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Research article

Neighbor sum distinguishing total choice number of IC-planar graphs with restrictive conditions

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Abstract: A neighbor sum distinguishing (NSD) total coloring ϕ of G is a proper total coloring such that $\sum_{z\in E_G(u)\cup\{u\}}\phi(z)\neq\sum_{z\in E_G(v)\cup\{v\}}\phi(z)$ for each edge $uv\in E(G)$. Pilśniak and Woźniak asserted that each graph with a maximum degree Δ admits an NSD total $(\Delta+3)$-coloring in 2015. In this paper, we prove that the list version of this conjecture holds for any IC-planar graph with $\Delta\geq 10$ but without five cycles by applying the discharging method, which improves the result of Zhang (NSD list total coloring of IC-planar graphs without five cycles).

Keywords: IC-planar graphs; neighbor sum distinguishing total choice number; combinatorial nullstellensatz; discharging method
Mathematics Subject Classification: 05C15

1. Introduction

Only simple graphs are considered in the paper. For any graph theory notation undefined here we follow [1].

Let $G=(V(G),E(G))$ be a simple graph. For a vertex $u\in V(G)$, we denote the set of edges incident with u by $E_G(u)$. The degree and the neighborhood of u are denoted by $d_G(u)$ and $N_G(u)$, respectively. We use $\delta(G)$ and $\Delta(G)$ (or Δ) to represent the minimum degree and the maximum degree of G, respectively.

Let $k\geq 0$ be an integer and $T(G)=V(G)\cup E(G)$. A neighbor sum distinguishing (NSD) total coloring of G is a mapping $\phi: T(G)\to\{1,2,\cdots,k\}$ such that for any two adjacent or incident elements $z_1,z_2\in T(G)$, $\phi(z_1)\neq\phi(z_2)$ and for every edge $uv\in E(G)$, $\sum_{z\in E_G(u)\cup\{u\}}\phi(z)\neq\sum_{z\in E_G(v)\cup\{v\}}\phi(z)$. The NSD total chromatic number of G is $\chi_\Sigma^t(G) = \min\{k \mid G$ has an NSD k-total coloring$\}$. In 2015, Pilśniak and Woźniak [2] introduced an important conjecture about the NSD total chromatic number as follows.

Conjecture 1.1. *[2] For every graph G, $\chi_\Sigma^t(G) \leq \Delta(G) + 3$.*

Conjecture 1.1 was proved to hold for many special classes of graphs, such as complete graphs, bipartite graphs, subcubic graphs [2], planar graphs with $\Delta \geq 10$ [3] and planar graphs with $\Delta \geq 7$ but without five cycles [4] and so on.

In 2008, Alberson [5] introduced the concept of IC-planar graphs. A graph is called an IC-planar graph if the graph has a drawing in the plane such that each edge is crossed at most once and two pairs of crossing edges share no common end vertex.

Conjecture 1.1 also holds for some special IC-planar graphs, such as IC-planar graphs with $\Delta \geq 12$ [6], IC-planar graphs with $\Delta \geq 7$ but without triangles [7] and IC-planar graphs with $\Delta \geq 10$ but without adjacent triangles [8].

A mapping L of G is called a k-list total assignment of G if it assigns to each member $z \in T(G)$ a set $L(z)$ with $|L(z)| = k$. For a k-list total assignment L of G, a mapping ϕ is called an NSD total L-coloring of G if the ϕ is an NSD total coloring of G and for each $z \in T(G)$, $\phi(z) \in L(z)$. The NSD total choice number of G is $\mathrm{ch}_\Sigma^{\mathrm{t}}(\mathrm{G}) = \min\{\mathrm{k} \mid G$ has an NSD total L-coloring for any k-list total assignment $L\}$.

Accordingly, the list version of Conjecture 1.1 is as follows.

Conjecture 1.2. *[2] For every graph G,* $\mathrm{ch}_\Sigma^{\mathrm{t}}(\mathrm{G}) \leq \Delta(\mathrm{G}) + 3$.

Obviously, Conjecture 1.2 implies Conjecture 1.1. Conjecture 1.2 was also shown to hold for many special classes of graphs, such as subcubic graphs [9, 10], planar graphs with $\Delta \geq 13$ [11], planar graphs with $\Delta \geq 8$ but without adjacent triangles [12] and IC-planar graphs with $\Delta \geq 14$ [13] and so on. Zhang [14] considered any IC-planar graph without five cycles and obtained the following result.

Theorem 1.1. *[14] For every IC-planar graph G without five cycles,*

$$\mathrm{ch}_\Sigma^{\mathrm{t}}(\mathrm{G}) \leq \max\{\Delta(\mathrm{G}) + 3, 14\}.$$

There are many results of neighbor distinguishing coloring. However given this kind of result the case of a small Δ is interesting and difficult. For the results in this regard, see [9, 10, 15, 16].

In this paper, we improve the result of Zhang [14] and obtain Theorem 1.2 as follows.

Theorem 1.2. *Let G be an IC-planar graph without five cycles. Then*

$$\mathrm{ch}_\Sigma^{\mathrm{t}}(\mathrm{G}) \leq \max\{\Delta(\mathrm{G}) + 3, 13\}.$$

2. Preliminaries

Let G be a simple graph. A vertex u of G is called an *ℓ-vertex* (*ℓ^+-vertex, ℓ^--vertex*) if $d_G(u) = \ell$ ($d_G(u) \geq \ell$, $d_G(u) \leq \ell$). We denote the number of ℓ-vertex (ℓ^+-vertex, ℓ^--vertex) adjacent to u by $n_G^{\ell}(u)$ ($n_G^{\ell^+}(u)$, $n_G^{\ell^-}(u)$). A cycle of length t (at least t, at most t) is called a *t-cycle* (*t^+-cycle, t^--cycle*). In particular, a $(d_G(v_1), d_G(v_2), d_G(v_3))$*-cycle* is a 3-cycle $v_1v_2v_3v_1$ when $d_G(v_1) \leq d_G(v_2) \leq d_G(v_3)$. Let X be a subset of $T(G)$ and $\psi : X \to \mathbb{R}$ be a mapping. For every $u \in V(G)$, set

$$m_\psi(u) = \Sigma_{z \in X \cap (E_G(u) \cup \{u\})} \psi(z).$$

For any k-list total assignment L of G and every $z \in T(G)$, set

$$S_{\psi}(z) = L(z) \setminus \{\psi(x) \mid x \in X \text{ and } x \text{ is adjacent or incident with } z \text{ in } G\}.$$

Lemma 2.1. *[17] For an arbitrary field $\mathbb{F}$, let $P = P(x_1, \cdots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \ldots, x_n]$ with degree $\deg(P) = \sum_{k=1}^{n} i_k$, where $i_k \geq 0$ is an integer. Assume that $c_P(x_1^{i_1}, \cdots, x_n^{i_n})$ is the coefficient of the monomial $x_1^{i_1} \ldots x_n^{i_n}$ in P. If $S_1, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_k| > i_k$ and $c_P(x_1^{i_1}, \cdots, x_n^{i_n}) \neq 0$, then there are $s_1 \in S_1, \ldots, s_n \in S_n$ satisfying $P(s_1, \ldots, s_n) \neq 0$.*

For finite real number sets $S_1, \cdots, S_m$, set

$$S_1 \oplus \cdots \oplus S_m = \{s_1 + \cdots + s_m \mid s_i \in S_i, s_i \neq s_j, \forall\, i \neq j\}.$$

Lemma 2.2. *[18] Let $m \geq 2$ be an integer and $S_1, \cdots, S_m$ be m finite real number sets with $|S_i| = n_i$ and $n_1 \geq \cdots \geq n_m$. Set*

$$n_1' = n_1 \textit{ and } n_i' = \min\{n_{i-1}' - 1, n_i\}, \textit{ for } 2 \leq i \leq m.$$

If $n_m' > 0$, then

$$|S_1 \oplus \cdots \oplus S_m| \geq \sum_{i=1}^{m} n_i' - \frac{1}{2}m(m+1) + 1.$$

3. Proof of Theorem 1.2

3.1. Unavoidable configurations

Let G be a counterexample to Theorem 1.2 with $E(G)$ being minimal. Set $k = \max\{\Delta(G) + 3, 13\}$. Let u be a 4^--vertex of G. For any k-list total assignment L, if $T(G) \setminus \{u\}$ exists a total coloring ϕ' such that for any adjacent or incident elements $z_1, z_2 \in T(G) \setminus \{u\}$, $\phi'(v_1) \neq \phi'(v_2)$; for any two adjacent vertices $v_1, v_2 \in V(G) \setminus \{u\}$, $m_{\phi'}(v_1) \neq m_{\phi'}(v_2)$ and for each $z \in T(G) \setminus \{u\}$, $\phi'(z) \in L(z)$, then there is a color in $L(u)$ to color u so that the resulting coloring ϕ obtained from ϕ' is an NSD total L-coloring of G since $|S_{\phi'}(u)| \geq k - 2d_G(u) \geq 5 > d_G(u)$, a contradiction. Thus, we will omit the colors of all 4^--vertices in the process of constructing an NSD total L-coloring of G in the following.

Below, we discuss some local configurations of the counterexample G. Theorem 1.1 implies Claim 3.1.

Claim 3.1. $\Delta(G) \leq 10$.

Claim 3.2. *Let u be an ℓ-vertex of G. Then each one of the following statements holds.*
(1) $n_G^{5^-}(u) = 0$ when $\ell \leq 5$.
(2) $n_G^{4^-}(u) \leq \ell - 6$ when $6 \leq \ell \leq 7$.
(3) $n_G^{3^-}(u) \leq \ell - 6$ when $8 \leq \ell \leq 9$, furthermore, $n_G^{4^-}(u) \leq \ell - 6$ when $n_G^{3^-}(u) \geq 1$.

Proof. (1) Suppose to be contrary that the 5^--vertex u is adjacent to one 5^--vertex v. Without loss of generality, set $d_G(u) = d_G(v) = 5$. For any k-list total assignment L of G, G' has an NSD total L'-coloring ϕ', where $G' = G - uv$ and L' is the restriction of L on G'. By erasing the colors on u and v from the coloring ϕ', we obtain the coloring ϕ''. Then

$$|S_{\phi''}(u)| \geq k - 2 \times (5 - 1) \geq 5,$$

$$|S_{\phi''}(v)| \ge k - 2\times(5-1) \ge 5,$$
$$|S_{\phi''}(uv)| \ge k - (5-1) - (5-1) \ge 5.$$

Let ϕ be a mapping on $T(G)$ with $\phi(u) = x_1, \phi(v) = x_2, \phi(uv) = y_1$ and $\phi(z) = \phi''(z)$ for every $z \in T(G) \setminus \{u, v, uv\}$, and set

$$P = P(x_1, x_2, y_1) \;=\; (x_1 - x_2)(x_1 - y_1)(x_2 - y_1) \prod_{z\in N_G(u)} (m_\phi(u) - m_\phi(z)) \cdot \prod_{z\in N_G(v)\setminus\{u\}} (m_\phi(v) - m_\phi(z)).$$

Then $\deg(P) = 12$. Set

$$P_0 = x_1^4 x_2^4 y_1^4.$$

Then $\deg(P_0) = \deg(P) = 12$ and $c_P(P_0) = -20$ via MATHEMATICA. By Lemma 2.1, there are $s_1 \in S_\phi(u), s_2 \in S_\phi(v)$ and $s_3 \in S_\phi(uv)$ such that $P(s_1, s_2, s_3) \neq 0$. Thus we can obtain an NSD total L-coloring ϕ of G with $\phi(u) = s_1, \phi(v) = s_2, \phi(uv) = s_3$ and $\phi(z) = \phi''(z)$, a contradiction.

(2) and (3) Set $N_G^{4^-}(u) = \{v_1, \ldots, v_i\}$ with $d_G(v_1) \le \cdots \le d_G(v_i) \le 4$. Obviously, (2) and (3) hold when $N_G^{4^-}(u) = \emptyset$. Below, let $N_G^{4^-}(u) \neq \emptyset$. Set $G_i' = G - \{uv_1, \ldots, uv_i\}$. For any k-list total assignment L of G, G_i' has an NSD total L'-coloring ϕ_i', where L' is the restriction of L on G_i'. By erasing the colors on $u, v_1, \ldots, v_i$ from the coloring ϕ_i', we obtain the coloring ϕ_i''. Then

$$|S_{\phi_i''}(u)| \ge k - 2\times(\ell - i),$$
$$|S_{\phi_i''}(uv_j)| \ge k - (\ell - i) - (d_G(v_j) - 1)\ (1 \le j \le i).$$

Suppose that (2) is false. Note that $6 \le \ell \le 7$. Fix $i = \ell - 5$. Then

$$|S_{\phi_i''}(u)| \ge k - 2\times(\ell - (\ell-5)) \ge 3,$$
$$|S_{\phi_i''}(uv_j)| \ge k - (\ell - (\ell-5)) - (d_G(v_j) - 1) \ge 5\ (1 \le j \le \ell - 5).$$

By lemma 2.2, we have

$$\begin{cases} |S_{\phi_i''}(u) \oplus S_{\phi_i''}(uv_1)| \ge 3 + 5 - \frac{1}{2}\times 2\times 3 + 1 > 6 - 1 & \text{if } \ell = 6. \\ |S_{\phi_i''}(u) \oplus S_{\phi_i''}(uv_1) \oplus S_{\phi_i''}(uv_2)| \ge 3 + 4 + 5 - \frac{1}{2}\times 3\times 4 + 1 > 7 - 2 & \text{if } \ell = 7. \end{cases}$$

Suppose that (3) is false. Note that $8 \le \ell \le 9$. Fix $i = \ell - 5$. Then $d_G(v_1) \le 3$ and

$$|S_{\phi_i''}(u)| \ge k - 2\times(\ell - (\ell-5)) \ge 3,$$
$$|S_{\phi_i''}(uv_1)| \ge k - (\ell - (\ell-5)) - (d_G(v_1) - 1) \ge 6,$$
$$|S_{\phi_i''}(uv_j)| \ge k - (\ell - (\ell-5)) - (d_G(v_j) - 1) \ge 5\ (2 \le j \le \ell - 5).$$

By lemma 2.2, we have

$$\begin{cases} |S_{\phi_i''}(u) \oplus S_{\phi_i''}(uv_1) \oplus \cdots \oplus S_{\phi_i''}(uv_3)| \ge \sum_{t=3}^{6} t - \frac{1}{2}\times 4\times 5 + 1 > 8 - 3 & \text{if } \ell = 8. \\ |S_{\phi_i''}(u) \oplus S_{\phi_i''}(uv_1) \oplus \cdots \oplus S_{\phi_i''}(uv_4)| \ge \sum_{t=2}^{6} t - \frac{1}{2}\times 5\times 6 + 1 > 9 - 4 & \text{if } \ell = 9. \end{cases}$$

Under each of the above two assumptions, there are $s_1 \in S_{\phi_i''}(u)$ to color u and $s_{j+1} \in S_{\phi_i''}(uv_j)$ $(1 \le j \le i)$ to color uv_j such that the resulting coloring ϕ obtained from ϕ_i'' satisfies $m_\phi(u) \neq m_\phi(z)$ for each $z \in N_G(u) \setminus \{v_1, \ldots, v_i\}$. Since $v_1, \ldots, v_i$ are 4^--vertices, ϕ is an NSD total L-coloring of G, a contradiction.

With the similar proof to that of Claim 3.2 (1), we can obtain the below Claim 3.3.

Claim 3.3. *Let C be an (ℓ_1, ℓ_2, ℓ_3)-cycle of G. Then $(\ell_1, \ell_2, \ell_3) \neq (5^-, 6^-, 6^-)$.*

Let H be a graph obtained from G by deleting all 2^--vertices. For each vertex $u \in V(H)$, we have

$$d_H(u) = d_G(u) - n_G^{2^-}(u).$$

By Claim 3.2, the following Claim 3.4 is immediate.

Claim 3.4. *Let u be an ℓ-vertex of H. Then each of the following statements holds.*
(1) $\ell \geq 3$.
(2) $\ell = d_G(u)$ *when* $3 \leq d_G(u) \leq 6$.
(3) $\ell \geq 6$ *when* $d_G(u) \geq 7$.
(4) $n_H^3(u) + n_H^4(u) \leq \ell - 6$ *when* $6 \leq \ell \leq 7$.
(5) $n_H^3(u) \leq \ell - 6$ *when* $8 \leq \ell \leq 9$.

By Claims 3.2 and 3.3, we can directly obtain the below Claim 3.5.

Claim 3.5. *Let C be an (ℓ_1, ℓ_2, ℓ_3)-cycle of H. Then*

$$(\ell_1, \ell_2, \ell_3) \in \{(3, 7^+, 7^+), (4, 7^+, 7^+), (5^+, 6^+, 6^+)\}.$$

On a planar graph, if the boundary of a face is a t-cycle (resp., t^+-cycle, t^--cycle, (ℓ_1, ℓ_2, ℓ_3)-cycle), then the face is called a *t-face* (resp., *t^+-face*, *t^--face*, *(ℓ_1, ℓ_2, ℓ_3)-face*). A 3-face with vertex set $\{v_1, v_2, v_3\}$ is denoted by $[v_1v_2v_3]$. A face is said to be *incident with* the vertices and edges in its boundary.

Through the rest, we assume that the counterexample G is embedded on a plane with every edge crossed by at most one other edge and the minimal crossings. By turning all crossings of G into new 4-vertices on the plane, we obtain a plane graph $G^\times$ called the *associated plane graph* of G. To make a difference, we call a vertex u *false vertex* if $u \in V(G^\times) \setminus V(G)$ and *real vertex* otherwise in $G^\times$. A face is called *false face* if it is incident with a false vertex and *real face* otherwise in $G^\times$.

Let $H^\times$ be the associated plane graph of H. For a vertex $u \in V(H)$, set

$$\begin{aligned} f_t(u) &= \text{the number of real 3-faces incident with } u, \\ f_\times(u) &= \text{the number of false 3-faces incident with } u. \end{aligned}$$

The follwing Claim 3.6 is immediate as G (and thus H) is an IC-planar graph without 5-cycles.

Claim 3.6. *Let u be a real vertex of $H^\times$ with $d_{H^\times}(u) \geq 4$. Then each of the following statements holds.*
(1) $0 \leq f_\times(u) \leq 2$.
(2) $f_t(u) \leq \lfloor \frac{2d_{H^\times}(u)}{3} \rfloor$ *if u is not adjacent to any false 4-vertex.*
(3) $f_t(u) \leq \lfloor \frac{2d_{H^\times}(u)-3}{3} \rfloor$ *if u is adjacent to one false 4-vertex and* $d_{H^\times}(u) \in \{0, 2\} (\mathrm{mod} 3)$.
(4) $f_t(u) \leq \lfloor \frac{2d_{H^\times}(u)-3}{3} \rfloor$ *if* $f_\times(u) = 2$ *and* $d_{H^\times}(u) \equiv 1 (\mathrm{mod} 3)$*;* $f_t(u) \leq \lfloor \frac{2d_{H^\times}(u)-6}{3} \rfloor$ *if* $f_\times(u) = 2$ *and* $d_{H^\times}(u) \in \{0, 2\} (\mathrm{mod} 3)$.
(5) $f_t(u) \leq \lfloor \frac{2d_{H^\times}(u)-4}{3} \rfloor$ *if u is adjacent to one 4-cycle in H.*

Set

$$n_{4^\times}(f) = \text{the number of false 4-vertices incident with the false face } f \text{ in } H^\times.$$

Claim 3.7. *Let f be a false t-face of $H^\times$. Then $n_{4^\times}(f) \leq \lfloor \frac{t}{3} \rfloor$.*

3.2. Discharging process

Let $\omega(H^{\times})$ be the number of connected components of $H^{\times}$. For every member $z \in V(H^{\times}) \cup F(H^{\times})$, assign the weight $w(z) = d_{H^{\times}}(z) - 4$. By Euler's formula, we have

$$\sum_{z \in V(H^{\times}) \cup F(H^{\times})} w(z) = \sum_{z \in V(H^{\times}) \cup F(H^{\times})} (d_{H^{\times}}(z) - 4) = -4(1 + \omega(H^{\times})).$$

In the following, we will apply the discharging method on $H^{\times}$ to prove that $H^{\times}$ (and thus H) does not exist. And so G does not exist. To redistribute weights among vertices and faces, and keep the total weights unchanged, we design the discharging rules are as follows:

(R1) Every real 3-vertex receives $\frac{1}{3}$ from its each neighbor.
(R2) Every false 3-face receives 1 from its incident false 4-vertex.
(R3) Let $[v_1 v_2 v_3]$ be a real (ℓ_1, ℓ_2, ℓ_3)-face in $H^{\times}$.

(R3.1) If $(\ell_1, \ell_2, \ell_3) = (3, 7^+, 7^+)$ or $(4, 7^+, 7^+)$, then $[v_1 v_2 v_3]$ receives $\frac{1}{2}$ from every incident real 7^+-vertex.
(R3.2) If $(\ell_1, \ell_2, \ell_3) = (5^+, 6^+, 6^+)$, then $[v_1 v_2 v_3]$ receives $\frac{1}{3}$ from every incident real 5^+-vertex.

(R4) Every false 4-vertex receives 1 from its incident false 5^+-face.
(R5) Let z be a false 4-vertex and x be a neighbor of z in $H^{\times}$.

(R5.1) Set $d_{H^{\times}}(x) = 6$. Then z receives $2 - \frac{1}{3} f_t(x)$.
(R5.2) Set $d_{H^{\times}}(x) = 7$. Then z receives $\frac{8}{3} - \frac{1}{3}(f_t(x) + n^3_{H^{\times}}(x))$.
(R5.3) Set $d_{H^{\times}}(x) = 8$. Then z receives $4 - \frac{1}{2} f_t(x) - \frac{1}{3} n^3_{H^{\times}}(x)$.
(R5.4) Set $d_{H^{\times}}(x) = 9$. Then z receives $5 - \frac{1}{2} f_t(x) - \frac{1}{3} n^3_{H^{\times}}(x)$.
(R5.5) Set $d_{H^{\times}}(x) = 10$. Then z receives $6 - \max\{\frac{1}{2} f_t(x) + \frac{1}{3} n^3_{H^{\times}}(x) | \, f_t(z) \le 6 \text{ and} f_t(x) + n^3_{H^{\times}}(x) \le 12\}$.

After redistributing weights by the discharging rules, we denote the new weight for each $z \in V(H^{\times}) \cup F(H^{\times})$ by $w'(z)$. Since the total weights keep unchanged, we have

$$\sum_{z \in V(H^{\times}) \cup F(H^{\times})} w'(z) = \sum_{z \in V(H^{\times}) \cup F(H^{\times})} w(z) = -4(1 + \omega(H^{\times})) < 0.$$

Thus, there is a nonempty set $Y \subseteq (V(H^{\times}) \cup F(H^{\times}))$ such that for every element $z \in Y$, we have

$$w'(z) < 0.$$

In the following, we will prove that such Y does not exist to obtain a contradiction.

Firstly, we choose arbitrarily an element z from $F(H^{\times})$. If z is a false 3-face, then $w'(z) = 3 - 4 + 1 = 0$ by (R2) since each false 3-face is incident with a false 4-vertex. If z is a real 3-face, then z is an (ℓ_1, ℓ_2, ℓ_3)-face with $(\ell_1, \ell_2, \ell_3) \in \{(3, 7^+, 7^+), (4, 7^+, 7^+), (5^+, 6^+, 6^+)\}$. Thus, by (R3), it is easy to verify $w'(z) \ge 0$. If z is a real 4^+-face or a false 4-face, then $w'(z) \ge 0$ since no rule is applied to it. If z is a false t^+-face with $t \ge 5$, then $w'(z) \ge t - 4 - \lfloor \frac{t}{3} \rfloor \ge 0$ by (R4) and Claim 3.7. Thus the nonempty set $Y \cap F(H^{\times}) = \emptyset$.

Next, we pick arbitrarily a false 4-vertex z from $V(H^{\times}) \setminus V(H)$. Set $N_{H^{\times}}(z) = \{u_1, u_2, u_3, u_4\}$. Then, up to isomorphism, the configuration of the induced subgraph $H^{\times}[\{z\} \cup N_{H^{\times}}(z)]$ is one of six configurations in Figure 1. Note that z is adjacent to at least two real 6^+-vertices by Claims 3.2 and 3.4.

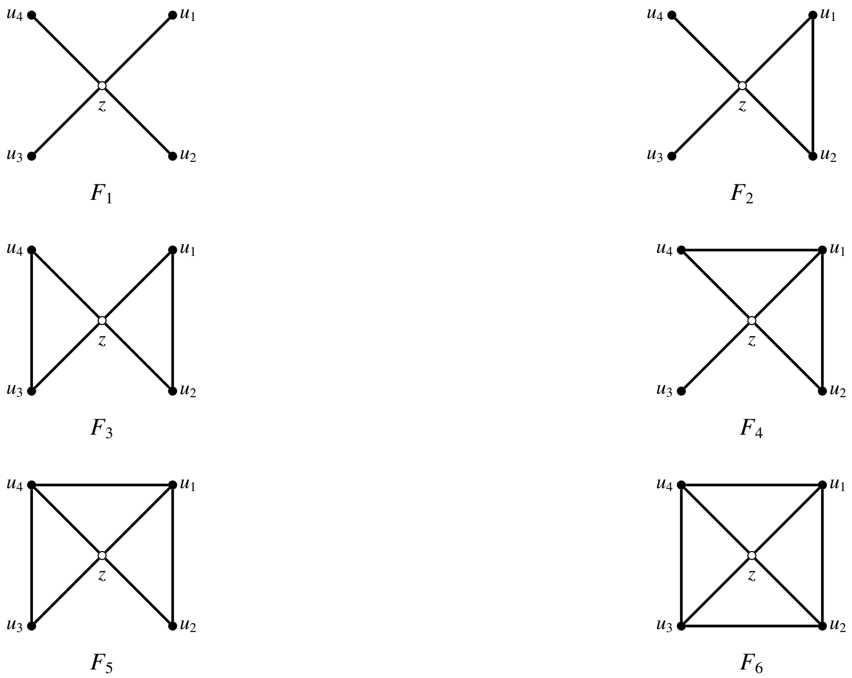

Figure 1. Six different configurations of $H^{\times}[\{z\} \cup N_{H^{\times}}(z)]$.

(1) Let the configuration of $H^{\times}[\{z\} \cup N_{H^{\times}}(z)]$ be F_1 in Figure 1. Since H is an IC-planar graph without 5-cycles, z is incident with at least two false 5^+-faces in $H^{\times}$. Thus, $w'(z) \geq 4-4+2-2\cdot\frac{1}{3} = \frac{4}{3} > 0$ by (R1) as z is adjacent to at most two real 3-vertices in $H^{\times}$.

(2) Let the configuration of $H^{\times}[\{z\} \cup N_{H^{\times}}(z)]$ be F_2 in Figure 1. Since H is an IC-planar graph without 5-cycles, z is incident with at least one false 5^+-face $H^{\times}$. Thus, $w'(z) \geq 4-4+1+2\times\min\{2-\frac{1}{3}\times 3, \frac{8}{3}-\frac{1}{3}\times 5, 4-\frac{1}{2}\times 4-\frac{1}{3}\times 2, 5-\frac{1}{2}\times 5-\frac{1}{3}\times 3, 6-\frac{1}{2}\times 6-6\cdot\frac{1}{3}\}-1-\frac{1}{3}\times 2 = \frac{4}{3} > 0$ by (R1), (R2), (R4), (R5) and Claims 3.1, 3.6 as z is adjacent to at most two real 3-vertices in $H^{\times}$.

(3) Let the configuration of $H^{\times}[\{z\} \cup N_{H^{\times}}(z)]$ be F_3 in Figure 1. Then $w'(z) \geq 4-4+2\times\min\{2-\frac{1}{3}\times 2, \frac{8}{3}-\frac{1}{3}\times 4, 4-\frac{1}{2}\times 4-\frac{1}{3}\times 2, 5-\frac{1}{2}\times 4-\frac{1}{3}\times 3, 6-\frac{1}{2}\times 5-6\cdot\frac{1}{3}\}-2\times 1-\frac{1}{3}\times 2 = 0$ by (R1), (R2), (R5) and Claims 3.1, 3.6.

(4) Let the configuration of $H^{\times}[\{z\} \cup N_{H^{\times}}(z)]$ be F_4 in Figure 1. Since H is an IC-planar graph without 5-cycles, z is incident with two false 5^+-faces in $H^{\times}$. Thus, $w'(z) \geq 4-4+2\times 1+2\times\min\{2-\frac{1}{3}\times 3, \frac{8}{3}-\frac{1}{3}\times 5, 4-\frac{1}{2}\times 4-\frac{1}{3}\times 2, 5-\frac{1}{2}\times 5-\frac{1}{3}\times 3, 6-\frac{1}{2}\times 6-6\cdot\frac{1}{3}\}-2-\frac{1}{3}\times 2 = \frac{4}{3} > 0$ by (R1), (R2), (R4), (R5) and Claims 3.1, 3.6.

(5) Let the configuration of $H^{\times}[\{z\} \cup N_{H^{\times}}(z)]$ be F_5 in Figure 1. Since H is an IC-planar graph without 5-cycles, z is incident with one false 5^+-face in $H^{\times}$. Note that u_i ($i = 1, 2, 3, 4$) is incident with at least one 4-cycle in H. Thus, $w'(z) \geq 4-4+1+2\times\min\{2-\frac{1}{3}\times 2, \frac{8}{3}-\frac{1}{3}\times 4, 4-\frac{1}{2}\times 4-\frac{1}{3}\times 2, 5-\frac{1}{2}\times 4-\frac{1}{3}\times 3, 6-\frac{1}{2}\times 5-6\cdot\frac{1}{3}\}-3-\frac{2}{3} = 0$ by (R1), (R2), (R4), (R5) and Claims 3.1, 3.6.

(6) Let the configuration of $H^{\times}[\{z\} \cup N_{H^{\times}}(z)]$ be F_6 in Figure 1. Note that z is adjacent to at least three real 6^+-vertices and at most one real 3-vertex by Claim 3.4.

(6.1) Assume that z is not adjacent to any real 3-vertex. Then $w'(z) \geq 4-4+3\times\min\{2-\frac{1}{3}\times 2, \frac{8}{3}-\frac{1}{3}\times 4, 4-\frac{1}{2}\times 3-\frac{1}{3}\times 2, 5-\frac{1}{2}\times 4-\frac{1}{3}\times 3, 6-\frac{1}{2}\times 5-6\cdot\frac{1}{3}\}-4 = 0$ by (R2), (R5) and Claims 3.1, 3.6.

(6.2) Assume that z is adjacent to one real 3-vertex. Then z is adjacent to three real 7^+-vertices by

Claim 3.4. Set $d_{H^\times}(u_1)=3$. If $d_{H^\times}(u_3)=7$, then u_3 is not adjacent to any real 3-vertex in $H^\times$ by Claim 3.4 since u_3 is adjacent to u_1 in H. Thus, $w'(z)\geq 4-4+(\frac{8}{3}-\frac{1}{3}\times 3)+2\times\min\{\frac{8}{3}-\frac{1}{3}\times 4, 4-\frac{1}{2}\times 3-\frac{1}{3}\times 2, 5-\frac{1}{2}\times 4-\frac{1}{3}\times 3, 6-\frac{1}{2}\times 5-6\cdot\frac{1}{3}\}-4-\frac{1}{3}=0$ by (R1), (R2), (R5) and Claims 3.1 and 3.6. If $8\leq d_{H^\times}(u_3)\leq 9$, then $w'(z)\geq 4-4+\min\{4-\frac{1}{2}\times 3-\frac{1}{3}\times 2, 5-\frac{1}{2}\times 4-\frac{1}{3}\times 3\}+2\times\min\{\frac{8}{3}-\frac{1}{3}\times 4, 4-\frac{1}{2}\times 3-\frac{1}{3}\times 2, 5-\frac{1}{2}\times 4-\frac{1}{3}\times 3, 6-\frac{1}{2}\times 5-6\cdot\frac{1}{3}\}-4-\frac{1}{3}=\frac{1}{6}>0$ by (R1), (R2), (R5) and Claims 3.1 and 3.6. If $d_{H^\times}(u_3)=10$, then $w'(z)\geq 4-4+(6-\frac{1}{2}\times 5-5\cdot\frac{1}{3})+2\times\min\{\frac{8}{3}-\frac{1}{3}\times 4, 4-\frac{1}{2}\times 3-\frac{1}{3}\times 2, 5-\frac{1}{2}\times 4-\frac{1}{3}\times 3, 6-\frac{1}{2}\times 5-6\cdot\frac{1}{3}\}-4-\frac{1}{3}=0$ by (R1), (R2), (R5) and Claims 3.1 and 3.6. By symmetry, we have $w'(z)\geq 0$ when $d_{H^\times}(u_2)=3$ or $d_{H^\times}(u_3)=3$ or $d_{H^\times}(u_4)=3$.

By the analysis above (1)–(6), the nonempty set $Y\cap(V(H^\times)\setminus V(H))=\emptyset$.

Finally, we choose arbitrarily a real vertex z from $V(H)$. Note that $3\leq d_{H^\times}(z)\leq 10$ by Claims 3.1 and 3.4. If z is a real 3-vertex, then $w'(z)=3-4+3\times\frac{1}{3}=0$ by (R1) since z has three neighbors. If z is a real 4-vertex, then $w'(z)=4-4=0$ since no rule is applied to it. If z is a real 5-vertex, then $f_t(z)\leq 3$ and $n^3_{H^\times}(z)+n^4_{H^\times}(z)=0$ by Claims 3.2, 3.4 and 3.6. Thus, $w'(z)\geq 5-4-3\times\frac{1}{3}=0$ by (R2). In the following, we consider that z is a real 6^+-vertex.

Assume that z is not adjacent to any false 4-vertex. Then $f_t(z)\leq\lfloor\frac{2d_{H^\times}(z)}{3}\rfloor$ by Claim 3.6. If $6\leq d_{H^\times}(z)\leq 9$, then $w'(z)\geq d_{H^\times}(z)-4-(d_{H^\times}(z)-6)\times\frac{1}{3}-\lfloor\frac{2d_{H^\times}(z)}{3}\rfloor\times\frac{1}{2}\geq 0$ by (R2) and Claims 3.4 and 3.6. If $d_{H^\times}(z)=10$, then $w'(z)\geq d_{H^\times}(z)-4-\max\{\frac{1}{2}f_t(z)+\frac{1}{3}n^3_{H^\times}(z)|f_t(z)\leq 6 \text{ and } f_t(z)+n^3_{H^\times}(z)\leq 13\}\geq\frac{2}{3}$ by Claim 3.6.

Assume that z is adjacent to one false 4-vertex. Note that $f_t(z)\leq 6$ and $f_t(z)+n^3_{H^\times}(z)\leq 12$ when $d_{H^\times}(z)=10$. By (R3), (R5) and Claim 3.4, it is easy to verify $w'(z)\geq 0$. Thus, the nonempty set $Y\cap V(H)=\emptyset$.

By the analysis above, the nonempty set $Y\cap(V(H^\times)\cup F(H^\times))=\emptyset$, This is a contradiction. The proof of Theorem 1.2 is completed.

4. Conclusions

Graph coloring is one of the important research contents of graph theory. In recent years, neighbor distinguishing colorings have gradually become one of the research hotspots of graph coloring. The paper is devoted to the study of neighbor sum distinguishing list total coloring of graphs. We proved that $\mathrm{ch}^{\mathrm{t}}_{\Sigma}(\mathrm{G})\leq\Delta(\mathrm{G})+3$ for every IC-planar graph with $\Delta\geq 10$ but without 5-cycles, which implies that Conjecture 1.2 holds for this class of graphs. There are many results about neighbor sum distinguishing list total coloring. According to currently known results, it yields a question for further research as follow:

Does Conjecture 1.2 hold for every IC-planar graph with $4\leq\Delta\leq 9$ but without 5-cycles?

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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