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Research article

High-dimensional Lehmer problem on Beatty sequences

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Abstract: Let q be a positive integer. For each integer a with $1 \leqslant a < q$ and $(a, q) = 1$, it is clear that there exists one and only one $\bar{a}$ with $1 \leqslant \bar{a} < q$ such that $a\bar{a} \equiv 1(q)$. Let k be any fixed integer with $k \geq 2, 0 < \delta_i \leq 1, i = 1, 2, \cdots, k$. $r_n(\delta_1, \delta_2, \cdots, \delta_k, \alpha, \beta, c; q)$ denotes the number of all k-tuples with positive integer coordinates $(x_1, x_2, \ldots, x_k)$ such that $1 \leq x_i \leq \delta_i q, (x_i, q) = 1, x_1 x_2 \cdots x_k \equiv c(q)$, and $x_1, x_2, \cdots, x_{k-1} \in B_{\alpha,\beta}$. In this paper, we consider the high-dimensional Lehmer problem related to Beatty sequences over incomplete intervals and give an asymptotic formula by the properties of Beatty sequences and the estimates for hyper Kloosterman sums.

Keywords: the Lehmer problem; Beatty sequence; exponential sum; asymptotic formula
Mathematics Subject Classification: 11B83, 11L05, 11N69

1. Introduction

Let q be a positive integer. For each integer a with $1 \leqslant a < q, (a, q) = 1$, we know that there exists one and only one $\bar{a}$ with $1 \leqslant \bar{a} < q$ such that $a\bar{a} \equiv 1(q)$. Let $r(q)$ be the number of integers a with $1 \leqslant a < q$ for which a and $\bar{a}$ are of opposite parity.

D. H. Lehmer (see [1]) posed the problem to investigate a nontrivial estimation for $r(q)$ when q is an odd prime. Zhang [2, 3] gave some asymptotic formulas for $r(q)$, one of which reads as follows:

$$r(q) = \frac{1}{2}\phi(q) + O\left(q^{\frac{1}{2}} d^2(q) \log^2 q\right).$$

Zhang [4] generalized the problem over short intervals and proved that

$$\sum_{\substack{a \leq N \\ a \in R(q)}} 1 = \frac{1}{2} N\phi(q) q^{-1} + O\left(q^{\frac{1}{2}} d^2(q) \log^2 q\right),$$

where

$$R(q) := \{a : 1 \leqslant a \leqslant q, (a, q) = 1, 2 \nmid a + \bar{a}\}.$$

Let $n \geqslant 2$ be a fixed positive integer, $q \geqslant 3$ and c be two integers with $(n, q) = (c, q) = 1$. Let $0 < \delta_1, \delta_2 \leq 1$. Lu and Yi [5] studied the Lehmer problem in the sense of short intervals as

$$r_n(\delta_1, \delta_2, c; q) := \sum_{\substack{a \leqslant \delta_1 q \\ a\bar{a} \equiv c \bmod q \\ n \nmid a + \bar{a}}} \sum_{\bar{a} \leqslant \delta_2 q} 1,$$

and obtained an interesting asymptotic formula,

$$r_n(\delta_1, \delta_2, c; q) = \left(1 - n^{-1}\right)\delta_1\delta_2\phi(q) + O\left(q^{\frac{1}{2}} d^6(q) \log^2 q\right).$$

Liu and Zhang [6] r-th residues and roots, and obtained two interesting mean value formulas. Guo and Yi [7] found the Lehmer problem also has good distribution properties on Beatty sequences. For fixed real numbers α and β, the associated non-homogeneous Beatty sequence is the sequence of integers defined by

$$\mathcal{B}_{\alpha,\beta} := (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty},$$

where $\lfloor t \rfloor$ denotes the integer part of any $t \in \mathbb{R}$. Such sequences are also called generalized arithmetic progressions. If α is irrational, it follows from a classical exponential sum estimate of Vinogradov [8] that $\mathcal{B}_{\alpha,\beta}$ contains infinitely many prime numbers; in fact, one has the asymptotic estimate

$$\#\left\{ \text{prime } p \leqslant x : p \in \mathcal{B}_{\alpha,\beta} \right\} \sim \alpha^{-1}\pi(x) \quad \text{as} \quad x \to \infty$$

where $\pi(x)$ is the prime counting function.

We define type $\tau = \tau(\alpha)$ for any irrational number α by the following definition:

$$\tau := \sup\left\{ t \in \mathbb{R} : \liminf_{n \to \infty} n^t \|\alpha n\| = 0 \right\}.$$

Based on the results obtained, we consider the high-dimensional Lehmer problem related to Beatty sequences over incomplete intervals in this paper. That is,

$$r_n(\delta_1, \delta_2, \cdots, \delta_k, c, \alpha, \beta; q) := \sideset{}{'}\sum_{\substack{x_1 \leqslant \delta_1 q \\ x_1 \cdots x_k \equiv c \bmod q \\ x_1, \cdots x_{k-1} \in B_{\alpha,\beta} \\ n \nmid x_1 + \cdots + x_k}} \cdots \sideset{}{'}\sum_{x_k \leqslant \delta_k q} 1, (0 < \delta_1, \delta_2, \cdots, \delta_k \leq 1),$$

and where k = 2, we get the result of [7].

By using the properties of Beatty sequences and the estimates for hyper Kloosterman sums, we obtain the following result.

Theorem 1.1. *Let $k \geq 2$ be a fixed positive integer, $q \geq n^3$ and c be two integers with $(n, q) = (c, q) = 1$, and $\delta_1, \delta_2, \cdots, \delta_k$ be real numbers satisfying $0 < \delta_1, \delta_2, \cdots, \delta_k \leq 1$. Let $\alpha > 1$ be an irrational number of finite type. Then, we have the following asymptotic formula:*

$$r_n(\delta_1, \delta_2, \cdots, \delta_k, c, \alpha, \beta; q) = \left(1 - n^{-1}\right)\alpha^{-(k-1)}\delta_1\delta_2 \cdots \delta_k \phi^{k-1}(q) + O(q^{k-1-\frac{1}{\tau+1}+\varepsilon}),$$

where $\phi(\cdot)$ is the Euler function, ε is a sufficiently small positive number, and the implied constant only depends on n.

Notation. In this paper, we denote by $\lfloor t \rfloor$ and $\{t\}$ the integral part and the fractional part of t, respectively. As is customary, we put

$$\mathbf{e}(t) := e^{2\pi i t} \quad \text{and} \quad \{t\} := t - \lfloor t \rfloor.$$

The notation $\|t\|$ is used to denote the distance from the real number t to the nearest integer; that is,

$$\|t\| := \min_{n \in \mathbb{Z}} |t - n|.$$

Let χ^0 be the principal character modulo q. The letter p always denotes a prime. Throughout the paper, ε always denotes an arbitrarily small positive constant, which may not be the same at different occurrences; the implied constants in symbols $O, \ll$ and $\gg$ may depend (where obvious) on the parameters α, n, ε but are absolute otherwise. For given functions F and G, the notations $F \ll G$, $G \gg F$ and $F = O(G)$ are all equivalent to the statement that the inequality $|F| \leqslant C|G|$ holds with some constant $C > 0$.

2. Preliminary lemmas

To complete the proof of the theorem, we need the following several definitions and lemmas.

Definition 2.1. *For an arbitrary set $\mathcal{S}$, we use $\mathbf{1}_{\mathcal{S}}$ to denote its indicator function:*

$$\mathbf{1}_{\mathcal{S}}(n) := \begin{cases} 1 & \text{if } n \in \mathcal{S}, \\ 0 & \text{if } n \notin \mathcal{S}. \end{cases}$$

We use $\mathbf{1}_{\alpha,\beta}$ to denote the characteristic function of numbers in a Beatty sequence:

$$\mathbf{1}_{\alpha,\beta}(n) := \begin{cases} 1 & \text{if } n \in \mathcal{B}_{\alpha,\beta}, \\ 0 & \text{if } n \notin \mathcal{B}_{\alpha,\beta}. \end{cases}$$

Lemma 2.2. *Let a, q be integers, $\delta \in (0, 1)$ be a real number, θ be a rational number. Let α be an irrational number of finite type τ and $H = q^{\varepsilon} > 0$. We have*

$$\sideset{}{'}\sum_{\substack{a \leqslant \delta q \\ a \in \mathcal{B}_{\alpha,\beta}}} 1 = \alpha^{-1} \delta \phi(q) + O\left((\phi(q))^{\frac{\tau}{\tau+1} + \varepsilon}\right),$$

and

$$\sum_{\substack{a \leqslant \delta q \\ a \in \mathcal{B}_{\alpha,\beta}}} \mathbf{e}(\theta a) = \alpha^{-1} \sum_{a \leqslant \delta_1 q} \mathbf{e}(\theta a) + O\left(\|\theta\|^{-1} q^{-\varepsilon} + q^{\varepsilon}\right).$$

Taking

$$H = \|\theta\|^{-\frac{1}{\tau+1} + \varepsilon},$$

we have

$$\sum_{\substack{a \leqslant \delta q \\ a \in B_{\alpha,\beta}}} \mathbf{e}(\theta a) = \alpha^{-1} \sum_{a \leqslant \delta_1 q} \mathbf{e}(\theta a) + O\left(\|\theta\|^{-\left(\frac{\tau}{\tau+1} + \varepsilon\right)}\right).$$

Proof. This is Lemma 2.4 and Lemma 2.5 of [7]. □

Lemma 2.3. *Let*

$$\mathbf{Kl}(r_1, r_2, \cdots, r_k; q) = \sum_{x_1 \leqslant q-1} \cdots \sum_{x_{k-1} \leqslant q-1} \mathbf{e}\left(\frac{r_1 x_1 + \cdots + r_{k-1}x_{k-1} + r_k \overline{x_1 \cdots x_{k-1}}}{p}\right).$$

Then

$$\mathbf{Kl}(r_1, r_2, \cdots, r_k; q) \ll q^{\frac{k-1}{2}} k^{\omega(q)} (r_1, r_k, q)^{\frac{1}{2}} \cdots (r_{k-1}, r_k, q)^{\frac{1}{2}}$$

where (a, b, c) *is the greatest common divisor of* a, b *and* c.

Proof. See [9]. □

Lemma 2.4. *Assume that* U *is a positive real number,* K *is a positive integer and that* a *and* b *are two real numbers. If*

$$a = \frac{s}{r} + \frac{\theta}{r^2}, \quad (r, s) = 1, r \geq 1, |\theta| \leq 1,$$

then

$$\sum_{k \leqslant K} \min(U, \frac{1}{||ak + b||}) \ll (\frac{K}{r} + 1)(U + r \log r).$$

Proof. The proof is given in [10]. □

3. Proof of theorem

We begin by the definition

$$r_n(\delta_1, \delta_2, \cdots, \delta_k, c, \alpha, \beta; q) = S_1 - S_2,$$

where

$$S_1 := \sum_{\substack{x_1 \leqslant \delta_1 q \\ x_1 \cdots x_k \equiv c \bmod q \\ x_1, \cdots x_{k-1} \in \mathcal{B}_{\alpha,\beta}}} \cdots \sum_{x_k \leqslant \delta_k q} 1,$$

and

$$S_2 := \sum_{\substack{x_1 \leqslant \delta_1 q \\ x_1 \cdots x_k \equiv c \bmod q \\ x_1, \cdots x_{k-1} \in \mathcal{B}_{\alpha,\beta} \\ n | x_1 + \cdots + x_k}} \cdots \sum_{x_k \leqslant \delta_k q} 1.$$

By the Definition 2.1, Lemma 2.2 and congruence properties, we have

$$\begin{aligned} S_1 &= \sum_{\substack{x_1 \leqslant \delta_1 q \\ x_1 \cdots x_k \equiv c \bmod q}} \cdots \sum_{x_k \leqslant \delta_k q} \mathbf{1}_{\alpha,\beta}(x_1) \cdots \mathbf{1}_{\alpha,\beta}(x_{k-1}) \\ &= \frac{1}{\phi(q)} \sum_{x_1 \leqslant \delta_1 q} \cdots \sum_{x_k \leqslant \delta_k q} \sum_{\chi \bmod q} \chi(x_1) \cdots \chi(x_k) \chi(\overline{c}) \mathbf{1}_{\alpha,\beta}(x_1) \cdots \mathbf{1}_{\alpha,\beta}(x_{k-1}) \\ &= S_{11} + S_{12}, \end{aligned}$$

where

$$S_{11} := \frac{1}{\phi(q)} \sideset{}{'}\sum_{x_1 \leqslant \delta_1 q} \cdots \sideset{}{'}\sum_{x_k \leqslant \delta_k q} \mathbf{1}_{\alpha,\beta}(x_1) \cdots \mathbf{1}_{\alpha,\beta}(x_{k-1}),$$

and

$$S_{12} := \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \chi(\overline{c}) \left(\sum_{x_1 \leqslant \delta_1 q} \cdots \sum_{x_k \leqslant \delta_k q} \chi(x_1) \cdots \chi(x_k) \mathbf{1}_{\alpha,\beta}(x_1) \cdots \mathbf{1}_{\alpha,\beta}(x_{k-1}) \right).$$

For S_2, it follows that

$$\begin{aligned} S_2 &= \frac{1}{\phi(q)} \sum_{\substack{x_1 \leqslant \delta_1 q \\ n \mid x_1 + \cdots + x_k}} \cdots \sum_{x_k \leqslant \delta_k q} \sum_{\chi \bmod q} \chi(x_1) \cdots \chi(x_k) \chi(\overline{c}) \mathbf{1}_{\alpha,\beta}(x_1) \cdots \mathbf{1}_{\alpha,\beta}(x_{k-1}) \\ &= S_{21} + S_{22}, \end{aligned}$$

where

$$S_{21} := \frac{1}{\phi(q)} \sideset{}{'}\sum_{\substack{x_1 \leqslant \delta_1 q \\ n \mid x_1 + \cdots + x_k}} \cdots \sideset{}{'}\sum_{x_k \leqslant \delta_k q} \mathbf{1}_{\alpha,\beta}(x_1) \cdots \mathbf{1}_{\alpha,\beta}(x_{k-1}),$$

and

$$S_{22} := \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \chi(\overline{c}) \sum_{\substack{x_1 \leqslant \delta_1 q \\ n \mid x_1 + \cdots + x_k}} \cdots \sum_{x_k \leqslant \delta_k q} \chi(x_1) \cdots \chi(x_{k-1}) \mathbf{1}_{\alpha,\beta}(x_1) \cdots \mathbf{1}_{\alpha,\beta}(x_{k-1}).$$

3.1. Estimation of S_{11}

From the classical bound

$$\sideset{}{'}\sum_{a \leqslant \delta q} 1 = \delta\phi(q) + O(d(q))$$

and Lemma 2.2, we have

$$\begin{aligned} S_{11} &= \frac{1}{\phi(q)} \left(\sideset{}{'}\sum_{x_1 \leqslant \delta_1 q} \mathbf{1}_{\alpha,\beta}(x_1) \right) \cdots \left(\sideset{}{'}\sum_{x_{k-1} \leqslant \delta_{k-1} q} \mathbf{1}_{\alpha,\beta}(x_{k-1}) \right) \left(\sideset{}{'}\sum_{x_k \leqslant \delta_k q} 1 \right) \\ &= \left(\delta_k + O\left(\frac{d(q)}{\phi(q)} \right) \right) \prod_{i=1}^{k-1} \left(\alpha^{-1} \delta_i \phi(q) + O\left((\phi(q))^{\frac{\tau}{\tau+1} + \varepsilon} \right) \right) \\ &= \alpha^{-(k-1)} \phi^{k-1}(q) \prod_{i=1}^{k-1} \delta_i + O\left(q^{k-1-\frac{1}{\tau+1}+\varepsilon} \right). \end{aligned} \tag{3.1}$$

3.2. *Estimation of* S_{21}

From Lemma 2.2, we obtain

$$
\begin{aligned}
S_{21} =& \frac{1}{\phi(q)}\left(\sum_{x_1\leqslant\delta_1 q}{}' \mathbf{1}_{\alpha,\beta}(x_1)\right)\cdots\left(\sum_{x_{k-1}\leqslant\delta_{k-1} q}{}' \mathbf{1}_{\alpha,\beta}(x_{k-1})\right)\left(\sum_{\substack{x_k\leqslant\delta_k q\\ n|x_k+(x_1+\cdots+x_{k-1})}}{}' 1\right)\\
=& \frac{1}{\phi(q)}\left(\sum_{x_1\leqslant\delta_1 q}{}' \mathbf{1}_{\alpha,\beta}(x_1)\right)\cdots\left(\sum_{x_{k-1}\leqslant\delta_{k-1} q}{}' \mathbf{1}_{\alpha,\beta}(x_{k-1})\right)\left(\sum_{\substack{x_k\leqslant\delta_k q\\ x_k\equiv-(x_1+\cdots+x_{k-1}) \bmod n}}\ \sum_{d|(x_k,q)}\mu(d)\right)\\
=& \frac{1}{\phi(q)}\left(\sum_{x_1\leqslant\delta_1 q}{}' \mathbf{1}_{\alpha,\beta}(x_1)\right)\cdots\left(\sum_{x_{k-1}\leqslant\delta_{k-1} q}{}' \mathbf{1}_{\alpha,\beta}(x_{k-1})\right)\left(\sum_{d|q}\mu(d)\sum_{\substack{x_k\leqslant\delta_k q\\ d|x_k\\ x_k\equiv-(x_1+\cdots+x_{k-1}) \bmod n}} 1\right)\\
=& \frac{1}{\phi(q)}\left(\sum_{x_1\leqslant\delta_1 q}{}' \mathbf{1}_{\alpha,\beta}(x_1)\right)\cdots\left(\sum_{x_{k-1}\leqslant\delta_{k-1} q}{}' \mathbf{1}_{\alpha,\beta}(x_{k-1})\right)\left(\sum_{d|q}\mu(d)\left(\frac{\delta_k q}{nd}+O(1)\right)\right)\\
=& \frac{1}{\phi(q)}\left(\frac{\delta_k\phi(q)}{n}+O\left(d(q)\right)\right)\prod_{i=1}^{k-1}\left(\alpha^{-1}\delta_i\phi(q)+O\left((\phi(q))^{\frac{\tau}{\tau+1}+\varepsilon}\right)\right)\\
=& \alpha^{-(k-1)}n^{-1}\phi^{k-1}(q)\prod_{i=1}^{k-1}\delta_i+O(q^{k-1-\frac{1}{\tau+1}+\varepsilon}).
\end{aligned}
\tag{3.2}
$$

3.3. *Estimation of* S_{22} *and* S_{12}

By the properties of exponential sums,

$$
\begin{aligned}
S_{22} =& \frac{1}{n\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\chi(\bar{c})\left(\sum_{x_1\leqslant\delta_1 q}\cdots\sum_{x_k\leqslant\delta_{k-1} q}\chi(x_1)\cdots\chi(x_k)\mathbf{1}_{\alpha,\beta}(x_1)\cdots\mathbf{1}_{\alpha,\beta}(x_{k-1})\right)\\
&\times\left(\sum_{l=1}^{n}\mathbf{e}(\frac{x_1+\cdots+x_k}{n}l)\right)\\
=& \frac{1}{n\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\chi(\bar{c})\sum_{l=1}^{n}\prod_{i=1}^{k-1}\left(\sum_{x_i\leqslant\delta_i q}\mathbf{1}_{\alpha,\beta}(x_i)\chi(x_i)\mathbf{e}(\frac{x_i}{n}l)\right)\left(\sum_{x_k\leqslant\delta_k q}\chi(x_k)\mathbf{e}(\frac{x_k}{n}l)\right).
\end{aligned}
\tag{3.3}
$$

Let

$$G(r,\chi) := \sum_{h=1}^{q}\chi(h)\mathbf{e}(\frac{rh}{q})$$

be the Gauss sum, and we know that for $\chi \neq \chi_0$,

$$\chi(x_i) = \frac{1}{q}\sum_{r=1}^{q} G(r,\chi)\mathbf{e}(-\frac{x_i r}{q}) = \frac{1}{q}\sum_{r=1}^{q-1} G(r,\chi)\mathbf{e}(-\frac{x_i r}{q}),$$

and

$$\frac{l}{n} - \frac{r}{q} \neq 0$$

for $1 \leqslant l \leqslant n, 1 \leqslant r \leqslant q-1$ and $(n,q) = 1$.

Therefore,

$$\sum_{x_k \leqslant \delta_k q} \chi(x_k)\mathbf{e}(\frac{x_k}{n} l) = \frac{1}{q}\sum_{r_k=1}^{q-1} G(r_k,\chi)\frac{f(\delta_k, l, r_k; n, q)}{\mathbf{e}(\frac{r_k}{q} - \frac{l}{h}) - 1}, \tag{3.4}$$

where

$$f(\delta, l, r; n, p) := 1 - \mathbf{e}\left((\frac{l}{n} - \frac{r}{q})\lfloor \delta q \rfloor\right)$$

and

$$|f(\delta_k, l, r_k; n, q)| \leqslant 2.$$

For $x_i (1 \leqslant i \leqslant k-1)$, using Lemma 2.2, we also have

$$\begin{aligned}
&\sum_{x_i \leqslant \delta_i q} \mathbf{1}_{\alpha,\beta}(x_i)\chi(x_i)\mathbf{e}(\frac{x_i}{n} l)\\
=&\frac{1}{q}\sum_{x_i \leqslant \delta_i q} \mathbf{1}_{\alpha,\beta}(x_i)\sum_{r_i=1}^{q-1} G(r_i,\chi)\,\mathbf{e}\left((\frac{l}{n} - \frac{r_i}{q})x_i\right)\\
=&\frac{1}{q}\sum_{r_i=1}^{q-1} G(r_i,\chi)\sum_{x_i \leqslant \delta_i q} \mathbf{1}_{\alpha,\beta}(x_i)\mathbf{e}\left((\frac{l}{n} - \frac{r_i}{q})x_i\right)\\
=&\frac{1}{q}\sum_{r_i=1}^{q-1} G(r_i,\chi)\left(\alpha^{-1}\sum_{a \leqslant \delta_i q}\mathbf{e}\left((\frac{l}{n} - \frac{r_i}{q})x_i\right) + O\left(\frac{q^{-\varepsilon}}{\|\frac{l}{n} - \frac{r_i}{q}\|} + q^{\varepsilon}\right)\right)\\
=&\frac{1}{q\alpha}\sum_{r_i=1}^{q-1} G(r_i,\chi)\left(\frac{f(\delta_i, l, r_i; n, q)}{\mathbf{e}(\frac{r_i}{q} - \frac{l}{n}) - 1} + O\left(\frac{q^{-\varepsilon}}{\|\frac{l}{n} - \frac{r_i}{q}\|} + q^{\varepsilon}\right)\right).
\end{aligned} \tag{3.5}$$

Let

$$\begin{aligned}
S_{23} &= \frac{1}{n\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi \neq \chi_0}} \chi(\overline{c})\sum_{l=1}^{n}\prod_{i=1}^{k-1}\left(\frac{1}{q\alpha}\sum_{r_i=1}^{q-1} G(r_i,\chi)\frac{f(\delta_i, l, r_i; n, q)}{\mathbf{e}(\frac{r_i}{q} - \frac{l}{n}) - 1}\right)\left(\frac{1}{q}\sum_{r_k=1}^{q-1} G(r_k,\chi)\frac{f(\delta_k, l, r_k; n, q)}{\mathbf{e}(\frac{r_k}{q} - \frac{l}{n}) - 1}\right)\\
&= \frac{1}{n\phi(q)q^k\alpha^{k-1}}\sum_{l=1}^{n}\sum_{r_1=1}^{q-1}\cdots\sum_{r_k=1}^{q-1}\frac{f(\delta_1, l, r_1; n, q)\cdots f(\delta_k, l, r_k; n, q)}{\left(\mathbf{e}(\frac{r_1}{q} - \frac{l}{n}) - 1\right)\cdots\left(\mathbf{e}(\frac{r_k}{q} - \frac{l}{n}) - 1\right)}\\
&\times \sum_{\substack{\chi \bmod q\\ \chi \neq \chi_0}} \chi(\overline{c})G(r_1,\chi)\cdots G(r_k,\chi).
\end{aligned} \tag{3.6}$$

From the definition of Gauss sum and Lemma 2.3, we know that

$$\begin{aligned}
&\sum_{\chi \bmod q} \chi(\overline{c}) G(r_1,\chi)\cdots G(r_k,\chi)\\
=&\sum_{h_1=1}^{q-1}\cdots\sum_{h_k=1}^{q-1}\sum_{\chi \bmod q}\chi(\overline{c})\chi(h_1)\cdots\chi(h_k)\mathbf{e}\Big(\frac{r_1h_1+\cdots+r_kh_k}{q}\Big)\\
=&\phi(q)\sum_{\substack{h_1=1\\ h_1\cdots h_k\equiv c \bmod q}}^{q-1}\cdots\sum_{h_k=1}^{q-1}\mathbf{e}\Big(\frac{r_1h_1+\cdots+r_kh_k}{q}\Big)\\
=&\phi(q)\sum_{h_1=1}^{q-1}\cdots\sum_{h_k=1}^{q-1}\mathbf{e}\Big(\frac{r_1h_1+\cdots r_{k-1}h_{k-1}+r_kc\overline{h_1\cdots h_{k-1}}}{q}\Big)\\
=&\phi(q)\mathbf{Kl}(r_1,r_2,\cdots,r_kc;q)\\
\ll&\phi(q)q^{\frac{k-1}{2}}k^{\omega(q)}(r_1,r_kc,q)^{\frac{1}{2}}\cdots(r_{k-1},r_kc,q)^{\frac{1}{2}}\\
\ll&\phi(q)q^{\frac{k-1}{2}}k^{\omega(q)}(r_1,q)\cdots(r_k,q)\,. \qquad (3.7)
\end{aligned}$$

By Mobius inversion, we get

$$G(r,\chi_0)=\sum_{h=1}^{q}{}'\,\mathbf{e}\Big(\frac{rh}{q}\Big)=\mu\left(\frac{q}{(r,q)}\right)\frac{\varphi(q)}{\varphi(q/(r,q))}\ll(r,q),$$

and

$$\chi_0(\overline{c})G(r_1,\chi_0)\cdots G(r_k,\chi_0)\ll(r_1,q)\cdots(r_k,q)\,.$$

Hence,

$$\begin{aligned}
&\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\chi(\overline{c})G(r_1,\chi)\cdots G(r_k,\chi)\\
=&\sum_{\chi \bmod q}\chi(\overline{c})G(r_1,\chi)\cdots G(r_k,\chi)-\chi_0(\overline{c})G(r_1,\chi_0)\cdots G(r_k,\chi_0)\\
\ll&\phi(q)q^{\frac{k-1}{2}}k^{\omega(q)}(r_1,q)\cdots(r_k,q)\,. \qquad (3.8)
\end{aligned}$$

From (3.8) we may deduce the following result:

$$\begin{aligned}
S_{23}\ll&\frac{k^{\omega(q)}}{nq^{\frac{k+1}{2}}\alpha^{k-1}}\sum_{l=1}^{n}\left(\sum_{r=1}^{q-1}\frac{(r,q)}{\left|\mathbf{e}(\frac{r}{q}-\frac{l}{n})-1\right|}\right)^k\\
\ll&\frac{k^{\omega(q)}}{nq^{\frac{k+1}{2}}\alpha^{k-1}}\sum_{l=1}^{n}\left(\sum_{r=1}^{q-1}\frac{(r,q)}{\left|\sin\pi(\frac{r}{q}-\frac{l}{n})\right|}\right)^k\\
\ll&\frac{k^{\omega(q)}}{nq^{\frac{k+1}{2}}\alpha^{k-1}}\sum_{l=1}^{n}\left(\sum_{r=1}^{q-1}\frac{(r,q)}{||\frac{r}{q}-\frac{l}{n}||}\right)^k
\end{aligned}$$

$$
\begin{aligned}
&= \frac{k^{\omega(q)}}{nq^{\frac{k+1}{2}}\alpha^{k-1}} \sum_{l=1}^{n} \left(\sum_{\substack{d|q \\ d<q}} \sum_{\substack{r \le q-1 \\ (r,q)=d}} \frac{d}{\|\frac{r}{q} - \frac{l}{n}\|} \right)^k \\
&= \frac{k^{\omega(q)}}{nq^{\frac{k+1}{2}}\alpha^{k-1}} \sum_{l=1}^{n} \left(\sum_{\substack{d|q \\ d<q}} d \sum_{\substack{m \le \frac{q-1}{d} \\ (m,q)=1}} \frac{1}{\|\frac{md}{q} - \frac{l}{n}\|} \right)^k \\
&= \frac{k^{\omega(q)}}{nq^{\frac{k+1}{2}}\alpha^{k-1}} \sum_{l=1}^{n} \left(\sum_{\substack{d|q \\ d<q}} d \sum_{k|q} \mu(k) \sum_{m \le \frac{q-1}{kd}} \frac{1}{\|\frac{mkd}{q} - \frac{l}{n}\|} \right)^k .
\end{aligned}
$$

It is easy to see

$$
\|\frac{mkd}{q} - \frac{l}{n}\| = \|\frac{mkn - l(q/d)}{(q/d)n}\| \ge \frac{1}{(q/d)n},
$$

and we obtain

$$
S_{23} \ll \frac{k^{\omega(q)}}{n\phi(q)q^{\frac{k+1}{2}}\alpha^{k-1}} \sum_{l=1}^{n} \left(\sum_{\substack{d|q \\ d<q}} d \sum_{k|q} \sum_{m \le \frac{q-1}{kd}} \min\left(\frac{qn}{d}, \frac{1}{\|\frac{mkd}{q} - \frac{l}{n}\|}\right) \right)^k .
$$

Let $kd/q = h_0/q_0$, where $q_0 \ge 1, (h_0, q_0) = 1$, and we will easily obtain $q/(kd) \le q_0 \le q/d$. By using Lemma 2.4 , we have

$$
\begin{aligned}
S_{23} &\ll \frac{k^{\omega(q)}}{nq^{\frac{k+1}{2}}\alpha^{k-1}} \sum_{l=1}^{n} \left(\sum_{\substack{d|q \\ d<q}} d \sum_{k|q} \left(\frac{(q-1)/(kd)}{q_0} + 1 \right) \left(\frac{qn}{d} + q_0 \log q_0\right) \right)^k \\
&\ll \frac{k^{\omega(q)}}{nq^{\frac{k+1}{2}}\alpha^{k-1}} \sum_{l=1}^{n} \left(\sum_{\substack{d|q \\ d<q}} d \sum_{k|q} \left(\frac{(q-1)/(kd)}{q/(kd)} + 1 \right) \left(\frac{qn}{d} + \frac{q}{d} \log \frac{q}{d}\right) \right)^k \\
&\ll \frac{k^{\omega(q)}q^{\frac{k-1}{2}}}{\alpha^{k-1}} \left(\sum_{\substack{d|q \\ d<q}} \sum_{k|q} n + \log q \right)^k \\
&\ll q^{\frac{k-1}{2}} d^{2k}(q)(\log q + n)^k.
\end{aligned}
$$

Let

$$
S_{24} := \frac{q^{(k-1)(-\varepsilon)}}{n\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \ne \chi_0}} \chi(\overline{c}) \sum_{l=1}^{n} \prod_{i=1}^{k-1} \left(\frac{1}{q\alpha} \sum_{r_i=1}^{q-1} G(r_i, \chi) \frac{1}{\|\frac{l}{n} - \frac{r_i}{q}\|} \right) \left(\frac{1}{q} \sum_{r_k=1}^{q-1} G(r_k, \chi) \frac{f(\delta_k, l, r_k; n, q)}{\mathbf{e}(\frac{r_k}{q} - \frac{l}{n}) - 1} \right)
$$

and

$$S_{25} := \frac{q^{(k-1)(\varepsilon)}}{n\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \chi(\overline{c}) \sum_{l=1}^{n} \prod_{i=1}^{k-1} \left(\frac{1}{q\alpha} \sum_{r_i=1}^{q-1} G(r_i, \chi) \right) \left(\frac{1}{q} \sum_{r_k=1}^{q-1} G(r_k, \chi) \frac{f(\delta_k, l, r_k; n, q)}{\mathbf{e}(\frac{r_k}{q} - \frac{l}{n}) - 1} \right).$$

By the same argument of S_{23}, it follows that

$$S_{24} \ll q^{\frac{k-1}{2} - \varepsilon} d^{2k}(q)(\log q + n)^k,$$

$$S_{25} \ll q^{\frac{k-3}{2} + \varepsilon}(\log q + n).$$

Since $n \ll q^{\frac{1}{3}}$, we have

$$S_{25} \ll S_{24} \ll S_{23} \ll q^{\frac{k-1}{2} + \varepsilon} n^k \ll q^{k-2+\varepsilon}. \tag{3.9}$$

Taking $n = 1$, we get

$$S_{12} \ll q^{\frac{k-1}{2} + \varepsilon}. \tag{3.10}$$

With (3.1), (3.2), (3.9) and (3.10), the proof is complete.

4. Conclusions

This paper considers the high-dimensional Lehmer problem related to Beatty sequences over incomplete intervals. And we give an asymptotic formula by the properties of Beatty sequences and the estimates for hyper Kloosterman sums.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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