AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 13663–13679.
DOI:10.3934/math.2023694
Received: 26 February 2023
Revised: 13 March 2023
Accepted: 20 March 2023
Published: 10 April 2023

Research article

On fixed-point approximations for a class of nonlinear mappings based on the JK iterative scheme with application

Junaid Ahmad[1,*], Kifayat Ullah[2], Hasanen A. Hammad[3,4] and Reny George[5,*]

[1] Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad 44000, Pakistan

[2] Department of Mathematical Sciences, University of Lakki Marwat, Lakki Marwat 28420, Khyber Pakhtunkhwa, Pakistan

[3] Department of Mathematics, Unaizah College of Sciences and Arts, Qassim University, Buraydah 52571, Saudi Arabia

[4] Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt

[5] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia

* **Correspondence:** Email: junaid.phdma126@iiu.edu.pk, r.kunnelchacko@psau.edu.sa.

Abstract: The goal of this manuscript is to introduce the JK iterative scheme for the numerical reckoning of fixed points in generalized contraction mappings. Also, weak and strong convergence results are investigated under this scheme in the setting of Banach spaces. Moreover, two numerical examples are given to illustrate that the JK iterative scheme is more effective than some other iterative schemes in the literature. Ultimately, as an application, the JK iterative scheme is applied to solve a discrete composite functional differential equation of the Volterra-Stieljes type.

Keywords: iteration scheme; convergence results; fixed point technique; nonexpansive mapping; uniformly convex Banach space
Mathematics Subject Classification: 47H09, 47H10

1. Introduction

Fixed-point methodology, in a completely novel way, provides the existence and approximation of solutions to those problems of analysis for which analytical methods fail [1–3]. This technique first expresses the given problem as a fixed-point problem; hence, the solution to the given problem becomes equivalent to finding the fixed point of a certain self-map or non-self-map. In 1922, Banach [4] invented the Banach contraction principle (BCP), which provides the existence and uniqueness of a fixed point for contractions and also suggests an approximation scheme for computing the approximate value of this fixed point. Banach and many other mathematicians studied the BCP in order to solve various problems. One of the important class of mappings in fixed point theory is the class of nonexpansive mappings, which properly includes the class of all contractions.

Browder [5] investigated the BCP in the context of a nonexpansive self-map and demonstrated that nonexpansive mappings admit a fixed point (sometimes unique, sometimes not) if the domain is convex closed bounded in a uniformly convex Banach space (UCBS), but unlike the BCP, the approximate value of a fixed point in this case cannot be obtained by implementing Picard's iterative scheme (see also, [6] and others). In 2008, Suzuki [7] introduced the class of Suzuki type nonexpansive (STN) mappings and proved that it is a proper generalization of nonexpansive mappings.

A mapping F on any subset G of a Banach space is known as a contraction if, for any $g, g' \in G$, there exists $b \in [0, 1)$ such that

$$\|Fg - Fg'\| \leq b\|g - g'\|. \tag{1.1}$$

We refer to F as a nonexpansive mapping when F satisfies (1.1) for $b = 1$. The point $v \in G$, which satisfies $Fv = v$ is called a fixed point of the mapping F. In this case, we use $Fix(F)$ to refer to the set $\{v \in G : v = Fv\}$. The self-map F is called STN if for each $g, g' \in G$,

$$\frac{1}{2}\|g - Fg\| \leq \|g - g'\| \Rightarrow \|Fg - Fg'\| \leq \|g - g'\|.$$

The class of STN mappings and the other classes discussed above have been extensively studied by several authors in recent years [8–15]. Thus, it is very natural to ask whether there is any new class of mappings that includes almost all of the above classes of mappings. A above question was answered by Pandey et al. [16] as follows: A given self-mapping F on the nonempty subset G of a Banach space is called Reich–Suzuki type nonexpansive (RSTN) if there exists $\gamma \in [0, 1)$ and for each $g, g' \in G$,

$$\frac{1}{2}\|g - Fg\| \leq \|g - g'\| \Rightarrow \|Fg - Fg'\| \leq \gamma\|g - Fg\| + \gamma\|g' - Fg'\| + (1 - 2\gamma)\|g - g'\|.$$

In this connection, see the early paper by Reich [17].

Remark 1.1. *Every STN mapping is RSTN with* $\gamma = 0$ *but the converse is not true.*

Recently, computational methods for different problems have gained the attention of many researchers. It is well known that the Picard iteration for a nonexpansive mapping may not converge to its fixed point (see, e.g., [3]). There are some other will-known iteration schemes converge to a fixed point of a given nonexpansive mapping and also enjoy a better rate of convergence as compared to the Picard iterative method.

Now, assume that G is any closed convex nonempty subset of a Banach space B, $i \geq 1$, $a_i, b_i, c_i \in (0, 1)$ and $F : G \to G$.

The Mann iterative scheme [18] is a sequence $\{z_i\}$ consisting of

$$\begin{cases} z_1 \in G, \\ z_{i+1} = (1 - a_i)z_i + a_iFz_i. \end{cases} \tag{1.2}$$

Ishikawa's iterative scheme [19] can be viewed as a two-step Mann iterative scheme [18], which is a sequence $\{z_i\}$ given by

$$\begin{cases} z_1 \in G, \\ s_i = (1 - b_i)z_i + b_i F z_i, \\ z_{i+1} = (1 - a_i)z_i + a_i F s_i. \end{cases} \tag{1.3}$$

The iterative process of Noor [9] is a three-step iteration that extends the Ishikawa [19] two-step iteration. The sequence of Noor iteration $\{z_i\}$ can be produced as follows:

$$\begin{cases} z_1 \in G, \\ u_i = (1 - c_i)z_i + c_i F z_i, \\ s_i = (1 - b_i)z_i + b_i F u_i, \\ z_{i+1} = (1 - a_i)z_i + a_i F s_i. \end{cases} \tag{1.4}$$

The iterative process of Agarwal et al. [8] is precisely independent but better than the Ishikawa iteration. The sequence of Agarwal iteration $\{z_i\}$ can be produced as follows:

$$\begin{cases} z_1 \in G, \\ s_i = (1 - b_i)z_i + b_i F z_i, \\ z_{i+1} = (1 - a_i)F z_i + a_i F s_i. \end{cases} \tag{1.5}$$

The following three-step iterative approximation scheme is suggested by Abbas and Nazir [20]:

$$\begin{cases} z_1 \in G, \\ u_i = (1 - c_i)z_i + c_i F z_i, \\ s_i = (1 - b_i)F z_i + b_i F u_i, \\ z_{i+1} = (1 - a_i)F s_i + a_i F u_i. \end{cases} \tag{1.6}$$

The Thakur et al. [10] process reads as follows:

$$\begin{cases} z_1 \in G, \\ u_i = (1 - b_i)z_i + b_i F z_i, \\ s_i = F((1 - a_i)z_i + a_i u_i), \\ z_{i+1} = F s_i. \end{cases} \tag{1.7}$$

Recently, Ahmad et al. [11] introduced the JK iteration process

$$\begin{cases} z_1 \in G, \\ u_i = (1 - b_i)z_i + b_i F z_i, \\ s_i = F u_i, \\ z_{i+1} = F((1 - a_i)F u_i + a_i F s_i). \end{cases} \tag{1.8}$$

Ahmad et al. [11] proved the convergence of the JK iterative process (1.8) in the setting of STN mappings. Furthermore, they demonstrated that the JK iterative scheme provided better approximation results than the previously introduced iterative schemes. Here, we study the JK (1.8) in the new setting of RSTN mappings. We then provide an example of RSTN mappings, which are not STN. Using this example, we collect some numerical results obtained by JK (1.8) and the above iterative schemes, and we observe the numerical accuracy of our main outcome. After this, we use a two-variable RSTN mapping and prove once again that the JK iterative scheme achieves faster convergence than the other leading iterative schemes in the literature. We provide a certain application of JK iteration to find the sought solution of a discrete composite functional differential equation of the Volterra-Stieljes type (DCFDEVST, for short) in the setting of Banach spaces.

2. Preliminaries

This section provides us some early results and facts, which are needed for the main results. We now consider a nonempty subset G that is essentially convex as well closed in a Banach space and assume that $q \in B$ is any fixed element. Suppose that $\{z_i\} \subseteq B$ is bounded. We define $r(q, \{z_i\})$ by

$$r(q, \{z_i\}) := \limsup_{i \to \infty} \|q - z_i\|.$$

We shall denote the asymptotic radius of $\{z_i\}$ with respect to G by $r(G, \{z_i\})$ and it reads as follows:

$$r(G, \{z_i\}) := \inf\{r(q, \{z_i\}) : q \in G\}.$$

We shall represent the asymptotic center of $\{z_i\}$ with respect to G by $A(G, \{z_i\})$ and it reads as follows:

$$A(G, \{z_i\}) := \{q \in G : r(q, \{z_i\}) = r(G, \{z_i\})\}.$$

More information on asymptotic centers and uniformly convex Banach spaces can be found in [21].

Definition 2.1. *[22] A Banach space B is called uniformly convex if for every $\varepsilon \in (0, 2]$, there is a real number $\lambda > 0$ such that for all $g, g' \in B$,*

$$\left.\begin{array}{c} \|g\| \leq 1 \\ \|g'\| \leq 1 \\ \|g - g'\| > \varepsilon \end{array}\right\} \Longrightarrow \frac{1}{2}\|g + g'\| \leq (1 - \lambda).$$

It should be noted that a Banach space B is said to satisfy Opial's property [23] if a sequence $\{t_i\}$ in B converges weakly to $s \in B$ and enjoys the following inequality:

$$\limsup_{i \to \infty} \|t_i - s\| < \limsup_{i \to \infty} \|t_i - g\| \text{ for all } g \in B - \{s\}.$$

The following fact is well known.

Lemma 2.1. *[24, 25] If G is a closed convex subset of a UCBS then the set $A(G, \{z_i\})$ is nonempty and convex if G is convex weakly compact and admits exactly only one element.*

The following lemma provides us the information about the scope of the class of Reich–Suzuki type mappings.

Lemma 2.2. *[26] Every STN self-map F of a nonempty subset of a Banach space is an RSTN mapping but the converse need not be true.*

The following important facts are also needed.

Lemma 2.3. *[26] Every RSTN self-map F of a nonempty subset of a Banach space having a nonempty fixed point set satisfies the following inequality:*

$$\|Fg - v\| \leq \|g - v\|, \ \ \forall v \in Fix(F) \ \ and \ \ g \in G.$$

Lemma 2.4. *[16] Every RSTN self-map F of a nonempty subset of a Banach space having a nonempty fixed point set satisfies the following inequality:*

$$\|g - Fg'\| \leq \frac{(\gamma + 3)}{(1 - \gamma)}\|g - Fg\| + \|g - g'\|, \ \ \forall g, g' \in G.$$

Lemma 2.5. *Suppose that B is a Banach space satisfying Opial's condition. Assume that F is a self-map of $G : B \to B$ and let $v \in G$. If F is RSTN, $\{z_i\}$ is weakly convergent to v and $\lim_{i\to\infty} \|z_i - Fz_i\| = 0$; then, $v \in Fix(F)$.*

Proof. From Lemma 2.4, we have

$$\|z_i - Fv\| \leq \frac{(\gamma + 3)}{(1 - \gamma)}\|z_i - Fz_i\| + \|z_i - v\|.$$

It follows that

$$\limsup_{i\to\infty} \|z_i - Fv\| \leq \limsup_{i\to\infty} \|z_i - v\|.$$

By Opial's property of B, we obtain that $Fv = v$, that is, $v \in Fix(F)$. This finishes the proof. □

The following lemma is a characterization of uniformly convex Banach spaces.

Lemma 2.6. *[27] Assume that B is any UCBS and $0 < e \leq s_i \leq f < 1$ for all $i \geq 1$ and fix $\xi \geq 0$. If $\{q_i\}$ and $\{p_i\}$ are sequences in B, then $\lim_{i\to\infty} \|q_i - p_i\| = 0$ provided that $\limsup_{i\to\infty} q_i \leq \xi$, $\limsup_{i\to\infty} p_i \leq \xi$ and $\lim_{i\to\infty} \|(1 - s_i)q_i + s_i p_i\| \leq \xi$.*

3. Convergence theorems of JK iterative algorithm

In this paper, our aim is to achieve some convergence results for RSTN self-mappings under the JK iterative scheme (1.8). We first provide a very basic lemma that will be used in each main result.

Lemma 3.1. *Assume that G denotes any subset of a UCBS B such that $F : G \to G$ forms an RSTN mapping with $Fix(F) \neq \emptyset$. In this case, if $\{z_i\}$ is generated by JK iteration (1.8), then for any arbitrary choice of $v \in Fix(F)$, $\lim_{i\to\infty} \|z_i - v\|$ exists.*

Proof. Suppose that $v \in Fix(F)$. By Lemma 2.3, one has

$$\begin{aligned}
\|u_i - v\| &= \|(1 - b_i)z_i + b_i F z_i - v\| \\
&\leq (1 - b_i)\|z_i - v\| + b_i\|Fz_i - v\| \\
&\leq (1 - b_i)\|z_i - v\| + b_i\|z_i - v\| \\
&\leq \|z_i - v\|,
\end{aligned}$$

and

$$\|s_i - v\| = \|Fu_i - v\| \leq \|u_i - v\|.$$

Hence

$$\begin{aligned}\|z_{i+1}-v\| &= \|F((1-a_i)Fu_i+a_iFs_i)-v\|\\ &\le \|(1-a_i)Fu_i+a_iFs_i-v\|\\ &\le (1-a_i)\|Fu_i-v\|+a_i\|Fs_i-v\|\\ &\le (1-a_i)\|u_i-v|+a_i\|s_i-v\|\\ &\le (1-a_i)\|u_i-v\|+a_i\|u_i-v\|\\ &= \|u_i-v\|\le\|z_i-v\|.\end{aligned}$$

Hence the real sequence $\{\|z_i - v\|\}$ is nonincreasing and bounded below. Accordingly, $\lim_{i\to\infty}\|z_i - v\|$ exists for every $v \in Fix(F)$. □

Theorem 3.1. *Assume that G denotes any subset of a UCBS B such that $F : G \to G$ forms an RSTN mapping. In this case, if $\{z_i\}$ is generated by JK iteration (1.8) then $\{z_i\}$ is bounded in G and $\lim_{i\to\infty}\|z_i - Fz_i\| = 0 \iff Fix(F) \neq \emptyset$.*

Proof. We assume that $\{z_i\}$ is bounded and $\lim_{i\to\infty}\|z_i - Fz_i\| = 0$. Choose $v \in A(G, \{z_i\})$. We shall prove that $Fv = v$. By Lemma 2.4, we have

$$\begin{aligned}r(Fv,\{z_i\}) &= \limsup_{i\to\infty}\|z_i-Fv\|\\ &\le \limsup_{i\to\infty}\left(\frac{(3+\gamma)}{(1-\gamma)}\|z_i-Fz_i\|+\|z_i-v\|\right)\\ &= \limsup_{i\to\infty}\|z_i-v\|\\ &= r(v,\{z_i\}).\end{aligned}$$

Consequently $Fv \in A(G, \{z_i\})$. According to Lemma 2.1, $A(G, \{z_i\})$ is a singleton; we conclude that $Fv = v$. Hence, $Fix(F) \neq \emptyset$.

Conversely, suppose that $Fix(F) \neq \emptyset$ and $v \in Fix(F)$. We are going to show that $\{z_i\}$ is bounded and $\lim_{i\to\infty}\|z_i - Fz_i\| = 0$. By Lemma 3.1, $\lim_{i\to\infty}\|z_i - v\|$ exists and $\{z_i\}$ is bounded. Put

$$\lim_{i\to\infty}\|z_i-v\|=\xi. \tag{3.1}$$

Now, we proved in Lemma 3.1 that $\|u_i - v\| \le \|z_i - v\|$. It follows that

$$\limsup_{i\to\infty}\|u_i-v\|\le\limsup_{i\to\infty}\|z_i-v\|=\xi. \tag{3.2}$$

Now, keeping in mind (3.1) and using Lemma 2.3, one has

$$\limsup_{i\to\infty}\|Fz_i-v\|\le\limsup_{i\to\infty}\|z_i-v\|=\xi. \tag{3.3}$$

Again looking into the proof of Lemma 3.1, we see that $\|z_{i+1} - v\| \le \|u_i - v\|$. It follows that

$$\xi=\liminf_{i\to\infty}\|z_{i+1}-v\|\le\liminf_{i\to\infty}\|u_i-v\|. \tag{3.4}$$

Now, from (3.2) and (3.4), we have

$$\xi = \lim_{i\to\infty} \|u_i - v\|. \tag{3.5}$$

Using (3.5), we have

$$\xi \;=\; \lim_{i\to\infty} \|u_i - v\| = \lim_{i\to\infty} \|(1-b_i)(z_i - v) + b_i(Fz_i - v)\|. \tag{3.6}$$

Now, using (3.1), (3.3) and (3.6) and applying Lemma 2.6 with $q_i = z_i - v$ and $p_i = Fz_i - v$, we get

$$\lim_{i\to\infty} \|Fz_i - z_i\| = 0.$$

This completes the proof. □

The next theorem establishes a strong convergence for RSTN mappings under a JK iterative scheme, where the proof is connected with the compactness property of G.

Theorem 3.2. *Assume that G denotes any compact convex subset of a UCBS B such that $F : G \to G$ forms a RSTN mapping with $Fix(F) \neq \emptyset$. In this case, if $\{z_i\}$ is generated by JK iteration (1.8), then $\{z_i\}$ essentially converges to a point of $Fix(T)$.*

Proof. Since the domain of G is a compact set and $\{z_i\} \subseteq G$, one can find a subsequence $\{z_{i_l}\}$ of $\{z_i\}$ such that $\lim_{l\to\infty} \|z_{i_l} - y\| = 0$ for some $y \in G$. By Lemma 2.4, we have

$$\|z_{i_l} - Fy\| \leq \frac{(3+\gamma)}{(1-\gamma)} \|z_{i_l} - Fz_{i_l}\| + \|z_{i_l} - y\|. \tag{3.7}$$

Moreover, Theorem 3.1 suggests $\lim_{l\to\infty} \|z_{i_l} - Fz_{i_l}\| = 0$. Now, using $\lim_{l\to\infty} \|z_{i_l} - Fz_{i_l}\| = 0$ and $\lim_{l\to\infty} \|z_{i_l} - y\| = 0$, we have from (3.7), $\lim_{l\to\infty} \|z_{i_l} - Fy\| = 0$. Now, the Banach space suggests the uniqueness of limits, that is, $Fy = y$. Hence, y is the element of the set $Fix(F)$. By Lemma 3.1, $\lim_{i\to\infty} \|z_i - y\|$ exists. It follows that y is the strong limit point of $\{z_i\}$. □

Now, we replace the compactness condition with another condition as follows:

Theorem 3.3. *Assume that G denotes any closed convex subset of a UCBS B such that $F : G \to G$ forms a RSTN mapping with $Fix(F) \neq \emptyset$. In this case, if $\{z_i\}$ is generated by JK iteration (1.8), then $\{z_i\}$ converges to a fixed point of F provided that* $\liminf_{i\to\infty} d(z_i, Fix(F)) = 0$.

Proof. Since the proof of this result is easy to establish, we omit it. □

We now suggest a strong convergence of $\{z_i\}$ under Condition (I). We first give the definition of Condition (I) which is originally due to Senter and Dotson [28].

Definition 3.1. *A self-map F of a subset G of a Banach space is said to satisfy Condition (I) if there exists a real-valued function η such that $\eta(c) = 0$ if and only if $c = 0$, $\eta(c) > 0$ for all $c > 0$ and $\|g - Fg\| \geq \eta(d(g, Fix(F)))$ for all $g \in G$.*

Theorem 3.4. *Assume that G denotes any closed convex subset of a UCBS B such that $F : G \to G$ forms a RSTN mapping with $Fix(F) \neq \emptyset$. In this case, if $\{z_i\}$ is generated by JK iteration (1.8), then $\{z_i\}$ converges to a fixed point of F provided that F satisfies Condition (I).*

Proof. Theorem 3.1 suggests that

$$\liminf_{i\to\infty} \|z_i - Fz_i\| = 0. \tag{3.8}$$

By Condition (I) of F, one has

$$\|z_i - Fz_i\| \geq \eta(d(z_i, Fix(F))). \tag{3.9}$$

From (3.8) and (3.9), we have

$$\liminf_{i\to\infty} \eta(d(z_i, Fix(F))) = 0.$$

Since $\eta(c) = 0$ implies $c = 0$, we get

$$\liminf_{i\to\infty} d(z_i, Fix(F)) = 0.$$

Now the conclusions of Theorem 3.3 provide us the strong convergence of $\{z_i\}$ to the point of $Fix(F)$. □

Now, we provide the weak convergence of $\{z_i\}$ under the Opial's condition. After the proof of this result, we shall close this section.

Theorem 3.5. *Assume that G denotes any closed convex subset of a UCBS B such that $F : G \to G$ forms a RSTN mapping with $Fix(F) \neq \emptyset$. In this case, if $\{z_i\}$ is generated by JK iteration (1.8), then $\{z_i\}$ essentially converges to a point of $Fix(T)$ if the Banach space has the Opial's condition.*

Proof. By Theorem 3.1, $\lim_{i\to\infty} \|z_i - Fz_i\| = 0$ and $\{z_i\}$ is bounded. Moreover, B is reflexive because of uniform convexity. Thus, using reflexivity of B, the existence of a weakly convergent subsequence $\{z_{i_s}\}$ of $\{z_i\}$ is ensured with some weak limit $g_1 \in G$. By Lemma 2.5, the point g_1 must be in $Fix(F)$. Now, we claim that the sequence $\{z_i\}$ also weakly converges to the point g_1. Assume the opposite, that is, there exists a subsequence $\{z_{i_t}\}$ of $\{z_i\}$ and a point $g_2 \in G$ such that $\{z_{i_t}\}$ converges weakly to g_2. Applying Lemma 2.5, we have $g_2 \in Fix(F)$. Using Lemma 3.1 and Opial's condition of B, we get

$$\begin{aligned}
\lim_{i\to\infty} \|z_i - g_1\| &= \lim_{s\to\infty} \|z_{i_s} - g_1\| < \lim_{s\to\infty} \|z_{i_s} - g_2\| \\
&= \lim_{i\to\infty} \|z_i - g_2\| = \lim_{t\to\infty} \|z_{i_t} - g_2\| \\
&< \lim_{t\to\infty} \|z_{i_t} - g_1\| = \lim_{i\to\infty} \|z_i - g_1\|.
\end{aligned}$$

The above contradiction implies that the point g_1 must be the weak limit of $\{z_i\}$. □

4. Numerical experiments

Now, we suggest two examples of RSTN self-mappings. One example is constructed on the two-dimensional Euclidean space. We prove that in both of these examples, the numerical effectiveness of our JK iterative scheme is far better than the many other iterative schemes of the literature.

Example 4.1. *Let $G = [5, 7]$ and let F be a map given by*

$$Fg = \begin{cases} \frac{g+20}{5}, & \text{if } g < 7, \\ 4, & \text{if } g = 7. \end{cases}$$

We want to show that F is an RSTN map. For this, we need to find at least one real number $\gamma \in [0,1)$ such that for all $g, g' \in G$, we have $|Fg - Fg'| \leq \gamma|g - Fg| + \gamma|g' - Fg'| + (1-2\gamma)|g - g'|$. To achieve the aim, we take $\gamma = \frac{1}{2}$. We shall divide the proof in some cases.

(i) *When $g, g' \in [5,7)$, then $Fg = \frac{g+20}{5}$ and $Fg' = \frac{g'+20}{5}$. Using the triangle inequality, we have*

$$\begin{aligned}
\gamma|g - Fg| + \gamma|g' - Fg'| + (1-2\gamma)|g - g'| &= \frac{1}{2}|g - (\frac{g+20}{5})| + \frac{1}{2}|g' - (\frac{g'+20}{5})| \\
&= \frac{1}{2}|\frac{4g-20}{5}| + \frac{1}{2}|\frac{4g'-20}{5}| \\
&\geq \frac{1}{2}|(\frac{4g-20}{5}) - (\frac{4g'-20}{5})| \\
&= \frac{1}{2}|\frac{4g-4g'}{5}| \\
&= \frac{4}{10}|g-g'| \\
&\geq \frac{1}{5}|g-g'| = |Fg - Fg'|.
\end{aligned}$$

(ii) *When $g \in [5,7)$ and $g' \in \{7\}$, then $Fg = \frac{g+20}{5}$ and $Fg' = 4$. Accordingly, we have*

$$\begin{aligned}
\gamma|g - Fg| + \gamma|g' - Fg'| + (1-2\gamma)|g - g'| &= \frac{1}{2}|g - (\frac{g+20}{5})| + \frac{1}{2}|7-4| \\
&= \frac{1}{2}|\frac{4g-20}{5}| + \frac{1}{2}|3| \\
&\geq \frac{1}{2}|3| \\
&> \frac{7}{5} \\
&\geq |\frac{g}{5}| = |Fg - Fg'|.
\end{aligned}$$

(iii) *When $g' \in [5,7)$ and $g \in \{7\}$, then $Fg' = \frac{g'+20}{5}$ and $Fg = 4$. Accordingly, we have*

$$\begin{aligned}
\gamma|g - Fg| + \gamma|g' - Fg'| + (1-2\gamma)|g - g'| &= \frac{1}{2}|7-4| + \frac{1}{2}|g' - (\frac{g'+20}{5})| \\
&= \frac{1}{2}|3| + \frac{1}{2}|\frac{4g'-20}{5}| \\
&\geq \frac{1}{2}|3| \\
&> \frac{7}{5} \\
&\geq |\frac{g'}{5}| = |Fg - Fg'|.
\end{aligned}$$

(iv) *When $g, g' \in \{7\}$ and $g \in \{7\}$, then $Fg = Fg' = 4$. Accordingly, we have*

$$\gamma|g - Fg| + \gamma|g' - Fg'| + (1-2\gamma)|g - g'| \geq 0 = |Fg - Fg'|.$$

Keeping the above cases in mind, one can conclude that F *is an RSTN map with* $\gamma = \frac{1}{2}$. *On the other hand,* F *is not STN. Because one can select the values* $g = 6$ *and* $g' = 7$, *then it is easy to show that* $\frac{1}{2}|g - Fg| < 1 = |g - g'|$ *and* $|Fg - Fg'| = 1.2 > 1 = |g - g'|$. *For every natural number* $i \geq 1$, *select the value of* $a_i = 0.86$ *and* $b_i = 0.50$ *and a starting value of* 6.9. *Then we can see in Table 1 that JK iteration is good as compared the earlier iterative schemes. In this case, the behavior of iterations is given in Figure 1.*

Table 1. Numerical values produced by some schemes.

i	JK (1.8)	Thakur (1.7)	Agarwal (1.5)
1	6.9	6.9	6.9
2	5.01422720000000	5.04985000000000	5.24920000000000
3	5.00010653327360	5.00130822144000	5.03270553000000
4	5.00000079772120	5.00003432773060	5.00429096632320
5	5.00000000597330	5.00000090075970	5.00056297478160
6	5.00000000004470	5.00000002363590	5.00007386229130
7	5.00000000000030	5.00000000062020	5.00000969073260
8	5	5.00000000001630	5.00000127142410
9	5	5.00000000000040	5.00000016681030
10	5	5	5.00000002188560
11	5	5	5.00000000287140
12	5	5	5.00000000037670
13	5	5	5.00000000004940
14	5	5	5.00000000000650
15	5	5	5.00000000000090
16	5	5	5.00000000000010
17	5	5	5

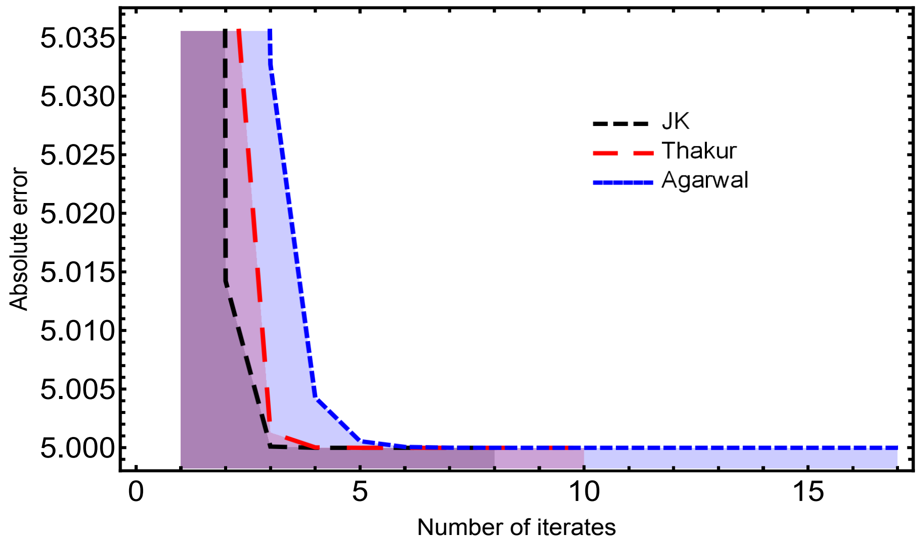

Figure 1. Comparison of various iterative methods with the iterative method JK.

Now, we suggest different values for the parameters a_i and b_i initial points and also we set the stopping criterion as $\|z_i - g^*\| < 10^{-15}$, and keep in mind that $g^* = 5$ is a unique fixed point of the self-map F. The numerical results are given in Tables 2–4.

Table 2. Choose $a_i = \frac{\sqrt{i+1}}{(i+5)^{\frac{1}{5}}}$ and $b_i = \frac{1}{\sqrt[3]{7i+10}}$.

Number of iterations to reach the fixed point.			
Intial value	Agarwal (1.5)	Thakur (1.7)	JK (1.8)
5.2	16	09	06
5.5	16	09	06
5.8	16	09	07
6.1	16	10	07
6.4	17	10	07
6.8	17	10	07

Table 3. Choose $a_i = \sqrt[8]{\frac{1}{4i+6}}$ and $b_i = \frac{i}{7i+11}$.

Number of iterations to reach the fixed point.			
Intial value	Agarwal (1.5)	Thakur (1.7)	JK (1.8)
5.2	19	10	08
5.5	20	10	09
5.8	20	11	09
6.1	20	11	09
6.4	21	11	09
6.8	21	11	09

Table 4. Choose $a_i = \sqrt[7]{\frac{i+1}{5i+2}}$ and $b_i = \sqrt{\frac{1}{3i+3}}$.

Number of iterations to reach the fixed point.			
Intial value	Agarwal (1.5)	Thakur (1.7)	JK (1.8)
5.2	18	10	08
5.5	19	10	08
5.8	19	10	08
6.1	19	10	08
6.4	20	10	08
6.8	21	10	08

We finish this section with an example. This example uses a subset of a two dimensional Euclidian space.

Example 4.2. *Let $G = [0,1] \times [0,1]$ and set a self–map on F by the following rule:*

$$F(g,h) = \begin{cases} (\frac{g}{3}, \frac{h}{4}) & \text{if } (g,h) \in [0, \frac{1}{2}) \times [0, \frac{1}{2}) \\ (\frac{g}{5}, \frac{h}{6}) & \text{if } (g,h) \in [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]. \end{cases}$$

We see that F is an RSTN mapping with the fixed point $(0,0)$*. In this case, the convergence of JK (1.8), Thakur (1.7) and Agarwal (1.5) iterative schemes toward the fixed point* $(0,0)$ *of the mapping F is shown in Table 5.*

Table 5. Numerical values produced by some schemes.

i	JK (1.8)	Thakur (1.7)	Agarwal (1.5)
1	$(0.1, 0.2)$	$(0.1, 0.2)$	$(0.1, 0.2)$
2	$(0.00316, 0.00277)$	$(0.00792, 0.00846)$	$(0.02378, 0.03388)$
3	$(0.00009, 0.00003)$	$(0.00062, 0.00035)$	$(0.00565, 0.00573)$
4	$(0, 0)$	$(0.00004, 0.00001)$	$(0.00134, 0.00097)$
5	$(0, 0)$	$(0, 0)$	$(0.00031, 0.00016)$
6	$(0, 0)$	$(0, 0)$	$(0.00007, 0.00002)$
7	$(0, 0)$	$(0, 0)$	$(0.00001, 0)$
8	$(0, 0)$	$(0, 0)$	$(0, 0)$
9	$(0, 0)$	$(0, 0)$	$(0, 0)$
10	$(0, 0)$	$(0, 0)$	$(0, 0)$

5. Application

The purpose of this section is to apply the JK iterative scheme for finding a weak solution of the DCFDEVST. Recently, in [29], (see, also, [30–34] and others) the authors studied the existence as well uniqueness of a weak solution for the (DCFDEVST). On the other hand, we know that, once the existence of a solution for a certain problem is established, the requested sought solution can be approximated by using a certain iterative scheme that is not always an easy task. In this research, we provide the JK iterative scheme approach to find the approximated solution for the DCFDEVST. To achieve the objective, we consider an RBS G with the norm $\|.\|_G$, where G^* denotes the dual space associated with G. Suppose $C[P, G]$, where $P = [0, U]$ is the space of continuous functions with the following norm:

$$\|z\|_C = \sup_{\xi \in P} \|z(\xi)\|_G, \ z \in C[P, G].$$

Now DCFDEVST is defined as follows:

$$\frac{d}{d\xi} z(\xi) = h_1\left(\xi, \int_0^{k(\xi)} h_2(\xi, s, z(s)) d_s L(\xi, s)\right), \xi \in P, \tag{5.1}$$

with the initial condition

$$z(0) = z_0. \tag{5.2}$$

Now, Problem (5.1) under the initial condition (5.2) will be considered under the following hypotheses:

(1) the mapping $k : P \to P$ is continuous and increasing such that $k(\xi) \leq \xi$;

(2) the mappings $h_1 : P \times G \to G$ and $h_2 : P \times P \times G \to G$ are weakly continuous and there exist $Q_1, Q_2 > 0$ such that

$$|F(h_1(\xi, x)) - h_1(\xi, y)| \leq Q_1 |F(x - y)|, \text{ and } |F(h_2(\xi, s, x)) - h_2(\xi, s, y)| \leq Q_2 |F(x - y)|,$$

for all $(\xi, x), (\xi, y) \in P \times G, \ F \in G^*$;

(3) there exists a continuous function $L : P \times \mathbb{R} \to \mathbb{R}$ such that

$$w = \max\{\sup|L(\xi, k(\xi))| + \sup|L(\xi, 0)|\}, \ \xi \in P;$$

(4) $Q_1 Q_2 w\xi < 1$.

In order to approximate the sought solution of Problem (5.1), it is sufficient to solve the following integral equation:

$$z(\xi) = z_0 + \int_0^{\xi} h_1\left(s, \int_0^{k(s)} h_2(s, \theta, z(\theta)) d_\theta L(s, \theta)\right) ds.$$

Next, we apply the iterative scheme (1.8) to approximate the sought solution to Problem (5.1) under the initial condition (5.2).

Theorem 5.1. *Under the hypotheses (1)–(4), Problem (5.1) under the initial condition (5.2) admits a unique solution $v \in C[P, G]$ and the sequence $\{z_i\}$ generated by the iterative scheme (1.8) is convergent to v under a suitable starting point.*

Proof. First, we set the function F on $C[P, G]$ by the following formula:

$$F(z(\xi)) = z_0 + \int_0^{\xi} h_1\left(s, \int_0^{k(s)} h_2(s, \theta, z(\theta)) d_\theta L(s, \theta)\right) ds.$$

Now

$$\begin{aligned} \|Fz_i - v\|_C &= \|Fz_i - Fv\|_C = \|\int_0^{\xi} h_1\left(s, \int_0^{k(s)} h_2(s, \theta, z_i(\theta)) d_\theta L(s, \theta)\right) ds \\ &\quad - \int_0^{\xi} h_1\left(s, \int_0^{k(s)} h_2(s, \theta, v(\theta)) d_\theta L(s, \theta)\right) ds)\|_C \\ &\leq \int_0^{\xi} Q_1 |F(\int_0^{k(s)} h_2(s, \theta, z_i(\theta)) d_\theta L(s, \theta) \\ &\quad - \int_0^{k(s)} h_2(s, \theta, v(\theta)) d_\theta L(s, \theta)) ds))| ds \\ &\leq Q_1 \int_0^{\xi} \int_0^{k(s)} |F(h_2(s, \theta, z_i(\theta)) - h_2(s, \theta, v(\theta)) d_\theta L(s, \theta)| ds \\ &\leq Q_1 \int_0^{\xi} \int_0^{k(s)} Q_2 |F\left(z_i(\theta) - v(\theta)) d_\theta L(s, \theta)\right)| ds \\ &= Q_1 Q_2 \|z_i - v\|_C \int_0^{\xi} \int_0^{k(s)} d_\theta L(s, \theta) ds \\ &= Q_1 Q_2 \|z_i - v\|_C \int_0^{\xi} (L(s, k(s) - L(s, 0)) ds \\ &= Q_1 Q_2 \|z_i - v\|_C \int_0^{\xi} ds = Q_1 Q_2 w\xi \|z_i - v\|_C \\ &\leq \|z_i - v\|_C. \end{aligned}$$

Hence we get

$$\|Fz_i - v\|_C \leq \|z_i - v\|_C. \tag{5.3}$$

Let $e_i = (1 - b_i)Fu_i + b_iFs_i$. Then similarly as above, we have

$$\|Fu_i - v\|_C \leq \|u_i - v\|_C, \tag{5.4}$$

$$\|Fs_i - v\|_C \leq \|s_i - v\|_C, \tag{5.5}$$

and

$$\|Fe_i - v\|_C \leq \|e_i - v\|_C. \tag{5.6}$$

Now using (5.3), we have

$$\begin{aligned} \|u_i - v\|_C &\leq \|(1 - b_i)z_i + b_iFz_i - v\|_C \\ &\leq (1 - b_i)\|z_i - v\|_C + b_i\|Fz_i - v\|_C \\ &\leq (1 - b_i)\|z_i - v\|_C + b_i\|z_i - v\|_C \\ &= \|z_i - v\|_C. \end{aligned}$$

Similarly using (5.4), we have

$$\begin{aligned} \|s_i - v\|_C &= \|Fu_i - v\|_C \\ &\leq \|u_i - v\|_C \\ &\leq \|z_i - v\|_C. \end{aligned}$$

Finally using (5.5) and (5.6), we have

$$\begin{aligned} \|z_{i+1} - v\|_C &= \|F((1 - b_i)Fu_i + b_iFs_i) - v\|_C \\ &= \|Fe_i - v\|_C \leq \|e_i - v\|_C \\ &= \|(1 - b_i)Fu_i + b_iFs_i - v\|_C \\ &\leq \|(1 - b_i)Fu_i + b_iFs_i - v\|_C \\ &\leq (1 - b_i)\|Fu_i - v\|_C + b_i\|Fs_i - v\|_C \\ &\leq (1 - b_i)\|u_i - v\|_C + b_i\|s_i - v\|_C \\ &\leq (1 - b_i)\|z_i - v\|_C + b_i\|Fz_i - v\|_C. \end{aligned}$$

Subsequently, we get

$$\|z_{i+1} - v\|_C \leq \|z_i - v\|_C.$$

Put $\|z_i - v\|_C = a_i$; then, we have

$$a_{i+1} \leq a_i \ \ \forall\, i \in \mathbb{N},$$

$$\Rightarrow \lim_{i\to\infty} a_i = 0.$$

It follows that $\|z_i - v\|_C \to 0$, that is, $z_i \to v$. Thus, we get that the sequence of JK iteration (1.8) is convergent to the unique solution v. This completes the proof. □

6. Conclusions

It is shown that the class of RSTN mappings is more general than the class of STN mappings, by Example 4.1. Also, several strong and weak convergence theorems under the JK iterative process were investigated under the class of RSTN mappings. In addition, under appropriate conditions, we proved that the JK iterative process is faster than the other well-known iterative schemes. So, our method improved and extended the results given by [11] in two ways: better rate of convergence and general setting of mappings. Finally, an application to find the solution to a special type of differential equation using our iterative scheme is provided.

Availability of data and materials

The data used to support the findings of this study are available from the corresponding author upon request.

Funding

Not applicable.

Author's contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgments

This study was supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

Conflict of interest

The authors declare that they have no competing interests.

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