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Research article

Hardy-Rogers type contraction in double controlled metric-like spaces

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Abstract: In this paper, we establish a new fixed point result for Hardy-Rogers type contractions in double controlled metric-like spaces. Our result generalizes many important theorems in the literature. We will provide an example to illustrate our results.

Keywords: b-metric spaces; controlled metric spaces; double controlled metric-like spaces; Hardy-Rogers contraction; fixed point
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1. Introduction

Numerous generalizations of the usual metric space have been established in the field of fixed point theory over the past few decades. Researchers have proved fixed point theorems for Banach type contractions, Kannan type contractions and several [1–3] other types of contraction mappings as a result of the discovery of these generalized metric spaces. By adding a constant to the right-hand side of the triangle inequality, Czerwik [4] proposed the concept of the fascinating generalized metric space known as b-metric space. The b-metric space has a different topology than the usual metric space. In 2017, Kamran et al. [5] extended the definition of b-metric spaces to extended b-metric spaces, and proved the related fixed point theorem; see [6–8]. In 2018, Mlaiki et al. [9] further generalized the extended b-metric spaces to controlled metric spaces by introducing a binary control function on the right side of the triangle inequality. They also established the corresponding Banach fixed point result in the same space. In 2019, Lattef [10] established a Kannan type fixed point result in controlled metric spaces. In 2020, Ahmad et al. [11] established a fixed point result for Reich type contractions in controlled metric spaces. As a further generalization of controlled metric spaces, Abdeljawad et al. [12] introduced double controlled metric type spaces (DCMTS for short) by employing two binary control functions on the right side of the triangle inequality. In 2020, Mlaiki [13] introduced double controlled metric-like spaces as a further generalization of double controlled metric type spaces (DCMTS for short). He established the corresponding Banach type and Kannan type fixed point results in the same space. He also provided some non-trivial examples and applications. The self distance need not be zero in double controlled metric-like spaces (DCMLS), which makes them significantly different from DCMTS. In [14], A. Tas established a fixed point result for Reich type contractions in DCMLS. Since the Reich type contraction is a generalization of Banach type and Kannan type contractions, it generalizes the results proved by Mlaiki in [13].

In this paper, we aim to establish a fixed point result for the Hardy-Rogers type contraction [15] in DCMLS. The Hardy-Rogers type contraction is a generalization of the Reich type contraction and various types of other contractions. It is interesting to explicitly state the several types of contractions mentioned above.

Remark 1.1. *Let $(\mathcal{X}, \zeta)$ be a complete metric space and $T : \mathcal{X} \to \mathcal{X}$ be a self-map, and $\mu, \tau \in \mathcal{X}$. The fixed point theory of the following contraction mappings is well studied in the literature.*

1) *Banach Type: $\zeta(F\mu, F\tau) \leq k\zeta(\mu, \tau)$ where $k \in (0, 1)$.*
2) *Kannan Type: $\zeta(F\mu, F\tau) \leq k[\zeta(\mu, F\mu) + \zeta(\tau, F\tau)]$ where $k \in (0, \frac{1}{2})$.*
3) *Chatterjee Type: $\zeta(F\mu, F\tau) \leq k[\zeta(\mu, F\tau) + \zeta(\tau, F\mu)]$ where $k \in (0, \frac{1}{2})$.*
4) *Reich Type: $\zeta(F\mu, F\tau) \leq \alpha\zeta(\mu, \tau) + \beta\zeta(\mu, F\mu) + \gamma\zeta(\tau, F\tau)]$ where $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$.*
5) *Hardy-Rogers Type: $\zeta(F\mu, F\tau) \leq \alpha\zeta(\mu, \tau) + \beta\zeta(\mu, F\mu) + \gamma\zeta(\tau, F\tau) + \delta\zeta(\mu, F\tau) + \omega\zeta(\tau, F\mu)$ where $\alpha, \beta, \gamma, \delta, \omega \in [0, 1)$ with $\alpha + \beta + \gamma + \delta + \omega < 1$.*

We observe that the Hardy-Rogers type contraction is a generalization of the Banach type (with $\alpha = k$, $\beta = \gamma = \delta = \omega = 0$), Kannan type (with $\alpha = 0$, $\beta = \gamma = k, \delta = \omega = 0$), Chatterjee type (with $\alpha = \beta = \gamma = 0$, $\delta = \omega = k$) and Reich type contractions (with $\delta = \omega = 0$).

2. Preliminaries

We begin with a definition of the extended b-metric spaces introduced by Kamran et al. [5].

Definition 2.1. *Let $\mathcal{X}$ be a non empty set and $\psi_1 : \mathcal{X} \times \mathcal{X} \to [1, +\infty)$. A function $\zeta : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ is called an extended b-metric type if it satisfies:*

1) *$\zeta(\mu, \tau) = 0$ if and only if $\mu = \tau$ for all $\mu, \tau \in \mathcal{X}$;*
2) *$\zeta(\mu, \tau) = \zeta(\tau, \mu)$ for all $\mu, \tau \in \mathcal{X}$;*
3) *$\zeta(\mu, \rho) \leq \psi_1(\mu, \tau)[\zeta(\mu, \tau) + \zeta(\tau, \rho)]$ for all $\mu, \tau, \rho \in \mathcal{X}$.*

The pair $(\mathcal{X}, \zeta)$ is called an extended b-metric space.

Nabil Mlaiki et al. [9] proposed the following new generalization of extended b-metric spaces called controlled metric type spaces.

Definition 2.2. *Let $\mathcal{X}$ be a non empty set and $\psi_1 : \mathcal{X} \times \mathcal{X} \to [1, +\infty)$. A function $\zeta : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ is called a controlled metric type if it satisfies*

1) *$\zeta(\mu, \tau) = 0$ if and only if $\mu = \tau$ for all $\mu, \tau \in \mathcal{X}$;*
2) *$\zeta(\mu, \tau) = \zeta(\tau, \mu)$ for all $\mu, \tau \in \mathcal{X}$;*
3) *$\zeta(\mu, \rho) \leq \psi_1(\mu, \tau)\zeta(\mu, \tau) + \psi_1(\tau, \rho)\zeta(\tau, \rho)$ for all $\mu, \tau, \rho \in \mathcal{X}$.*

The pair $(\mathcal{X}, \zeta)$ is called controlled a metric type space.

In [12], Thabed Abdeljawad et al. proposed the following generalization of controlled metric type space and named it a double controlled type metric space (DCMTS).

Definition 2.3. *(DCMTS) Let $\mathcal{X}$ be a non empty set and $\psi_1, \psi_2 : \mathcal{X} \times \mathcal{X} \to [1, +\infty)$. A function $\zeta : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ is called a double controlled metric type if it satisfies:*

1) $\zeta(\mu, \tau) = 0$ if and only if $\mu = \tau$ for all $\mu, \tau \in \mathcal{X}$;
2) $\zeta(\mu, \tau) = \zeta(\tau, \mu)$ for all $\mu, \tau \in \mathcal{X}$;
3) $\zeta(\mu, \rho) \leq \psi_1(\mu, \tau)\zeta(\mu, \tau) + \psi_2(\tau, \rho)\zeta(\tau, \rho)$ for all $\mu, \tau, \rho \in \mathcal{X}$.

The pair $(\mathcal{X}, \zeta)$ is called a double controlled metric type space.

Example 2.1. *[12] Let $\mathcal{X} = [0, +\infty)$. Define $\zeta : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ by*

$$\zeta(\mu, \tau) = \begin{cases} 0, & \text{if and only if} \quad \mu = \tau, \\ \frac{1}{\mu}, & \text{if } \mu \geq 1 \text{ and } \tau \in [0, 1), \\ \frac{1}{\tau}, & \text{if } \tau \geq 1 \text{ and } \mu \in [0, 1), \\ 1, & \text{if not.} \end{cases}$$

Consider $\psi_1, \psi_2 : \mathcal{X}^2 \to [1, +\infty)$ as

$$\psi_1(\mu, \tau) = \begin{cases} \mu, & \text{if } \mu, \tau \geq 1, \\ 1, & \text{if not,} \end{cases} \quad \text{and} \quad \psi_2(\mu, \tau) = \begin{cases} 1, & \text{if } \mu, \tau < 1, \\ \max\{\mu, \tau\}, & \text{if not.} \end{cases}$$

The pair $(\mathcal{X}, \zeta)$ is a double controlled metric type space.

In [13], Mlaiki et al. generalized double controlled metric type spaces (DCMTS) to double controlled metric-like spaces (DCMLS) by weakening the condition (2.3) in Definition 2.3 such that the self-distance is not required to be zero.

Definition 2.4. *(DCMLS) Let $\mathcal{X}$ be a non empty set and $\psi_1, \psi_2 : \mathcal{X} \times \mathcal{X} \to [1, +\infty)$. A function $\zeta : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ is called a double controlled metric type if it satisfies:*

1) $\zeta(\mu, \tau) = 0$ implies $\mu = \tau$;
2) $\zeta(\mu, \tau) = \zeta(\tau, \mu)$ for all $\mu, \tau \in \mathcal{X}$;
3) $\zeta(\mu, \rho) \leq \psi_1(\mu, \tau)\zeta(\mu, \tau) + \psi_2(\tau, \rho)\zeta(\tau, \rho)$ for all $\mu, \tau, \rho \in \mathcal{X}$.

The pair $(\mathcal{X}, \zeta)$ is called a double controlled metric type space.

Every *DCMLS* is a *DCMTS*; however, the converse is not true in general, as shown by the following example.

Example 2.2. *[13] Let $\mathcal{X} = [0, +\infty)$. Define $\zeta : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ by*

$$d(\mu, \tau) = \begin{cases} 0, & \text{if } \mu = \tau \neq 0, \\ \frac{1}{2}, & \text{if } \mu = \tau = 0, \\ \frac{1}{\mu}, & \text{if } \mu \geq 1 \text{ and } \tau \in [0, 1), \\ \frac{1}{\tau}, & \text{if } \tau \geq 1 \text{ and } \mu \in [0, 1), \\ 1, & \text{otherwise.} \end{cases}$$

Let $\psi_1, \psi_2 : \mathcal{X} \times \mathcal{X} \to [1, +\infty)$ *be*

$$\psi_1(\mu, \tau) = \begin{cases} \mu, & \text{if } \mu, \tau \geq 1, \\ \tau, & \text{otherwise,} \end{cases}$$

and

$$\psi_2(\mu, \tau) = \begin{cases} \mu, & \text{if } \mu, \tau < 1, \\ \max\{\mu, \tau\}, & \text{otherwise.} \end{cases}$$

Then, it is easy to verify that $(\mathcal{X}, d)$ *is a double controlled metric-like space. However,* $(\mathcal{X}, d)$ *is not a double controlled metric type space.*

Example 2.3. *[13] Let* $\mathcal{X} = \{a, b, c\}$ *and define* $\zeta : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ *by*

$$\begin{aligned} &\zeta(a, a) = \zeta(b, b) = 0, \zeta(c, c) = \frac{1}{10}, \\ &\zeta(a, b) = \zeta(b, a) = 1, \\ &\zeta(a, c) = \zeta(c, a) = \frac{1}{2}, \\ &\zeta(b, c) = \zeta(c, b) = \frac{2}{5}. \end{aligned}$$

Define $\psi_1, \psi_2 : \mathcal{X} \times \mathcal{X} \to [1, +\infty)$,

$$\begin{aligned} &\psi_1(a, a) = \psi_1(b, b) = \psi_1(c, c) = \psi_1(a, c) = 1, \psi_1(a, b) = \frac{11}{10}, \psi_1(b, c) = \frac{8}{5}, \\ &\psi_2(a, a) = \psi_2(b, b) = \psi_2(c, c) = 1, \psi_2(a, b) = \frac{11}{10}, \psi_2(a, c) = \frac{3}{2}, \psi_2(b, c) = \frac{5}{4}. \end{aligned}$$

Then, it is easy to verify $(\mathcal{X}, \zeta)$ *is a double controlled metric-like space but not a double controlled metric type space.*

We recall the topology of double controlled metric-like spaces.

Definition 2.5. *[13] Let* $(\mathcal{X}, \zeta)$ *be a double controlled metric-like space. For each sequence* $\{\theta_n\} \in \mathcal{X}$*, we say*

1) *that* $\{\theta\}$ *is a Cauchy sequence if* $\lim_{n,m\to\infty} d(\theta_n, \theta_m)$ *exists and is finite,*
2) *that* $\{\theta_n\}$ *converges to* θ *if* $\lim_{n\to\infty} d(\theta_n, \theta) = 0$,
3) *that* $(\mathcal{X}, \theta)$ *is complete if every Cauchy sequence in* $\mathcal{X}$ *is convergent to some point in* $\mathcal{X}$.

Mlaiki [13] established the following Banach and Kannan type fixed point results in $DCMLS$.

Theorem 2.1. *Let* $(\mathcal{X}, \zeta)$ *be a complete double controlled metric like space (DCMLS) with* $\psi_1, \psi_2 : \mathcal{X} \times \mathcal{X} \to [1, +\infty)$*. Let* $F : \mathcal{X} \to \mathcal{X}$ *be a mapping such that*

$$\zeta(F\mu, F\tau) \leq k\zeta(\mu, \tau),$$

for all $\mu, \tau \in \mathcal{X}$*, where* $k \in (0, 1)$*. For* $\theta_0 \in \mathcal{X}$*, take* $\theta_n = F^n\theta_0$*. Suppose that*

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\psi_1(\theta_{i+1}, \theta_{i+2})}{\psi_1(\theta_i, \theta_{i+1})} \psi_2(\theta_{i+1}, \theta_m) < \frac{1}{k}.$$

In addition, assume that, for every $\mu \in \mathcal{X}$*, we have that* $\lim_{n\to\infty} \psi_1(\mu, \theta_n)$ *and* $\lim_{n\to\infty} \psi_2(\theta_n, \mu)$ *exist and are finite. Then,* F *has a unique fixed point.*

Theorem 2.2. *Let (X, ζ) be a complete double controlled metric like space (DCMLS) with $\psi_1, \psi_2 : X \times X \to [1, +\infty)$. Let $F : X \to X$ be a mapping such that*

$$\zeta(F\mu, F\tau) \leq k[\zeta(\mu, F\mu) + \zeta(\tau, F\tau)].$$

for all $\mu, \tau \in X$, where $k \in (0, \frac{1}{2})$. For $\theta \in X$, take $\theta = F^n\theta_0$. Suppose that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\psi_1(\theta_{i+1}, \theta_{i+2})}{\psi_1(\theta_i, \theta_{i+1})} \psi_2(\theta_{i+1}, \theta_m) < \frac{1-k}{k}.$$

In addition, assume that, for every $\mu \in X$, we have that $\lim_{n\to\infty} \psi_1(\mu, \theta_n)$ exists and is finite, and $\lim_{n\to\infty} \psi_2(\theta_n, \mu) < \frac{1}{k}$. Then, F has a fixed point. Moreover, if $\zeta(\theta, \theta) = 0$ for every fixed point θ, then the fixed point is unique.

In [14], A. Tas generalized Theorems 2.1 and 2.2 by proving the following theorem for the Reich type contraction.

Theorem 2.3. *Let (X, ζ) be a complete double controlled metric like space (DCMLS) with $\psi_1, \psi_2 : X \times X \to [1, +\infty)$. Let $F : X \to X$ be a mapping satisfying the Reich condition, that is,*

$$\zeta(F\mu, F\tau) \leq \alpha\zeta(\mu, \tau) + \beta\zeta(\mu, F\mu) + \gamma\zeta(\tau, F\tau),$$

where $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$. Let $\frac{\alpha+\beta}{1-\gamma} = h$ for all $\mu, \tau \in X$. For $\theta \in X$, take $\theta_n = F^n\theta_0$. Suppose that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\psi_1(\theta_{i+1}, \theta_{i+2})}{\psi_1(\theta_i, \theta_{i+1})} \psi_2(\theta_{i+1}, \theta_m) < \frac{1}{h}.$$

In addition, assume that, for every $\mu \in X$, we have that $\lim_{n\to\infty} \psi_1(\mu, \theta_n)$ exists finitely, and $\lim_{n\to\infty} \psi_2(\mu, \theta_n) < \frac{1}{\gamma}$. Then, F has a unique fixed point.

Our goal is to generalize Theorem 2.3 in double controlled metric like spaces (DCMLS) for Hardy-Rogers type contractions. By Remark 1.1, we see that the Reich type contraction is a special case of the Hardy-Rogers contraction.

3. Main result

The following result is analogous to the Hardy-Rogers type fixed point theorem.

Theorem 3.1. *Let (X, ζ) be a complete double controlled metric like space (DCMLS) with $\psi_1, \psi_2 : X \times X \to [1, +\infty)$. Let $F : X \to X$ be a mapping satisfying the Hardy-Rogers contraction, that is,*

$$\zeta(F\mu, F\tau) \leq \alpha\zeta(\mu, \tau) + \beta\zeta(\mu, F\mu) + \gamma\zeta(\tau, F\tau) + \delta\zeta(\mu, F\tau) + \omega\zeta(\tau, F\mu), \tag{3.1}$$

for all $\mu, \tau \in X$, and $\alpha, \beta, \gamma, \delta, \omega \in [0, 1), \alpha + \beta + \gamma + \delta + \omega < 1$. For $\theta_0 \in X$, define a sequence $\{\theta_n\}$ by $\theta_n = F^n\theta_0$. Suppose that the following conditions are satisfied:

1) $\sup_{m\geq 1} \lim_{i\to\infty} \dfrac{[\alpha + \beta + \delta\psi_1(x_i, \theta_{i+1})]\psi_2(\theta_{i+1}, \theta_m)}{1 - \gamma - \delta\psi_2(\theta_{i+1}, \theta_{i+2}) - \omega\psi_1(\theta_{i+1}, \theta_{i+2}) - \omega\psi_2(\theta_{i+2}, \theta_{i+1})} \dfrac{\psi_1(\theta_{i+1}, \theta_{i+2})}{\psi_1(\theta_i, \theta_{i+1})} < 1;$

2) $\lim_{n\to\infty} \prod_{k=0}^{n} \frac{\psi_1(\theta_n, \theta_{n+1})[\alpha+\beta+\delta\psi_1(x_{n-k-1}, \theta_{n-k})]}{1-\gamma-\delta\psi_2(\theta_{n-k}, \theta_{n-k+1}) - \omega\psi_1(\theta_{n-k}, \theta_{n-k+1}) - \omega\psi_2(\theta_{n-k+1}, \theta_{n-k})} = 0$*;*

3) $1-\gamma-\delta\psi_2(\theta_n, \theta_{n+1}) - \omega\psi_1(\theta_n, \theta_{n+1}) - \omega\psi_2(\theta_{n+1}, \theta_n) > 0$*;*

4) For every $\mu \in \mathcal{X}$*,* $\lim_{n\to\infty} \psi_1(\mu, \theta_n)$*,* $\lim_{n\to\infty} \psi_1(\theta_n, \mu)$*,* $\lim_{n\to\infty} \psi_2(\mu, \theta_n)$ *exist finitely, and* $\lim_{n\to\infty} \psi_2(\theta_n, \mu) < \frac{1}{\gamma+\delta}$.

Then, F has a fixed point. In addition, if $\alpha+\delta+\omega \neq 1$ *and* $\beta = \gamma = 0$*, then the fixed point of F is unique.*

Proof. Let $\theta_0 \in \mathcal{X}$. Define a sequence $\{\theta_n\}$ in $\mathcal{X}$ with $\theta_n = F^n\theta_0$ so that $\theta_{n+1} = F\theta_n$ for all $n \in \mathbb{N}$. Letting $\mu = \theta_{n-1}$ and $\tau = \theta_n$ in the Hardy-Rogers contraction, we have

$$\begin{aligned}\zeta(\theta_n, \theta_{n+1}) &= d(F(\theta_{n-1}), F\theta_n)\\ &\leq \alpha\zeta(\theta_{n-1}, \theta_n) + \beta\zeta(\theta_{n-1}, F\theta_{n-1}) + \gamma\zeta(\theta_n, F\theta_n)\\ &+ \delta\zeta(\theta_{n-1}, F\theta_n) + \omega\zeta(\theta_n, F\theta_{n-1})\\ &\leq \alpha\zeta(\theta_{n-1}, \theta_n) + \beta\zeta(\theta_{n-1}, \theta_n) + \gamma\zeta(\theta_n, \theta_{n+1})\\ &+ \delta\zeta(\theta_{n-1}, \theta_{n+1}) + \omega\zeta(\theta_n, \theta_n).\end{aligned} \tag{3.2}$$

Note that in DCMLS, $\zeta(\theta_n, \theta_n) \neq 0$ in general. By applying the triangle inequality, we obtain

$$\zeta(\theta_n, \theta_n) \leq \psi_1(\theta_n, \theta_{n+1})\zeta(\theta_n, \theta_{n+1}) + \psi_2(\theta_{n+1}, \theta_n)\zeta(\theta_{n+1}, \theta_n) \tag{3.3}$$

and

$$\zeta(\theta_{n-1}, \theta_{n+1}) \leq \psi_1(\theta_{n-1}, \theta_n)\zeta(\theta_{n-1}, \theta_n) + \psi_2(\theta_n, \theta_{n+1})\zeta(\theta_n, \theta_{n+1}). \tag{3.4}$$

Using inequality (3.3) and (3.4) in (3.2), we get

$$\begin{aligned}\zeta(\theta_n, \theta_{n+1}) &\leq \alpha\zeta(\theta_{n-1}, \theta_n) + \beta\zeta(\theta_{n-1}, \theta_n) + \gamma\zeta(\theta_n, \theta_{n+1})\\ &+ \delta[\psi_1(\theta_{n-1}, \theta_n)\zeta(\theta_{n-1}, \theta_n) + \psi_2(\theta_n, \theta_{n+1})\zeta(\theta_n, \theta_{n+1})]\\ &+ \omega[\psi_1(\theta_n, \theta_{n+1})\zeta(\theta_n, \theta_{n+1}) + \psi_2(\theta_{n+1}, \theta_n)\zeta(\theta_{n+1}, \theta_n)].\end{aligned} \tag{3.5}$$

Rearranging inequality (3.5), we get

$$\begin{aligned}&(1-\gamma-\delta\psi_2(\theta_n, \theta_{n+1}) - \omega\psi_1(\theta_n, \theta_{n+1}) - \omega\psi_2(\theta_{n+1}, \theta_n))\zeta(\theta_n, \theta_{n+1})\\ &\qquad \leq [\alpha+\beta+\delta\psi_1(\theta_{n-1}, \theta_n)]\zeta(\theta_{n-1}, \theta_n).\end{aligned} \tag{3.6}$$

By the condition (3.1) of Theorem 3.1, we have

$$1-\gamma-\delta\psi_2(\theta_n, \theta_{n+1}) - \omega\psi_1(\theta_n, \theta_{n+1}) - \omega\psi_2(\theta_{n+1}, \theta_n) > 0,$$

and thus we obtain

$$\zeta(\theta_n, \theta_{n+1}) \leq \frac{\alpha+\beta+\delta\psi_1(\theta_{n-1}, \theta_n)}{1-\gamma-\delta\psi_2(\theta_n, \theta_{n+1}) - \omega\psi_1(\theta_n, \theta_{n+1}) - \omega\psi_2(\theta_{n+1}, \theta_n)}\zeta(\theta_{n-1}, \theta_n). \tag{3.7}$$

For convenience, we let

$$R_n = \frac{\alpha+\beta+\delta\psi_1(\theta_{n-1}, \theta_n)}{1-\gamma-\delta\psi_2(\theta_n, \theta_{n+1}) - \omega\psi_1(\theta_n, \theta_{n+1}) - \omega\psi_2(\theta_{n+1}, \theta_n)}, \tag{3.8}$$

so that we have

$$\begin{aligned}\zeta(\theta_n,\theta_{n+1}) &\le R_n\zeta(\theta_{n-1},\theta_n)\\ &\le R_nR_{n-1}\zeta(\theta_{n-2},\theta_{n-1})\\ &\le R_nR_{n-1}R_{n-2}\zeta(\theta_{n-3},\theta_{n-2})\\ &\vdots\\ &\le R_nR_{n-1}R_{n-2}\dots R_0d(\theta_0,\theta_1)\\ &\le \prod_{k=0}^{n} R_{n-k}d(\theta_0,\theta_1).\end{aligned} \tag{3.9}$$

We further let

$$P_n=\prod_{k=0}^{n} R_{n-k}, \tag{3.10}$$

so that inequality (3.9) becomes

$$\zeta(\theta_n,\theta_{n+1})\le P_n\zeta(\theta_0,\theta_1). \tag{3.11}$$

Now, for all $m,n\in\mathbb{N}$ with $m>n$, we have

$$\begin{aligned}\zeta(\theta_n,\theta_m) &\le \psi_1(\theta_n,\theta_{n+1})\zeta(\theta_n,\theta_{n+1})+\psi_2(\theta_{n+1},\theta_m)\zeta(\theta_{n+1},\theta_m)\\ &\le \psi_1(\theta_n,\theta_{n+1})\zeta(\theta_n,\theta_{n+1})\\ &+\psi_2(\theta_{n+1},\theta_m)\psi_1(\theta_{n+1},\theta_{n+2})\zeta(\theta_{n+1},\theta_{n+2})\\ &+\psi_2(\theta_{n+1},\theta_m)\psi_2(\theta_{n+2},\theta_m)\zeta(\theta_{n+2},\theta_m)\\ &\le \psi_1(\theta_n,\theta_{n+1})\zeta(\theta_n,\theta_{n+1})\\ &+\psi_2(\theta_{n+1},\theta_m)\psi_1(\theta_{n+1},\theta_{n+2})\zeta(\theta_{n+1},\theta_{n+2})\\ &+\psi_2(\theta_{n+1},\theta_m)\psi_2(\theta_{n+2},\theta_m)\psi_1(\theta_{n+2},\theta_{n+3})\zeta(\theta_{n+2},\theta_{n+3})\\ &+\psi_2(\theta_{n+1},\theta_m)\psi_2(\theta_{n+2},\theta_m)\psi_1(\theta_{n+3},\theta_m)\zeta(\theta_{n+3},\theta_m)\\ &\vdots\\ &\le \psi_1(\theta_n,\theta_{n+1})\zeta(\theta_n,\theta_{n+1})\\ &+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\psi_2\left(\theta_j,\theta_m\right)\right)\psi_1(\theta_i,\theta_{i+1})\zeta(\theta_i,\theta_{i+1})\\ &+\prod_{i=n+1}^{m-1}\psi_2(\theta_i,\theta_m)\zeta(\theta_{m-1},\theta_m).\end{aligned} \tag{3.12}$$

Using inequality (3.11) in (3.12), we get

$$\begin{aligned}
\zeta(\theta_n,\theta_m) &\leq \psi_1(\theta_n,\theta_{n+1}) P_n \zeta(\theta_0,\theta_1) \\
&\quad + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\psi_2\left(\theta_j,\theta_m\right)\right)\psi_1(\theta_i,\theta_{i+1}) P_i \zeta(\theta_0,\theta_1) \\
&\quad + \prod_{i=n+1}^{m-1}\psi_2(\theta_i,\theta_m) P_{m-1}\zeta(\theta_0,\theta_1) \\
&\leq \psi_1(\theta_n,\theta_{n+1}) P_n \zeta(\theta_0,\theta_1) \\
&\quad + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\psi_2\left(\theta_j,\theta_m\right)\right)\psi_1(\theta_i,\theta_{i+1}) P_i \zeta(\theta_0,\theta_1).
\end{aligned} \tag{3.13}$$

We used the fact that $\psi_1(\mu,\tau) \geq 1$ and $\psi_2(\mu,\tau) \geq 1$. Let

$$\mathcal{S}_p = \sum_{i=0}^{p}\left(\prod_{j=0}^{i}\psi_2\left(\theta_j,\theta_m\right)\right)\psi_1(\theta_i,\theta_{i+1}) P_i \zeta(\theta_0,\theta_1). \tag{3.14}$$

Therefore, inequality (3.13) becomes,

$$\zeta(\theta_n,\theta_m) \leq [\psi_1(\theta_n,\theta_{n+1}) P_n + (\mathcal{S}_{m-1} - \mathcal{S}_n)]\zeta(\theta_0,\theta_1). \tag{3.15}$$

Now, consider

$$G_i = \left(\prod_{j=0}^{n}\psi_2\left(\theta_j,\theta_m\right)\right)\psi_1(\theta_i,\theta_{i+1}) P_i \zeta(\theta_0,\theta_1), \tag{3.16}$$

so that we have

$$\frac{G_{i+1}}{G_i} = \frac{P_{i+1}}{P_i}\psi_2(\theta_{i+1},\theta_m)\frac{\psi_1(\theta_{i+1},\theta_{i+2})}{\psi_1(\theta_i,\theta_{i+1})}. \tag{3.17}$$

We further have

$$\begin{aligned}
\frac{P_{i+1}}{P_i} &= \frac{\prod_{k=0}^{i+1} R_{i+1-k}}{\prod_{k=0}^{i} R_{i-k}} \\
&= \frac{R_{i+1}R_iR_{i-1}R_{i-2}R_{i-2}\ldots R_1R_0}{R_iR_{i-1}R_{i-2}R_{i-2}\ldots R_1R_0} \\
&= R_{i+1} \\
&= \frac{\alpha+\beta+\delta\psi_1(x_i,\theta_{i+1})}{1-\gamma-\delta\psi_2(\theta_{i+1},\theta_{i+2})-\omega\psi_1(\theta_{i+1},\theta_{i+2})-\omega\psi_2(\theta_{i+2},\theta_{i+1})}.
\end{aligned} \tag{3.18}$$

Using Eq (3.18) in (3.17), we obtain

$$\frac{G_{i+1}}{G_i} = \frac{[\alpha+\beta+\delta\psi_1(\theta_i,\theta_{i+1})]\psi_2(\theta_{i+1},\theta_m)}{1-\gamma-\delta\psi_2(\theta_{i+1},\theta_{i+2})-\omega\psi_1(\theta_{i+1},\theta_{i+2})-\omega\psi_2(\theta_{i+2},\theta_{i+1})}\frac{\psi_1(\theta_{i+1},\theta_{i+2})}{\psi_1(\theta_i,\theta_{i+1})}. \tag{3.19}$$

By the condition (3.1) of Theorem 3.1, we have $\frac{G_{i+1}}{G_i} < 1$. Therefore, by the ratio test, the limit of the sequence $\{\mathcal{S}_n\}$ exists finitely, which implies that $\{\mathcal{S}_n\}$ is Cauchy.

By condition (3.1), we further obtain

$$\begin{aligned}\lim_{n\to\infty}\psi_1(\theta_n,\theta_{n+1})P_n &= \lim_{n\to\infty}\psi_1(\theta_n,\theta_{n+1})\prod_{k=0}^{n}R_{n-k}\\ &= \lim_{n\to\infty}\prod_{k=0}^{n}\frac{\psi_1(\theta_n,\theta_{n+1})(\alpha+\beta+\delta\psi_1(x_{n-k-1},\theta_{n-k}))}{1-\gamma-\delta\psi_2(\theta_{n-k},\theta_{n-k+1})-\omega\psi_1(\theta_{n-k},\theta_{n-k+1})-\omega\psi_2(\theta_{n-k+1},\theta_{n-k})}=0.\end{aligned} \tag{3.20}$$

Letting m, n tend to infinity in (3.15), we get

$$\lim_{m,n\to\infty}\zeta(\theta_n,\theta_m)=0. \tag{3.21}$$

So, the sequence $\{\theta_n\}$ is Cauchy. Since $\mathcal{X}$ is a complete double controlled metric-like space, there exists $s \in \mathcal{X}$ such that $\{\theta_n\}$ converges to s, so

$$\lim_{n\to\infty}\zeta(\theta_n,s)=0. \tag{3.22}$$

Next, we prove that s is a fixed point of F, that is, $F(s) = s$.

$$\begin{aligned}\zeta(s,Fs) &\le \psi_1(s,\theta_{n+1})\zeta(s,\theta_{n+1})+\psi_2(\theta_{n+1},Fs)\zeta(\theta_{n+1},Fs)\\ &= \psi_1(s,\theta_{n+1})\zeta(s,\theta_{n+1})+\psi_2(\theta_{n+1},Fs)\zeta(F\theta_n,Fs)\\ &\le \psi_1(s,\theta_{n+1})\zeta(s,\theta_{n+1})+\psi_2(\theta_{n+1},Fs)[\alpha\zeta(\theta_n,s)+\beta\zeta(\theta_n,F\theta_n)\\ &+\gamma\zeta(s,Fs)+\delta\zeta(\theta_n,Fs)+\omega\zeta(s,F\theta_n)]\\ &= \psi_1(s,\theta_{n+1})\zeta(s,\theta_{n+1})+\psi_2(\theta_{n+1},Fs)[\alpha\zeta(\theta_n,s)+\beta\zeta(\theta_n,\theta_{n+1})\\ &+\gamma\zeta(s,Fs)+\delta\zeta(\theta_n,Fs)+\omega\zeta(s,\theta_{n+1})]\\ &= \psi_1(s,\theta_{n+1})\zeta(s,\theta_{n+1})+\psi_2(\theta_{n+1},Fs)[\alpha\zeta(\theta_n,s)+\beta\zeta(\theta_n,\theta_{n+1})\\ &+\gamma\zeta(s,Fs)+\delta[\psi_1(\theta_n,s)\zeta(\theta_n,s)+\psi_2(s,Fs)\zeta(s,Fs)]\\ &+\omega\zeta(s,\theta_{n+1})].\end{aligned} \tag{3.23}$$

Using the condition (3.1), and letting n tend to infinity in (3.23), we obtain

$$\zeta(s,Fs)\le\lim_{n\to\infty}\psi_2(\theta_{n+1},Fs)(\gamma+\delta)\zeta(s,Fs). \tag{3.24}$$

Suppose that $F(s) \neq s$. By the condition (3.1) of Theorem 3.1, we have $\lim_{n\to\infty}\psi_2(\theta_{n+1},Fs) < \frac{1}{\gamma+\delta}$. Thus the inequality (3.24) implies

$$0<\zeta(s,Fs)<\zeta(s,Fs), \tag{3.25}$$

which is a contradiction. This implies that $F(s) = s$.

Let $\alpha + \delta + \omega \neq 1$ and $\beta = \gamma = 0$. Suppose that F has two fixed points, s and t. Then,

$$\begin{aligned}\zeta(s,t)=\zeta(Fs,Ft)&\le\alpha\zeta(s,t)+\beta\zeta(s,Fs)+\gamma\zeta(t,Ft)+\delta\zeta(s,Ft)+\omega\zeta(t,Fs)\\ &=\alpha\zeta(s,t)+\beta\zeta(s,s)+\gamma\zeta(t,t)+\delta\zeta(s,t)+\omega\zeta(t,s),\end{aligned} \tag{3.26}$$

which implies

$$(1-\alpha-\delta-\omega)\zeta(s,t) \leq \beta\zeta(s,s)+\gamma\zeta(t,t). \tag{3.27}$$

With $\alpha+\delta+\omega \neq 1$ and $\beta=\gamma=0$, the inequality (3.27) implies that $\zeta(s,t)=0$, which further implies that $s=t$. □

Next, we state some special cases of Theorem 3.1.

Corollary 3.1. *With $\delta=\omega=0$ in Theorem 3.1, we obtain Theorem 2.3 for the Reich contraction. Note that the condition (3.1) of Theorem 3.1 reduces to $\lim_{n\to\infty}\left(\frac{\alpha+\beta}{1-\gamma}\right)^n = 0$, which is used in the proof of Theorem 2.3.*

Corollary 3.2. *With $\beta=\gamma=\delta=\omega=0$ in Theorem 3.1, we obtain Theorem 2.1 for the Banach contraction.*

Corollary 3.3. *With $\alpha=\delta=\omega=0, \beta=\gamma\in[0,\frac{1}{2})$ in Theorem 3.1, we obtain Theorem 2.2 for the Kannan contraction.*

Corollary 3.4. *Since DCMLS are a generalization of DCMTS, Theorem 3.1 holds in DCMTS as well.*

4. Applications

Finally, we provide few applications of our proven result.

Example 4.1. *Let $\mathcal{X}=\{3,1,5\}$. Consider a map $\zeta:\mathcal{X}\times\mathcal{X}\longrightarrow[0,\infty)$ defined by:*

$\zeta(\mu,\tau)$	3	1	5
3	0	10	6
1	10	0	4
5	6	4	1

Given $\psi_1,\psi_2:\mathcal{X}\times\mathcal{X}\longrightarrow[1,+\infty)$ as

$\psi_1(\mu,\tau)$	3	1	5
3	1	$\frac{11}{10}$	1
1	$\frac{11}{10}$	1	$\frac{6}{5}$
5	1	$\frac{6}{5}$	1

and

$\psi_2(\mu,\tau)$	3	1	5
3	1	$\frac{11}{10}$	$\frac{7}{6}$
1	$\frac{11}{10}$	1	1
5	$\frac{7}{6}$	1	1

it is easy to verify that $(\mathcal{X},\zeta)$ is a complete double controlled metric like space with control functions ψ_1 and ψ_2.

Define a function $F:\mathcal{X}\to\mathcal{X}$ by

$$F\mu = \begin{cases} 2, & \text{if } \mu=3, \\ 1, & \text{if } \mu\in\{1,5\}. \end{cases}$$

Then, F has a fixed point.

Proof. We take $\alpha = \frac{1}{5}, \beta = \frac{1}{3}, \gamma = \frac{1}{20}, \delta = \frac{1}{20}, \omega = \frac{1}{30}$. Now, consider the following three cases to prove the condition (3.1) of Theorem 3.1

Case 1. $\mu = 3, \tau = 1, \zeta(F3, F1) = \zeta(5,1) = 4 \leq \frac{77}{15} = \frac{1}{5}\zeta(3,1)+\frac{1}{3}\zeta(3,5)+\frac{1}{10}\zeta(1,1)+\frac{1}{10}\zeta(3,1)+\frac{1}{30}\zeta(1,5)$.

Case 2. $\mu = 3, \tau = 5, \zeta(F3, F5) = \zeta(5,1) = 4 < \frac{139}{30} = \frac{1}{5}\zeta(3,5)+\frac{1}{3}\zeta(3,5)+\frac{1}{10}\zeta(5,1)+\frac{1}{10}\zeta(3,1)+\frac{1}{30}\zeta(5,5)$.

Case 3. $\mu = 1, \tau = 5, \zeta(F1, F5) = \zeta(1,1) = 0 < \frac{11}{5} = \frac{1}{5}\zeta(1,5)+\frac{1}{3}\zeta(1,1)+\frac{1}{10}\zeta(5,1)+\frac{1}{10}\zeta(1,1)+\frac{1}{30}\zeta(5,1)$.

Let $\theta_0 = 1 \in \mathcal{X}$. Then, $\theta_n = F^n\theta_0 = 1$ for all $n \geq 1$. Therefore, we have $\psi_1(\theta_i, \theta_j) = \psi_1(1,1) = 1$ and $\psi_2(\theta_i, \theta_j) = \psi_2(1,1) = 1$ for all indices i and j. The condition (3.1) of Theorem 3.1 becomes

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha+\beta+\delta}{1-\gamma-\delta-\omega-\omega} = \frac{19}{22} < 1.$$

The condition (3.1) of Theorem 3.1 becomes

$$\lim_{n\to\infty}\left(\frac{\alpha+\beta+\delta}{1-\gamma-\delta-\omega-\omega}\right)^n = \lim_{n\to\infty}\left(\frac{19}{22}\right)^n = 0.$$

The condition (3.1) of Theorem 3.1 becomes

$$1-\gamma-\delta\psi_2(\theta_n,\theta_{n+1})-\omega\psi_1(\theta_n,\theta_{n+1})-\omega\psi_2(\theta_{n+1},\theta_n) = \frac{11}{15} \neq 0.$$

It is easy to verify that for every $\mu \in \mathcal{X}$, $\lim_{n\to\infty}\psi_1(\mu,\theta_n) = \psi_1(\mu,1)$, $\lim_{n\to\infty}\psi_1(\theta_n,\mu) = \psi_1(1,\mu)$, $\lim_{n\to\infty}\psi_2(\mu,\theta_n) = \psi_2(\mu,1)$ exist finitely, and $\lim_{n\to\infty}\psi_2(\theta_n,\mu) = \psi_2(1,\mu) < \frac{1}{\gamma+\delta} = 5$ for every $\mu \in \mathcal{X}$. Therefore, all the conditions of Theorem 3.1 are satisfied, and we conclude that F has a fixed point given by $\mu = 1$. □

Example 4.2. *Consider the space of all continuous real valued functions $\mathcal{X} = C[0,1]$, and $\zeta(r(\mu), h(\mu)) : \mathcal{X}\times\mathcal{X} \longrightarrow [0,+\infty)$ is defined as*

$$\zeta(r(\mu), h(\mu)) = \sup_{\mu\in[0,1]}|r(\mu)-h(\mu)|^2.$$

Define the control functions $\psi_1, \psi_2 : \mathcal{X}\times\mathcal{X} \to [1,+\infty)$ by

$$\psi_1(r(\mu), h(\mu)) = 1 + \sup_{\mu\in[0,1]}|r(\mu)h(\mu)|$$

and

$$\psi_2(r(\mu), h(\mu)) = 1, \textit{for all } r, h \in \mathcal{X}.$$

It is not difficult to see that $(\mathcal{X}, \zeta)$ is a complete double controlled metric like space.

Theorem 4.1. *Let $\mathcal{X} = C[0,1]$ be the complete double controlled metric like space given in Example 4.2. Consider the following Fredholm integral equation:*

$$r(\mu) = \int_0^1 l(\mu, \omega, r(\mu))d\omega, \tag{4.1}$$

where $l(\mu,\omega,r(\mu)) : [0,1]\times[0,1] \longrightarrow \mathbb{R}$ is a given continuous function satisfying the following condition for all $r(\mu), h(\mu) \in \mathcal{X}, \mu, \omega \in [0,1]$:

$$|l(\mu,\omega,r(\mu)) - l(\mu,\omega,h(\mu))| \leq \sqrt{H(\mu)}$$

where

$$H(\mu) = \alpha d(r(\mu), h(\mu)) + \beta d(r(\mu), Fr(\mu)) + \gamma d(h(\mu), Fh(\mu)),$$

$F(r(\mu)) = \int_0^1 l(\mu, \omega, r(\mu))d\omega$, *and* $\alpha, \beta, \gamma, \in [0, 1), \alpha + \beta + \gamma < 1$.

Then, the integral Eq (4.1) *has a unique solution.*

Proof. Let $F : C[0, 1] \longrightarrow C[0, 1]$ be defined by $F(r(\mu)) = \int_0^1 l(\mu, \omega, r(\mu))d\omega$, and then

$$\begin{aligned}
\zeta(Fr(\mu), Fh(\mu)) &= \sup_{\mu\in[0,1]} |Fr(\mu) - Fh(\mu)|^2 \\
&= \sup_{\mu\in[0,1]} |\int_0^1 l(\mu, \omega, r(\mu))d\omega - \int_0^1 l(\mu, \omega, h(\mu))d\omega|^2 \\
&\leq \sup_{\mu\in[0,1]} \int_0^1 |l(\mu, \omega, r(\mu))d\omega - l(\mu, \omega, h(\mu))|^2 d\omega \\
&\leq \sup_{\mu\in[0,1]} \int_0^1 | \sqrt{H(\mu)}|^2 d\omega \\
&\leq \sup_{\mu\in[0,1]} |H(\mu)| \int_0^1 d\omega \\
&\leq \sup_{\mu\in[0,1]} H(\mu) \\
&\leq \alpha d(r(\mu), h(\mu)) + \beta d(r(\mu), Fr(\mu)) + \gamma d(h(\mu), Fh(\mu)).
\end{aligned} \tag{4.2}$$

It is not difficult to verify the other conditions of Theorem 3.1. Therefore, there is a function $r \in C[0, 1]$ such that $Fr = r$. This implies that the integral Eq (4.1) has a solution. □

5. Conclusions

In this paper, we have established a Hardy-Rogers type contraction mapping theorem in the setting of double controlled metric like spaces. We have obtained some of the classical results as a special case of our proven result. Following that, we presented an example to demonstrate the veracity of our main result. Given that the study of various contraction mappings and the study of generalized metric spaces are two key research fields in fixed point theory, we propose some open problems for future work.

i) Consider replacing the Hardy-Rogers contraction given by condition (3.1) with some non-trivial rational contraction generalizing other types of contractions, like Ciric contraction, Jaggi and Das type contraction, etc.
ii) Establishing Theorem (3.1) in other generalized metric spaces like double controlled quasi metric spaces, M-metric spaces, triple controlled metric spaces, rectangular metric spaces, and so on.
iii) Consider placing the constants $\alpha, \beta, \gamma, \delta, \omega$ in Theorem 3.1 with some special functions.
iv) Establishing new and non-trivial applications of Theorem 3.1.

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Conflict of interest

The authors declare that they have no competing interests.

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