AIMS Mathematics, 8(6): 13593–13622.
DOI:10.3934/math.2023690
Received: 16 February 2023
Revised: 20 March 2023
Accepted: 20 March 2023
Published: 10 April 2023

Mathematics

http://www.aimspress.com/journal/Math

Research article

Concentration of solutions for double-phase problems with a general nonlinearity

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Abstract: In this paper, we study the following problems with a general nonlinearity:

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), & \text{in } \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $2 \le p < q < N$, the potential V is a positive continuous function having a local minimum. $f : \mathbb{R} \to \mathbb{R}$ is a C^1 subcritical nonlinearity. Under some proper assumptions of V and f, we obtain the concentration of positive solutions with the local minimum of V by applying the penalization method for above equation. We must note that the monotonicity of $\frac{f(s)}{s^{p-1}}$ and the so-called Ambrosetti-Rabinowitz condition are not required.

Keywords: double-phase problems; penalization method; variational methods
Mathematics Subject Classification: 35A15, 35B38, 35J60

1. Introduction and main results

In this paper, we investigate the concentration of positive solutions for the following double-phase problems with a general nonlinearity:

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $2 \le p < q < N$, the potential V is a positive continuous function having a local minimum. $f : \mathbb{R} \to \mathbb{R}$ is a C^1 subcritical nonlinearity.

The content of this paper is closely related to the double phase problems, we briefly introduce the development of this research. It is common knowledge that the first contributions to this field are due to Ball [9], in relationship with problems in nonlinear elasticity and composite materials. The double-phase problem (1.1) is motivated by numerous local and nonlocal models arising in mathematical physics. For example, we can refer to Born-Infeld equation [11, 12] which appears in electromagnetism, electrostatics and electrodynamics as a model based on a modification of Maxwell's Lagrangian density:

$$-\operatorname{div}\left(\frac{\nabla u}{(1-2|\nabla u|^2)^{1/2}}\right) = h(u) \text{ in } \Omega.$$

In fact, according to the Taylor formula, we obtain that

$$(1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3}{2\cdot 2^2}x^2 + \frac{5!!}{3!\cdot 2^3}x^3 + \cdots + \frac{(2n-3)!!}{(n-1)!2^{n-1}}x^{n-1} + \cdots \text{ for } |x| < 1.$$

Taking $x = 2|\nabla u|^2$ and adopting the first order approximation, we get a particular case of the problem (1.1) for $p = 2$ and $q = 4$.

When $p = q$, the problem (1.1) becomes p-Laplace equation:

$$-\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u = f(u) \text{ in } \mathbb{R}^N. \tag{1.2}$$

Elliptic problems like (1.2), in the semilinear case which corresponds to $p = 2$, arise from the problem of obtaining standing waves for the nonlinear Schrödinger equations given by

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2}\Delta\psi - V(x)\psi + f(\psi) = 0, \quad (t, x) \in \mathbb{R}\times\mathbb{R}^N,$$

where i is the imaginary unit and $\hbar$ is the Planck constant. Further backgrounds for these equations can be found in [13, 41] and references therein. Gloss in [23] studied existence and asymptotic behavior of positive solutions by using penalization for quasilinear elliptic equations of (1.2). Note that, f is a subcritical nonlinearity without Ambrosetti-Rabinowitz condition((AR) condition in short):

$$0 < \theta\int_0^u f(s)\mathrm{d}s \le f(u)u \text{ for } \theta \in (p, p^*).$$

In [21], by using a variational approach based on the local mountain-pass theorem, the author proved the existence and concentration of positive bound states of the equation involving critical growth: $f(u) = g(u) + u^{p^*-1}$ in (1.2). However this g requires the (AR) condition and the monotonicity of $\frac{f(s)}{s^{p-1}}$. In [24], He and Li studied the following elliptic problem:

$$\begin{cases} -\varepsilon^p\Delta_p u + V(z)|u|^{p-2}u - f(u) = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \quad u > 0 \text{ in } \Omega, \quad N > p > 2, \end{cases}$$

where Ω is a possibly unbounded domain in $\mathbb{R}^N$ with empty or smooth boundary, ε is a small parameter. $f \in C^1(\mathbb{R}^+, \mathbb{R})$ is of subcritical and $V : \mathbb{R}^N \to \mathbb{R}$ is a locally Hölder continuous function. As a result, they obtained the existence and concentration of weak solutions by penalization method.

When $\varepsilon = 1$ in problem (1.1), the main interest in this general class of problems has been due to the fact that they arise from applications in physics and related sciences, such as biophysics, plasma physics and chemical reaction, see for example [16]. In last decade, many authors paid their attention to seek positive solutions, bounded states solutions, multiple solutions, see for instance [15, 33, 36, 43] and the references therein. Perera-Squassina [37] studied double phase problems and stated an existence result which was proved under different conditions by using Morse theory in terms of critical groups. The corresponding eigenvalue problem of the double phase operator with Dirichlet boundary condition was analyzed by Colasuonno-Squassina [17] who proved the existence and properties of related variational eigenvalues. According to variational methods, Liu-Dai [34] treated double phase problems and proved existence and multiplicity results.

Ambrosio in [4] dealt with the following problem

$$(-\Delta)_p^s u + (-\Delta)_q^s u + |u|^{p-2}u + |u|^{q-2}u = \lambda h(x) f(u) + |u|^{q^*-2}u \qquad \text{in } \mathbb{R}^N.$$

Using suitable variational arguments and concentration-compactness lemma, the authors proved the existence of a nontrivial non-negative solution for λ sufficiently large. Note that [4] dealt with the constant potential, and then in [27], under proper assumptions, Isernia proved the existence of a positive solution and a negative ground state solution for the following class of fractional $p\&q$-Laplacian problems with potentials vanishing at infinity:

$$(-\Delta)_p^s u + (-\Delta)_q^s u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = K(x) f(u) \quad \text{in } \mathbb{R}^N.$$

Recently, Alves, Ambrosio and Isernia [1] by applying minimax theorems and the Ljusternik-Schnirelmann theory, they investigated the existence, multiplicity and concentration of nontrivial solutions for (1.1) provided that ε is sufficiently small. Costa and Figueiredo [19] used Mountain Pass Theorem and the penalization arguments introduced by Del Pino and Felmer's associated to Lions' Concentration and Compactness Principle to overcome the lack of compactness, and then showed existence and concentration results for (1.1). Ambrosio and Rădulescu in [7] considered the following class of fractional problems with unbalanced growth:

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)\left(|u|^{p-2}u + |u|^{q-2}u\right) = \lambda f(u), & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}\left(\mathbb{R}^N\right) \cap W^{s,q}\left(\mathbb{R}^N\right), u > 0, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

Applying suitable variational and topological arguments, they obtained multiple positive solutions for $\varepsilon > 0$ that were sufficiently small as well as related concentration properties, in relationship with the set where the potential V attains its minimum. In [45], Zhang et al. investigated the following perturbed double phase problem with competing potentials:

$$\begin{cases} -\epsilon^p \Delta_p u - \epsilon^q \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = K(x) f(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \ u > 0, & \text{in } \mathbb{R}^N, \end{cases}$$

where $(1 < p < q < N)$. The authors assumed that the potentials V, K and the nonlinearity f are continuous but are not necessarily of class C^1. Under some natural hypotheses, using topological and variational tools from Nehari manifold analysis and Ljusternik-Schnirelmann category theory, they studied the existence of positive ground state solutions. Moreover, they determined two concrete sets related to the potentials V and K as the concentration positions and described the concentration of ground state solutions as $\epsilon \to 0$. Zhang et al. in [49] studied the following double phase problem

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)\left(|u|^{p-2}u + |u|^{q-2}u\right) = \lambda f(u) + |u|^{q_s^*-2}u, & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}\left(\mathbb{R}^N\right) \cap W^{s,q}\left(\mathbb{R}^N\right), u > 0, & \text{in } \mathbb{R}^N. \end{cases}$$

They established the existence of multiple positive solutions as well as related concentration properties. In [48], Zhang et al. considered the singularly perturbed double phase problems with nonlocal reaction, they got the concentration result. For more results of the existence and concentration of solutions, we refer to [8,30,46,47] and the references therein.

It is worth mentioning that all the works above assumed that the nonlinearity satisfied Ambrosetti-Rabinowitz condition, so the authors can use Nehari manifold to obtain the concentration and multiplicity properties of solutions. [7,49] are associated with fractional, but this it also true for $s = 1$. On the other hand, the nonlinearities in these equations are not general, so by being motivated by the above works, it is quite natural to ask if $f(u)$ is a general nonlinearity which satisfies Beresticky-Lions type assumptions, does the same results established for double phase problem? In the present paper, we give an affirmative answer to this question.

Before stating our main result, we shall introduce the main hypotheses. Assume that the potential $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function fulfilling the following conditions which are always called del Pino-Felmer type [20] conditions.

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ such that $V_1 := \inf_{x\in\mathbb{R}^N} V(x) > 0$.

(V_2) There exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 := \inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x)$$

with $V_0 > 0$, and $0 \in \mathcal{M} := \{x \in \Lambda : V(x) = V_0\}$.

Moreover, the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(t) = 0$ for $t \leq 0$, and satisfies the following hypotheses:

(f_1) $\lim_{t\to 0} \frac{f(t)}{t^{p-1}} = 0$;

(f_2) There exists $\nu \in (q, q^*)$ such that $\lim_{|t|\to+\infty} \dfrac{f(t)}{t^{\nu-1}} < \infty$;

(f_3) There exists $T > 0$ such that $F(T) > \dfrac{V_0}{p}T^p + \dfrac{V_0}{q}T^q$.

Theorem 1.1. *Assume that* $(V_1) - (V_2)$ *and* $(f_1) - (f_3)$ *are satisfied. Then, for small* $\varepsilon > 0$, *there exists a positive solution* u_ε *to* (1.1) *such that* u_ε *has a maximum point* x_ε *satisfying*

$$\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathcal{M}) = 0$$

and for any such x_ε, the function $v_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$ converges uniformly as $\varepsilon \to 0$ (up to a subsequence) to a least energy solution of

$$\begin{cases} -\Delta_p u - \Delta_q u + V_0(|u|^{p-2}u + |u|^{q-2}u) = f(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover, we have

$$u_\varepsilon(x) \leq C_1 e^{-C_2|x-x_\varepsilon|} \text{ for all } x \in \mathbb{R}^N,\ C_1,\ C_2 > 0.$$

We note that, to the best of our knowledge, there is no result on the existence and concentration of positive bound state solutions for double-phase problems with Berestycki-Lions nonlinearity.

We use a truncation approach to aprove our result. The main difficulties in the proof of Theorem 1.1 lie in two aspects:

(1) The nonlinearity $f(u)$ does not satisfy (AR) condition and the fact that the function $\frac{f(u)}{u^{q-1}}$ is not increasing for $u > 0$ prevent us from obtaining a bounded Palais-Smale sequence and using the Nehari manifold, respectively. Moreover, the arguments in [20] can not be applied in this paper;

(2) The unboundedness of the domain $\mathbb{R}^N$ leads to the lack of compactness.

As we will see later, the above two aspects prevent us from using the variational method in a standard way. In order to get over the above two difficulties, inspired by [13, 25], we recover the compactness by penalization method which was first introduced in [14].

The plan of this paper is the following. In Section 2, we define some function spaces. Section 3 is devoted to study ground state solution for the limit problem of (1.1), and we give the proof of Theorem 1.1 in the last section.

2. Variational setting

In this section, we fix the notations and recall some results for the uses later.

Let $u : \mathbb{R}^N \mapsto \mathbb{R}$. For $1 < p < q$, let us define $D^{1,p}(\mathbb{R}^N) = \overline{C^\infty(\mathbb{R}^N)}^{|\nabla \cdot|_p}$. We denote the following fractional Sobolev space

$$W^{1,p}(\mathbb{R}^N) := \{u : |u|_p < +\infty, |\nabla u|_p < +\infty\}$$

equipped with the natural norm

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} :=\Big(|\nabla u|_p^p + |u|_p^p\Big)^{1/p},$$

where $|\cdot|_p^p := \int_{\mathbb{R}^N} |\cdot|^p \mathrm{d}x$.

For all $u, v \in W^{1,p}(\mathbb{R}^N)$,

$$\langle u, v\rangle_{W^{1,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \Big(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\Big)\,\mathrm{d}x.$$

In this work we need to introduce the following Banach space

$$X = W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$$

equipped with the norm

$$\|u\|_X := \|u\|_{W^{1,p}(\mathbb{R}^N)} + \|u\|_{W^{1,q}(\mathbb{R}^N)}.$$

Note that $W^{1,r}(\mathbb{R}^N)$ is a separable reflexive Banach space for all $r \in (1, +\infty)$, and so X is a separable reflexive Banach space.

For any fixed $\varepsilon \geq 0$, we also introduce the following Banach space

$$X_\varepsilon := \left\{u \in X : \int_{\mathbb{R}^N} V(\varepsilon x)(|u|^p + |u|^q)\mathrm{d}x < +\infty\right\}$$

equipped with the norm

$$\|u\|_{X_\varepsilon} := \|u\|_{V_\varepsilon,p} + \|u\|_{V_\varepsilon,q},$$

where $\|u\|_{V_\varepsilon,t}^t = \int_{\mathbb{R}^N} (|\nabla u|^t + V(\varepsilon x)|u|^t)\, \mathrm{d}x$ for all $t \in \{p, q\}$. When $V(x) = V_0$, we denote the following Banach space

$$X_0 := \left\{ u \in X : \int_{\mathbb{R}^N} V_0(|u|^p + |u|^q)\mathrm{d}x < +\infty \right\}$$

equipped with the norm

$$\|u\|_{X_0} := \|u\|_{V_0,p} + \|u\|_{V_0,q},$$

where $\|u\|_{V_0,t}^t = \int_{\mathbb{R}^N} (|\nabla u|^t + V_0|u|^t)\, \mathrm{d}x$ for all $t \in \{p, q\}$. Finally, we consider

$$X_{\mathrm{rad}}\left(\mathbb{R}^N_+\right) := \{u \in X_0 : u(x) = u(|x|)\}.$$

Lemma 2.1. (see [42, Theorem 2.8]) (General Minimax Principle) *Let X be a Banach space. Let M_0 be a closed subspace of the metric space M and $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma := \{\gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0\}.$$

If $\varphi \in C^1(X, \mathbb{R})$ satisfies

$$a := \sup_{\gamma_0 \in \Gamma_0} \sup_{u \in M_0} \varphi(\gamma_0(u)) < c := \inf_{\gamma \in \Gamma} \sup_{u \in M} \varphi(\gamma(u)) < \infty,$$

then, for every $\varepsilon \in (0, (c-a)/2), \delta > 0$ and $\gamma \in \Gamma$ such that $\sup_M \varphi \circ \gamma \le c + \varepsilon$, there exists $u \in X$ such that

(a) $c - 2\varepsilon \le \varphi(u) \le c + 2\varepsilon$,
(b) $\mathrm{dist}(u, \gamma(M)) \le 2\delta$,
(c) $\|\varphi'(u)\| \le \frac{8\varepsilon}{\delta}$.

3. The limiting problem

First of all, in order to make functional of the limiting problem equation to be C^1 and let it is a meaningful functional on X_0, we modify f as in [10]. Let $\widehat{f} : \mathbb{R} \to \mathbb{R}$ be define as follows:

(i) if $f(t) > 0$ for all $t \ge \widehat{T}$, put $\widehat{f}(t) := f(t)$,
(ii) if there exists $\tau_0 \ge \widehat{T}$ such that $f(\tau_0) = 0$, we put

$$\widehat{f}(t) := \begin{cases} f(t), & \text{for } t < \tau_0, \\ 0, & \text{for } t \ge \tau_0, \end{cases}$$

where $\widehat{T} := \sup\{t \in [0, T] : f(t) > V_0 t^{p-1} + V_0 t^{q-1}\}$.

It is clear that $\widehat{f}$ satisfies the same assumptions as f and

$$0 \le \liminf_{t\to\infty} \frac{\widehat{f}(t)}{t^p} \le \limsup_{t\to\infty} \frac{\widehat{f}(t)}{t^p} < \infty.$$

At the same time, note that, if (ii) occurs and u is a solution to (1.1) with $\widehat{f}(t)$, we can use $(u-\tau_0)_+$ as test function to obtain that $u \le \tau_0$ in $\mathbb{R}^N$, then u is solution to (1.1) with $f(t)$. From now on, we replace f by $\widehat{f}$ and keep the same notation $f(t)$.

In this section we focus on the following limiting problem associated with (1.1) :

$$\begin{cases} -\Delta_p u - \Delta_q u + V_0(|u|^{p-2}u + |u|^{q-2}u) = f(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

We define the energy functional for the limiting problem (3.1) by

$$I_{V_0}(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} V_0|u|^p \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} V_0|u|^q \mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

In view of [38], if $u \in X_0$ is a weak solution to problem (3.1), then we have the following Pohozăev identity:

$$\begin{aligned} P_{V_0}(u) &= \frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x + \frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x \\ &\quad + \frac{N}{p}\int_{\mathbb{R}^N} V_0|u|^p \mathrm{d}x + \frac{N}{q}\int_{\mathbb{R}^N} V_0|u|^q \mathrm{d}x - N\int_{\mathbb{R}^N} F(u)\mathrm{d}x. \end{aligned}$$

Lemma 3.1. *I_{V_0} possesses the Mountain-Pass geometry.*

Proof. By (f_1) and (f_2), for all $t \in \mathbb{R}$ we get

$$|f(t)| \le \varepsilon|t|^{p-1} + C_\varepsilon|t|^{q^*-1}$$

and

$$|F(t)| \le \frac{\varepsilon}{p}|t|^{p-1} + \frac{C_\varepsilon}{q}|t|^{q^*}.$$

So, for $2 \le p < q < N$, we have

$$\begin{aligned} I_{V_0}(u) &\ge \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + V_0|u|^p)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla u|^q + V_0|u|^q)\mathrm{d}x - \frac{\varepsilon}{p}|u|_p^p - \frac{C_\varepsilon}{q^*}|u|_{q^*}^{q^*} \\ &= \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla u|^q + V_0|u|^q)\mathrm{d}x + \frac{V_0-\varepsilon}{pV_0}\int_{\mathbb{R}^N}(|\nabla u|^p + V_0|u|^p)\mathrm{d}x - \frac{C_\varepsilon}{q^*}|u|_{q^*}^{q^*} \\ &= \frac{1}{q}\Big[\int_{\mathbb{R}^N}(|\nabla u|^p + V_0|u|^p)\mathrm{d}x + \int_{\mathbb{R}^N}(|\nabla u|^q + V_0|u|^q)\mathrm{d}x\Big] - \frac{C_\varepsilon}{q^*}|u|_{q^*}^{q^*} \\ &= \frac{1}{q}(\|u\|_{V_0,p}^p + \|u\|_{V_0,q}^q) - C\|u\|_{X_0}^{q^*}, \end{aligned}$$

where we used $\varepsilon = (\frac{1}{p} - \frac{1}{q})pV_0$. Hence, there exist $\rho, \delta > 0$ such that

$$\begin{aligned} I_{V_0}(u) &\ge \frac{1}{q}(\|u\|_{V_0,p}^q + \|u\|_{V_0,q}^q) - C\|u\|_{X_0}^{q^*} \\ &\ge \frac{1}{2^{q-1}q}\|u\|_{X_0}^q - C\|u\|_{X_0}^{q^*} \end{aligned}$$

$$\geq \delta$$

for $\|u\|_{X_0} = \rho$.

Now, for all $R > 0$, we define

$$w_R(x, y) := \begin{cases} T & \text{if } (x) \in B_R^+(0), \\ T\left(R + 1 - \sqrt{|x|}\right) & \text{if } (x) \in B_{R+1}^+(0) \backslash B_R^+(0), \\ 0 & \text{if } (x) \in \mathbb{R}_+^N \backslash B_{R+1}^+(0). \end{cases}$$

It is easy to see that $w_R \in X_{\text{rad}}\left(\mathbb{R}_+^N\right)$. We note that, according to (f_3), for $R > 0$ large enough it holds

$$\int_{\mathbb{R}^N} \left[F\left(w_R(x)\right) - \frac{V_0}{p} w_R^p(x) - \frac{V_0}{q} w_R^q(x)\right] \mathrm{d}x \geq 0.$$

Now, fix such an $R > 0$ and consider $w_{R,\theta}(x) := w_R\left(\frac{x}{e^\theta}\right)$. Then,

$$\begin{aligned} I_{V_0}\left(w_{R,\theta}\right) =& \frac{1}{p} e^{(N-p)\theta} \int_{\mathbb{R}_+^N} |\nabla u|^p \mathrm{d}x + \frac{1}{q} e^{(N-q)\theta} \int_{\mathbb{R}_+^N} |\nabla u|^q \mathrm{d}x \\ & - e^{N\theta} \int_{\mathbb{R}^N} \left[F\left(w_R(x)\right) - \frac{V_0}{p} w_R^p(x) - \frac{V_0}{q} w_R^q(x)\right] \mathrm{d}x \\ & \to -\infty \text{ as } \theta \to \infty. \end{aligned}$$

This ends the proof. □

Hence, according to Lemma 3.1, we can define the Mountain-Pass level of I_{V_0} by

$$c_{V_0} := \inf_{\gamma \in \Gamma_{V_0}} \sup_{t \in [0,1]} I_{V_0}(\gamma(t)), \tag{3.2}$$

where the set of paths is defined as

$$\Gamma_{V_0} := \{\gamma \in C([0, 1], X_0) : \gamma(0) = 0 \text{ and } I_{V_0}(\gamma(1)) < 0\}. \tag{3.3}$$

Obviously, $c_{V_0} > 0$. Moreover, similar to [3], we note that

$$c_{V_0} = c_{V_0,\text{rad}},$$

where

$$c_{V_0,\text{rad}} := \inf_{\gamma \in \Gamma_{V_0,\text{rad}}} \max_{t \in [0,1]} I_{V_0}(\gamma(t))$$

and

$$\Gamma_{V_0,\text{rad}} := \left\{\gamma \in C\left([0, 1], X_{\text{rad}}\left(\mathbb{R}_+^N\right)\right) : \gamma(0) = 0, I_{V_0}(\gamma(1)) < 0\right\}.$$

Next, we will construct a (PS) sequence $\{w_n\}_{n=1}^\infty$ for I_{V_0} at the level c_{V_0} that satisfies $I'_{V_0}(w_n) \to 0$ as $n \to \infty$, i.e.,

Proposition 3.1. *There exists a sequence $\{w_n\}_{n=1}^\infty$ in X_0 such that, as $n \to \infty$,*

$$I_{V_0}(w_n) \to c_{V_0}, \;\; I'_{V_0}(w_n) \to 0, P_{V_0}(w_n) \to 0. \tag{3.4}$$

Proof. Let we define $\widetilde{I}_{V_0}(\theta,u) := (I_{V_0}\circ\Phi)(\theta,u)$ for $(\theta,u)\in\mathbb{R}\times X_{\text{rad}}\left(\mathbb{R}^N_+\right)$, where $\Phi(\theta,u) := u\left(\frac{x}{e^\theta}\right)$. Here $\mathbb{R}\times X_{\text{rad}}\left(\mathbb{R}^N_+\right)$ is equipped with the standard norm

$$\|(\theta,u)\|_{\mathbb{R}\times X_0} := \left(|\theta|^2+\|u\|^2_{X_0}\right)^{\frac{1}{2}}.$$

According to Lemma 3.1 that $\widetilde{I}_{V_0}$ has a mountain pass geometry, so we can define the mountain pass level of $\widetilde{I}_{V_0}$

$$\tilde{c}_{V_0} := \inf_{\widetilde{\gamma}\in\widetilde{\Gamma}_{V_0}}\max_{t\in[0,1]}\widetilde{I}_{V_0}(\widetilde{\gamma}(t)),$$

where

$$\widetilde{\Gamma}_{V_0} := \left\{\widetilde{\gamma}\in C\left([0,1],\mathbb{R}\times X_{\text{rad}}\left(\mathbb{R}^N_+\right)\right) : \widetilde{\gamma}(0)=(0),\widetilde{I}_{V_0}(\widetilde{\gamma}(1))<0\right\}.$$

It is easy to prove that $\widetilde{c}_{V_0}=c_{V_0}$(see [6,26]). Then according to Lemma 2.1, we obtain that there exists a sequence $(\theta_n,u_n)\subset\mathbb{R}\times X_{\text{rad}}(\mathbb{R}^N_+)$ such that, as $n\to\infty$,

(*i*) $(I_{V_0}\circ\Phi)(\theta_n,u_n)\to c_{V_0}$,
(*ii*) $(I_{V_0}\circ\Phi)'(\theta_n,u_n)\to 0$ in $(\mathbb{R}\times X_{\text{rad}}(\mathbb{R}^N_+))'$,
(*iii*) $\theta_n\to 0$.

In fact, we only take $\varepsilon=\varepsilon_n=\frac{1}{n^2},\delta=\delta_n=\frac{1}{n}$ in Lemma 2.1, (i) and (ii) follow by (a) and (c) in Lemma 2.1. In view of (3.2) and (3.3), for $\varepsilon=\varepsilon_n:=\frac{1}{n^2}$, it is easy to find that $\gamma_n\in\Gamma_{V_0}$ such that $\sup\limits_{t\in[0,1]} I_{V_0}(\gamma_n(t))\le c_{V_0}+\frac{1}{n^2}$. We define $\widetilde{\gamma}_n(t):=(0,\gamma_n(t))$, then we have

$$\sup_{t\in[0,1]}(I_{V_0}\circ\Phi)(\widetilde{\gamma}_n(t))=\sup_{t\in[0,1]}I_{V_0}(\gamma_n(t))\le c_{V_0}+\frac{1}{n^2}.$$

From (b) of Lemma 2.1, there exists $(\theta_n,u_n)\in\mathbb{R}\times X_0$ such that

$$\operatorname*{dist}_{\mathbb{R}\times X_0}\left((\theta_n,u_n),(0,\gamma_n(t))\right)\le\frac{2}{n},$$

which implies that (iii) holds true. Here, we used the notation

$$\operatorname*{dist}_{\mathbb{R}\times X_0}((\theta,u),A) := \inf_{(\tau,v)\in\mathbb{R}\times X_0}\left(|\theta-\tau|^2+\|u-v\|^2_{X_0}\right)^{\frac{1}{2}}$$

for $A\subset\mathbb{R}\times X_0$. Now, for $(h,w)\in\mathbb{R}\times X_0$, it holds

$$\left\langle(I_{V_0}\circ\Phi)'(\theta_n,u_n),(h,w)\right\rangle=\left\langle I'_{V_0}(\Phi(\theta_n,u_n)),\Phi'(\theta_n,w)\right\rangle+P_{V_0}(\Phi(\theta_n,u_n))h. \tag{3.5}$$

Then, taking $h=1$ and $w=0$ in (3.5), we obtain that

$$P_{V_0}(\Phi(\theta_n,u_n))\to 0.$$

Moreover, for all $v\in X_0$, we only take $w(x,y)=v\left(e^{\theta_n}x,e^{\theta_n}y\right)$ and $h=0$ in (3.5), it follows from (ii) and (iii) that

$$\left\langle I'_{V_0}(\Phi(\theta_n,u_n)),v\right\rangle=o(1)\left\|v\left(e^{\theta_n}x,e^{\theta_n}y\right)\right\|_{X_0}=o(1)\|v\|_{X_0}.$$

Therefore, $w_n:=\Phi(\theta_n,u_n)$ is the sequence that fulfills the desired properties. □

Lemma 3.2. *Every sequence* (w_n) *satisfying* (3.4) *is bounded in* X_0.

Proof. According to (3.4), it is easy to see that

$$\begin{aligned} c_{V_0} + o_n(1) &= I_{V_0}(w_n) - \frac{1}{N} P_{V_0}(w_n) \\ &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla w_n|^p \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} V_0|w_n|^p \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} V_0|w_n|^q \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} F(w_n)\mathrm{d}x - \frac{1}{N}\left(\frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla w_n|^p \mathrm{d}x + \frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x \right. \\ &\quad \left. + \frac{N}{p}\int_{\mathbb{R}^N} V_0|w_n|^p \mathrm{d}x + \frac{N}{q}\int_{\mathbb{R}^N} V_0|w_n|^q \mathrm{d}x - N\int_{\mathbb{R}^N} F(w_n)\mathrm{d}x\right) \\ &= \frac{1}{N}\left(\int_{\mathbb{R}^N} |\nabla w_n|^p \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x\right). \end{aligned}$$

So we get that $\int_{\mathbb{R}^N} |\nabla w_n|^p \mathrm{d}x$ and $\int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x$ are bounded in $\mathbb{R}$. On the other hand, $P(w_n) = o_n(1)$ and $(f_1) - (f_2)$ yield

$$\begin{aligned} &\frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla w_n|^p \mathrm{d}x + \frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x + \frac{N}{p}\int_{\mathbb{R}^N} V_0|w_n|^p \mathrm{d}x + \frac{N}{q}\int_{\mathbb{R}^N} V_0|w_n|^q \mathrm{d}x \\ = {} & N\int_{\mathbb{R}^N} F(w_n)\mathrm{d}x + o_n(1) \\ \leq {} & N\delta\, |w_n(\cdot,0)|_p^p + NC_\delta\, |w_n(\cdot,0)|_{q^*}^{q^*} + o_n(1). \end{aligned}$$

Choosing $\delta > 0$ sufficiently small and using the boundedness of $\left(|w_n(\cdot,0)|_{q^*}\right)$, we can deduce that $\left(|w_n|_p\right)$ and $\left(|w_n|_q\right)$ are bounded in $\mathbb{R}$. In conclusion, (w_n) is bounded in X_0. □

The following lemma is a version of Lions' concentration-compactness lemma.

Lemma 3.3. (see [42]) *Let* $2 \leq p < \xi < q^*$. *Assume* $\{u_n\}$ *is a bounded sequence in* X_0 *which satisfies*

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^\xi \mathrm{d}x = 0$$

for some $R > 0$. Then $u_n \to 0$ in $L^\xi(\mathbb{R}^N)$ for $\xi \in (p, q^*)$.

Lemma 3.4. *There exist a sequence* $(x_n) \subset \mathbb{R}^N$ *and constants* $R > 0, \beta > 0$ *such that*

$$\int_{\Gamma_R^0(x_n)} w_n^2(x)\mathrm{d}x \geq \beta,$$

where (w_n) is the sequence given in Proposition 3.1.

Proof. By contradiction, we assume that the thesis is not true. Then, according to Lemma 3.3, we deduce that

$$w_n(\cdot) \to 0 \text{ in } L^\xi\left(\mathbb{R}^N\right), \quad \forall \xi \in (p, q^*). \tag{3.6}$$

Consequently, by using $(f_1)-(f_2)$, we have that

$$\int_{\mathbb{R}^N} f\left(w_n(x)\right) w_n(x)\mathrm{d}x = o_n(1).$$

According to $\left\langle I'_{V_0}\left(w_n\right), w_n\right\rangle = o_n(1)$, we can obtain that

$$\int_{\mathbb{R}^N} |\nabla w_n|^p \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} V_0|w_n|^p \mathrm{d}x + \int_{\mathbb{R}^N} V_0|w_n|^q \mathrm{d}x - \int_{\mathbb{R}^N} f(w_n)w_n \mathrm{d}x = o_n(1),$$

and so we deduce that $\|w_n\|_{X_0} \to 0$. Therefore, $I_{V_0}\left(w_n\right) \to 0$ and this leads to a contradiction because $c_{V_0} > 0$. □

Now we define

$$\mathcal{T}_{V_0} := \left\{u \in X\left(\mathbb{R}^N\right) \setminus \{0\} : I'_{V_0}(u) = 0, \max_{x\in\mathbb{R}^N} u(x) = u(0)\right\},$$

$$b_{V_0} := \inf_{u\in\mathcal{T}_{V_0}} I_{V_0}(u),$$

and

$$\mathcal{S}_{V_0} := \left\{u \in \mathcal{T}_{V_0} : I_{V_0}(u) = b_{V_0}\right\}.$$

Lemma 3.5. *There exists $u \in \mathcal{S}_{V_0}$.*

Proof. Assume that (w_n) is the sequence given by Proposition 3.1. Let $\widetilde{w}_n(x) := w_n(x + x_n)$, where x_n is given by Lemma 3.4. Due to Lemma 3.3, (w_n) is bounded in $X_{\mathrm{rad}}(\mathbb{R}^N)$, that is $\|w_n\|_{X_{\mathrm{rad}}(\mathbb{R}^N)} \leq C$ for all $n \in \mathbb{N}$. Going if necessary to a subsequence, we can assume that $\widetilde{w}_n \rightharpoonup \widetilde{w}$ in $X_{\mathrm{rad}}(\mathbb{R}^N)$ for some $\widetilde{w} \in X_{\mathrm{rad}}(\mathbb{R}^N) \setminus \{0\}$ and we obtain that

$$\widetilde{w}_n(x) \to \widetilde{w}(x) \text{ in } L^{\xi}(\mathbb{R}^N) \text{ for any } \xi \in (p, q^*).$$

So

$$\int_{\mathbb{R}^N} f(\widetilde{w_n})\widetilde{w_n} \to \int_{\mathbb{R}^N} f(\widetilde{w})\widetilde{w}. \tag{3.7}$$

Moreover, $\widetilde{w}$ satisfies

$$(-\Delta)_p\widetilde{w} + (-\Delta)_q\widetilde{w} + V_0(|\widetilde{w}|^{p-2}\widetilde{w} + |\widetilde{w}|^{q-2}\widetilde{w}) = f(\widetilde{w}) \text{ in } \mathbb{R}^N. \tag{3.8}$$

Therefore, we have

$$\frac{1}{p}\int_{\mathbb{R}^N} |\nabla\widetilde{w}|^p \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla\widetilde{w}|^q \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} V_0|\widetilde{w}|^p \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} V_0|\widetilde{w}|^q \mathrm{d}x = \int_{\mathbb{R}^N} F(\widetilde{w})\mathrm{d}x,$$

and

$$\begin{aligned} &\frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla\widetilde{w}|^p \mathrm{d}x + \frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla\widetilde{w}|^q \mathrm{d}x + \frac{N}{p}\int_{\mathbb{R}^N} V_0|\widetilde{w}|^p \mathrm{d}x + \frac{N}{q}\int_{\mathbb{R}^N} V_0|\widetilde{w}|^q \mathrm{d}x \\ = \; & N\int_{\mathbb{R}^N} F(\widetilde{w})\mathrm{d}x. \end{aligned}$$

From (3.7) we can see that

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla\widetilde{w}|^p\mathrm{d}x+\int_{\mathbb{R}^N}|\nabla\widetilde{w}|^q\mathrm{d}x+\int_{\mathbb{R}^N}V_0|\widetilde{w}|^p\mathrm{d}x+\int_{\mathbb{R}^N}V_0|\widetilde{w}|^q\mathrm{d}x\\
\le\ &\liminf_{n\to\infty}\left[\int_{\mathbb{R}^N}|\nabla\widetilde{w}_n|^p\mathrm{d}x+\int_{\mathbb{R}^N}|\nabla\widetilde{w}_n|^q\mathrm{d}x+\int_{\mathbb{R}^N}V_0|\widetilde{w}_n|^p\mathrm{d}x+\int_{\mathbb{R}^N}V_0|\widetilde{w}_n|^q\mathrm{d}x\right]\\
\le\ &\limsup_{n\to\infty}\left[\int_{\mathbb{R}^N}|\nabla\widetilde{w}_n|^p\mathrm{d}x+\int_{\mathbb{R}^N}|\nabla\widetilde{w}_n|^q\mathrm{d}x+\int_{\mathbb{R}^N}V_0|\widetilde{w}_n|^p\mathrm{d}x+\int_{\mathbb{R}^N}V_0|\widetilde{w}_n|^q\mathrm{d}x\right]\\
=\ &\limsup_{n\to\infty}\left[\int_{\mathbb{R}^N}|\nabla w_n|^p\mathrm{d}x+\int_{\mathbb{R}^N}|\nabla w_n|^q\mathrm{d}x+\int_{\mathbb{R}^N}V_0|w_n|^p\mathrm{d}x+\int_{\mathbb{R}^N}V_0|w_n|^q\mathrm{d}x\right]\\
=\ &\limsup_{n\to\infty}\int_{\mathbb{R}^N}f(w_n)w_n\mathrm{d}x\\
=\ &\limsup_{n\to\infty}\int_{\mathbb{R}^N}f(\widetilde{w}_n)\widetilde{w}_n\mathrm{d}x\\
=\ &\int_{\mathbb{R}^N}f(\widetilde{w})\widetilde{w}\mathrm{d}x\\
=\ &\int_{\mathbb{R}^N}|\nabla\widetilde{w}|^p\mathrm{d}x+\int_{\mathbb{R}^N}|\nabla\widetilde{w}|^q\mathrm{d}x+\int_{\mathbb{R}^N}V_0|\widetilde{w}|^p\mathrm{d}x+\int_{\mathbb{R}^N}V_0|\widetilde{w}|^q\mathrm{d}x,
\end{aligned}$$

which implies that $\|\widetilde{w}_n\|_{X_0}\to\|\widetilde{w}\|_{X_0}$ and thus $\widetilde{w}_n\to\widetilde{w}$ in X_0. Therefore, by $I_{V_0}(w_n)=I_{V_0}(\widetilde{w}_n)\to c_{V_0}$ and $I'_{V_0}(w_n)=I'_{V_0}(\widetilde{w}_n)\to 0$, we have that $I_{V_0}(\widetilde{w})=c_{V_0}$ and $I'_{V_0}(\widetilde{w})=0$. Since $\widetilde{w}\neq 0$, we deduce that $c_{V_0}\ge b_{V_0}$.

Now, let $w\in X_0\backslash\{0\}$ be any solution to (3.1). Define

$$w_t(x):=\begin{cases} w\left(\frac{x}{t}\right) & \text{for } t>0,\\ 0 & \text{for } t=0.\end{cases}$$

Using the fact that w satisfies the Pohozǎev identity, we get

$$\begin{aligned}
I_{V_0}(w_t(x)) =\ &\frac{1}{p}\int_{\mathbb{R}^N}|\nabla w_t(x)|^p\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}|\nabla w_t(x)|^q\mathrm{d}x+\frac{1}{p}\int_{\mathbb{R}^N}V_0|w_t(x)|^p\mathrm{d}x\\
&+\frac{1}{q}\int_{\mathbb{R}^N}V_0|w_t(x)|^q\mathrm{d}x-\int_{\mathbb{R}^N}F(w_t(x))\mathrm{d}x\\
=\ &\frac{t^{N-p}}{p}\int_{\mathbb{R}^N}|\nabla w|^p\mathrm{d}x+\frac{t^{N-q}}{q}\int_{\mathbb{R}^N}|\nabla w|^q\mathrm{d}x+\frac{1}{p}t^N\int_{\mathbb{R}^N}V_0|w|^p\mathrm{d}x\\
&+\frac{1}{q}t^N\int_{\mathbb{R}^N}V_0|w|^q\mathrm{d}x-t^N\int_{\mathbb{R}^N}F(w)\mathrm{d}x\\
=\ &\frac{t^{N-p}}{p}\int_{\mathbb{R}^N}|\nabla w|^p\mathrm{d}x+\frac{t^{N-q}}{q}\int_{\mathbb{R}^N}|\nabla w|^q\mathrm{d}x-\frac{N-p}{Np}t^N\int_{\mathbb{R}^N}|\nabla w|^p\mathrm{d}x\\
&-\frac{N-q}{Nq}t^N\int_{\mathbb{R}^N}|\nabla w|^q\mathrm{d}x,
\end{aligned}$$

and differentiating with respect to t we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}I_{V_0}(w_t(x)) = \frac{N-p}{p}t^{N-p-1}\int_{\mathbb{R}^N}|\nabla w|^p\mathrm{d}x+\frac{N-q}{q}t^{N-q-1}\int_{\mathbb{R}^N}|\nabla w|^q\mathrm{d}x$$

$$
\begin{aligned}
& -\frac{N-p}{p}t^{N-1}\int_{\mathbb{R}^N}|\nabla w|^p\mathrm{d}x-\frac{N-q}{q}t^{N-1}\int_{\mathbb{R}^N}|\nabla w|^q\mathrm{d}x \\
= & \frac{N-p}{p}t^{N-p-1}\left(1-t^p\right)\int_{\mathbb{R}^N}|\nabla w|^p\mathrm{d}x+\frac{N-q}{q}t^{N-q-1}\left(1-t^q\right)\int_{\mathbb{R}^N}|\nabla w|^q\mathrm{d}x,
\end{aligned}
$$

so we obtain that

$$
\frac{\mathrm{d}}{\mathrm{d}t}I_{V_0}(w_t(x))>0\quad\forall t\in(0,1),\quad\frac{\mathrm{d}}{\mathrm{d}t}I_{V_0}(w_t(x))<0\quad\forall t\in(1,\infty),
$$

which implies that

$$
\max_{t\geq 0}I_{V_0}(w_t(x))=I_{V_0}(w_1(x))=I_{V_0}(w).
$$

Therefore, we have that

$$
\begin{aligned}
I_{V_0}(w_t(x)) = & \frac{t^{N-p}}{p}\int_{\mathbb{R}^N}|\nabla w|^p\mathrm{d}x+\frac{t^{N-q}}{q}\int_{\mathbb{R}^N}|\nabla w|^q\mathrm{d}x-\frac{N-p}{Np}t^N\int_{\mathbb{R}^N}|\nabla w|^p\mathrm{d}x \\
& -\frac{N-q}{Nq}t^N\int_{\mathbb{R}^N}|\nabla w|^q\mathrm{d}x\to-\infty,
\end{aligned}
$$

as $t\to\infty$. After a suitable scale change in t, we obtain that $w_t(x)\in\Gamma_{V_0}$. By the definition of c_{V_0}, we see that $I_{V_0}(w_t(x))\geq c_{V_0}$. Since w is arbitrary, we have that $b_{V_0}\geq c_{V_0}$ and this implies that $b_{V_0}=c_{V_0}$.

Choosing $w^-=\min\{w,0\}$ as test function of (3.1) we can deduce that $w\geq 0$ in $\mathbb{R}^N$. By $(f_1)-(f_2)$ and using a Moser iteration argument (see [6]), we obtain that $w\in L^\infty\left(\mathbb{R}^N\right)$. According to Corollary 2.1 in Ambrosio and Rădulescu [7], we can see that $w\in C^\sigma(\mathbb{R}^N)$ for some $\sigma\in(0,1)$. Similar to the proof of Theorem 1.1-(ii) in Jarohs [28], we obtain that $w>0$ in $\mathbb{R}^N$. Note that, the methods of [6] and [7] are still applicable to this article, so they are directly quoted here. □

Remark 3.1. *For $m>0$, we use the notation*

$$
I_m(u)=\frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\mathrm{d}x+\frac{m}{p}\int_{\mathbb{R}^N}|u|^p\mathrm{d}x+\frac{m}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x-\int_{\mathbb{R}^N}F(u)\mathrm{d}x,
$$

and denote by c_m the corresponding mountain pass level. It is standard to verify that if $m_1>m_2$ then $c_{m_1}>c_{m_2}$.

In what follows, we aim to prove that $\mathcal{S}_{V_0}$ is compact in X_0.

Lemma 3.6. *$\mathcal{S}_{V_0}$ is compact in X_0.*

Proof. For any $U\in\mathcal{S}_{V_0}$, we have that

$$
\begin{aligned}
c_{V_0}+o_n(1) = & I_{V_0}(U)-\frac{1}{N}P_{V_0}(U) \\
= & \frac{1}{p}\int_{\mathbb{R}^N}|\nabla U|^p\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x+\frac{1}{p}\int_{\mathbb{R}^N}V_0|U|^p\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}V_0|U|^q\mathrm{d}x \\
& -\int_{\mathbb{R}^N}F(U)\mathrm{d}x-\frac{1}{N}\left(\frac{N-p}{p}\int_{\mathbb{R}^N}|\nabla U|^p\mathrm{d}x+\frac{N-q}{q}\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x\right. \\
& \left.+\frac{N}{p}\int_{\mathbb{R}^N}V_0|U|^p\mathrm{d}x+\frac{N}{q}\int_{\mathbb{R}^N}V_0|U|^q\mathrm{d}x-N\int_{\mathbb{R}^N}F(U)\mathrm{d}x\right)
\end{aligned}
$$

$$= \frac{1}{N}\left(\int_{\mathbb{R}^N} |\nabla U|^p \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla U|^q \mathrm{d}x\right).$$

So $\mathcal{S}_{V_0}$ is bounded in X_0.

For any sequence $\{U_k\} \subset \mathcal{S}_{V_0}$, up to a subsequence, we can assume that there is a $U_0 \in X_0$ such that

$$U_k \rightharpoonup U_0 \text{ in } X_0 \tag{3.9}$$

and U_0 satisfies

$$-\Delta_p U_0 - \Delta_q U_0 + V_0(|U_0|^{p-2}U_0 + |U_0|^{q-2}U_0) = f(U_0), \text{ in } \mathbb{R}^N, \ U_0 \geq 0.$$

Next, we will prove that U_0 is nontrivial. Note that, up to a subsequence, we have

$$U_k \to U_0 \text{ in } L^t_{\text{loc}}(\mathbb{R}^N), \ t \in (p, q^*). \tag{3.10}$$

From (3.10), $\left\{U_k^t\right\}$ is uniformly integrable in any bounded domain in $\mathbb{R}^N$. By Lemma 2.2 (i) in [25], $\|U_k\|_{L^\infty_{\text{loc}}(\mathbb{R}^N)} \leq C$. In view of [31], $\exists \alpha \in (0,1)$ such that $\|U_k\|_{C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)} \leq C$. Due to $\{U_k\} \subset \mathcal{S}_{V_0}$, by Lemma 3.5, we have that $U_k > 0$. It is easy to prove that $\liminf\limits_{k\to\infty} \|U_k\|_\infty > 0$ because of $\lim\limits_{t\to 0} \frac{f(t)}{t^{p-1}} = 0$. In fact, by using U_k satisfies (3.1), we have that

$$-\Delta_p U_k - \Delta_q U_k + V_0(|U_k|^{p-2}U_k + |U_k|^{q-2}U_k) = f(U_k),$$

that is

$$\begin{aligned} &\frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla U_k|^p \mathrm{d}x + \frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla U_k|^q \mathrm{d}x + \frac{V_0 N}{p}\int_{\mathbb{R}^N} |U_k|^p \mathrm{d}x + \frac{V_0 N}{q}\int_{\mathbb{R}^N} |U_k|^q \mathrm{d}x \\ = \ & N\int_{\mathbb{R}^N} F(U_k)\mathrm{d}x. \end{aligned}$$

According to $\lim\limits_{t\to 0} \frac{f(t)}{t^{p-1}} = 0$, we obtain that, $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$f(t) < \varepsilon t^{p-1} \text{ for } |t| < \delta,$$

then $F(U_k) < \frac{\varepsilon}{p}|U_k|_p^p$. Assume by contradiction, we have $\liminf\limits_{k\to\infty} \|U_k\|_\infty = 0$, then for δ given above, we have $|U_k| < \delta$. Therefore,

$$\begin{aligned} &\frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla U_k|^p \mathrm{d}x + \frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla U_k|^q \mathrm{d}x \\ = \ & N\int_{\mathbb{R}^N} F(U_k)\mathrm{d}x - \frac{V_0 N}{p}\int_{\mathbb{R}^N} |U_k|^p \mathrm{d}x - \frac{V_0 N}{q}\int_{\mathbb{R}^N} |U_k|^q \mathrm{d}x < 0, \end{aligned}$$

which leads to a contradiction. Noting that $U_k(0) = \|U_k\|_\infty$, we know that $U_0 \not\equiv 0$. Therefore, we can find that $\exists C_0 > 0$ such that $U_k(0) \geq C_0 > 0$, then $U_0(0) \geq C_0 > 0$, this means that U_0 is nontrivial. Similar to the proof of Lemma 3.5, we can check that $J_{V_0}(U_0) = c_{V_0}$ and $U_k \to U_0$ in X_0. This completes the proof that $\mathcal{S}_{V_0}$ is compact in X_0. □

4. Proof of Theorem 1.1

The energy functional corresponding to (1.1) is

$$I_\varepsilon(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + V(\varepsilon x)|u|^p)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla u|^q + V(\varepsilon x)|u|^q)\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

We define

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \Lambda/\varepsilon \\ \varepsilon^{-1} & \text{if } x \notin \Lambda/\varepsilon \end{cases}$$

and

$$Q_\varepsilon(v) = \left(\int_{\mathbb{R}^N} \chi_\varepsilon v^p - 1\right)_+^2.$$

Finally, set $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ be given by

$$J_\varepsilon(v) = I_\varepsilon(v) + Q_\varepsilon(v).$$

Note that this type of penalization was first introduced in [14]. It is standard to prove that $J_\varepsilon \in C^1(X_\varepsilon, \mathbb{R})$. In order to find solutions of (1.1) which concentrate around the local minimum of V in Λ as $\varepsilon \to 0$, we only look for the critical points of J_ε for which Q_ε is zero.

Let $c_{V_0} = I_{V_0}(U)$ for $U \in \mathcal{S}_{V_0}$ and $10\delta = \operatorname{dist}\left\{\mathcal{M}, \mathbb{R}^N\backslash\Lambda\right\}$, we fix a $\beta \in (0, \delta)$ and a cut-off function $\varphi \in C_c^\infty\left(\mathbb{R}^N\right)$ such that $0 \le \varphi \le 1, \varphi(x) = 1$ for $|x| \le \beta, \varphi(x) = 0$ for $|x| \ge 2\beta$ and $|\nabla\varphi| \le \frac{C}{\beta}$. Also, setting $\varphi_\varepsilon(y) = \varphi(\varepsilon y)$. We will look for a solution of (1.1) near the set

$$Y_\varepsilon := \left\{\varphi(\varepsilon x - x')\, U(x - (x'/\varepsilon)) : x' \in \mathcal{M}^\beta, U \in \mathcal{S}_{V_0}\right\}$$

for sufficiently small $\varepsilon > 0$, where $\mathcal{M}^\beta := \left\{y \in \mathbb{R}^N : \inf_{z\in\mathcal{M}} |y - z| \le \beta\right\}$. Moreover, for $A \subset X_\varepsilon$, we use the notation

$$A^a := \left\{u \in X_\varepsilon : \inf_{v\in A} \|u - v\|_{X_\varepsilon} \le a\right\}.$$

For $U \in \mathcal{S}_{V_0}$ arbitrary but fixed, we define $W_{\varepsilon,t}(x) := \varphi(\varepsilon x)U\left(\frac{x}{t}\right)$, we will show that J_ε possesses the Mountain-Pass geometry.

Let $U_t(x) := U(\frac{x}{t})$, similar to the proof of Lemma 3.1, we obtain that

$$\begin{aligned} I_{V_0}(U_t) &= \frac{t^{N-p}}{p}\int_{\mathbb{R}^N}|\nabla U|^p\mathrm{d}x + \frac{t^{N-q}}{q}\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x + \frac{t^N}{p}\int_{\mathbb{R}^N}V_0|U|^p\mathrm{d}x \\ &\quad + \frac{t^N}{q}\int_{\mathbb{R}^N}V_0|U|^q\mathrm{d}x - t^N\int_{\mathbb{R}^N}F(U)\mathrm{d}x \\ &\to -\infty \text{ as } t \to \infty, \end{aligned}$$

so there exists $t_0 > 0$ such that $I_{V_0}(U_{t_0}) < -3$.

It is easy to check that $Q_\varepsilon(W_{\varepsilon,t_0}) = 0$, then from the Dominated Convergence Theorem we have, for $\varepsilon > 0$ small,

$$J_\varepsilon(W_{\varepsilon,t_0}) = I_\varepsilon(W_{\varepsilon,t_0})$$

$$
\begin{aligned}
&= \frac{1}{p}\int_{\mathbb{R}^N}|\nabla W_{\varepsilon,t_0}|^p\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}|\nabla W_{\varepsilon,t_0}|^q\mathrm{d}x+\frac{1}{p}\int_{\mathbb{R}^N}V(\varepsilon x)|W_{\varepsilon,t_0}|^p\mathrm{d}x\\
&\quad+\frac{1}{q}\int_{\mathbb{R}^N}V(\varepsilon x)|W_{\varepsilon,t_0}|^q\mathrm{d}x-\int_{\mathbb{R}^N}F(W_{\varepsilon,t_0})\mathrm{d}x\\
&\overset{\widetilde{x}=\frac{x}{t_0}}{=} \frac{t_0^{N-p}}{p}\int_{\mathbb{R}^N}\left|\varepsilon t_0^2\nabla\varphi(\varepsilon t_0\widetilde{x})U(\widetilde{x})+\varphi(\varepsilon\widetilde{x})\nabla U(\widetilde{x})\right|^p\mathrm{d}\widetilde{x}\\
&\quad+\frac{t_0^{N-q}}{q}\int_{\mathbb{R}^N}\left|\varepsilon t_0^2\nabla\varphi(\varepsilon t_0\widetilde{x})U(\widetilde{x})+\varphi(\varepsilon t_0\widetilde{x})\nabla U(\widetilde{x})\right|^q\mathrm{d}\widetilde{x}\\
&\quad+\frac{t_0^N}{p}\int_{\mathbb{R}^N}V(\varepsilon t_0\widetilde{x})|\varphi(\varepsilon t_0\widetilde{x})U(\widetilde{x})|^p\mathrm{d}\widetilde{x}\\
&\quad+\frac{t_0^N}{q}\int_{\mathbb{R}^N}V(\varepsilon t_0\widetilde{x})|\varphi(\varepsilon t_0\widetilde{x})U(\widetilde{x})|^q\mathrm{d}\widetilde{x}\\
&\quad-t^N\int_{\mathbb{R}^N}F(\varphi(\varepsilon t_0\widetilde{x})U(\widetilde{x})\mathrm{d}\widetilde{x}\\
&= I_{V_0}(U_{t_0})+o(1)<-2. \qquad (4.1)
\end{aligned}
$$

By using (f_1) and (f_2), for $2\le p<q<N$, we have

$$
\begin{aligned}
J_\varepsilon(u) &\ge I_\varepsilon(u)\\
&\ge \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p+V_\varepsilon|u|^p)\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}(|\nabla u|^q+V_\varepsilon|u|^q)\mathrm{d}x-\frac{\varepsilon}{p}|u|_p^p-\frac{C_\varepsilon}{q^*}|u|_{q^*}^{q^*}\\
&\ge \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla u|^q+V_\varepsilon|u|^q)\mathrm{d}x+\frac{V_0-\varepsilon}{pV_0}\int_{\mathbb{R}^N}(|\nabla u|^p+V_\varepsilon|u|^p)\mathrm{d}x-\frac{C_\varepsilon}{q^*}|u|_{q^*}^{q^*}\\
&= \frac{1}{q}\Big[\int_{\mathbb{R}^N}(|\nabla u|^p+V_\varepsilon|u|^p)\mathrm{d}x+\int_{\mathbb{R}^N}(|\nabla u|^q+V_\varepsilon|u|^q)\mathrm{d}x\Big]-\frac{C_\varepsilon}{q^*}|u|_{q^*}^{q^*}\\
&= \frac{1}{q}(\|u\|_{V_\varepsilon,p}^p+\|u\|_{V_\varepsilon,q}^q)-C\|u\|_{X_\varepsilon}^{q^*},
\end{aligned}
$$

where we used $\varepsilon=(\frac{1}{p}-\frac{1}{q})pV_0$. Hence, there exist $\rho,\delta>0$ such that, for $\|u\|_{X_0}=\rho$,

$$
\begin{aligned}
J_\varepsilon(u) &\ge \frac{1}{q}(\|u\|_{V_\varepsilon,p}^q+\|u\|_{V_\varepsilon,q}^q)-C\|u\|_{X_\varepsilon}^{q^*}\\
&\ge \frac{1}{2^{q-1}q}\|u\|_{X_\varepsilon}^q-C\|u\|_{X_\varepsilon}^{q^*}\\
&\ge \delta.
\end{aligned}
$$

Hence, we can define the Mountain-Pass value of J_ε as follows,

$$c_\varepsilon:=\inf_{\gamma\in\Gamma_\varepsilon}\max_{s\in[0,1]}J_\varepsilon(\gamma(s)),$$

where $\Gamma_\varepsilon:=\{\gamma\in C([0,1],X_\varepsilon)\mid\gamma(0)=0,\gamma(1)=W_{\varepsilon,t_0}\}$.

Lemma 4.1. *There holds*

$$\overline{\lim_{\varepsilon\to0}}\,c_\varepsilon\le c_{V_0}.$$

Proof. Denote $W_{\varepsilon,0} = \lim_{t\to 0} W_{\varepsilon,t}$ in X_ε sense, then it is easy to see that $W_{\varepsilon,0} = 0$. Therefore, let $\gamma(s) := W_{\varepsilon,st_0}(0 \le s \le 1\ )$, we obtain that $\gamma(s) \in \Gamma_\varepsilon$, then

$$c_\varepsilon \le \max_{s\in[0,1]} J_\varepsilon(\gamma(s)) = \max_{t\in[0,t_0]} J_\varepsilon\left(W_{\varepsilon,t}\right),$$

and we only need to prove that

$$\overline{\lim_{\varepsilon\to 0}} \max_{t\in[0,t_0]} J_\varepsilon\left(W_{\varepsilon,t}\right) \le c_{V_0}.$$

In fact, similar to (4.1), we obtain that

$$\begin{aligned}\max_{t\in[0,t_0]} J_\varepsilon\left(W_{\varepsilon,t}\right) &= \max_{t\in[0,t_0]} I_{V_0}\left(U_t^*\right) + o(1)\\ &\le \max_{t\in[0,\infty)} I_{V_0}\left(U_t^*\right) + o(1) = I_{V_0}\left(U^*\right) + o(1) = c_{V_0} + o(1).\end{aligned}$$

This finishes the proof. □

Lemma 4.2. *There holds*

$$\underline{\lim_{\varepsilon\to 0}} c_\varepsilon \ge c_{V_0}.$$

Proof. Assuming by contradiction that $\underline{\lim}_{\varepsilon\to 0} c_\varepsilon < c_{V_0}$, then there exist $\delta_0 > 0, \varepsilon_n \to 0$ and $\gamma_n \in \Gamma_{\varepsilon_n}$ such that $J_{\varepsilon_n}\left(\gamma_n(s)\right) < c_{V_0} - \delta_0$ for $s \in [0,1]$. We could fix an ε_n such that

$$\frac{1}{p} V_0 \varepsilon_n \left(1 + \left(1 + c_{V_0}\right)^{1/2}\right) < \min\left\{\delta_0, 1\right\}. \tag{4.2}$$

Due to $I_{\varepsilon_n}\left(\gamma_n(0)\right) = 0$ and $I_{\varepsilon_n}\left(\gamma_n(1)\right) \le J_{\varepsilon_n}\left(\gamma_n(1)\right) = J_{\varepsilon_n}\left(W_{\varepsilon_n,t_0}\right) < -2$, we can look for an $s_n \in (0,1)$ such that $I_{\varepsilon_n}\left(\gamma_n(s)\right) \ge -1$ for $s \in [0, s_n]$ and $I_{\varepsilon_n}\left(\gamma_n\left(s_n\right)\right) = -1$. Moreover, for any $s \in [0, s_n]$, we have that

$$Q_{\varepsilon_n}\left(\gamma_n(s)\right) = J_{\varepsilon_n}\left(\gamma_n(s)\right) - I_{\varepsilon_n}\left(\gamma_n(s)\right) \le 1 + c_{V_0} - \delta_0,$$

this implies that

$$\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon_n)} \gamma_n^p(s)\mathrm{d}x \le \varepsilon_n\left(1 + \left(1 + c_{V_0}\right)^{1/2}\right) \text{ for } s \in \left[0, s_n\right].$$

So for $s \in [0, s_n]$, we have

$$\begin{aligned}
& I_{\varepsilon_n}\left(\gamma_n(s)\right)\\
= \ & I_{V_0}\left(\gamma_n(s)\right) + \frac{1}{p}\int_{\mathbb{R}^N} \left(V\left(\varepsilon_n x\right) - V_0\right)\gamma_n^p(s)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} \left(V\left(\varepsilon_n x\right) - V_0\right)\gamma_n^q(s)\mathrm{d}x\\
\ge \ & I_{V_0}\left(\gamma_n(s)\right) + \frac{1}{p}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon_n)} \left(V\left(\varepsilon_n x\right) - V_0\right)\gamma_n^p(s)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon_n)} \left(V\left(\varepsilon_n x\right) - V_0\right)\gamma_n^q(s)\mathrm{d}x\\
\ge \ & I_{V_0}\left(\gamma_n(s)\right) + \frac{1}{p}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon_n)} \left(V\left(\varepsilon_n x\right) - V_0\right)\gamma_n^p(s)\mathrm{d}x\\
\ge \ & I_{V_0}\left(\gamma_n(s)\right) - \frac{1}{p} V_0 \varepsilon_n\left(1 + \left(1 + c_{V_0}\right)^{1/2}\right).
\end{aligned}$$

Then

$$
\begin{aligned}
I_{V_0}\left(\gamma_n\left(s_n\right)\right) &\leq I_{\varepsilon_n}\left(\gamma_n\left(s_n\right)\right)+\frac{1}{p} V_0 \varepsilon_n\left(1+\left(1+c_{V_0}\right)^{1/2}\right) \\
&=-1+\frac{1}{p} V_0 \varepsilon_n\left(1+\left(1+c_{V_0}\right)^{1/2}\right)<0,
\end{aligned}
$$

and recalling (3.2), we obtain that

$$
\max_{s \in[0, s_n]} I_{V_0}\left(\gamma_n(s)\right) \geq c_{V_0}.
$$

Therefore, we get that

$$
\begin{aligned}
c_{V_0}-\delta_0 &\geq \max_{s \in[0,1]} J_{\varepsilon_n}\left(\gamma_n(s)\right) \geq \max_{s \in[0,1]} I_{\varepsilon_n}\left(\gamma_n(s)\right) \geq \max_{s \in[0, s_n]} I_{\varepsilon_n}\left(\gamma_n(s)\right) \\
&\geq \max_{s \in[0, s_n]} I_{V_0}\left(\gamma_n(s)\right)-\frac{1}{p} V_0 \varepsilon_n\left(1+\left(1+c_{V_0}\right)^{1/2}\right),
\end{aligned}
$$

that is $0<\delta_0 \leq \frac{1}{p} V_0 \varepsilon_n\left(1+\left(1+c_{V_0}\right)^{1/2}\right)$, which contradicts (4.2). As desired. □

By using Lemma 4.1 and Lemma 4.2, we have

$$
\lim_{\varepsilon \rightarrow 0}\left(\max_{s \in[0,1]} J_{\varepsilon}\left(\gamma_{\varepsilon}(s)\right)-c_{\varepsilon}\right)=0,
$$

where $\gamma_{\varepsilon}(s)=W_{\varepsilon, s t_0}$ for $s \in[0,1]$. Denote

$$
\tilde{c}_{\varepsilon}:=\max_{s \in[0,1]} J_{\varepsilon}\left(\gamma_{\varepsilon}(s)\right),
$$

it is easy to see that $c_{\varepsilon} \leq \tilde{c}_{\varepsilon}$ and

$$
\lim_{\varepsilon \rightarrow 0} c_{\varepsilon}=\lim_{\varepsilon \rightarrow 0} \tilde{c}_{\varepsilon}=c_{V_0}.
$$

Now define

$$
J_{\varepsilon}^{\alpha}=\left\{u \in X_{\varepsilon} \mid J_{\varepsilon}(u) \leq \alpha\right\},
$$

and for a set $A \subset X_{\varepsilon}$ and $\alpha>0$, let $A^{\alpha} \equiv\left\{u \in X_{\varepsilon} \mid \inf_{v \in A}\|u-v\|_{\varepsilon} \leq \alpha\right\}$.

Lemma 4.3. *Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be such that $\lim_{i \rightarrow \infty} \varepsilon_i=0$ and $\{u_{\varepsilon_i}(\cdot)\} \subset Y_{\varepsilon_i}^d$ such that*

$$
\lim_{i \rightarrow \infty} J_{\varepsilon_i}\left(u_{\varepsilon_i}(\cdot)\right) \leq c_{V_0} \text { and } \lim_{i \rightarrow \infty} J_{\varepsilon_i}^{\prime}\left(u_{\varepsilon_i}(\cdot)\right)=0.
$$

Then, for sufficiently small $d>0$, there exists, up to a subsequence, $\{y_i\}_{i=1}^{\infty} \subset \mathbb{R}^N$, $x \in \mathcal{M}$, $U \in \mathcal{S}_{V_0}$ such that

$$
\lim_{i \rightarrow \infty}\left|\varepsilon_i y_i-x\right|=0 \text { and } \lim_{i \rightarrow \infty}\left\|u_{\varepsilon_i}(\cdot)-\varphi_{\varepsilon_i}\left(\cdot-y_i\right) U\left(\cdot-y_i\right)\right\|_{X_{\varepsilon_i}}=0.
$$

Proof. For convenience' sake, we write ε for ε_i. According to the compactness of $\mathcal{S}_{V_0}$ and $\mathcal{M}^{\beta}$, there exist $Z \in \mathcal{S}_{V_0}$ and $x \in \mathcal{M}^{\beta}$ such that

$$
\left\|u_{\varepsilon}(\cdot)-\varphi_{\varepsilon}(\cdot-\frac{x}{\varepsilon}) Z(\cdot-\frac{x}{\varepsilon})\right\|_{X_{\varepsilon}} \leq 2 d \tag{4.3}
$$

for small $\varepsilon > 0$. Note that, we denote $u_\varepsilon^1(\cdot) = \varphi_\varepsilon(\cdot - \frac{x}{\varepsilon})u_\varepsilon(\cdot)$ and $u_\varepsilon^2 = u_\varepsilon - u_\varepsilon^1$.

As a first step in the proof of this lemma we will check that

$$J_\varepsilon(u_\varepsilon) \geq J_\varepsilon\left(u_\varepsilon^1\right) + J_\varepsilon\left(u_\varepsilon^2\right) + O(\varepsilon). \tag{4.4}$$

Suppose there exist $x_\varepsilon \in B(\frac{x}{\varepsilon}, \frac{2\beta}{\varepsilon})\backslash B(\frac{x}{\varepsilon}, \frac{\beta}{\varepsilon})$ and $R > 0$ satisfying $\liminf_{\varepsilon\to 0} \int_{B(x_\varepsilon,R)} (u_\varepsilon)^2\, dy > 0$. Going if necessary to a subsequence, we can assume that $\varepsilon x_\varepsilon \to x_0$ with x_0 in the closure of $B(x, 2\beta)\backslash B(x, \beta)$ and that $u_\varepsilon(\cdot + x_\varepsilon) \rightharpoonup \widetilde{W}$ in X_ε for some $\widetilde{W} \in X_\varepsilon$. Moreover, note that $\widetilde{W}$ satisfies

$$(-\Delta)_p\widetilde{W} + (-\Delta)_q\widetilde{W} + V(x_0)(|\widetilde{W}|^{p-2}\widetilde{W} + |\widetilde{W}|^{q-2}\widetilde{W}) = f(\widetilde{W}) \in X_\varepsilon.$$

According to definition, $I_{V(x_0)}(\widetilde{W}) \geq c_{V(x_0)}$. For large $R > 0$, by using Fatou's lemma, we also have that

$$\liminf_{\varepsilon\to 0} \int_{B(x_\varepsilon,R)} |\nabla u_\varepsilon|^p\, dy \geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\widetilde{W}|^p dy \tag{4.5}$$

and

$$\liminf_{\varepsilon\to 0} \int_{B(x_\varepsilon,R)} |\nabla u_\varepsilon|^q\, dy \geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\widetilde{W}|^q dy. \tag{4.6}$$

Now, recalling from Remark 3.1 that $c_a > c_b$ if $a > b$, we see that $c_{V(x_0)} \geq c_{V_0}$ because of $V(x_0) \geq V_0$. According to Pohozǎev identity, for any $U \in \mathcal{S}_{V_0}$,

$$\frac{1}{N}\left(\int_{\mathbb{R}^N} |\nabla U|^p dx + \int_{\mathbb{R}^N} |\nabla U|^q dx\right) = I_{V_0}(U). \tag{4.7}$$

Thus, from (4.5), (4.6) and (4.7) we get that

$$\liminf_{\varepsilon\to 0} \int_{B(x_\varepsilon,R)} |\nabla u_\varepsilon|^p\, dy + \liminf_{\varepsilon\to 0} \int_{B(x_\varepsilon,R)} |\nabla u_\varepsilon|^q\, dy \geq \frac{N}{2} I_{V(x_0)}(\widetilde{W}) \geq \frac{N}{2} c_{V_0} > 0.$$

Then, taking $d > 0$ sufficiently small, we get a contradiction with (4.3), so there does not exist such a sequence $\{x_\varepsilon\}_\varepsilon$ and we deduce from a result of Lemma 3.3 that

$$\liminf_{\varepsilon\to 0} \int_{B(x/\varepsilon, 2\beta/\varepsilon)\backslash B(x/\varepsilon, \beta/\varepsilon)} |u_\varepsilon|^t\, dy = 0,$$

where $t \in (p, q^*)$. As a consequence, we can deduce using $(f_1), (f_2)$ that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \left(F(u_\varepsilon) - F\left(u_\varepsilon^1\right) - F\left(u_\varepsilon^2\right)\right) dy = 0.$$

At this point, we write

$$J_\varepsilon(u_\varepsilon) \geq J_\varepsilon\left(u_\varepsilon^1\right) + J_\varepsilon\left(u_\varepsilon^2\right) - \int_{\mathbb{R}^N} \left(F(u_\varepsilon) - F\left(u_\varepsilon^1\right) - F\left(u_\varepsilon^2\right)\right) dy + O(\varepsilon).$$

Hence, the inequality (4.4) holds.

Next, we estimate $J_\varepsilon\left(u_\varepsilon^2\right)$. Due to $\{u_\varepsilon\}_\varepsilon$ is bounded, it is easy to see from (4.3) that $\left\|u_\varepsilon^2\right\|_\varepsilon \leq 4d$ for small $\varepsilon > 0$. By using Sobolev's inequality, for some $C > 0$, we have that

$$\begin{aligned}
J_\varepsilon\left(u_\varepsilon^2\right) &\geq I_\varepsilon\left(u_\varepsilon^2\right) \\
&\geq \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u_\varepsilon^2|^p + V_\varepsilon|u_\varepsilon^2|^p)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla u_\varepsilon^2|^q + V_\varepsilon|u_\varepsilon^2|^q)\mathrm{d}x - \frac{\varepsilon}{p}|u_\varepsilon^2|_p^p - C_\varepsilon|u_\varepsilon^2|_{q^*}^{q^*} \\
&\geq \frac{1}{2p}\int_{\mathbb{R}^N}(|\nabla u_\varepsilon^2|^p + V_\varepsilon|u_\varepsilon^2|^p)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla u_\varepsilon^2|^q + V_\varepsilon|u_\varepsilon^2|^q)\mathrm{d}x - C_\varepsilon|u_\varepsilon^2|_{q^*}^{q^*} \\
&\geq \frac{1}{2p}\int_{\mathbb{R}^N}(|\nabla u_\varepsilon^2|^p + V_\varepsilon|u_\varepsilon^2|^p)\mathrm{d}x + \frac{1}{2q}\int_{\mathbb{R}^N}(|\nabla u_\varepsilon^2|^q + V_\varepsilon|u_\varepsilon^2|^q)\mathrm{d}x - C_\varepsilon|u_\varepsilon^2|_{q^*}^{q^*} \\
&\geq \frac{1}{2q}\int_{\mathbb{R}^N}(|\nabla u_\varepsilon^2|^p + V_\varepsilon|u_\varepsilon^2|^p)\mathrm{d}x + \frac{1}{2q}\int_{\mathbb{R}^N}(|\nabla u_\varepsilon^2|^q + V_\varepsilon|u_\varepsilon^2|^q)\mathrm{d}x - C_\varepsilon|u_\varepsilon^2|_{q^*}^{q^*} \\
&\geq \frac{1}{2q}\left\|u_\varepsilon^2\right\|_\varepsilon^q - C\left\|u_\varepsilon^2\right\|_\varepsilon^{q^*} \\
&\geq \left\|u_\varepsilon^2\right\|_\varepsilon^q\left(\frac{1}{2q} - C(4d)^{q^*-q}\right).
\end{aligned}$$

In particular, taking $d > 0$ small enough, we can assume that $J_\varepsilon\left(u_\varepsilon^2\right) \geq 0$.

Now let $W_\varepsilon(y) = u_\varepsilon^1(y + \frac{x}{\varepsilon})$. Going if necessary to a subsequence, we can assume that, $W_\varepsilon \rightharpoonup W$ in X_ε for some W. Moreover W satisfies

$$(-\Delta)_p W(y) + (-\Delta)_q W(y) + V(x)(|W(y)|^{p-2}W(y) + |W(y)|^{q-2}W(y)) = f(W(y)), y \in \mathbb{R}^N.$$

According to the maximum principle, we obtain that W is positive. Let us prove that $W_\varepsilon \to W$ in X_ε. Suppose there exist $R > 0$ and a sequence $\{z_\varepsilon\}_\varepsilon$ with $z_\varepsilon \in B(\frac{x}{\varepsilon}, \frac{2\beta}{\varepsilon})$ satisfying

$$\liminf_{\varepsilon\to 0}\left|z_\varepsilon - \frac{x}{\varepsilon}\right| = \infty \quad \text{and} \quad \liminf_{\varepsilon\to 0}\int_{B(z_\varepsilon,R)}\left(u_\varepsilon^1\right)^2\mathrm{d}y > 0.$$

We can assume that $\varepsilon z_\varepsilon \to z_0 \in \Lambda$ as $\varepsilon \to 0$. Then we have $\widetilde{W}_\varepsilon(y) = u_\varepsilon^1(y + z_\varepsilon)$ converges weakly to $\widetilde{W}$ in X_ε satisfying

$$(-\Delta)_p\widetilde{W} + (-\Delta)_q\widetilde{W} + V(z_0)(|\widetilde{W}|^{p-2}\widetilde{W} + |\widetilde{W}|^{q-2}\widetilde{W}) = f(\widetilde{W}), \text{ for } y \in \mathbb{R}^N.$$

At this point as before we get a contradiction, then by using $(f_1), (f_2)$ and Lemma 3.3 we obtain that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} F\left(W_\varepsilon\right)\mathrm{d}x \to \int_{\mathbb{R}^N} F(W)\mathrm{d}x. \tag{4.8}$$

It follows from the weak convergence of W_ε to W in X_ε that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} J_\varepsilon\left(u_\varepsilon^1\right) \\
\geq\ &\liminf_{\varepsilon\to 0} I_\varepsilon\left(u_\varepsilon^1\right) \\
=\ &\liminf_{\varepsilon\to 0}\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla W_\varepsilon(y)|^p + V_\varepsilon|W_\varepsilon(y)|^p)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla W_\varepsilon(y)|^q + V_\varepsilon|W_\varepsilon(y)|^q)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
& -\int_{\mathbb{R}^N} F\left(W_\varepsilon(y)\right) \mathrm{d}y \\
\geq\ & \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla W|^p+V_0|W|^p)\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}(|\nabla W|^q+V_0|W|^q)\mathrm{d}x-\int_{\mathbb{R}^N} F\left(W\right)\mathrm{d}y \\
\geq\ & c_{V_0}.
\end{aligned} \tag{4.9}
$$

On the other hand, due to $\lim_{\varepsilon\to 0} J_\varepsilon\left(u_\varepsilon\right)\leq c_{V_0}$, $J_\varepsilon\left(u_\varepsilon^2\right)\geq 0$ and because of (4.4), we have

$$
\limsup_{\varepsilon\to 0} J_\varepsilon\left(u_\varepsilon^1\right)\leq c_{V_0}. \tag{4.10}
$$

Combining (4.9) and (4.10), we obtain that $J_\varepsilon(W)=c_{V_0}$. Similar to [29], we can obtain that $x\in\mathcal{M}$. At this point it is clear that $W(y)=U(y-z)$ with $U\in\mathcal{S}_{V_0}$ and $z\in\mathbb{R}^N$.

Finally, by using (4.8) and (4.10) and the fact that $V(y)\geq V_0$ on Λ, it follows from (4.9) that

$$
\begin{aligned}
\int_{\mathbb{R}^N}(|\nabla W|^p+V_0|W|^p)\,\mathrm{d}y &\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\left|\nabla u_\varepsilon^1(y)\right|^p+V(\varepsilon y)|u_\varepsilon^1(y)|^p\right)\mathrm{d}y \\
&\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\left|\nabla u_\varepsilon^1(y)\right|^p+V_0|u_\varepsilon^1(y)|^p\right)\mathrm{d}y \\
&\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}(|\nabla W_\varepsilon(y)|^p+V_0|W_\varepsilon(y)|^p)\,\mathrm{d}y
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^N}(|\nabla W|^q+V_0|W|^q)\,\mathrm{d}y &\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\left|\nabla u_\varepsilon^1(y)\right|^q+V(\varepsilon y)|u_\varepsilon^1(y)|^q\right)\mathrm{d}y \\
&\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\left|\nabla u_\varepsilon^1(y)\right|^q+V_0|u_\varepsilon^1(y)|^q\right)\mathrm{d}y \\
&\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}(|\nabla W_\varepsilon(y)|^q+V_0|W_\varepsilon(y)|^q)\,\mathrm{d}y.
\end{aligned}
$$

Moreover, by using weak lower semi-continuity, we prove the strong convergence of u_ε^1 to W in X_ε. In particular, setting $y_\varepsilon=x/\varepsilon+z$ we obtain $u_\varepsilon^1\to\varphi_\varepsilon\left(\cdot-y_\varepsilon\right)U\left(\cdot-y_\varepsilon\right)$ strongly in X_ε. This means that $u_\varepsilon^1\to\varphi_\varepsilon\left(\cdot-y_\varepsilon\right)U\left(\cdot-y_\varepsilon\right)$ strongly in X_ε.

In order to conclude the proof of the Lemma, it suffices to show that $u_\varepsilon^2\to 0$ in X_ε. Now, using (4.4), $\lim_{\varepsilon\to 0} J_\varepsilon\left(u_\varepsilon^1\right)=c_{V_0}$ and the estimation of $J_\varepsilon\left(u_\varepsilon^2\right)$, we have that for some $C>0$

$$
c_{V_0}\geq\lim_{\varepsilon\to 0} J_\varepsilon\left(u_\varepsilon\right)\geq c_{V_0}+\left\|u_\varepsilon^2\right\|_{X_\varepsilon}^q\left(\frac{1}{2q}-C(4d)^{q^*-q}\right)+O(\varepsilon).
$$

This proves that $u_\varepsilon^2\to 0$ in X_ε, which completes the proof. □

Lemma 4.4. *For sufficiently small $d_1>d_2>0$, there exist constants $\omega>0$ and $\varepsilon_0>0$ such that $\left|J_\varepsilon'(u)\right|\geq\omega$ for $u\in J_\varepsilon^{\tilde{c}_\varepsilon}\cap\left(Y_\varepsilon^{d_1}\backslash Y_\varepsilon^{d_2}\right)$ and $\varepsilon\in(0,\varepsilon_0)$.*

Proof. To the contrary, we can suppose that for small $d_1>d_2>0$, there exist $\{\varepsilon_i\}_{i=1}^\infty$ with $\lim_{i\to\infty}\varepsilon_i=0$ and $u_{\varepsilon_i}\in Y_{\varepsilon_i}^{d_1}\backslash Y_{\varepsilon_i}^{d_2}$ satisfying $\lim_{i\to\infty} J_{\varepsilon_i}\left(u_{\varepsilon_i}\right)\leq c_{V_0}$ and $\lim_{i\to\infty}\left|J_{\varepsilon_i}'\left(u_{\varepsilon_i}\right)\right|=0$. Note that, for convenience' sake, we write ε for ε_i. By using Lemma 4.3, there exists $\{y_\varepsilon\}_\varepsilon\subset\mathbb{R}^N$ such that for some $U\in\mathcal{S}_{V_0}$ and $x\in\mathcal{M}$,

$$
\lim_{\varepsilon\to 0}|\varepsilon y_\varepsilon-x|=0 \quad\text{and}\quad \lim_{\varepsilon\to 0}\left\|u_\varepsilon-\varphi_\varepsilon\left(\cdot-y_\varepsilon\right)U\left(\cdot-y_\varepsilon\right)\right\|_\varepsilon=0.
$$

According to the definition of Y_ε, we obtain that $\lim\limits_{\varepsilon\to 0}\text{dist}\,(u_\varepsilon, Y_\varepsilon) = 0$. This contradicts that $u_\varepsilon \notin Y_\varepsilon^{d_2}$, and completes the proof. □

According to Lemma 4.4, we fix a $d > 0$ and corresponding $\omega > 0$ and $\varepsilon_0 > 0$ such that $\left|J'_\varepsilon(u)\right| \geq \omega$ for $u \in J_\varepsilon^{\tilde{c}_\varepsilon} \cap \left(Y_\varepsilon^d \backslash Y_\varepsilon^{d/2}\right)$ and $\varepsilon \in (0, \varepsilon_0)$. Then, we obtain the following Lemma.

Lemma 4.5. *There exists $\alpha > 0$ such that for sufficiently small $\varepsilon > 0$, $J_\varepsilon(\gamma_\varepsilon(s)) \geq c_\varepsilon - \alpha$ implies that $\gamma_\varepsilon(s) \in Y_\varepsilon^{d/2}$ where $\gamma_\varepsilon(s) = W_{\varepsilon, st_0}(s)$.*

Proof. Due to $\text{supp}\,(\gamma_\varepsilon(s)) \subset \mathcal{M}_\varepsilon^{2\beta}$ for each $s \in [0, 1]$, it follows that $J_\varepsilon(\gamma_\varepsilon(s)) = I_\varepsilon(\gamma_\varepsilon(s))$. Moreover, we see from a change of variables that

$$\begin{aligned}
I_\varepsilon(\gamma_\varepsilon(s)) &= \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla\gamma_\varepsilon(s)|^p + V_\varepsilon|\gamma_\varepsilon(s)|^p)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla\gamma_\varepsilon(s)|^q + V_\varepsilon|\gamma_\varepsilon(s)|^q)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} F(\gamma_\varepsilon(s))\mathrm{d}x \\
&= \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla\gamma_\varepsilon(s)|^p + V_0|\gamma_\varepsilon(s)|^p)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla\gamma_\varepsilon(s)|^q + V_0|\gamma_\varepsilon(s)|^q)\mathrm{d}x \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^N}(V_\varepsilon(x) - V_0)|\gamma_\varepsilon(s)|^p)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}(V_\varepsilon(x) - V_0)|\gamma_\varepsilon(s)|^q)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} F(\gamma_\varepsilon(s))\mathrm{d}x \\
&= \frac{(st_0)^{N-p}}{p}\int_{\mathbb{R}^N}|\nabla U|^p\mathrm{d}x + \frac{(st_0)^{N-q}}{q}\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x + \frac{(st_0)^N}{p}\int_{\mathbb{R}^N} V_0|U|^p\mathrm{d}x \\
&\quad + \frac{(st_0)^N}{q}\int_{\mathbb{R}^N} V_0|U|^q\mathrm{d}x - (st_0)^N\int_{\mathbb{R}^N} F(U)\mathrm{d}x + O(\varepsilon).
\end{aligned}$$

Then by using the Pohozǎev identity, we have that

$$\begin{aligned}
J_\varepsilon(\gamma_\varepsilon(s)) &= I_\varepsilon(\gamma_\varepsilon(s)) \\
&= \frac{(st_0)^{N-p}}{p}\int_{\mathbb{R}^N}|\nabla U|^p\mathrm{d}x + \frac{(st_0)^{N-q}}{q}\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x - \frac{N-p}{Np}(st_0)^N\int_{\mathbb{R}^N}|\nabla U|^p\mathrm{d}x \\
&\quad - \frac{N-q}{Nq}(st_0)^N\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x + O(\varepsilon) \\
&= \left(\frac{(st_0)^{N-p}}{p} - \frac{N-2}{Np}(st_0)^N\right)\int_{\mathbb{R}^N}|\nabla U|^p\mathrm{d}x \\
&\quad + \left(\frac{(st_0)^{N-q}}{q} - \frac{N-q}{Nq}(st_0)^N\right)\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x + O(\varepsilon).
\end{aligned}$$

Note that

$$c_{V_0} = \max_{t\in(0,\infty)}\left(\frac{t^{N-p}}{p} - \frac{N-2}{Np}t^N\right)\int_{\mathbb{R}^N}|\nabla U|^p\mathrm{d}x + \left(\frac{t^{N-q}}{q} - \frac{N-q}{Nq}t^N\right)\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x$$

and $\lim_{\varepsilon\to 0} c_\varepsilon = c_{V_0}$. Then, since, denoting $g_1(t) = \frac{t^{N-p}}{p} - \frac{N-p}{Np}t^N, g_2(t) = \frac{t^{N-q}}{q} - \frac{N-q}{Nq}t^N$,

$$g_1'(t)\begin{cases} > 0 & \text{for } t \in (0,1), \\ = 0 & \text{for } t = 1, \\ < 0 & \text{for } t > 1, \end{cases} \qquad g_2'(t)\begin{cases} > 0 & \text{for } t \in (0,1), \\ = 0 & \text{for } t = 1, \\ < 0 & \text{for } t > 1. \end{cases}$$

Then we have $g_1''(1) = p - N < 0$ and $g_2''(1) = q - N < 0$, the conclusion follows. □

Lemma 4.6. *For sufficiently fixed small $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$ such that $J_\varepsilon'(u_n) \to 0$ as $n \to \infty$.*

Proof. According to Lemma 4.5, there exists $\alpha > 0$ such that for sufficiently small $\varepsilon > 0$, $J_\varepsilon(\gamma_\varepsilon(s)) \geq c_\varepsilon - \alpha$ implies that $\gamma_\varepsilon(s) \in Y_\varepsilon^{d/2}$. If Lemma 4.6 does not hold for sufficiently small $\varepsilon > 0$, there exists $a(\varepsilon) > 0$ such that $\left|J_\varepsilon'(u)\right| \geq a(\varepsilon)$ on $Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$. Also we know from Lemma 4.4 that there exists $\omega > 0$, independent of $\varepsilon > 0$, such that $\left|J_\varepsilon'(u)\right| \geq \omega$ for $u \in J_\varepsilon^{\tilde{c}_\varepsilon} \cap \left(Y_\varepsilon^d \backslash Y_\varepsilon^{d/2}\right)$. Thus, recalling that $\lim_{\varepsilon\to 0}(c_\varepsilon - \tilde{c}_\varepsilon) = 0$, by a deformation argument, for sufficiently small $\varepsilon > 0$, it is possible to construct a path $\gamma \in \Gamma_\varepsilon$ satisfying $J_\varepsilon(\gamma(s)) < c_\varepsilon$, $s \in [0,1]$. This contradiction proves the Lemma. □

Lemma 4.7. *For sufficiently small fixed $\varepsilon > 0$, J_ε has a critical point $u_\varepsilon \in Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$.*

Proof. Let $\varepsilon > 0$ be fixed, small enough. According to Lemma 4.6, there exists a sequence $\{u_{n,\varepsilon}\}_{n=1}^{\infty} \subset Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$ such that $\left|J_\varepsilon'(u_{n,\varepsilon})\right| \to 0$ as $n \to \infty$. Since Y_ε^d is bounded, we can assume that $u_{n,\varepsilon} \rightharpoonup u_\varepsilon$ in X_ε as $n \to \infty$. Similar to [14, Proposition 3], we obtain that

$$\lim_{R\to\infty} \sup_{n\geq 1} \int_{|x|\geq R} \left(\left|\nabla u_{n,\varepsilon}\right|^p + V_\varepsilon \left|u_{n,\varepsilon}\right|^p\right) \mathrm{d}x = 0 \tag{4.11}$$

and

$$\lim_{R\to\infty} \sup_{n\geq 1} \int_{|x|\geq R} \left(\left|\nabla u_{n,\varepsilon}\right|^q + V_\varepsilon \left|u_{n,\varepsilon}\right|^q\right) \mathrm{d}x = 0, \tag{4.12}$$

which immediately implies that $u_{n,\varepsilon} \to u_\varepsilon$ in $L^r\left(\mathbb{R}^N\right)(p \leq r < q^*)$ as $n \to \infty$. Moreover, by using $(f_1) - (f_2)$, we have $\sup \left\|f(u_{n,\varepsilon})\right\| < \infty$. Then, for any $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$,

$$\int_{\mathbb{R}^N} f(u_{n,\varepsilon})(u_{n,\varepsilon} - u_\varepsilon)\varphi \mathrm{d}x \to 0 \quad \text{ as } n \to \infty.$$

Then, similar to [23, Proposition 5.3], $u_{n,\varepsilon} \to u_\varepsilon$ strongly in X_ε as $n \to \infty$. Thus, $J_\varepsilon'(u_\varepsilon) = 0$ in X_ε and $u_\varepsilon \in Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$. This completes the proof. □

Next, we use a Moser iteration argument [35] to obtain a fundamental L^∞-estimate.

Lemma 4.8. *Let (u_n) be the sequence defined as in Lemma 4.3. Then, $J_{\varepsilon_n}(u_n) \to c_{V_0}$ in $\mathbb{R}$ as $n \to \infty$, and there is some sequence $\{\hat{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $\hat{u}_n(\cdot) := u_n(\cdot + \hat{y}_n) \in L^\infty\left(\mathbb{R}^N\right)$ and $|\hat{u}_n|_{L^\infty(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$.*

Proof. Proceeding as in the proof of Lemma 4.1 and Lemma 4.2, we know that $J_{\varepsilon_n}(u_n) \to c_{V_0}$ in $\mathbb{R}$ as $n \to \infty$. Then, we can use Lemma 4.3 to deduce that there is a sequence $\{\hat{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $\hat{u}_n(\cdot) := u_n(\cdot + \hat{y}_n) \to \hat{u}(\cdot) \in X_0$ and $y_n := \varepsilon_n \hat{y}_n \to y_0 \in \mathcal{M}$ as $n \to \infty$. For any $L > 0$ and $\beta > 1$ we introduce the function

$$\psi(\hat{u}_n) := \hat{u}_n \hat{u}_{n,L}^{q(\beta-1)} \in X_{\varepsilon_n}, \text{ where } \hat{u}_{n,L} := \min\{\hat{u}_n, L\}.$$

Choosing $\psi(\hat{u}_n)$ as test function, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla \hat{u}|^{p-1} \hat{u}_n(x) \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla \hat{u}|^{q-1} \hat{u}_n(x) \mathrm{d}x \\
&+ \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |\hat{u}_n|^{p-2} \hat{u}_n \psi(\hat{u}_n) \mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |\hat{u}_n|^{q-2} \hat{u}_n \psi(\hat{u}_n) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} f(\varepsilon_n x + y_n, \hat{u}_n) \psi(\hat{u}_n) \mathrm{d}x.
\end{aligned}$$

According to the growth of f, we see that for any $\sigma > 0$ there exists $C_\sigma > 0$ such that

$$|f(t)| \leqslant \sigma |t|^p + C_\sigma |t|^{q^*} \text{ for all } t \in \mathbb{R}.$$

Using (V_1) and taking $\sigma \in (0, V_0)$, together with the above relations, we can conclude that

$$\int_{\mathbb{R}^N} |\nabla \hat{u}_n|^{p-1} \hat{u}_n(x) \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla \hat{u}_n|^{q-1} \hat{u}_n(x) \mathrm{d}x \leqslant C \int_{\mathbb{R}^N} |\hat{u}_n|^{q^*} \hat{u}_{n,L}^{q(\beta-1)} \mathrm{d}x \tag{4.13}$$

for some constant $C > 0$.

Now, let us introduce the following functions

$$\varphi(t) := \frac{|t|^q}{q} \quad \text{and} \quad \Upsilon(t) := \int_0^t (\psi'(\tau))^{\frac{1}{q}} \mathrm{d}\tau.$$

We first observe that ψ is an increasing function, so we have that

$$(a-b)(\psi(a) - \psi(b)) \geqslant 0 \quad \text{for all } a, b \in \mathbb{R}. \tag{4.14}$$

Then by using (4.14) and the Jensen inequality, we can obtain that

$$\varphi'(a-b)(\psi(a) - \psi(b)) \geqslant |\Upsilon(a) - \Upsilon(b)|^q \text{ for all } a, b \in \mathbb{R}. \tag{4.15}$$

Obviously, we have

$$\Upsilon(\hat{u}_n) \geqslant \frac{1}{\beta} \hat{u}_n \hat{u}_{n,L}^{\beta-1}. \tag{4.16}$$

Therefore, by using (4.13), (4.14), (4.15) and (4.16), we can look for some constant $C > 0$ such that

$$\left|\hat{u}_n \hat{u}_{n,L}^{\beta-1}\right|_{q^*}^q \leqslant C\beta^q \int_{\mathbb{R}^N} \hat{u}_n^{q^*} \hat{u}_{n,L}^{q(\beta-1)} \mathrm{d}x. \tag{4.17}$$

Choose $\beta = \frac{q^*}{q}$ and let $R > 0$ large enough. According to $\hat{u}_n \to \hat{u}$ in X_0 as $n \to \infty$ with the Hölder inequality, we can obtain that there exists some constant $C > 0$ such that

$$\left[\int_{\mathbb{R}^N} \left(\hat{u}_n \hat{u}_{n,L}^{\frac{q^*-q}{q}}\right)^{q^*} \mathrm{d}x\right]^{\frac{q}{q^*}} \leqslant C\beta^q \int_{\mathbb{R}^N} R^{q^*-q} \hat{u}_n^{q^*} \mathrm{d}x + C\epsilon \left[\int_{\mathbb{R}^N} \left(\hat{u}_n \hat{u}_{n,L}^{\frac{q^*-q}{q}}\right)^{q^*} \mathrm{d}x\right]^{\frac{q}{q^*}}.$$

We choose a fixed $\epsilon \in (0, 1/C)$ and deduce that

$$\left[\int_{\mathbb{R}^N} \left(\hat{u}_n \hat{u}_{n,L}^{\frac{q^*-q}{q}}\right)^{q^*} \mathrm{d}x\right]^{\frac{q}{q^*}} \leqslant C\beta^q \int R^{q^*-q} \hat{u}_n^{q^*} \mathrm{d}x < +\infty.$$

In the above inequality, we pass to the limit as $L \to +\infty$ and we can obtain $\hat{u}_n \in L^{\frac{q^*}{q}}\left(\mathbb{R}^N\right)$.

Due to $0 \leqslant \hat{u}_{n,L} \leqslant \hat{u}_n$, then in (4.17) we pass to the limit as $L \to +\infty$ and we obtain that

$$|\hat{u}_n|_{\beta q^*}^{\beta q} \leqslant C\beta^q \int_{\mathbb{R}^N} \hat{u}_n^{q^*+q(\beta-1)} \mathrm{d}x.$$

The fact means that

$$\left(\int_{\mathbb{R}^N} \hat{u}_n^{\beta q^*} \mathrm{d}x\right)^{\frac{1}{q^*(\beta-1)}} \leqslant \left(C^{1/q}\beta\right)^{\frac{1}{\beta-1}} \left[\int_{\mathbb{R}^N} \hat{u}_n^{q^*+q(\beta-1)} \mathrm{d}x\right]^{\frac{1}{q(\beta-1)}}.$$

Next, we consider the sequence $\{\beta_m\}_{m\geqslant 1} \subset \mathbb{R} (m \in \mathbb{N})$ which satisfies the following relation:

$$q^* + q\left(\beta_{m+1} - 1\right) = \beta_m q^* \quad \text{and} \quad \beta_1 = \frac{q^*}{q}.$$

It follows that

$$\beta_{m+1} = \beta_1^m \left(\beta_1 - 1\right) + 1,$$

and so we have that

$$\lim_{m\to\infty} \beta_m = +\infty.$$

Define

$$T_m := \left(\int_{\mathbb{R}^N} \hat{u}_n^{\beta_m q^*} dx\right)^{\frac{1}{q^*\left(\beta_{m-1}\right)}},$$

then we have

$$T_{m+1} \leqslant \left(C^{1/q}\beta_{m+1}\right)^{\frac{1}{\beta_{m+1}-1}} T_m.$$

Obviously, by using a standard iteration argument we obtain that

$$T_{m+1} \leqslant \prod_{k=1}^{m} \left(C^{1/q}\beta_{k+1}\right)^{\frac{1}{\beta_{k+1}-1}} T_1 \leqslant \bar{C} T_1, \text{ where } \bar{C} \text{ is independent of } m.$$

According to the above inequality we pass to the limit as $m \to \infty$ and then we deduce that $|\hat{u}_n|_{L^\infty(\mathbb{R}^N)} \leqslant C$ uniformly in $n \in \mathbb{N}$. □

Proof of Theorem 1.1. According to Lemma 4.7, there exist $d > 0$ and $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, J_ε has a critical point $u_\varepsilon \in Y_\varepsilon^d \cap \Gamma_\varepsilon^{\tilde{c}_\varepsilon}$. Since u_ε satisfies

$$(-\Delta)_p u_\varepsilon + (-\Delta)_q u_\varepsilon + V(\varepsilon x)(|u_\varepsilon|^{p-2}u_\varepsilon + |u_\varepsilon|^{q-2}u_\varepsilon) = f(u_\varepsilon) + 4\left(\int_{\mathbb{R}^N} \chi_\varepsilon u_\varepsilon^p \mathrm{d}x - 1\right)_+ \chi_\varepsilon u_\varepsilon \text{ in } \mathbb{R}^N$$

and $f(t) = 0$ for $t \leq 0$, we have that $u_\varepsilon > 0$ in $\mathbb{R}^N$. Moreover, by elliptic estimates through Moser iteration scheme, that is Lemma 4.8, we obtain that $\{\|u_\varepsilon\|_{L^\infty}\}_\varepsilon$ is bounded. Now by using Lemma 4.3, we have

$$\lim_{\varepsilon\to 0}\left[\frac{1}{p}\left(\int_{\mathbb{R}^N\setminus\mathcal{M}_\varepsilon^{2\delta}} |\nabla u_\varepsilon|^p + V_\varepsilon\,(u_\varepsilon)^p\,\mathrm{d}x\right) + \frac{1}{q}\left(\int_{\mathbb{R}^N\setminus\mathcal{M}_\varepsilon^{2\delta}} |\nabla u_\varepsilon|^q + V_\varepsilon\,(u_\varepsilon)^q\,\mathrm{d}x\right)\right] = 0,$$

and thus, by elliptic estimates (see [22]), we have that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N\setminus\mathcal{M}_\varepsilon^{2\delta})} = 0.$$

Similar to [44], it is easy to check that there exist $C, c > 0$, independent of $u \in \mathcal{S}_{V_0}$ such that

$$u(x) \leq C\exp(-c|x|).$$

In fact, by using the Radial Lemma ([10], Radial Lemma A.IV) we obtain

$$u(x) \leq C\frac{\|u\|_{L^p}}{|x|^{N/p}} \quad \text{for all } x \neq 0,$$

where $C = C(N, p)$. Thus $\lim_{|x|\to\infty} u(x) = 0$ uniformly for $u \in \mathcal{S}_{V_0}$. By the comparison principle there exist $C, c > 0$, independent of $u \in \mathcal{S}_{V_0}$ such that

$$u(x) \leq C\exp(-c|x|) \quad \text{for all } x \in \mathbb{R}^N.$$

According to a comparison principle, for some $C, c > 0$, we obtain that

$$u_\varepsilon(x) \leq C\exp\left(-c\,\mathrm{dist}\left(x, \mathcal{M}_\varepsilon^{2\delta}\right)\right).$$

This implies that $Q_\varepsilon(u_\varepsilon) = 0$ and thus u_ε satisfies (1.1). Finally let x_ε be a maximum point of u_ε. By Lemma 3.6 and Lemma 4.3, we readily deduce that $\varepsilon x_\varepsilon \to x$ for some $x \in \mathcal{M}$ as $\varepsilon \to 0$, and that for some $C, c > 0$,

$$u_\varepsilon(x) \leq C\exp(-c\,|x - x_\varepsilon|).$$

This completes the proof.

Acknowledgments

L. Wang was supported by National Natural Science Foundation of China (No. 12161038), Science and Technology project of Jiangxi provincial Department of Education (Grant No. GJJ212204, GJJ2200635) and Natural Science Foundation program of Jiangxi Provincial (20202BABL211004). D. Zhou was partially supported by the National Natural Science Foundation of China (No. 12071113).

Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

References

1. C. O. Alves, V. Ambrosio, T. Isernia, Existence, multiplicity and concentration for a class of fractional $p\&q$ Laplacian problems in $\mathbb{R}^N$, *Commun. Pur. Appl. Anal.*, **18** (2019), 2009–2045. https://doi.org/10.3934/cpaa.2019091
2. V. Ambrosio, Concentration phenomena for a fractional relativistic Schrödinger equation with critical growth, arXiv: 2105.13632.
3. V. Ambrosio, Concentration phenomena for a class of fractional Kirchhoff equations in $\mathbb{R}^N$ with general nonlinearities, *Nonlinear Anal.*, **195** (2020), 111761. https://doi.org/10.1016/j.na.2020.111761
4. V. Ambrosio, Fractional $p\&q$ Laplacian problems in $\mathbb{R}^N$ with critical growth, *Z. Anal. Anwend.*, **39** (2020), 289–314. https://doi.org/10.4171/zaa/1661
5. V. Ambrosio, The nonlinear fractional relativistic Schrödinger equation: existence, multiplicity, decay and concentration results, *Discrete Cont. Dyn.*, **41** (2021), 5659–5705. https://doi.org/10.3934/dcds.2021092
6. V. Ambrosio, Mountain pass solutions for the fractional Berestycki-Lions problem, *Adv. Differential Equations*, **23** (2018), 455–488. https://doi.org/10.57262/ade/1516676484
7. V. Ambrosio, V. D. Rădulescu, Fractional double-phase patterns: concentration and multiplicity of solutions, *J. Math. Pure. Appl.*, **142** (2020), 101–145. https://doi.org/10.1016/j.matpur.2020.08.011
8. V. Ambrosio, T. Isernia, V. D. Rădulescu, Concentration of positive solutions for a class of fractional p-Kirchhoff type equations, *P. Roy. Soc. Edinb. A*, **151** (2021), 601–651. https://doi.org/10.1017/prm.2020.32
9. J. M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, *Philos. Trans. R. Soc. A*, **306** (1982), 557–611. https://doi.org/10.1098/rsta.1982.0095
10. H. Berestycki, P. L. Lions, Nonlinear scalar field equations, I, Existence of a ground state, *Arch. Rational. Mech. Anal.*, **82** (1983), 347–375. https://doi.org/10.1007/BF00250556
11. D. Bonheure, P. d'Avenia, A. Pomponio, On the electrostatic Born-Infeld equation with extended charges, *Commun. Math. Phys.*, **346** (2016), 877–906. https://doi.org/10.1007/s00220-016-2586-y
12. M. Born, L. Infeld, Foundations of the new field theory, *Nature*, **132** (1933), 1004.
13. J. Byeon, L. Jeanjean, Standing waves for nonlinear Schrödinger equations with a general nonlinearity, *Arch. Rational. Mech. Anal.*, **185** (2007), 185–200. https://doi.org/10.1007/s00205-006-0019-3
14. J. Byeon, Z. Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, *Arch. Rational Mech. Anal.*, **165** (2002), 295–316. https://doi.org/10.1007/s00205-002-0225-6

15. S. Barile, G. M. Figueiredo, Existence of least energy positive, negative and nodal solutions for a class of $p\&q$-problems with potentials vanishing at infinity, *J. Math. Anal. Appl.*, **427** (2015), 1205–1233. https://doi.org/10.1016/j.jmaa.2015.02.086

16. L. Cherfils, V. Il'yasov, On the stationary solutions of generalized reaction diffusion equations with $p\&q$-laplacian, *Commun. Pur. Appl. Anal.*, **4** (2005), 9–22. https://doi.org/10.3934/cpaa.2005.4.9

17. F. Colasuonno, M. Squassina, Eigenvalues for double phase variational integrals, *Ann. Mat. Pur. Appl.*, **195** (2016), 1917–1959. https://doi.org/10.1007/s10231-015-0542-7

18. G. S. Costa, G. M. Figueiredo, Existence and concentration of nodal solutions for a subcritical $p\&q$ equation, *Commun. Pur. Appl. Anal.*, **19** (2020), 5077–5095. https://doi.org/10.3934/cpaa.2020227

19. G. S. Costa, G. M. Figueiredo, Existence and concentration of positive solutions for a critical p&q equation, *Adv. Nonlinear Anal.*, **11** (2022), 243–267. https://doi.org/10.1515/anona-2020-0190

20. M. del Pino, P. L. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var.*, **4** (1996), 121–137. https://doi.org/10.1007/BF01189950

21. J. M. do Ó, On existence and concentration of positive bound states of p-Laplacian equations in $\mathbb{R}^N$ involving critical growth, *Nonlinear Anal. Theor.*, **62** (2005), 777–801. https://doi.org/10.1016/j.na.2005.03.093

22. D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Heidelberg: Springer, 2001. https://doi.org/10.1007/978-3-642-61798-0

23. E. Gloss, Existence and concentration of bound states for a p-Laplacian equation in $\mathbb{R}^N$, *Adv. Nonlinear Stud.*, **10** (2010), 273–296. https://doi.org/10.1515/ans-2010-0203

24. Y. He, G. Li, The existence and concentration of weak solutions to a class of p-Laplacian type problems in unbounded domains, *Sci. China Math.*, **57** (2014), 1927–1952. https://doi.org/10.1007/s11425-014-4830-2

25. Y. He, G. Li, Standing waves for a class of Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents, *Calc. Var.*, **54** (2015), 3067–3106. https://doi.org/10.1007/s00526-015-0894-2

26. J. Hirata, N. Ikoma, K. Tanaka, Nonlinear scalar field equations in $\mathbb{R}^N$: mountain pass and symmetric mountain pass approaches, *Topol. Methods Nonlinear Anal.*, **35** (2010), 253–276.

27. T. Isernia, Fractional $p\&q$-laplacian problems with potentials vanishing at infinity, *Opuscula Math.*, **40** (2020), 93–110. https://doi.org/10.7494/opmath.2020.40.1.93

28. S. Jarohs, Strong comparison principle for the fractional p-Laplacian and applications to starshaped rings, *Adv. Nonlinear Stud.*, **18** (2018), 691–714. https://doi.org/10.1515/ans-2017-6039

29. L. Jeanjean, K. Tanaka, A remark on least energy solutions in $\mathbb{R}^N$, *Proc. Amer. Math. Soc.*, **131** (2003), 2399–2408. https://doi.org/10.1090/S0002-9939-02-06821-1

30. C. Ji, V. D. Rădulescu, Multiplicity and concentration of solutions to the nonlinear magnetic Schrödinger equation, *Calc. Var.*, **59** (2020), 115. https://doi.org/10.1007/s00526-020-01772-y

31. H. Jin, W. Liu, J. Zhang, Singularly perturbed fractional Schrödinger equation involving a general critical nonlinearity, *Adv. Nonlinear Stud.*, **18** (2018), 487–499. https://doi.org/10.1515/ans-2018-2015

32. T. Jin, Y. Y. Li, J. Xiong, On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions, *J. Eur. Math. Soc.*, **16** (2014), 1111–1171. https://doi.org/10.4171/JEMS/456

33. G. Li, X. Liang, The existence of nontrivial solutions to nonlinear elliptic equation of *p*-*q*-Laplacian type on $\mathbb{R}^N$, *Nonlinear Anal.*, **71** (2009), 2316–2334. https://doi.org/10.1016/j.na.2009.01.066

34. W. Liu, G. Dai, Existence and multiplicity results for double phase problem, *J. Differ. Equations*, **265** (2018), 4311–4334. https://doi.org/10.1016/j.jde.2018.06.006

35. J. Moser, A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations, *Commun. Pure Appl. Math.*, **13** (1960), 457–468. https://doi.org/10.1002/cpa.3160130308

36. D. Mugnai, N. S. Papageorgiou, Wang's multiplicity result for superlinear (p,q)-equations without the Ambrosetti-Rabinowitz condition, *Trans. Amer. Math. Soc.*, **366** (2014), 4919–4937. https://doi.org/10.1090/S0002-9947-2013-06124-7

37. K. Perera, M. Squassina, Existence results for double-phase problems via Morse theory, *Commun. Contemp. Math.*, **20** (2018), 1750023. https://doi.org/10.1142/S0219199717500237

38. P. Pucci, J. Serrin, A general variational identity, *Indiana Univ. Math. J.*, **35** (1986), 681–703. https://doi.org/10.1512/iumj.1986.35.35036

39. L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Commun. Pure Appl. Math.*, **60** (2007), 67–112. https://doi.org/10.1002/cpa.20153

40. V. C. Z. Sissa, P. H. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^N$, *Commun. Pure Appl. Math.*, **45** (1992), 1217–1269. https://doi.org/10.1002/cpa.3160451002

41. W. A. Strauss, Mathematical aspects of classical nonlinear field equations, In: *Lecture notes in physics*, Heidelberg: Springer, 1979, 123–149. https://doi.org/10.1007/3-540-09246-3_7

42. M. Willem, *Minimax theorems*, MA: Birkhäuser Boston, 1996. https://doi.org/10.1007/978-1-4612-4146-1

43. M. Wu, Z. Yang, A class of *p*-*q*-Laplacian type equation with potentials eigenvalue problem in $\mathbb{R}^N$, *Bound. Value Probl.*, **2009** (2009), 185319. https://doi.org/10.1155/2009/185319

44. J. Zhang, D. G. Costa, J. M. do Ó, Semiclassical states of p-Laplacian equations with a general nonlinearity in critical case, *J. Math. Phys.*, **57** (2016), 071504. https://doi.org/10.1063/1.4959220

45. J. Zhang, W. Zhang, V. D. Rădulescu, Double phase problems with competing potentials: concentration and multiplication of ground states, *Math. Z.*, **301** (2022), 4037–4078. https://doi.org/10.1007/s00209-022-03052-1

46. W. Zhang, S. Yuan, L. Wen, Existence and concentration of ground-states for fractional Choquard equation with indefinite potential, *Adv. Nonlinear Anal.*, **11** (2022), 1552–1578. https://doi.org/10.1515/anona-2022-0255

47. W. Zhang, J. Zhang, Multiplicity and concentration of positive solutions for fractional unbalanced double phase problems, *J. Geom. Anal.*, **32** (2022), 235. https://doi.org/10.1007/s12220-022-00983-3

48. W. Zhang, J. Zhang, V. D. Rădulescu, Concentrating solutions for singularly perturbed double phase problems with nonlocal reaction, *J. Differ. Equations*, **347** (2023), 56–103. https://doi.org/10.1016/j.jde.2022.11.033
49. Y. Zhang, X. Tang, V. D. Rădulescu, Concentration of solutions for fractional double-phase problems: critical and supercritical cases, *J. Differ. Equations*, **302** (2021), 139–184. https://doi.org/10.1016/j.jde.2021.08.038