Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 13725–13746.
DOI: 10.3934/math.2023697
Received: 09 October 2022
Revised: 25 January 2023
Accepted: 07 February 2023
Published: 10 April 2023

Research article

A new fourth-order grouping iterative method for the time fractional sub-diffusion equation having a weak singularity at initial time

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Abstract: A new fourth-order explicit grouping iterative method is constructed for the numerical solution of the fractional sub-diffusion equation. The discretization of the equation is based on fourth-order finite difference method. Captive fractional discretization having functions with a weak singularity at $t = 0$ is used for time and similarly, the space derivative is approximated with the help of fourth-order approximation. Furthermore, the convergence and stability of the scheme are analyzed. Finally, the accuracy and validity are investigated by some numerical examples.

Keywords: sub-diffusion equation; Crank-Nicolson; grouping method; convergence and stability; fourth-order FDM; captive fractional discretisation
Mathematics Subject Classification: 35R11, 65N06

1. Introduction

Fractional-order differential equations (FDM) have many applications in various fields of engineering and science, such as chemical and physical phenomena [1–5]. For instance, the fractional-order diffusion equation is used to describe anomalous diffusion phenomena in the transport process through disordered and complex systems including fractal media, fractional kinetic equations regarding slow diffusion, and movement of small molecular along the concentration space [6].

In this article, the two-dimensional (2-D) time-fractional sub-diffusion equation (FSDE) with the weak singularity at initial time $t = 0$ is considered as follows:

$$\frac{\partial^\alpha V}{\partial t^\alpha} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + G(x, y, t), \tag{1.1}$$

with subject to conditions

$$V(x, y, 0) = \zeta_0(x, y),$$

and

$$\begin{aligned}&V(0,y,t)=\zeta_1(x,y,t),\ V(L,y,t)=\zeta_2(x,y,t),\\&V(x,0,t)=\zeta_3(x,y,t),\ V(x,L,t)=\zeta_4(x,y,t),\\&0\le x,y\le M,\ 0\le t\le N,\end{aligned}$$

where $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4$ are known functions and $\alpha \in (0,1)$.

The FSDE can be obtained from the anomalous diffusion system by replacing the time derivative with a fractional derivative α where $0 < \alpha < 1$. The FSDE is an important class of fractional partial differential equations (PDEs), which is mainly used in the modeling of fractional random walk, the phenomenon of wave propagation, diffusion unification, etc. [7, 8].

The Caputo derivative of order α is

$$ {}_0^C D_t^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(x)}{(t-x)^{\alpha}} dx. \tag{1.2}$$

The Caputo derivative approximated using the $L1$ gives the accuracy $2-\alpha$ [9, 10], but the presence of kernel $(t-x)^{\alpha}$ produces solutions for Eq (1.1) with the weak-singularity at initial time $t = 0$, which increases the computation cost and give low convergence rate for the approximate methods on uniform meshes [11]. Therefore, to increase the convergence rate many researchers solved the FSDE using different high-order numerical methods. Based on the chronology, several numerical methods are proposed for the solution of 2-D time FSDE (1.1), for example, Cui [12] proposed high-order alternating direction implicit (ADI) method, and it is unconditionally stable and convergent with convergence order $O(\nu^{\alpha} + h_0^4)$. Zhuang and Liu [13] proposed an unconditionally stable and convergent implicit difference scheme. Zhang et al. [14] used the Crank-Nicolson-type compact ADI scheme and proved the unconditional stability and convergence having convergence order $O(\nu^{min[2-\frac{\alpha}{2},2\alpha]} + h_0 1^4 + h_0 2^4)$ in H_0^1 norm. Ji and Sun [15] used a high-order numerical scheme to solve (1.1), and proved convergence in $L_1(L_\infty)$-norm and unconditional stability by the energy method. Wang et al. [16] solved 2-D FSDE using C-N alternating direction implicit finite difference method (FDM), where the fractional derivative is discretized using Riemann-Liouville fractional definition and to improve its temporal accuracy they used the Richardson extrapolation algorithm. Also, they proved its unique solvability, unconditional stability, and convergence $O(\nu^{2\gamma} + h_0 x^4 + h_0 y^4)$ of the scheme. Zhai and Feng [17] presented three different compact schemes for a 2-D time-fractional diffusion equation. The Caputo fractional definition is used for fractional derivative. All the schemes are fourth-order accurate for space and second-order accurate for the time variable. The stability of all the schemes is analyzed using Fourier analysis, which shows that two schemes are unconditionally stable, and the third one is conditionally stable.

The advantage of high-order schemes is that it produces more accurate results but at the same time increase the execution timings because of the escalated computational complexity of the scheme. Similarly, the advantage of explicit group methods over the standard point methods that the it considered quarter grid points of the solution domain and the points are considered as iterative points in the iterative process which reduces the computational complexity of the proposed method and hence reduce the execution time per iteration. Since, the computational complexity is greatly reduced using the explicit group method the 2-D time-fractional advection-diffusion, hyperbolic telegraph fractional differential, and fractional diffusion equations etc. [18–22] with second-order accuracy, therefore, we proposed the grouping strategy with uniform grids for the solution of the 2-D time FSDE with fourth-order accuracy. The purpose of this paper is to solve 2-D time FSDE with the fourth-order explicit group method (FEGM).

The paper is organized as follows: In Section 2, the derivation of the group explicit method from the finite difference method is presented. Section 3 discussed the stability of the proposed scheme, and the convergence of the proposed scheme is presented in Section 4. To show the efficiency of the proposed method, some numerical examples with discussion are presented in Section 5, and finally, Section 6 consists of the conclusion.

2. The fourth-order grouping scheme

First, let us define some notations:

$$\begin{aligned}
&\delta_x^2 V_{i,j}^k = V_{i+1,j}^k - 2V_{i,j}^k + V_{i-1,j}^k,\ V_{i,j}^{k+\frac{1}{2}} = \frac{V_{i,j}^{k+1} + V_{i,j}^k}{2},\\
&x_i = ih_0,\ y_j = jh_0,\ i, j = 0, 1, 2, 3..., M,\\
&t_k = kv,\ k = 0, 1, 2, 3..., N,
\end{aligned}$$

where $h_0 = \Delta x = \Delta y = \frac{L}{M}$ represents the space step and $v = \frac{T}{N}$ time step.

Since, the Taylor series expansion with respect to x is

$$V_{i+1,j}^k = V_{i,j}^k + \frac{h_0}{1!} V_x|_{i,j}^k + \frac{h_0^2}{2!} V_{xx}|_{i,j}^k + \frac{h_0^3}{3!} V|_{i,j}^k + ..., \tag{2.1}$$

$$V_{i-1,j}^k = V_{i,j}^k - \frac{h_0}{1!} V_x|_{i,j}^k + \frac{h_0^2}{2!} V_{xx}|_{i,j}^k - \frac{h_0^3}{3!} V_{xxx}|_{i,j}^k + ... \tag{2.2}$$

By adding Eqs (2.1) and (2.2) and after rearranging, we get

$$\begin{aligned}
\frac{\partial^2 V_{i,j}^k}{\partial x^2} &= \frac{V_{i+1,j}^k - 2u_{i,j}^k + V_{i-1,j}^k}{h_0^2} + \frac{2h_0^2}{4!} \frac{\partial^4 V_{i,j}^k}{\partial x^4} + ...\\
&= \frac{V_{i+1,j}^k - 2V_{i,j}^k + V_{i-1,j}^k}{h_0^2} + O(h_0^2).
\end{aligned} \tag{2.3}$$

Therefore, the Taylor series expansions at points $u_{i+1,j}^k$ and $u_{i,j+1}^k$ are

$$\frac{\delta_x^2}{h_0^2} V_{i,j}^k = \frac{\partial^2 V}{\partial x^2}|_{i,j}^k - \frac{h_0^2}{12} \frac{\partial^4 V}{\partial x^4}|_{i,j}^k - \frac{h_0^4}{360} \frac{\partial^6 V}{\partial x^6}|_{i,j}^k + O(h_0^6), \tag{2.4}$$

$$\frac{\delta_y^2}{h_0^2} V_{i,j}^k = \frac{\partial^2 V}{\partial y^2}|_{i,j}^k - \frac{h_0^2}{12} \frac{\partial^4 V}{\partial y^4}|_{i,j}^k + \frac{h_0^4}{360} \frac{\partial^6 V}{\partial y^6}|_{i,j}^k + O(h_0^6). \tag{2.5}$$

The difference operator δ_x^2, which maintain the three-point stencil is given by [23]

$$\frac{\delta_x^2}{h_0^2(1 + \frac{1}{12}\delta_x^2)} V_{i,j}^k = \frac{\partial^2 V}{\partial x^2}\Big|_{i,j}^k - \frac{h_0^4}{240} \frac{\partial^4 V}{\partial x^4}\Big|_{i,j}^k + O(h_0^6), \tag{2.6}$$

and

$$\frac{\delta_y^2}{h_0^2(1+\frac{1}{12}\delta_y^2)}V_{i,j}^k = \left.\frac{\partial^2 V}{\partial y^2}\right|_{i,j}^k - \frac{h_0^4}{240}\left.\frac{\partial^4 V}{\partial y^4}\right|_{i,j}^k + O(h_0^6). \tag{2.7}$$

The fractional derivative is approximated using the Central difference formula as [24]

$$\begin{aligned}
\frac{\partial^\alpha}{\partial t^\alpha}V(x,y,t)\Big|_{i,j}^{k+\frac{1}{2}} =& \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{k+\frac{1}{2}}} V_t(x,y,\varepsilon)(t_{k+\frac{1}{2}}-\varepsilon)^{-\alpha}\partial\varepsilon\\
=& \frac{1}{\Gamma(1-\alpha)}\left[\int_0^{t_{k+\frac{1}{2}}} V_t(x,y,\varepsilon)\left((k+\frac{1}{2})\nu-\varepsilon\right)^{-\alpha}\partial\varepsilon\right.\\
&\left.+\int_{t_k}^{t_{t_{k+\frac{1}{2}}}}\left(\frac{V_{i,j}^{k+1}-V_{i,j}^k}{\nu}+O(\nu)\right)\left((k+\frac{1}{2})\nu-\varepsilon\right)^{-\alpha}\partial\varepsilon\right]\\
=& \frac{1}{\Gamma(1-\alpha)}\sum_{s_0=1}^{k}\left[\int_{(s_0-1)\nu}^{s\nu}\frac{V_{i,j}^{s_0}-V_{i,j}^{s_0-1}}{\nu}+(\varepsilon-t_{s_0-\frac{1}{2}})V_{tt}(x_i,y_j,c_{s_0})\right.\\
&\left.\times((k+\frac{1}{2}\nu-\varepsilon)^{-\alpha}\partial\varepsilon)\right]+\int_{k\nu}^{(k+\frac{1}{2})\nu}\left(\frac{V_{i,j}^{k+1}-V_{i,j}^k}{\nu}+O(\nu)\right)\left((k+\frac{1}{2})\nu-\varepsilon\right)^{-\alpha}\partial\varepsilon\\
=& \frac{1}{\Gamma(1-\alpha)}\sum_{s_0=1}^{k}\frac{V_{i,j}^{s_0}-V_{i,j}^{s_0-1}}{\nu}\int_{(s_0-1)\nu}^{s\nu}\left((k+\frac{1}{2})\nu-\varepsilon\right)^{-\alpha}\partial\varepsilon\\
&+\frac{1}{\Gamma(1-\alpha)}\sum_{s_0=1}^{k}\int_{(s-1)\nu}^{s\nu}(\varepsilon-t_{s_0-\frac{1}{2}})V_{tt}(x_i,y_j,c_{s_0})\left((k+\frac{1}{2})\nu-\varepsilon\right)^{-\alpha}\partial\varepsilon\\
&+\frac{1}{\Gamma(1-\alpha)}\left[\frac{V_{i,j}^{k+1}-V_{i,j}^k}{\nu}+O(\nu)\right]\int_{k\nu}^{(k+\frac{1}{2})\nu}\left[\left((k+\frac{1}{2})\nu-\varepsilon\right)^{-\alpha}-\varepsilon\right]\partial\varepsilon\\
=& \frac{1}{\nu^\alpha(1-\alpha)\Gamma(1-\alpha)}\sum_{s_0=1}^{k}\left[V_{i,j}^{s_0}-V_{i,j}^{s_0-1}\right]\left[(k-s_0+\frac{3}{2})^{1-\alpha}-(k-s_0+\frac{1}{2})^{1-\alpha}\right]\\
&+\frac{1}{\nu^\alpha(1-\alpha)\Gamma(1-\alpha)}(V_{i,j}^{k+1}-V_{i,j}^k)\frac{1}{2^{1-\alpha}}\\
&+\frac{1}{\Gamma(1-\alpha)}\sum_{s_0=1}^{k}\int_{(s-1)\nu}^{s\nu}(\varepsilon-t_{s_0-\frac{1}{2}})V_{tt}(x_i,y_j,c_{s_0})\left((k+\frac{1}{2})\nu-\varepsilon\right)^{-\alpha}\partial\varepsilon\\
&+\frac{1}{\Gamma(1-\alpha)(1-\alpha)2^{1-\alpha}}O(\nu)^{2-\alpha}.
\end{aligned}$$

Therefore, after some simplifications, the Crank-Nicolson (C-N) Caputo fractional derivative

$$\frac{\partial^\alpha}{\partial t^\alpha}\ V(x,y,t)\Big|_{i,j}^{k+\frac{1}{2}} = a_1\ V\Big|_{i,j}^k+\sum_{s=1}^{k-1}(a_{k-s+1}-a_{k-s})\ V\Big|_{i,j}^s-a_k\ V\Big|_{i,j}^0+\sigma\frac{V_{i,j}^{k+1}+V_{i,j}^k}{2^{1-\alpha}}+O(\nu^{2-\alpha}), \tag{2.8}$$

$$\sigma=\frac{1}{\nu^\alpha\Gamma(2-\alpha)},\ a_s=\sigma\left((s+\frac{1}{2})^{1-\alpha}-(s-\frac{1}{2})^{1-\alpha}\right),\ s=0,1,2,...,k.$$

Now using Eqs (2.6)–(2.8) and C-N or standard point (SP) scheme at $V(x_i, y_j, t_{k+\frac{1}{2}})$, the standard fourth-order finite difference scheme for Eq (1.1) is as follows:

$$
\begin{aligned}
&a_1 V_{i,j}^k + \sum_{s=1}^{k-1}(a_{k-s+1} - a_{k-s})V_{i,j}^s - a_k V_{i,j}^0 + \sigma \frac{V_{i,j}^{k+1} + V_{i,j}^k}{2^{1-\alpha}} \\
=&\left(1 + \frac{1}{12}\delta_x^2\right)^{-1} \times \frac{\delta_x^2}{h_0^2} V_{i,j}^{k+\frac{1}{2}} + \left(1 + \frac{1}{12}\delta_y^2\right)^{-1} \frac{\delta_y^2}{h_0^2} V_{i,j}^{k+\frac{1}{2}} + f_{i,j}^{k+\frac{1}{2}} + O(\nu^{2-\alpha} + h_0^4).
\end{aligned}
\tag{2.9}
$$

Substituting the values of δ_x^2, δ_y^2 and $V_{i,j}^{k+\frac{1}{2}}$ into Eq (2.9), and after rearranging we get the standard point SP compact scheme:

$$
\begin{aligned}
\lambda_1 V_{i,j}^{k+1} =& \lambda_2 (V_{i+1,j}^{k+1} + V_{i-1,j}^{k+1} + V_{i,j+1}^{k+1} + V_{i,j-1}^{k+1}) + \lambda_3 (V_{i+1,j+1}^{k+1} + V_{i-1,j+1}^{k+1} \\
&+ V_{i+1,j-1}^{k+1} + V_{i-1,j-1}^{k+1}) + \lambda_4 V_{i,j}^k + \lambda_5 (V_{i+1,j}^k + V_{i-1,j}^k + V_{i,j+1}^k \\
&+ V_{i,j-1}^k) + \lambda_6 (V_{i+1,j+1}^k + V_{i-1,j+1}^k + V_{i+1,j-1}^k + V_{i-1,j-1}^k) \\
&+ \frac{25}{18} h_0^2 f_{i,j}^{k+\frac{1}{2}} + \frac{5}{36} h_0^2 (f_{i+1,j}^{k+\frac{1}{t}} + f_{i-1,j}^{k+\frac{1}{t}} + f_{i,j+1}^{k+\frac{1}{t}} + f_{i,j-1}^{k+\frac{1}{t}}) \\
&+ \frac{h_0^2}{72} (f_{i+1,j+1}^{k+\frac{1}{2}} + f_{i-1,j+1}^{k+\frac{1}{2}} + f_{i+1,j-1}^{k+\frac{1}{2}} + f_{i-1,j-1}^{k+\frac{1}{2}}) \\
&- \sum_{s=1}^{k-1}(a_{k-s+1} - a_{k-s}) \left(\frac{25}{18} h_0^2 V_{i,j}^s + \frac{5}{36} h_0^2 (V_{i+1,j}^s + V_{i-1,j}^s + V_{i,j+1}^s \right. \\
&\left. + V_{i,j-1}^s) + \frac{h_0^2}{72} (V_{i+1,j+1}^s + V_{i-1,j+1}^s + V_{i+1,j-1}^s + V_{i-1,j-1}^s) \right) + O(\nu^{2-\alpha} + h_0^4),
\end{aligned}
\tag{2.10}
$$

where

$$
\begin{aligned}
&g_0 = \frac{\sigma}{2^{1-\alpha}}, \; g_1 = a_1 - g_0, \; \lambda_1 = \frac{1}{72}(240 + 100 h_0^2 g_0), \; \lambda_2 = \frac{1}{72}(48 - 10 h_0^2 g_0), \\
&\lambda_3 = \frac{1}{72}(12 - h_0^2 g_0), \; \lambda_4 = -\frac{1}{72}(240 + 100 h_0^2 g_1), \; \lambda_5 = \frac{1}{72}(48 - 10 h_0^2 g_1), \\
&\lambda_6 = \frac{1}{72}(12 - h_0^2 g_1).
\end{aligned}
$$

Now, using Eq (2.10) will give the following system for the group of four points:

$$
\begin{bmatrix}
\lambda_1 & -\lambda_2 & -\lambda_3 & -\lambda_2 \\
-\lambda_2 & \lambda_1 & -\lambda_2 & -\lambda_3 \\
-\lambda_3 & -\lambda_2 & \lambda_1 & -\lambda_2 \\
-\lambda_2 & -\lambda_3 & -\lambda_2 & \lambda_1
\end{bmatrix}
\begin{bmatrix}
V_{i,j}^{k+1} \\ V_{i+1,j}^{k+1} \\ V_{i+1,j+1}^{k+1} \\ V_{i,j+1}^{k+1}
\end{bmatrix}
=
\begin{bmatrix}
rhs_{i,j} \\ rhs_{i+1,j} \\ rhs_{i+1,j+1} \\ rhs_{i,j+1}
\end{bmatrix},
\tag{2.11}
$$

where

$$
\begin{aligned}
rhs_{i,j} =& \lambda_2 (V_{i-1,j}^{k+1} + V_{i,j-1}^{k+1}) + \lambda_3 (V_{i-1,j+1}^{k+1} + V_{i+1,j-1}^{k+1} + V_{i-1,j-1}^{k+1}) + \lambda_4 V_{i,j}^k \\
&+ \lambda_5 (V_{i+1,j}^k + V_{i-1,j}^k + V_{i,j+1}^k + V_{i,j-1}^k) + \lambda_6 (V_{i+1,j+1}^k + V_{i-1,j+1}^k \\
&+ V_{i+1,j-1}^k + V_{i-1,j-1}^k) + \frac{25}{18} h_0^2 f_{i,j}^{k+\frac{1}{2}} + \frac{5}{36} h_0^2 (f_{i+1,j}^{k+\frac{1}{2}} + f_{i-1,j}^{k+\frac{1}{2}} + f_{i,j+1}^{k+\frac{1}{2}} \\
&+ f_{i,j-1}^{k+\frac{1}{2}}) + \frac{h_0^2}{72} (f_{i+1,j+1}^{k+\frac{1}{2}} + f_{i-1,j+1}^{k+\frac{1}{2}} + f_{i+1,j-1}^{k+\frac{1}{2}} + f_{i-1,j-1}^{k+\frac{1}{2}}) - F_{i,j},
\end{aligned}
$$

$$
\begin{aligned}
rhs_{i+1,j} =& \lambda_2(V^{k+1}_{i+2,j} + V^{k+1}_{i+1,j-1}) + \lambda_3(V^{k+1}_{i+2,j+1} + V^{k+1}_{i+2,j-1} + V^{k+1}_{i,j-1}) + \lambda_4 V^k_{i+1,j} \\
&+ \lambda_5(V^k_{i+2,j} + V^k_{i,j} + V^k_{i+1,j+1} + V^k_{i+1,j-1}) + \lambda_6(V^k_{i+2,j+1} + V^k_{i,j+1} \\
&+ V^k_{i+2,j-1} + V^k_{i,j-1}) + \frac{25}{18}h_0^2 f^{k+\frac{1}{2}}_{i+1,j} + \frac{5}{36}h_0^2(f^{k+\frac{1}{2}}_{i+2,j} + f^{k+\frac{1}{2}}_{i,j} + f^{k+\frac{1}{2}}_{i+1,j+1} \\
&+ f^{k+\frac{1}{2}}_{i+1,j-1}) + \frac{h_0^2}{72}(f^{k+\frac{1}{2}}_{i+2,j+1} + f^{k+\frac{1}{2}}_{i,j+1} + f^{k+\frac{1}{2}}_{i+2,j-1} + f^{k+\frac{1}{2}}_{i,j-1}) - F_{i+1,j},
\end{aligned}
$$

$$
\begin{aligned}
rhs_{i+1,j+1} =& \lambda_2(V^{k+1}_{i+2,j+1} + V^{k+1}_{i+1,j+2}) + \lambda_3(V^{k+1}_{i+2,j+2} + V^{k+1}_{i,j+2} + V^{k+1}_{i+2,j}) + \lambda_4 V^k_{i+1,j+1} \\
&+ \lambda_5(V^k_{i+2,j+1} + V^k_{i,j+1} + V^k_{i+1,j+2} + V^k_{i+1,j}) + \lambda_6(V^k_{i+2,j+2} + V^k_{i,j+2} \\
&+ V^k_{i+2,j} + V^k_{i,j}) + \frac{25}{18}h_0^2 f^{k+\frac{1}{2}}_{i+1,j+1} + \frac{5}{36}h_0^2(f^{k+\frac{1}{2}}_{i+2,j+1} + f^{k+\frac{1}{2}}_{i,j+1} + f^{k+\frac{1}{2}}_{i+1,j+2} \\
&+ f^{k+\frac{1}{2}}_{i+1,j}) + \frac{h_0^2}{72}(f^{k+\frac{1}{2}}_{i+2,j+2} + f^{k+\frac{1}{2}}_{i,j+2} + f^{k+\frac{1}{2}}_{i+2,j} + f^{k+\frac{1}{2}}_{i,j}) - F_{i+1,j+1},
\end{aligned}
$$

$$
\begin{aligned}
rhs_{i,j+1} =& \lambda_2(V^{k+1}_{i-1,j+1} + V^{k+1}_{i,j+2}) + \lambda_3(V^{k+1}_{i+1,j+2} + V^{k+1}_{i-1,j+2} + V^{k+1}_{i-1,j}) + \lambda_4 V^k_{i,j+1} \\
&+ \lambda_5(V^k_{i+1,j+1} + V^k_{i-1,j+1} + V^k_{i,j+2} + V^k_{i,j}) + \lambda_6(V^k_{i+1,j+2} + V^k_{i-1,j+2} \\
&+ V^k_{i+1,j} + V^k_{i-1,j}) + \frac{25}{18}h_0^2 f^{k+\frac{1}{2}}_{i,j+1} + \frac{5}{36}h_0^2(f^{k+\frac{1}{2}}_{i+1,j+1} + f^{k+\frac{1}{2}}_{i-1,j+1} + f^{k+\frac{1}{2}}_{i,j+2} \\
&+ f^{k+\frac{1}{2}}_{i,j}) + \frac{h_0^2}{72}(f^{k+\frac{1}{2}}_{i+1,j+2} + f^{k+\frac{1}{2}}_{i-1,j+2} + f^{k+\frac{1}{2}}_{i+1,j} + f^{k+\frac{1}{2}}_{i-1,j}) - F_{i,j+1},
\end{aligned}
$$

and

$$
\begin{aligned}
F_{i,j} = \sum_{s=1}^{k-1}(a_{k-s+1} - a_{k-s})\Bigg(&\frac{25}{18}h_0^2 V^s_{i,j} + \frac{5}{36}h_0^2(V^s_{i+1,j} + V^s_{i-1,j} + V^s_{i,j+1} \\
&+V^s_{i,j-1}) + \frac{h_0^2}{72}(V^s_{i+1,j+1} + V^s_{i-1,j+1} + V^s_{i+1,j-1} + V^s_{i-1,j-1})\Bigg).
\end{aligned}
$$

Similarly, the inverted matrix equation (2.11) will give explicit group equation

$$
\begin{bmatrix} V^{k+1}_{i,j} \\ V^{k+1}_{i+1,j} \\ V^{k+1}_{i+1,j+1} \\ V^{k+1}_{i,j+1} \end{bmatrix} = \frac{1}{d}\begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_2 \\ \phi_2 & \phi_1 & \phi_2 & \phi_3 \\ \phi_3 & \phi_2 & \phi_1 & \phi_2 \\ \phi_2 & \phi_3 & \phi_2 & \phi_1 \end{bmatrix}\begin{bmatrix} rhs_{i,j} \\ rhs_{i+1,j} \\ rhs_{i+1,j+1} \\ rhs_{i,j+1} \end{bmatrix}, \tag{2.12}
$$

where

$$
\begin{aligned}
&\phi_1 = \lambda_1^3 - 2\lambda_1\lambda_2^2 - 2\lambda_2^2\lambda_3 - \lambda_1\lambda_3^2,\ \phi_2 = \lambda_1^2\lambda_2 + 2\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3^2, \\
&\phi_3 = 2\lambda_1\lambda_2^2 + \lambda_1^2\lambda_3 + 2\lambda_2^2\lambda_3 - \lambda_3^3,\ d = (-4\lambda_2^2 + (\lambda_1 - \lambda_3)^2)(\lambda_1 + \lambda_3)^2.
\end{aligned}
$$

In the proposed method, firstly, group of four points are computed for the different iterations using Eq (2.12) till the required convergence is attained. After the required convergence, the SP compact scheme Eq (2.10) is used directly once for computing of reaming points. Figures 1 and 2 show the grid points on the x-y plane for FDM and FEGM at various time levels when $m = 9$ respectively.

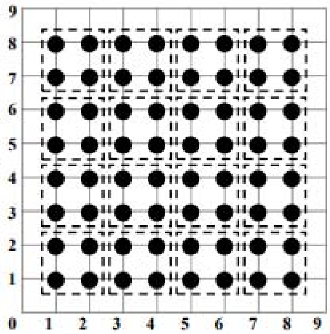

Figure 1. Four points in computation of grouping method.

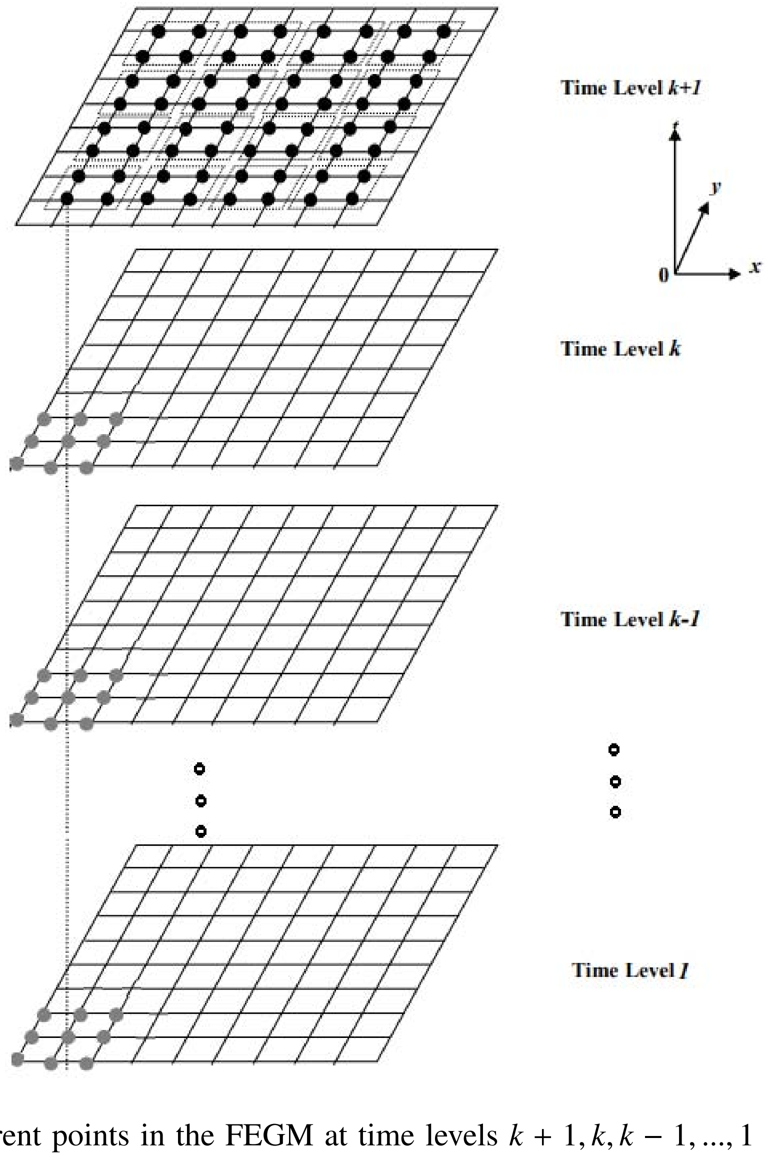

Figure 2. Different points in the FEGM at time levels $k+1, k, k-1, ..., 1$ with mesh size $m = 9$.

3. Stability of the FEGM

In this section, the stability of the proposed method is discussed.

The Eq (2.12) can also be written as

$$\begin{aligned} &AV^1 = BV^0 + f^{\frac{1}{2}},\ k = 0, \\ &AV^{k+1} = BV^k - h_0^2 \sum_{s=1}^{k-1} (a_{k-s+1} - a_{k-s}) CV^s + h_0^2 f^{k+\frac{1}{2}},\ k > 0, \end{aligned} \tag{3.1}$$

where

$$A = \begin{bmatrix} R_1 & R_3 & & & 0 \\ R_2 & R_1 & R_3 & & \\ & R_2 & R_1 & & \\ & & & \ddots & R_3 \\ 0 & & & R_2 & R_1 \end{bmatrix},\ B = \begin{bmatrix} P_1 & P_3 & & & 0 \\ P_2 & P_1 & P_3 & & \\ & P_2 & P_1 & & \\ & & & \ddots & P_3 \\ 0 & & & P_2 & P_1 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} G_1 & G_3 & & & \\ G_2 & G_1 & G_3 & & \\ & G_2 & G_1 & & \\ & & & \ddots & G_3 \\ & & & G_2 & G_1 \end{bmatrix},\ C = \begin{bmatrix} Q_1 & Q_3 & & & 0 \\ Q_2 & Q_1 & Q_3 & & \\ & Q_2 & Q_1 & & \\ & & & \ddots & Q_3 \\ 0 & & & Q_2 & Q_1 \end{bmatrix},\ f = \begin{bmatrix} K_1 \\ K_1 \\ \vdots \\ K_1 \\ K_1 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} G_6 & G_4 & & & \\ G_8 & G_6 & G_4 & & \\ & G_8 & G_6 & & \\ & & & \ddots & G_4 \\ & & & G_8 & G_6 \end{bmatrix},\ R_3 = \begin{bmatrix} G_7 & G_9 & & & \\ G_5 & G_7 & G_9 & & \\ & G_5 & G_7 & & \\ & & & \ddots & G_9 \\ & & & G_5 & G_7 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} H_1 & H_3 & & & \\ H_2 & H_1 & H_3 & & \\ & H_2 & H_1 & & \\ & & & \ddots & H_3 \\ & & & H_2 & H_1 \end{bmatrix},\ P_2 = \begin{bmatrix} H_6 & H_4 & & & \\ H_8 & H_6 & H_4 & & \\ & H_8 & H_6 & & \\ & & & \ddots & H_4 \\ & & & H_8 & H_6 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} H_7 & H_9 & & & \\ H_5 & H_7 & H_9 & & \\ & H_5 & H_7 & & \\ & & & \ddots & H_9 \\ & & & H_5 & H_7 \end{bmatrix},\ Q_1 = \begin{bmatrix} L_1 & L_3 & & & \\ L_2 & L_1 & L_3 & & \\ & L_2 & L_1 & & \\ & & & \ddots & L_3 \\ & & & L_2 & L_1 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} L_6 & L_4 & & & \\ L_8 & L_6 & L_4 & & \\ & L_8 & L_6 & & \\ & & & \ddots & L_4 \\ & & & L_8 & L_6 \end{bmatrix},\ Q_3 = \begin{bmatrix} L_7 & L_9 & & & \\ L_5 & L_7 & L_9 & & \\ & L_5 & L_7 & & \\ & & & \ddots & L_9 \\ & & & L_5 & L_7 \end{bmatrix},\ K_1 = \begin{bmatrix} W_1 \\ W_1 \\ \vdots \\ W_1 \\ W_1 \end{bmatrix},$$

$$G_1=\begin{bmatrix}\lambda_1 & -\lambda_2 & -\lambda_3 & -\lambda_2\\ -\lambda_2 & \lambda_1 & -\lambda_2 & -\lambda_3\\ -\lambda_3 & -\lambda_2 & \lambda_1 & -\lambda_2\\ -\lambda_2 & -\lambda_3 & -\lambda_2 & \lambda_1\end{bmatrix},\ G_2=\begin{bmatrix}0&0&-\lambda_3&-\lambda_2\\0&0&-\lambda_2&-\lambda_3\\0&0&0&0\\0&0&0&0\end{bmatrix},\ G_3=\begin{bmatrix}0&0&0&0\\0&0&0&0\\-\lambda_3&-\lambda_2&0&0\\-\lambda_2&-\lambda_3&0&0\end{bmatrix},$$

$$G_4=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&-\lambda_3&0&0\end{bmatrix},\ G_5=\begin{bmatrix}0&0&0&0\\0&0&0&-\lambda_3\\0&0&0&0\\0&0&0&0\end{bmatrix},\ G_6=\begin{bmatrix}0&-\lambda_2&-\lambda_3&0\\0&0&0&0\\0&0&0&0\\0&-\lambda_3&-\lambda_2&0\end{bmatrix},$$

$$G_7=\begin{bmatrix}0&0&0&0\\-\lambda_2&0&0&-\lambda_3\\-\lambda_3&0&0&-\lambda_2\\0&0&0&0\end{bmatrix},\ G_8=\begin{bmatrix}0&0&-\lambda_3&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\ G_9=\begin{bmatrix}0&0&0&0\\0&0&0&0\\-\lambda_3&0&0&0\\0&0&0&0\end{bmatrix},$$

$$H_1=\begin{bmatrix}\lambda_4&\lambda_5&\lambda_6&\lambda_5\\ \lambda_5&\lambda_4&\lambda_5&\lambda_6\\ \lambda_6&\lambda_5&\lambda_4&\lambda_5\\ \lambda_5&\lambda_6&\lambda_5&\lambda_4\end{bmatrix},\ H_2=\begin{bmatrix}0&0&\lambda_6&\lambda_5\\0&0&\lambda_5&\lambda_6\\0&0&0&0\\0&0&0&0\end{bmatrix},\ H_3=\begin{bmatrix}0&0&0&0\\0&0&0&0\\ \lambda_6&\lambda_5&0&0\\ \lambda_5&\lambda_6&0&0\end{bmatrix},$$

$$H_4=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&\lambda_6&0&0\end{bmatrix},\ H_5=\begin{bmatrix}0&0&0&0\\0&0&0&\lambda_6\\0&0&0&0\\0&0&0&0\end{bmatrix},\ H_6=\begin{bmatrix}0&\lambda_5&\lambda_6&0\\0&0&0&0\\0&0&0&0\\0&\lambda_6&\lambda_5&0\end{bmatrix},$$

$$H_7=\begin{bmatrix}0&0&0&0\\ \lambda_5&0&0&\lambda_6\\ \lambda_6&0&0&\lambda_5\\0&0&0&0\end{bmatrix},\ H_8=\begin{bmatrix}0&0&\lambda_6&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\ H_9=\begin{bmatrix}0&0&0&0\\0&0&0&0\\ \lambda_6&0&0&0\\0&0&0&0\end{bmatrix},$$

$$L_1=\frac{1}{18}\begin{bmatrix}25&\frac{5}{2}&\frac{1}{4}&\frac{5}{4}\\ \frac{5}{2}&25&\frac{5}{2}&\frac{1}{4}\\ \frac{1}{4}&\frac{5}{2}&25&\frac{5}{2}\\ \frac{5}{2}&\frac{1}{4}&\frac{5}{2}&25\end{bmatrix},\ L_2=\frac{1}{18}\begin{bmatrix}0&0&\frac{1}{4}&\frac{5}{4}\\0&0&\frac{5}{2}&\frac{1}{4}\\0&0&0&0\\0&0&0&0\end{bmatrix},\ L_3=\frac{1}{18}\begin{bmatrix}0&0&0&0\\0&0&0&0\\ \frac{1}{4}&\frac{5}{4}&0&0\\ \frac{5}{4}&\frac{1}{4}&0&0\end{bmatrix},$$

$$L_4=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&\frac{1}{72}&0&0\end{bmatrix},\ L_5=\begin{bmatrix}0&0&0&0\\0&0&0&\frac{1}{72}\\0&0&0&0\\0&0&0&0\end{bmatrix},\ L_6=\frac{1}{18}\begin{bmatrix}0&\frac{5}{2}&\frac{1}{4}&0\\0&0&0&0\\0&0&0&0\\0&\frac{1}{4}&\frac{5}{2}&0\end{bmatrix},$$

$$L_7=\frac{1}{18}\begin{bmatrix}0&0&0&0\\ \frac{5}{2}&0&0&\frac{1}{4}\\ \frac{1}{4}&0&0&\frac{5}{2}\\0&0&0&0\end{bmatrix},\ L_8=\begin{bmatrix}0&0&\frac{1}{72}&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\ L_9=\begin{bmatrix}0&0&0&0\\0&0&0&0\\ \frac{1}{72}&0&0&0\\0&0&0&0\end{bmatrix},\ W_1=\begin{bmatrix}f_{i,j}\\ f_{i+1,j}\\ f_{i+1,j+1}\\ f_{i,j+1}\end{bmatrix}.$$

It can observe that Eq (3.1) form the particular structure as

$$[A_{(N-2)^2\times(N-2)^2}]V^{k+1}=[B_{(N-2)^2\times(N-2)^2}]V^k-h_0^2\sum_{s=1}^{k-1}(a_{k-s+1}-a_{k-s})[C_{(N-2)^2\times(N-2)^2}]V^s+h_0^2f^{k+\frac{1}{2}}.$$

Proposition 3.1. *The proposed scheme Eq (2.12) is unconditionally stable.*

Proof. Let $V_{i,j}^k$ represents approximate and $v_{i,j}^k$ represents exact solutions for the time FSDE respectively, then the error is defined as $\epsilon_{i,j}^k = v_{i,j}^k - V_{i,j}^k$. So, from Eq (3.1),

$$\begin{aligned} &AE^1 = BE^0,\ k = 0, \\ &AE^{k+1} = BE^k - h_0^2 \sum_{s=1}^{k-1} (a_{k-s+1} - ak - s) CE^s,\ k > 0, \end{aligned} \tag{3.2}$$

where

$$E^{k+1} = \begin{bmatrix} E_1^{k+1} \\ E_2^{k+1} \\ \vdots \\ E_{m-2}^{k+1} \\ E_{m-1}^{k+1} \end{bmatrix},\ E_i^{k+1} = \begin{bmatrix} \epsilon_1^{k+1} \\ \epsilon_2^{k+1} \\ \vdots \\ \epsilon_{m-2}^{k+1} \\ \epsilon_{m-1}^{k+1} \end{bmatrix},\ \epsilon_i^{k+1} = \begin{bmatrix} \epsilon_{i,j}^{k+1} \\ \epsilon_{i+1,j}^{k+1} \\ \epsilon_{i+1,j+1}^{k+1} \\ \epsilon_{i,j+1}^{k+1} \end{bmatrix},\ i, j = 1, 2, ..., m-1.$$

From Eq (3.1) we know

$$A = G_1 I + (G_2 + G_3) Q + G_6 I + (G_4 + G_8) Q + G_7 I + (G_5 + G_9) Q, \tag{3.3}$$

$$B = H_1 I + (H_2 + H_3) Q + H_6 I + (H_4 + H_8) Q + H_7 I + (H_5 + H_9) Q, \tag{3.4}$$

$$C = L_1 I + (L_2 + L_3) Q + L_6 I + (L_4 + L_8) Q + L_7 I + (L_5 + L_9) Q, \tag{3.5}$$

where I and Q are two matrices, I represents identity matrix and Q represents unity values having each diagonal forthwith above and below the main diagonal, and elsewhere zero.

Suppose maximum eigenvalues are represented with ψ, χ and η for the matrices A, B and C respectively, then using Mathematica software, we get

$$\begin{aligned} &\psi = \frac{9}{8}\left(g_1 h_0^2 + 4\right), \\ &\chi = (\frac{29}{6} + \frac{79}{72} h_0^2 g_0), \\ &\eta = \frac{121}{72}. \end{aligned} \tag{3.6}$$

From Eq (3.2), when k=0,

$$\begin{aligned} &E^1 = A^{-1} B E^0, \\ &\left\|E^1\right\| \le \left\|A^{-1}B\right\| \left\|E^0\right\| = \frac{132 + 121 h^2 g_1}{348 + 79 h^2 g_0} \left\|E^0\right\|. \end{aligned}$$

But since $a_1 = \sigma((\frac{3}{2})^{1-\alpha} - (\frac{1}{2})^{1-\alpha}) = g_0(3^{1-\alpha} - 1)$ and $g_1 = a_1 - g_0 = g_0(3^{1-\alpha} - 2)$. Also we know that $3^{1-\alpha} < 3$, so,

$$\begin{aligned} &3^{1-\alpha} - 2 < 1, \\ &g_0(3^{1-\alpha} - 2) < g_0, \qquad \because g_0 > 0, \\ &g_1 < g_0. \end{aligned}$$

Hence,

$$\left\|E^1\right\| \le \left\|E^0\right\|, \quad \because g_0 > g_1.$$

Suppose

$$\|E^r\| \le \left\|E^0\right\|, \quad r = 2, 3, ..., k, \tag{3.7}$$

and for $r = k + 1$,

$$\begin{aligned}
\left\|E^{k+1}\right\| &= \left\|A^{-1}\left(BE^k - h_0^2\sum_{s=1}^{k-1}(a_{k-s+1} - a_{k-s})CE^s)\right)\right\| \\
&\le \left\|A^{-1}B\right\|\left\|E^k\right\| + h_0^2\sum_{s=1}^{k-1}(a_{k-s+1} - a_{k-s})\left\|A^{-1}C\right\|\|E^s\| \\
&\le \left(\left\|A^{-1}B\right\| + h_0^2\sum_{s=1}^{k-1}(a_{k-s+1} - a_{k-s})\left\|A^{-1}C\right\|\right)\left\|E^0\right\| \qquad \text{by using Eq. (3.7)} \\
&= \left(\frac{324 + 81h_0^2g_1}{348 + h_0^2g_0} + \frac{121h_0^2(a_k - a_1)}{72(348 + 79h_0^2g_0)}\right)\left\|E^0\right\| \\
&= \left(\frac{324 + 81h_0^2g_1 + 1.68h_0^2(a_k - a_1)}{348 + 79h_0^2g_0}\right)\left\|E^0\right\| \\
&= \left(\frac{132 + (72.02)h^2(g_1 + (a_k - a_1))}{348 + 79h_0^2g_0}\right)\left\|E^0\right\|.
\end{aligned}$$

$$\left\|E^{k+1}\right\| \le \left\|E^0\right\|, \qquad \because (a_k - a_1) < 0.$$

So, by mathematical induction, we prove that FEGM is unconditionally stable. □

4. Convergence analysis

Suppose $e_{i,j}^{k+\frac{1}{2}}$, $e_{i+1,j}^{k+\frac{1}{2}}$, $e_{i+1,j+1}^{k+\frac{1}{2}}$ and $e_{i,j+1}^{k+\frac{1}{2}}$ represent different truncation errors, then,

$$R^{k+\frac{1}{2}} = \{R_{1,1}^{k+\frac{1}{2}}, R_{1,2}^{k+\frac{1}{2}}, ..., R_{1,\frac{M_2-1}{4}}^{k+\frac{1}{2}}, R_{2,1}^{k+\frac{1}{2}}, R_{2,2}^{k+\frac{1}{2}}, ..., R_{\frac{M_1-1}{4},\frac{M_2-1}{4}}^{k+\frac{1}{2}}\},$$

where

$$R_{i,j}^{k+\frac{1}{2}} = \{e_{i,j}^{k+\frac{1}{2}}, e_{i+1,j}^{k+\frac{1}{2}}, e_{i+1,j+1}^{k+\frac{1}{2}}, e_{i,j+1}^{k+\frac{1}{2}}\},\ i, j = \{1, 2, ..., \frac{M-1}{4}\},$$

so from Eq (2.10) we have

$$\left\|R^{k+\frac{1}{2}}\right\| \le \varphi_0(\nu^{2-\gamma} + h_0^4), \tag{4.1}$$

where φ_0 is a constant.

Proposition 4.1. *The FEGS equation (2.12) is unconditionally convergent with the order of convergence* $O(\nu^{2-\alpha} + h^4)$.

Proof. Since from Eq (4.1),

$$\left\|R^{(k-1)+\frac{1}{2}}\right\| \leq \varphi_0(\nu^{2-\alpha} + h_0^4), \tag{4.2}$$

then,

$$\begin{aligned}&\left\|R^{(k-1)+\frac{1}{2}}\right\| - \left\|R^{k+\frac{1}{2}}\right\| \leq 0,\\ &\left\|R^{(k-1)+\frac{1}{2}}\right\| \leq \left\|R^{k+\frac{1}{2}}\right\|.\end{aligned} \tag{4.3}$$

Since $E^0 = 0$, then from Eq (2.10), we have

$$\begin{aligned}&AE^1 = R^{\frac{1}{2}},\ k = 0,\\ &AE^{k+1} = BE^k - h_0^2 \sum_{s=1}^{k-1}(a_{k-s+1} - a_{k-s})CE^s + R^{k+\frac{1}{2}},\ k > 0.\end{aligned} \tag{4.4}$$

When $k = 0$,

$$\begin{aligned}&AE^1 = R^{\frac{1}{2}},\\ &\left\|E^1\right\| \leq \left\|A^{-1}\right\| \left\|R^{\frac{1}{2}}\right\| = \frac{1}{\lambda_1 + 2\lambda_2 + \lambda_3} \left\|R^{\frac{1}{2}}\right\| = \frac{1}{348 + 79h_0^2 g_0} \left\|R^{\frac{1}{2}}\right\|,\\ &\left\|E^1\right\| \leq \mu_0 \left\|R^{\frac{1}{2}}\right\|,\ \text{where } \mu_0 = \frac{1}{348 + 79h_0^2 g_0} \ \text{ and } \ \mu_0 \in (0, 1),\\ &\left\|E^1\right\| \leq \left\|R^{\frac{1}{2}}\right\|.\end{aligned}$$

Assume that

$$\|E^s\| \leq \left\|R^{(s-1)+\frac{1}{2}}\right\|,\ s = 2, 3, ..., k, \tag{4.5}$$

and now from Eq (4.4),

$$AE^{k+1} = BE^k - h_0^2 \sum_{s=1}^{k-1}(a_{k-s+1} - a_{k-s})CE^s + R^{k+\frac{1}{2}}. \tag{4.6}$$

By taking norm function on both sides of Eq (4.6),

$$\begin{aligned}\left\|E^{k+1}\right\| &\leq \left\|A^{-1}B\right\| \left\|E^k\right\| - h_0^2 \sum_{s=1}^{k-1}(a_{k-s+1} - a_{k-s}) \left\|A^{-1}C\right\| \|E^s\| + \left\|A^{-1}\right\| \left\|R^{k+\frac{1}{2}}\right\|\\ &\leq \left(\frac{324 + 81h_0^2 g_1}{348 + 79h_0^2 g_0} + \frac{1.68h_0^2(a_k - a_1)}{348 + 79h_0^2 g_0} + \frac{1}{348 + 79h_0^2 g_0}\right) \left\|R^{k+\frac{1}{2}}\right\| \text{ (by using Eqs (4.3) and (4.5))}\\ &= \left(\frac{325 + 81h_0^2 g_1 + 1.68h_0^2(a_k - a_1)}{348 + 79h_0^2 g_0}\right) \left\|R^{k+\frac{1}{2}}\right\|\\ &= \gamma \left\|R^{k+\frac{1}{2}}\right\|,\end{aligned}$$

where $\gamma = \frac{133+121h^2 g_1+1.68h_0^2(a_k-a_1)}{348+79h_0^2 g_0}$, but since $h \in (0, 1)$, $g_0 > g_1$, and $(a_1 - a_k) < 0$, then $\gamma \in (0, 1)$, therefore,

$$\left\|E^{k+1}\right\| \leq \left\|R^{k+\frac{1}{2}}\right\| \leq \varphi_0(\nu^{2-\alpha} + h_0^4).$$

Therefore, we get

$$\left\|E^{k+1}\right\| \leq \varphi_0(\nu^{2-\alpha} + h_0^4),\ \forall k = 0, 1, 2..., N-1.$$

Thus the proposed scheme is conditionally stable. □

5. Solvability of the proposed scheme

The proposed scheme can be written in matrix form:

$$\begin{aligned}
&\mathcal{G}_1 V^1 = \mathcal{G}_2 V^0 + \mathcal{G}_3 \Upsilon^{\frac{1}{2}},\ \ k = 0,\\
&\mathcal{G}_1 V^{k+1} = \mathcal{G}_2 V^k + \mathcal{G}_3 \Upsilon^{k+\frac{1}{2}} - \sum_{s=1}^{k-1}(a_{k-s+1} - a_{k-s})\mathcal{G}_3 V^s,\ \ k \geq 1,\\
&V_{i,j}^0 = b_0(x_i, y_j),\ 1 \leq i \leq M,\ 1 \leq j \leq M,\\
&V_{0,j}^k = b_1(0, y_j),\ 1 \leq j \leq M,\ 0 \leq k \leq N,\\
&V_{L,j}^k = b_2(L, y_j),\ 1 \leq j \leq M,\ 0 \leq k \leq N,\\
&V_{i,0}^k = b_3(x_i, 0),\ 1 \leq i \leq M,\ 0 \leq k \leq N,\\
&V_{i,L}^k = b_4(x_i, L),\ 1 \leq i \leq M,\ 0 \leq k \leq N,
\end{aligned} \tag{5.1}$$

where

$$\mathcal{G}_1 = \begin{bmatrix}
\lambda_1 & -\lambda_2 & & \cdots & -\lambda_2 & -\lambda_3 & \cdots & & 0\\
-\lambda_2 & \lambda_1 & -\lambda_2 & & -\lambda_3 & -\lambda_2 & -\lambda_3 & & \vdots\\
 & -\lambda_2 & \lambda_1 & -\lambda_2 & \cdots & -\lambda_3 & -\lambda_2 & -\lambda_3 & \\
\vdots & \cdots & -\lambda_2 & \lambda_1 & -\lambda_2 & & -\lambda_3 & -\lambda_2 & -\lambda_3\\
 & & & -\lambda_2 & \lambda_1 & -\lambda_2 & & -\lambda_3 & -\lambda_2\\
\vdots & \cdots & & & -\lambda_2 & \lambda_1 & -\lambda_2 & & \\
 & & & & & -\lambda_2 & \lambda_1 & -\lambda_2 & \\
 & & & & & & -\lambda_2 & \lambda_1 & -\lambda_2\\
0 & & \cdots & & \cdots & & & -\lambda_2 & \lambda_1
\end{bmatrix},$$

$$\mathcal{G}_2 = \begin{bmatrix}
\lambda_4 & \lambda_5 & & \cdots & \lambda_5 & \lambda_6 & \cdots & & 0\\
\lambda_5 & \lambda_4 & \lambda_5 & & \lambda_6 & \lambda_5 & \lambda_6 & & \vdots\\
 & \lambda_5 & \lambda_4 & \lambda_5 & \cdots & \lambda_6 & \lambda_5 & \lambda_6 & \\
\vdots & \cdots & \lambda_5 & \lambda_4 & \lambda_5 & & \lambda_6 & \lambda_5 & \lambda_6\\
 & & & \lambda_5 & \lambda_4 & \lambda_5 & & \lambda_6 & \lambda_5\\
\vdots & \cdots & & & \lambda_5 & \lambda_4 & \lambda_5 & & \\
 & & & & & \lambda_5 & \lambda_4 & \lambda_5 & \\
 & & & & & & \lambda_5 & \lambda_4 & \lambda_5\\
0 & & \cdots & & \cdots & & & \lambda_5 & \lambda_4
\end{bmatrix},$$

$$\mathcal{G}_3 = \begin{bmatrix} \rho_1 & \rho_2 & & \cdots & \rho_2 & \rho_3 & \cdots & & 0 \\ \rho_2 & \rho_1 & \rho_2 & & \rho_3 & \rho_2 & \rho_3 & & \vdots \\ & \rho_2 & \rho_1 & \rho_2 & \cdots & \rho_3 & \rho_2 & \rho_3 & \\ \vdots & \cdots & \rho_2 & \rho_1 & \rho_2 & & \rho_3 & \rho_2 & \rho_3 \\ & & & \rho_2 & \rho_1 & \rho_2 & & \rho_3 & \rho_2 \\ \vdots & \cdots & & & \rho_2 & \rho_1 & \rho_2 & & \\ & & & & & \rho_2 & \rho_1 & \rho_2 & \\ & & & & & & \rho_2 & \rho_1 & \rho_2 \\ 0 & & \cdots & & \cdots & & & \rho_2 & \rho_1 \end{bmatrix},$$

$$\Upsilon^k = [\Upsilon_0^k, \Upsilon_1^k, \Upsilon_2^k, ..., \Upsilon_n^k]^T,\ \Upsilon^{k+\frac{1}{2}} = f(x_i, y_j, t_{k+\frac{1}{2}}),\ \rho_1 = \frac{25h_0^2}{18},\ \rho_2 = \frac{5h_0^2}{36} \text{ and } \rho_3 = \frac{h_0^2}{72}.$$

Proposition 5.1. *The difference equation (2.12) is uniquely solvable.*

Proof. Since $\lambda_1 = \frac{1}{72}(240 + 100h_0^2 g_0)$, $\lambda_2 = \frac{1}{72}(48 - 10h_0^2 g_0)$, $\lambda_3 = \frac{1}{72}(12 - h_0^2 g_0)$, and $h_0, g_0 > 0$, then,

$$|\lambda_1| = \frac{10}{3} + \frac{25h_0^2 g_0}{18}$$

and

$$3|\lambda_2| + 2|\lambda_3| \le \frac{7}{3} + \frac{4}{9}h_0^2 g_0 < \frac{10}{3} + \frac{25}{18}h_0^2 g_0 = |\lambda_1|.$$

Hence, $|\lambda_1| > 3|\lambda_2| + 2|\lambda_1|$, which shows that matrix G_1 is strictly diagonally dominant and G_1 is non-singular. This completes the proof. □

6. Numerical experiments and discussion

The proposed method is simulated using the Intel Core i-7, 2.40GHz GHz, 6GB of RAM with Windows 8 using Mathematica software, and the experiments were done using the proposed method with SOR iterative technique as an acceleration factor ($\omega = 1.8$) with different mesh sizes ($n = 10, 14, 18, 22, 30$) and different time steps. Furthermore, throughout the experiments, the L_∞-norm convergence criteria $\zeta = 10^{-5}$ is used. Also, the C_2-order and C_1-order of convergence are used for the computational order of spatial and temporal convergence using [25]

$$C_2 - order = log_2\left(\frac{||L_\infty\,(16\nu, 2h_0)||}{||L_\infty\,(\nu, h_0)||}\right), \tag{6.1}$$

$$C_1 - order = log_2\left(\frac{||\ L_\infty(2\nu, h_0)\ ||}{||\ L_\infty(\nu, h_0)\ ||}\right), \tag{6.2}$$

where L_∞ is the maximum error.

Some examples are presented below to show the efficiency of FEGM.

Problem 1. [26]

$$\frac{\partial^\alpha V}{\partial t^\alpha} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \left(\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + 2t^2\right)\sin(x)\sin(y), \quad 0 < \text{x,y} < 1, \quad 0 < t \le 1,$$

with initial and Dirichlet boundary conditions.

The analytic solution for Problem 1 is

$$V(x, y, t) = t^2 \sin(x) \sin(y).$$

Problem 2. [12]

$$\frac{\partial^2 V}{\partial t^2} = \frac{\partial^2 V}{\partial x^1} + \frac{\partial^2 V}{\partial y^2} + \left(\Gamma(2+\alpha) - 2t^{1+\alpha}\right) e^{x+y}, \quad 0 < \text{x,y} < 1, \quad 0 < t \leq 1,$$

with initial and Dirichlet boundary conditions.

The analytic solution for Problem 2 is

$$V(x, y, t) = e^{x+y} t^{1+\alpha}.$$

The number of iterations, error analysis, and execution times are shown for the comparison between FEGM and SP methods from Tables 1–5. The execution times in FEGM are decreased by (4.7 − 28.49)%, (2 − 23)%, (9.16 − 28.13)%, (8.9 − 25.39)% and (6.98 − 27.79)% compared to SP method in Tables 1–5 respectively. Similarly, in Table 6, the comparison between the proposed method and high-order standard point method [27] is presented, which shows the proposed method gives better results. Figures 3 and 4 represent the exact and approximate solution for Problem 1, respectively, which depicts the effectiveness of the FEGM. Likewise, In Figure 5, the compression of execution times between the SP and proposed methods are shown, which shows the proposed method is efficient in terms of execution timings. Table 7 shows the computational complexity per iteration, while the computational effort is shown in Tables 8 and 9, which depict that the FEGM requires less number of operations during computations as compared to the standard SP method. Tables 10 and 11 represent the spatial convergence order for Problems 1 and 2, respectively. Similarly, Tables 12 and 13 represent the temporal convergence order for the first and second Problem respectively, which depict the experimental and theoretical convergence orders in agreement.

Table 1. Numerical results for the Problem 1, when $\alpha = 0.5$.

h_0/ν	No. of iteration		Execution time		Maximum-error		Average-error	
	FEGM	SP	FEGM	SP	FEGM	SP	FEGM	SP
$h_0 = \nu = \frac{1}{10}$	43	49	7.39	7.76	1.6404×10^{-4}	1.653×10^{-4}	7.9408×10^{-5}	7.3891×10^{-5}
$h_0 = \nu = \frac{1}{18}$	41	48	66.19	79.82	6.6571×10^{-5}	6.5544×10^{-5}	2.8320×10^{-5}	2.9081×10^{-5}
$h_0 = \nu = \frac{1}{22}$	42	54	148.1	207.1	4.5739×10^{-5}	4.6717×10^{-5}	1.9997×10^{-5}	2.085×10^{-5}
$h_0 = \nu = \frac{1}{30}$	43	56	530.45	658.68	3.0559×10^{-5}	3.2268×10^{-5}	1.1887×10^{-5}	1.2180×10^{-5}

Table 2. Numerical results for the Problem 1, when $\alpha = 0.75$.

h_0/ν	No. of iteration		Execution time		Maximum-error		Average-error	
	FEGM	SP	FEGM	SP	FEGM	SP	FEGM	SP
$h_0 = \nu = \frac{1}{10}$	44	50	7.75	7.94	2.1534×10^{-4}	2.2065×10^{-4}	1.0469×10^{-4}	1.0634×10^{-4}
$h_0 = \nu = \frac{1}{14}$	44	43	27.36	27.28	1.2736×10^{-4}	1.2786×10^{-4}	5.9175×10^{-5}	6.0624×10^{-5}
$h_0 = \nu = \frac{1}{18}$	42	48	68.29	80.18	8.0887×10^{-5}	8.5709×10^{-5}	3.0710×10^{-5}	3.9889×10^{-5}
$h_0 = \nu = \frac{1}{22}$	41	55	150.40	197.87	6.1207×10^{-5}	6.2773×10^{-5}	2.8554×10^{-5}	2.8809×10^{-5}
$h_0 = \nu = \frac{1}{30}$	42	56	523.25	648.55	3.7373×10^{-5}	4.4510×10^{-5}	1.6460×10^{-5}	1.7874×10^{-5}

Table 3. Numerical results for the Problem 2, where $\alpha = 0.1$.

h_0/ν	No. of iteration		Execution time		Maximum-error		Average-error	
	FEGM	SP	FEGM	SP	FEGM	SP	FEGM	SP
$h_0 = \nu = \frac{1}{10}$	44	50	7.75	7.94	2.1534×10^{-4}	2.2065×10^{-4}	1.0469×10^{-4}	1.0634×10^{-4}
$h_0 = \nu = \frac{1}{14}$	44	43	27.36	27.28	1.2736×10^{-4}	1.2786×10^{-4}	5.9175×10^{-5}	6.0624×10^{-5}
$h_0 = \nu = \frac{1}{18}$	42	48	68.29	80.18	8.0887×10^{-5}	8.5709×10^{-5}	3.0710×10^{-5}	3.9889×10^{-5}
$h_0 = \nu = \frac{1}{22}$	41	55	150.40	197.87	6.1207×10^{-5}	6.2773×10^{-5}	2.8554×10^{-5}	2.8809×10^{-5}
$h_0 = \nu = \frac{1}{30}$	42	56	523.25	648.55	3.7373×10^{-5}	4.4510×10^{-5}	1.6460×10^{-5}	1.7874×10^{-5}

Table 4. Numerical results for the Problem 2, where $\alpha = 0.5$.

ν/h_0	No. of iteration		Execution time		Maximum-error		Average-error	
	FEGM	SP	FEGM	SP	FEGM	SP	FEGM	SP
$h_0 = \nu = \frac{1}{10}$	46	52	7.65	8.72	2.9442×10^{-4}	2.8915×10^{-4}	1.3091×10^{-4}	1.2849×10^{-4}
$h_0 = \nu = \frac{1}{14}$	48	47	30.93	29.53	1.3893×10^{-4}	1.4377×10^{-4}	3.7322×10^{-5}	3.9455×10^{-5}
$h_0 = \nu = \frac{1}{18}$	48	52	77.2	84.75	8.5499×10^{-5}	8.5075×10^{-5}	2.4058×10^{-5}	2.4340×10^{-5}
$h_0 = \nu = \frac{1}{22}$	47	57	172.53	203.76	5.5120×10^{-5}	5.5018×10^{-5}	1.9837×10^{-5}	1.7813×10^{-5}
$h_0 = \nu = \frac{1}{30}$	47	65	588.74	789.14	2.9259×10^{-5}	2.9669×10^{-5}	1.2478×10^{-5}	1.2549×10^{-5}

Table 5. Numerical results for the Problem 2, where $\alpha = 0.75$.

h_0/ν	No. of iteration		Execution time		Maximum-error		Average-error	
	FEGM	SP	FEGM	SP	FEGM	SP	FEGM	SP
$h_0 = \nu = \frac{1}{10}$	47	53	7.92	8.82	2.2868×10^{-4}	2.2917×10^{-4}	1.0194×10^{-4}	1.0189×10^{-4}
$h_0 = \nu = \frac{1}{14}$	48	48	31.23	30.46	2.3640×10^{-4}	2.3848×10^{-4}	1.1941×10^{-4}	1.2003×10^{-4}
$h_0 = \nu = \frac{1}{18}$	49	52	78.34	85.3	2.1361×10^{-4}	2.1290×10^{-4}	1.0621×10^{-4}	1.0863×10^{-4}
$h_0 = \nu = \frac{1}{22}$	48	57	169.15	207.6	1.9686×10^{-4}	1.8908×10^{-4}	9.5981×10^{-5}	9.5644×10^{-5}
$h_0 = \nu = \frac{1}{30}$	48	65	599.97	775.49	1.5244×10^{-4}	1.4938×10^{-4}	7.3782×10^{-5}	7.3922×10^{-5}

Table 6. Comparison of the Proposed method with standard point method [27] for Example 2 when $\alpha = 0.5$.

h_0/ν	No. of iteration		Maximum-error		Average-error	
	FEGM	[27]	FEGM	[27]	FEGM	[27]
$h_0 = \nu = \frac{1}{10}$	43	53	1.6404×10^{-4}	1.2428×10^{-2}	7.9408×10^{-5}	8.8490×10^{-3}
$h_0 = \nu = \frac{1}{18}$	41	52	6.6571×10^{-5}	7.1213×10^{-3}	2.8320×10^{-5}	3.6917×10^{-3}
$h_0 = \nu = \frac{1}{22}$	42	55	4.7739×10^{-5}	2.6959×10^{-3}	1.9997×10^{-5}	1.3580×10^{-3}
$h_0 = \frac{1}{65}, \nu = \frac{1}{35}$	41	58	2.5368×10^{-5}	2.0605×10^{-3}	5.0090×10^{-6}	1.0285×10^{-3}

Table 7. The number of computing operations required for the FEGM and SP Technique.

Technique	operations per iteration	
	+/-	*/ ÷
SP	$(26 + 8(k-1))m^2$	$(8 + 4(k-1))m^2$
FEGM	$(28 + 8(k-1))(m-1)^2$ $+ (26+8(k-1))(2m-1)$	$(12 + 4(k-1))(m-1)^2$ $+ (8+4(k-1))(2m-1)$

Table 8. The total computation effort for the Problem 1, where $\alpha = \frac{1}{2}$.

h_0/ν	SP method		FEGM	
	Number of iteration	Total operations	Number of iteration	Total operations
$h_0 = \nu = \frac{1}{10}$	49	695800	43	631498
$h_0 = \nu = \frac{1}{18}$	48	3701376	41	3232686
$h_0 = \nu = \frac{1}{22}$	54	7474896	42	5924940
$h_0 = \nu = \frac{1}{30}$	56	19252800	43	15000378

Table 9. The total computation effort for the Problem 2, where $\alpha = \frac{3}{4}$.

k/m	SP method		FEGM	
	Number of iteration	Total operations	Number of iteration	Total operations
$h_0 = \nu = \frac{1}{10}$	53	752600	47	690242
$h_0 = \nu = \frac{1}{18}$	52	4009824	49	3863454
$h_0 = \nu = \frac{1}{22}$	57	7890168	48	6771360
$h_0 = \nu = \frac{1}{30}$	65	22347000	48	16744608

Table 10. The spatial convergence order for the Problem 1.

$\alpha = 0.4$			$\alpha = 0.5$		
ν/h_0	Maximum error	C_2-order	ν/h_0	Maximum error	C_2-order
$\nu = h_0 = 0.5$	1.4394×10^{-4}	—	$\nu = h_0 = 0.5$	3.1088×10^{-4}	—
$\nu = 0.031, h_0 = 0.25$	1.0910×10^{-5}	3.72	$\nu = 0.031, h_0 = 0.25$	2.2509×10^{-5}	3.78
$\nu = h_0 = 0.25$	2.1929×10^{-4}	—	$\nu = h_0 = 0.25$	3.546×10^{-4}	—
$\nu = 0.016, h_0 = 0.12$	9.1371×10^{-6}	4.58	$\nu = 0.016, h_0 = 0.12$	2.1414×10^{-5}	4.04
$\alpha = 0.6$			$\alpha = 0.8$		
ν/h_0	Maximum error	C_2-order	ν/h_0	Maximum error	C_2-order
$\nu = h_0 = 0.5$	5.2435×10^{-4}	—	$\nu = h_0 = 0.5$	9.9004×10^{-4}	—
$\nu = 0.031, h_0 = 0.25$	2.7191×10^{-5}	4.26	$\nu = 0.031, h_0 = 0.25$	9.9004×10^{-5}	4.13
$\nu = h_0 = 0.25$	5.0421×10^{-4}	—	$\nu = h_0 = 0.25$	7.2673×10^{-4}	—
$\nu = 0.016, h_0 = 0.12$	3.1980×10^{-5}	3.97	$\nu = 0.016, h_0 = 0.12$	3.7243×10^{-5}	4.28

Table 11. The spatial convergence order for the Problem 2.

$\alpha = 0.7$			$\alpha = 0.8$		
ν/h_0	Maximum error	C_2-order	ν/h_0	Maximum error	C_2-order
$\nu = h_0 = 0.5$	1.6072×10^{-3}	—	$\nu = h_0 = 0.5$	3.4202×10^{-3}	—
$\nu = 0.031$, $h_0 = 0.25$	1.0543×10^{-4}	3.93	$\nu = 0.031$, $h_0 = 0.25$	1.7636×10^{-4}	4.27
$\nu = h_0 = 0.25$	1.3545×10^{-3}	—	$\nu = h_0 = 0.25$	1.6955×10^{-3}	—
$\nu = 0.016$, $h_0 = 0.12$	5.9784×10^{-5}	4.50	$\nu = 0.016$, $h_0 = 0.12$	8.2806×10^{-5}	4.35
$\alpha = 0.3$			$\alpha = 0.5$		
ν/h_0	Maximum error	C_2-order	ν/h_0	Maximum error	C_2-order
$\nu = h_0 = 0.5$	3.832×10^{-3}	—	$\nu = h_0 = 0.5$	8.6771×10^{-4}	—
$\nu = 0.031$, $h_0 = 0.25$	3.2105×10^{-4}	3.57	$\nu = 0.031$, $h_0 = 0.25$	4.8772×10^{-5}	4.15
$\nu = h_0 = 0.25$	4.5916×10^{-4}	—	$\nu = h_0 = 0.25$	4.1734×10^{-4}	—
$\nu = 0.016$, $h_0 = 0.12$	2.5978×10^{-5}	4.14	$\nu = 0.016$, $h_0 = 0.12$	2.3009×10^{-5}	4.18

Table 12. Temporal convergence order for the Problem 1, when $h_0 = \frac{1}{8}$.

ν	$\alpha = 0.3$		$\alpha = 0.8$	
	L_∞	C_1- Order	L_∞	C_1- Order
$\nu = \frac{1}{10}$	7.8675×10^{-4}	—	2.1609×10^{-4}	—
$\nu = \frac{1}{20}$	3.2208×10^{-5}	1.28	7.1112×10^{-5}	1.60
$\nu = \frac{1}{40}$	1.7253×10^{-5}	1.29	2.6203×10^{-5}	1.44
$\nu = \frac{1}{80}$	6.2057×10^{-6}	1.72	8.3392×10^{-6}	1.65

Table 13. Temporal convergence order for the Problem 2, when $h_0 = \frac{1}{8}$.

ν	$\alpha = 0.1$		$\alpha = 0.8$	
	L_∞	C_1- Order	L_∞	C_1- Order
$\nu = \frac{1}{10}$	4.7691×10^{-4}	—	2.1429×10^{-4}	—
$\nu = \frac{1}{20}$	1.6086×10^{-4}	1.56	8.2679×10^{-5}	1.37
$\nu = \frac{1}{40}$	5.1170×10^{-5}	1.65	3.2827×10^{-5}	1.33
$\nu = \frac{1}{80}$	1.7848×10^{-5}	1.51	1.3010×10^{-5}	1.51

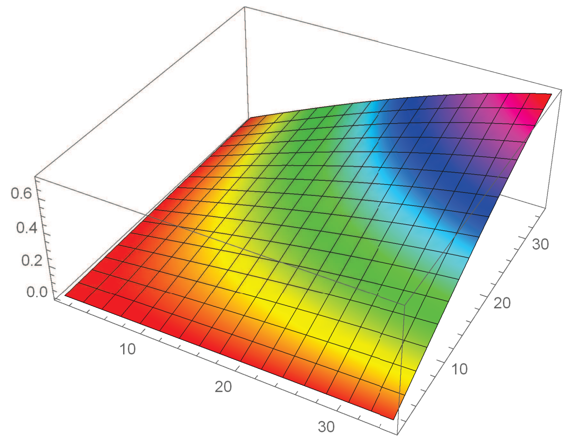

Figure 3. Approximate solution for the Problem 1, where $h_0 = \nu = \frac{1}{35}$.

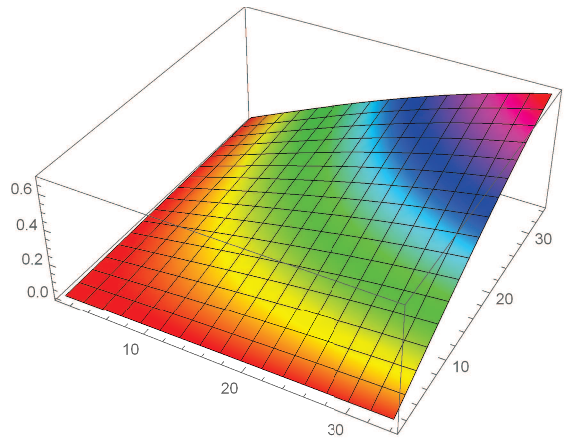

Figure 4. Exact solution for the Problem 1, where $h_0 = \nu = \frac{1}{35}$.

(a) Problem 1 when $\alpha = 0.5$

(b) Problem 2 when $\alpha = 0.1$

Figure 5. Execution time (in sec) for different mesh sizes for the Problems 1 and 2.

7. Conclusions

In this article, the 2-D fractional sub-diffusion equation is solved using the fractional explicit group method with weak singularity at initial time $t = 0$, where the standard point finite difference scheme is used for the development of the fourth-order grouping scheme. The fractional explicit group method reduces the computational complexity and execution time by comparing it with the standard point fourth-order method without deteriorating the accuracy of the solutions. Furthermore, the unconditional stability and convergence of the proposed scheme are proved using the matrix analysis via mathematical induction, which confirms the feasibility and reliability of the new formulation.

Acknowledgments

This work was supported by the Ministry of Higher Education under Fundamental Research Grant Scheme (FRGS/1/2019/STG06/UTM/02/13, FRGS/1/2020/STG05/UTM/02/12).

Conflict of interest

We declare no conflicts of interest in this paper.

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