tag{5.23}$$

This inequality and Chebyshev's inequality imply (3.3). □

6. Conclusions

In this article, we consider the problem of Euler–Maruyama approximate solutions of SDEs with multiple delay functions. The main result of this article is Theorem 2.1. In the theorem, we obtain the rate of convergence of approximate solutions to the exact solutions. We have applied the results to obtain confidence interval estimations for the approximate solutions. This information improves the understanding of gaps between exact solutions of SDEs with multiple delays by applying stochastic analysis and numerical solutions through simulation studies.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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