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Research article

Some integral inequalities for harmonical cr-h-Godunova-Levin stochastic processes

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Abstract: An important part of optimization is the consideration of convex and non-convex functions. Furthermore, there is no denying the connection between the ideas of convexity and stochastic processes. Stochastic processes, often known as random processes, are groups of variables created at random and supported by mathematical indicators. Our study introduces a novel stochastic process for center-radius (cr) order based on harmonic h-Godunova-Levin ($\mathcal{GL}$) in the setting of interval-valued functions ($\mathcal{IVFS}$). With some interesting examples, we establish some variants of Hermite-Hadamard ($\mathcal{H.H}$) types inequalities for generalized interval-valued harmonic cr-h-Godunova-Levin stochastic processes.

Keywords: Jensen inequality; Hermite-Hadamard inequality; Godunova-Levin function; cr-order relation; stochastic h-convex
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1. Introduction

Interval analysis offers a variety of practical strategies for working with ambiguous data. This approach can be applied to models with data that are inaccurate because they were collected using unreliable measurement techniques. Interval analysis is a type of set-valued analysis that is used in mathematical analysis and general topology. We can handle interval uncertainty in some deterministic real-world phenomena using this technique. For the first time in numerical analysis, interval analysis was introduced in Moore's acclaimed book the mathematics of numerical analysis, see Ref. [1]. There has been the extensive application of interval analysis over the last fifty years in a variety of fields, such as the following: computer graphics [2], interval differential equation [3], automatic error analysis [4], and neural network output optimization [5], etc.

The concept of convexity is fundamental to many branches of mathematics and science, such as probability theory, economics, optimal control theory, fuzzy analysis, and natural and applied sciences. Furthermore, generalized convexity can also be a powerful tool for solving a variety of nonlinear analysis, applied analysis, and math and physics problems. The optimality conditions of diffeomorphic functions are characterised by variational inequalities, whose origins can be traced back to Euler, Lagrange, and Newton. As a counterpart to the arithmetic means, we have harmonic means. Among other applications, harmonic means are found in electrical circuit theory. By adding up the reciprocals of the individual resistance values of parallel resistors and considering the reciprocal of their combined value, we can obtain the total resistance of the set. In addition, harmonic means are used in parallel algorithms that solve a wide range of issues, see Ref. [6]. The study of convexity with integral problems is a particularly fascinating field. Integral inequalities have recently proven helpful for both qualitative and quantitative assessments of convexity. In mathematics, the Hermite-Hadamard inequality is well known for being the first geometric interpretation of convex maps. A famous double inequality is defined as follows:

$$\Phi\left(\frac{f+g}{2}\right) \leq \frac{1}{g-f}\int_f^g \Phi(\delta)d\delta \leq \frac{\Phi(f)+\Phi(g)}{2}, \tag{1.1}$$

where $\Phi : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex function on interval I and $f, g \in I$ with $f < g$. This function, which has been improved, generalized, and utilizing h-convexity, covers convexity classes of different sorts, see Refs. [7–17]. There is a class of convex functions called harmonic convex which were introduced by Anderson et al. [18]. Using the harmonic variational inequality, Noor [19] has shown that the optimality conditions of the differentiable harmonic convex functions on the harmonic convex set. A number of generalizations of integral inequalities apply to harmonic convex functions with applications across a variety of dimensions, see Refs. [20–28]. Furthermore, stochastic convexity must be understood in order to construct numerical estimates of existing probabilistic quantities, which is crucial in statistics and probability. In the beginning, Nikodem's 1980 work on convex stochastic processes, see Ref. [29]. Numerous examples of stochastic convexity applications were provided by Shaked et al. in [30]. In 1992, Skowronski revised the authors' earlier findings once more while also introducing some fresh ideas on convex stochastic processes and obtaining some additional findings, see Ref. [31]. In 2012, Kotrys extended a well-known double inequality called $\mathcal{H}.\mathcal{H}$ inequality to convex stochastic processes, see Ref. [32]. In 2015, Nelson Merentes and his co-authors utilized Varoşanec [33], concept of h-convexity and updated earlier findings generated by many writers in the context of h-convex stochastic processes. By describing h-convex stochastic processes, they develop $\mathcal{H}.\mathcal{H}$, Schur, and Jensen type inequalities in that work, see Ref. [34]. These inequalities for convex stochastic processes have undergone some recent developments, see Refs. [35–42]. Furthermore, Mevlut Tunc and the authors listed in [43, 44] created inequalities of the Ostrowski type for both h-convexity and h-convex stochastic processes, respectively.

Bhunia [45] developed the center-radius order in 2014 based on the radius and midpoints of the interval. Based on the ideas of center-radius order, the following authors created these inequalities for harmonical cr-h-convex and cr-h-Godunova-Levin functions in 2022, see Refs. [46–51]. By providing interesting examples, center-radius order relations about harmonical cr-h-Godunova-Levin functions can provide more precise inequality terms and can be demonstrated to be valid. The application of total order relations to convexity and inequality is therefore crucial for understanding. This order relation is somewhat different to calculate compared to the other order relations used in interval analysis to create inequalities; we can compute it using the midpoint and centre of the interval.

The novelty of the present study is the first time stochastic processes have been used in conjunction with interval analysis; it serves as a starting point for researchers interested in this field. We also observe that by applying this approach, the inequality term derived from the center-radius order relations using stochastic processes provides much more precise results than other partial order relations of this type. In order to verify the validity of our claim, we analyze interesting examples in which the interval difference between the end points is much closer. More importantly, we know that stochastic processes can be applied to interval analysis in many different ways, see Refs. [52–56]. Researchers examined gradient descent as an optimal method for strongly convex stochastic optimization in [57]. When considering terminal wealth with budget constraints, a continuous-time financial portfolio selection model with expected utility maximization boils down to solving a convex stochastic optimization problem, see Ref. [58].

Inspired by Refs. [34,43,44,49,51]. We create various $\mathcal{H}.\mathcal{H}$ inequalities in the context of $\mathcal{IVFS}$ by fusing center-radius order relation with harmonic h-$\mathcal{GL}$-convex stochastic process. The study provides several examples in addition to the conclusions.

2. Backgrounds and preliminaries

As it relates to concepts that have been utilised but not defined, see Refs. [7, 49]. It will be very helpful if you are familiar with a few fundamental arithmetic ideas related to interval analysis as you process the remaining portions of the article.

$$[\iota] = [\underline{\iota}, \overline{\iota}] \qquad (z \in \mathbf{R}, \underline{\Omega} \leqq z \leqq \overline{\Omega};\ z \in \mathbf{R}),$$
$$[\Omega] = [\underline{\Omega}, \overline{\Omega}] \qquad (z \in \mathbf{R}, \underline{\iota} \leqq z \leqq \overline{\Omega};\ z \in \mathbf{R}),$$
$$[\iota] + [\Omega] = [\underline{\iota}, \overline{\iota}] + [\underline{\Omega}, \overline{\Omega}] = [\underline{\iota} + \underline{\Omega}, \overline{\iota} + \overline{\Omega}]$$

and

$$\delta\Omega = \delta[\underline{\Omega}, \overline{\Omega}] = \begin{cases} \left[\delta\underline{\Omega}, \delta\overline{\Omega}\right], & (\delta > 0); \\ \{0\}, & (\delta = 0); \\ \left[\delta\overline{\Omega}, \delta\underline{\Omega}\right], & (\delta < 0), \end{cases}$$

where $\delta \in \mathbf{R}$.

Let $\mathbf{R}_I$ and $\mathbf{R}_I^+$ be the collection of all and positive intervals of $\mathbf{R}$, respectively. The following will discuss several algebraic properties of interval arithmetic.

Let $\Omega = [\underline{\Omega}, \bar{\Omega}] \in \mathbf{R}_I$, then $\Omega_c = \frac{\overline{\Omega}+\underline{\Omega}}{2}$ and $\Omega_r = \frac{\overline{\Omega}-\underline{\Omega}}{2}$ are basically the center and radius of interval Ω. A cr form of interval Ω can be expressed as:

$$\Omega = \langle \Omega_c, \Omega_r \rangle = \left\langle \frac{\overline{\Omega} + \underline{\Omega}}{2}, \frac{\overline{\Omega} - \underline{\Omega}}{2} \right\rangle.$$

The formulas we employ to establish an interval's radius and centre are as follows:

Definition 2.1. (See [49]) The cr-order relation for $\Omega = [\underline{\Omega}, \overline{\Omega}] = \langle \Omega_c, \Omega_r \rangle$, $\iota = [\underline{\iota}, \bar{\iota}] = \langle \iota_c, \iota_r \rangle \in \mathbf{R}_I$ represented as (see Figure 1).

$$\Omega \preceq_{cr} \iota \Longleftrightarrow \begin{cases} \Omega_c < \iota_c, & \text{if } \Omega_c \neq \iota_c; \\ \Omega_r \leq \iota_r, & \text{if } \Omega_c = \iota_c. \end{cases}$$

For these intervals $\Omega, \iota \in \mathbf{R}_I$, we have either $\Omega \preceq_{cr} \iota$ or $\iota \preceq_{cr} \Omega$. Riemann integral for $\mathcal{IVFS}$ are represented as:

Definition 2.2. (See [49]) Let $\Theta : [f, g]$ be an **IVF** such that $\Theta = [\underline{\Theta}, \overline{\Theta}]$. Then Θ is Riemann integrable (**IR**) on $[f, g]$ iff $\underline{\Theta}$ and $\overline{\Theta}$ are **IR** on $[f, g]$, that is,

$$(\mathbf{IR}) \int_f^g \Theta(\mathrm{s}) d\mathrm{s} = \left[(\mathbf{R}) \int_f^g \underline{\Theta}(\mathrm{s}) d\mathrm{s}, (\mathbf{R}) \int_f^g \overline{\Theta}(\mathrm{s}) ds \right].$$

The pack of all (**IR**) $\mathcal{IVFS}$ on $[f, g]$ is represented by $\mathbf{IR}_{([f,g])}$. The pack of all center-radius order $\mathcal{IVFS}$ are denoted by cr-$\mathcal{IVFS}$.

Theorem 2.1. (See [49]) Let $\Theta, \Phi : [f, g]$ be $\mathcal{IVFS}$ given by $\Theta = [\underline{\Theta}, \overline{\Theta}]$ and $\Phi = [\underline{\Phi}, \overline{\Phi}]$. If $\Theta(\mathrm{s}) \preceq_{cr} \Phi(\mathrm{s})$, for all $s \in [f, g]$, then

$$\int_f^g \Theta(\mathrm{s}) d\mathrm{s} \preceq_{\mathbf{cr}} \int_f^g \Phi(\mathrm{s}) d\mathrm{s}.$$

We will now give an example and a few thought-provoking instances to back up the aforementioned theorem (see Figure 2).

Example 2.1. Conider $\Theta = [z^2, z^2 + 2]$ and $\Phi = [z, 2z]$, then, $\forall\, z \in [0, 1]$.

$$\Phi_{\mathbf{C}} = \frac{3z}{2}, \Phi_{\mathbf{R}} = \frac{z}{2}, \Theta_{\mathbf{C}} = z^2 + 1 \quad \text{and} \quad \Theta_{\mathbf{R}} = 1.$$

From Definition 2.1, we have $\Phi(z) \preceq_{cr} \Theta(z)$, $\forall\, z \in [0, 1]$.

Since,

$$\int_0^1 [z, 2z] dz = \left[\frac{1}{2}, 1 \right]$$

and

$$\int_0^1 [z^2, z^2 + 2] dz = \left[\frac{1}{3}, \frac{7}{3} \right].$$

From Theorem 2.1, we have

$$\int_0^1 \Phi(z) dz \preceq_{cr} \int_0^1 \Theta(z) dz.$$

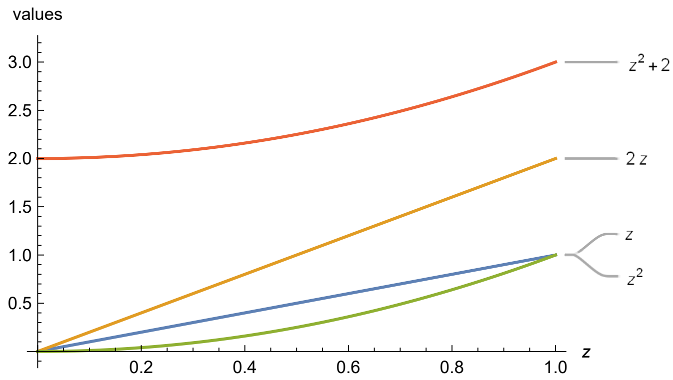

Figure 1. Graph shows that Definition 2.1 is valid.

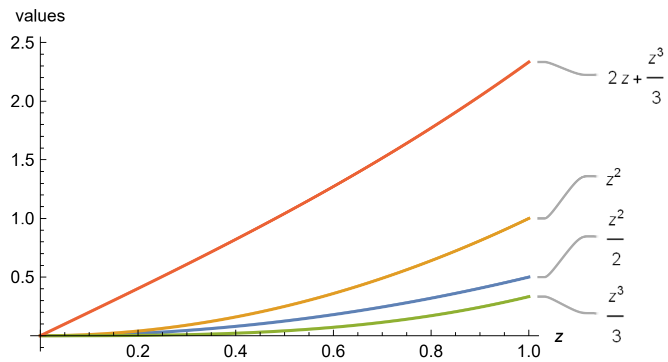

Figure 2. Graph shows that Theorem 2.1 holds.

Some novel definitions and properties

Definition 2.3. Define $(\Omega, \mathbb{A}, \mathcal{P})$ be a probability space (**PBS**). A function $\Phi : \Omega \to \mathbf{R}$ is said to be random variable if they satisfy the axioms of $\mathbb{A}$-measurable. A function $\Phi : I \times \Omega \to \mathbf{R}$ where $I \subseteq \mathbf{R}$ is known as stochastic process if, $\forall\ f \in I$ the function $\Phi(f, .)$ is a random variable.

Properties of stochastic process

A stochastic process $\Phi : I \times \Omega \to \mathbf{R}$ is

- Continuous in interval I, if $\forall\ f_o \in I$, we have

$$P - \lim_{f \to f_o} \Phi(f, .) = \Phi(f_o, .)$$

  where $P - \lim$ represent the limit in probability space.
- Mean square continuous over interval I, if $\forall\ f_o \in I$, we have

$$\lim_{f \to f_o} \mathbf{E}\left[(\Phi(f, .) - \Phi(f_o, .))^2\right] = 0$$

  where $\mathbf{E}\left[\Phi(f, .)\right]$ represent the expectation of random variable $\Phi(f, .)$.

- Mean-square differentiable at some point f, if one has random variable $\Phi' : I \times \Omega \to \mathbf{R}$, then this holds

$$\Phi'(f,.) = P - \lim_{f\to f_o} \frac{\Phi(f,.) - \Phi(f_o,.)}{f - f_o}.$$

- Mean square integral in interval I, if $\forall\ f \in I$, with $\mathbf{E}[\Phi(f,.)] < \infty$. Let $[f,g] \subseteq I$, $f = s_o < s_1 < s_2 ... < s_k$ is a partition of $[f,g]$. Consider $\eta_n \in [s_{n-1}, s_n]$, $\forall\, n = 1,...,k$. A random variable $S : \Omega \to \mathbf{R}$ is mean-square integral of the stochastic process Φ over interval $[f,g]$, if this holds

$$\lim_{k\to\infty} \mathbf{E}\left[\left(\sum_{n=1}^{k} \Phi(\eta_n,.)(s_n - s_{n-1}) - S(.)\right)^2\right] = 0.$$

In that case, we write it as

$$S(.) = \int_f^g \Phi(s,.)ds\ (a.e). \tag{2.1}$$

Definition 2.4. (See [49,51]) Consider $h : [0,1] \to \mathbf{R}^+$. We say that $\Phi : [f,g] \to \mathbf{R}^+$ is called h-convex function, or that $\Phi \in SX(\text{cr-h}, [f,g], \mathbf{R}^+)$, if $\forall\ f_1, g_1 \in [f,g]$ and $\delta \in [0,1]$, we have

$$\Phi(\delta f_1 + (1-\delta)g_1) \leq\ h(\delta)\Phi(f_1) + h(1-\delta)\Phi(g_1). \tag{2.2}$$

In (2.2), if "$\leq$" is replaced with "$\geq$", then it is called h-concave function or $\Phi \in SV(\text{cr-h}, [f,g], \mathbf{R}^+)$.

Definition 2.5. (See [49,51]) Consider $h : (0,1) \to \mathbf{R}^+$. We say that $\Phi : [f,g] \to \mathbf{R}^+$ is called h-$\mathcal{GL}$ function, or that $\Phi \in SGX$ $(\text{cr-h}, [f,g], \mathbf{R}^+)$, if $\forall\ f_1, g_1 \in [f,g]$ and $\delta \in [0,1]$, we have

$$\Phi(\delta f_1 + (1-\delta)g_1) \leq \frac{\Phi(f_1)}{h(\delta)} + \frac{\Phi(g_1)}{h(1-\delta)}. \tag{2.3}$$

In (2.3), if "$\leq$" is replaced with "$\geq$", then it is called h-$\mathcal{GL}$ concave function or $\Phi \in SGV(\text{cr-h}, [f,g], \mathbf{R}^+)$.

Definition 2.6. (See [34,51]). Consider $h : [0,1] \to \mathbf{R}^+$. We say that $\Phi : I \times \Omega \to \mathbf{R}^+$ is called h-convex stochastic process, or that $\Phi \in SPX(\text{cr-h}, I, \mathbf{R}^+)$, if $\forall\ f_1, g_1 \in I$ and $\delta \in (0,1)$, we have

$$\Phi(\delta f_1 + (1-\delta)g_1,.) \leq\ h(\delta)\Phi(f_1,.) + h(1-\delta)\Phi(g_1,.). \tag{2.4}$$

In (2.4), if "$\leq$" is replaced with "$\geq$", then it is called h-concave stochastic process or $\Phi \in SPV(\text{cr-h}, I, \mathbf{R}^+)$.

Definition 2.7. (See [34,51]) Consider $h : (0,1) \to \mathbf{R}^+$. We say that $\Phi : I \times \Omega \to \mathbf{R}^+$ is called h-$\mathcal{GL}$ convex stochastic process, or that $\Phi \in SGPX(\text{cr-h}, I, \mathbf{R}^+)$, if $\forall\ f_1, g_1 \in I$ and $\delta \in (0,1)$, we have

$$\Phi(\delta f_1 + (1-\delta)g_1,.) \leq \frac{\Phi(f_1,.)}{h(\delta)} + \frac{\Phi(g_1,.)}{h(1-\delta)}. \tag{2.5}$$

In (2.5), if "$\leq$" is replaced with "$\geq$", then it is called h-$\mathcal{GL}$-concave stochastic process or $\Phi \in SGPV(\text{cr-h}, I, \mathbf{R}^+)$.

Definition 2.8. (See [49,51]) Consider $h : [0,1] \to \mathbf{R}^+$. We say that $\Phi = [\underline{\Phi}, \overline{\Phi}] : [f,g] \to \mathbf{R}_{\mathbf{I}}^+$ is called cr-h-convex function, or that $\Phi \in SX(\text{cr-h}, [f,g], \mathbf{R}_{\mathbf{I}}^+)$, if $\forall\ f_1, g_1 \in [f,g]$ and $\delta \in [0,1]$, we have

$$\Phi(\delta f_1 + (1-\delta)g_1) \preceq_{cr}\ h(\delta)\Phi(f_1) + h(1-\delta)\Phi(g_1). \tag{2.6}$$

In (2.6), if "$\preceq_{cr}$" is replaced with "$\succeq_{cr}$", then it is called cr-h-concave function or $\Phi \in SV(\text{cr-h}, [f,g], \mathbf{R}_{\mathbf{I}}^+)$.

Definition 2.9. (See [49,51]) Consider $h : (0,1) \to \mathbf{R}^+$. We say that $\Phi = [\underline{\Phi}, \overline{\Phi}] : [f,g] \to \mathbf{R}_{\mathbf{I}}^+$ is called cr-h-$\mathcal{GL}$ convex function, or that $\Phi \in SGX(\text{cr-h}, [f,g], \mathbf{R}_{\mathbf{I}}^+)$, if $\forall\ f_1, g_1 \in [f,g]$ and $\delta \in (0,1)$, we have

$$\Phi(\delta f_1 + (1-\delta)g_1) \preceq_{cr} \frac{\Phi(f_1)}{h(\delta)} + \frac{\Phi(g_1)}{h(1-\delta)}. \tag{2.7}$$

In (2.7), if "$\preceq_{cr}$" is replaced with "$\succeq_{cr}$", then it is called cr-h-$\mathcal{GL}$-concave function or $\Phi \in SGV$(cr-h, $[f,g], \mathbf{R}_\mathbf{I}^+$).

Definition 2.10. (See [49, 51]). Consider $h : (0,1) \to \mathbf{R}^+$. We say that $\Phi = [\underline{\Phi}, \overline{\Phi}] : [f,g] \to \mathbf{R}_\mathbf{I}^+$ is called harmonic cr-h-$\mathcal{GL}$ convex function, or that $\Phi \in SGX$(cr-h, $[f,g], \mathbf{R}_\mathbf{I}^+$), if $\forall\ f_1, g_1 \in [f,g]$ and $\delta \in (0,1)$, we have

$$\Phi\left(\frac{f_1 g_1}{\delta f_1 + (1-\delta)g_1}\right) \preceq_{cr} \frac{\Phi(f_1)}{h(\delta)} + \frac{\Phi(g_1)}{h(1-\delta)}. \tag{2.8}$$

In (2.8), if "$\preceq_{cr}$" is replaced with "$\succeq_{cr}$", then it is called harmonic cr-h-$\mathcal{GL}$-concave function or $\Phi \in SGV$(cr-h, $[f,g], \mathbf{R}_\mathbf{I}^+$).

Remark 2.1. ***Geometric interpretation***
Now let's take a look at harmonic Godunova-Levin convex functions from a geometric perspective. Consider f_1, g_1 from the domain of Φ, and consider the point $\frac{f_1 g_1}{\delta f_1 + (1-\delta)g_1}$, with $\delta \in (0,1)$. We will notice that $(1-\delta)\Phi(f_1) + \delta\Phi(g_1)$ gives us the weighted average of $\Phi(f_1)$ and $\Phi(g_1)$, where $\Phi\left(\frac{f_1 g_1}{\delta f_1 + (1-\delta)g_1}\right)$ gives the output at the point $\frac{f_1 g_1}{\delta f_1 + (1-\delta)g_1}$. So, for harmonic Godunova-Levin convex function Φ the value of the function Φ at $\frac{f_1 g_1}{\delta f_1 + (1-\delta)g_1}$ whose initial point is f_1 and terminal point is g_1 is less than or equal to the chord joining the points $(f_1, \Phi(f_1))$ and $(g_1, \Phi(g_1))$.

Remark 2.2. *In comparison with ordinary convex functions, harmonic Godunova-Levin functions behave quite differently and have more properties. For clarity, see the following three examples, which are not convex but are harmonically Godunova-Levin convex on the interval $(0, \infty)$. As a result, we conclude that this is a more generalized and also larger class of convex functions that cover a broader range of functions (see Figure 3).*

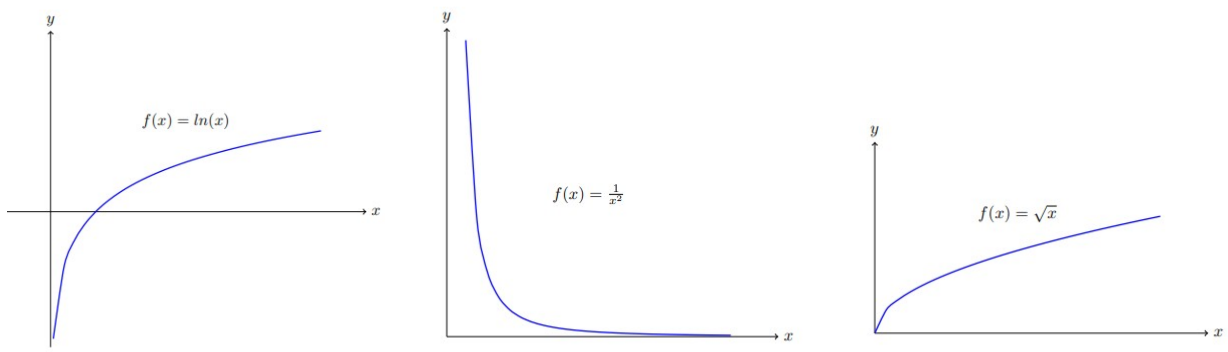

Figure 3. The following three functions are harmonic Godunova-Levin convex on the interval $(0, \infty)$.

Now let's introduce the concept for stochastic process for cr-$\mathcal{IVFS}$.

Definition 2.11. (See [34, 51]) Consider $h : [0,1] \to \mathbf{R}^+$. We say that stochastic process $\Phi = [\underline{\Phi}, \overline{\Phi}] : I \times \Omega \to \mathbf{R}_\mathbf{I}^+$ where $[f,g] \subseteq I$ is called h-convex stochastic process for cr-$\mathcal{IVFS}$ or that $\Phi \in SPX$(cr-h, $[f,g], \mathbf{R}_\mathbf{I}^+$), if $\forall\ f_1, g_1 \in [f,g]$ and $\delta \in [0,1]$, we have

$$\Phi(\delta f_1 + (1-\delta)g_1, .) \preceq_{cr}\ h(\delta)\Phi(f_1, .) + h(1-\delta)\Phi(g_1, .). \tag{2.9}$$

In (2.9), if "$\preceq_{cr}$" is replaced with "$\succeq_{cr}$", then it is called h-concave stochastic process for cr-$\mathcal{IVFS}$ or $\Phi \in SPV(\text{cr-h}, [f, g], \mathbf{R}_{\mathbf{I}}^{+})$.

Definition 2.12. (See [34, 51]) Consider $h : (0, 1) \to \mathbf{R}^{+}$. We say that stochastic process $\Phi = [\underline{\Phi}, \overline{\Phi}] : I \times \Omega \to \mathbf{R}_{\mathbf{I}}^{+}$ where $[f, g] \subseteq I$ is called harmonic h-$\mathcal{GL}$-convex stochastic process for cr-$\mathcal{IVFS}$ or that $\Phi \in SGHPX(\text{cr-h}, [f, g], \mathbf{R}_{\mathbf{I}}^{+})$, if $\forall\ f_1, g_1 \in [f, g]$ and $\delta \in (0, 1)$, we have

$$\Phi\left(\frac{f_1 g_1}{\delta f_1 + (1-\delta) g_1}, .\right) \preceq_{cr} \frac{\Phi(f_1, .)}{h(\delta)} + \frac{\Phi(g_1, .)}{h(1-\delta)}. \tag{2.10}$$

In (2.10), if "$\preceq_{cr}$" is replaced with "$\succeq_{cr}$", then it is called harmonic h-$\mathcal{GL}$-concave stochastic process for cr-$\mathcal{IVFS}$ or $\Phi \in SGHPV(\text{cr-h}, [f, g], \mathbf{R}_{\mathbf{I}}^{+})$.

Remark 2.3. *(i) If $h = 1$, Definition 2.12 becomes a stochastic process for harmonical-cr-P-function.*

(ii) If $h(\delta) = \frac{1}{h(\delta)}$, Definition 2.12 becomes a stochastic process for harmonical-cr-convex function.

(iii) If $h(\delta) = \delta$, Definition 2.12 becomes a stochastic process for harmonical cr-$\mathcal{GL}$ function.

(iv) If $h = \delta^{s}$, Definition 2.12 becomes a stochastic process for harmonical cr-s-$\mathcal{GL}$ function.

3. Hermite-Hadamard inequality for harmonical cr-h-$\mathcal{GL}$ stochastic process

This section developed the $\mathcal{H}.\mathcal{H}$ inequalities for a harmonically stochastic process for center-radius interval order relation for the class of Godunova-Levin function.

Theorem 3.1. *Let $h : (0, 1) \to \mathbf{R}^{+}$ and $h\left(\frac{1}{2}\right) \neq 0$. A function $\Theta : I \times \Omega \to \mathbf{R}_{I}{}^{+}$ is h-$\mathcal{GL}$-convex stochastic process as well as mean square integrable for cr-$\mathcal{IVFS}$. For every f,g $\in [f, g] \subseteq I$, $(f < g)$, if $\Theta \in SGHPX(cr\text{-}h, [f, g], \mathbf{R}_{I}{}^{+})$ and $\Theta \in \mathbf{IR}_{I}$. Almost everywhere, the following inequality is satisfied*

$$\frac{\left[h\left(\frac{1}{2}\right)\right]}{2}\Theta\left(\frac{2fg}{f+g}, .\right) \preceq_{cr} \frac{fg}{g-f}\int_{f}^{g}\frac{\Theta(\varepsilon, .)}{\varepsilon^{2}}d\varepsilon \preceq_{cr} [\Theta(f, .) + \Theta(g, .)]\int_{0}^{1}\frac{d\varrho}{h(\varrho)}. \tag{3.1}$$

Proof. Since $\Theta \in SGHPX(\text{cr-h}, [f, g], R_{I}{}^{+})$, we have

$$h\left(\frac{1}{2}\right)\Theta\left(\frac{2fg}{f+g}, .\right) \preceq_{cr} \Theta\left(\frac{fg}{\varrho f + (1-\varrho) g}, .\right) + \Theta\left(\frac{fg}{(1-\varrho) f + \varrho g}, .\right).$$

With integration over $(0, 1)$, we have

$$\begin{aligned}
h\left(\frac{1}{2}\right)\Theta\left(\frac{2fg}{f+g}, .\right) &\preceq_{cr} \left[\int_{0}^{1}\Theta\left(\frac{fg}{\varrho f + (1-\varrho) g}, .\right)d\varrho + \int_{0}^{1}\Theta\left(\frac{fg}{(1-\varrho) f + \varrho g}, .\right)d\varrho\right] \\
&= \left[\int_{0}^{1}\underline{\Theta}\left(\frac{fg}{\varrho f + (1-\varrho) g}, .\right)d\varrho + \int_{0}^{1}\underline{\Theta}\left(\frac{fg}{(1-\varrho) f + \varrho g}, .\right)d\varrho,\right. \\
&\quad \left.\int_{0}^{1}\overline{\Theta}\left(\frac{fg}{\varrho f + (1-\varrho) g}, .\right)d\varrho + \int_{0}^{1}\overline{\Theta}\left(\frac{fg}{(1-\varrho) f + \varrho g}, .\right)d\varrho\right] \\
&= \left[\frac{2fg}{g-f}\int_{f}^{g}\frac{\underline{\Theta}(\varepsilon, .)}{\varepsilon^{2}}d\varepsilon, \frac{2fg}{g-f}\int_{f}^{g}\frac{\overline{\Theta}(\varepsilon, .)}{\varepsilon^{2}}d\varepsilon\right] \\
&= \frac{2fg}{g-f}\int_{f}^{g}\frac{\Theta(\varepsilon, .)}{\varepsilon^{2}}d\varepsilon.
\end{aligned} \tag{3.2}$$

By Definition 2.12, we have

$$\Theta\left(\frac{fg}{\varrho f+(1-\varrho)g},.\right)\preceq_{cr}\frac{\Theta(f,.)}{h(\varrho)}+\frac{\Theta(g,.)}{h(1-\varrho)}.$$

With integration over (0,1), we have

$$\int_0^1\Theta\left(\frac{fg}{\varrho f+(1-\varrho)g},.\right)d\varrho\preceq_{cr}\Theta(f,.)\int_0^1\frac{d\varrho}{h(\varrho)}+\Theta(g,.)\int_0^1\frac{d\varrho}{h(1-\varrho)}.$$

Accordingly,

$$\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)}{\varepsilon^2}d\varepsilon\preceq_{cr}[\Theta(f,.)+\Theta(g,.)]\int_0^1\frac{d\varrho}{h(\varrho)}.\tag{3.3}$$

Adding (3.2) and (3.3), results are obtained as expected

$$\frac{h\left(\frac{1}{2}\right)}{2}\Theta\left(\frac{2fg}{f+g},.\right)\preceq_{cr}\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)}{\varepsilon^2}d\varepsilon\preceq_{cr}[\Theta(f,.)+\Theta(g,.)]\int_0^1\frac{d\varrho}{h(\varrho)}.$$

□

Remark 3.1.
- *If $h(\varrho)=1$, Theorem 3.1 becomes result for stochastic process harmonically cr-P-function:*

$$\frac{1}{2}\Theta\left(\frac{2fg}{f+g},.\right)\preceq_{cr}\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)}{\varepsilon^2}d\varepsilon\preceq_{cr}[\Theta(f,.)+\Theta(g,.)].$$

- *If $h(\varrho)=\frac{1}{\varrho}$, Theorem 3.1 becomes result for stochastic process harmonically cr-convex function:*

$$\Theta\left(\frac{2fg}{f+g},.\right)\preceq_{cr}\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)}{\varepsilon^2}d\varepsilon\preceq_{cr}\frac{[\Theta(f,.)+\Theta(g,.)]}{2}.$$

- *If $h(\varrho)=\frac{1}{(\varrho)^s}$, Theorem 3.1 becomes result for stochastic process harmonically cr-s-convex function:*

$$2^{s-1}\Theta\left(\frac{2fg}{f+g},.\right)\preceq_{cr}\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)}{\varepsilon^2}d\varepsilon\preceq_{cr}\frac{[\Theta(f,.)+\Theta(g,.)]}{s+1}.$$

Example 3.1. Let $[f,g]=[1,2]$, $h(\varrho)=\frac{1}{\varrho}$, $\forall\,\varrho\in(0,1)$. $\Theta:[f,g]\to {R_I}^+$ is defined as

$$\Theta(\varepsilon,.)=\left[\frac{-1}{\varepsilon^4}+3,\frac{1}{\varepsilon^4}+4\right]$$

where

$$\frac{h\left(\frac{1}{2}\right)}{2}\Theta\left(\frac{2fg}{f+g},.\right)=\Theta\left(\frac{4}{3}\right)=\left[\frac{687}{256},\frac{1105}{256}\right],$$

$$\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)}{\varepsilon^2}d\varepsilon=2\left[\int_1^2\left(\frac{3\varepsilon^4-1}{\varepsilon^6}\right)d\varepsilon,\int_1^2\left(\frac{4\varepsilon^4+1}{\varepsilon^6}\right)d\varepsilon\right]=\left[\frac{418}{160},\frac{702}{160}\right],$$

$$[\Theta(f,.)+\Theta(g,.)]\int_0^1\frac{d\varrho}{h(\varrho)}=\left[\frac{79}{32},\frac{145}{32}\right].$$

As a result,

$$\left[\frac{687}{256},\frac{1105}{256}\right]\preceq_{cr}\left[\frac{418}{160},\frac{702}{160}\right]\preceq_{cr}\left[\frac{79}{32},\frac{145}{32}\right].$$

This verifies the Theorem 3.1.

Theorem 3.2. *Let $h:(0,1)\rightarrow \mathbf{R}^{+}$ and $h\left(\frac{1}{2}\right)\neq 0$. A function $\Theta: I\times\Omega\rightarrow \mathbf{R}_I{}^{+}$ is h-$\mathcal{GL}$-convex stochastic process as well as mean square integrable for cr-$\mathcal{IVFS}$. For every f,g $\in[f,g]\subseteq I$, $(f<g)$, if $\Theta\in SGHPX(cr\text{-}h,[f,g],\mathbf{R}_I{}^{+})$ and $\Theta\in\mathbf{IR}_I$. Almost everywhere, the following inequality is satisfied*

$$\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\Theta\left(\frac{2fg}{f+g},.\right)\preceq_{cr}\triangle_1\preceq_{cr}\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)}{\varepsilon^2}d\varepsilon\preceq_{cr}\triangle_2$$
$$\preceq_{cr}\left\{[\Theta(f,.)+\Theta(g,.)]\left[\frac{1}{2}+\frac{1}{h(\frac{1}{2})}\right]\right\}\int_0^1\frac{d\varrho}{h(\varrho)},$$

where

$$\triangle_1=\frac{[h(\frac{1}{2})]}{4}\left[\Theta\left(\frac{4fg}{f+3g},.\right)+\Theta\left(\frac{4fg}{g+3f},.\right)\right],$$
$$\triangle_2=\left[\Theta\left(\frac{2fg}{f+g},.\right)+\frac{\Theta(f,.)+\Theta(g,.)}{2})\right]\int_0^1\frac{d\varrho}{h(\varrho)}.$$

Proof. Consider $\left[f,\frac{2fg}{f+g}\right]$, we have

$$\Theta\left(\frac{4fg}{f+3g},.\right)\preceq_{cr}\frac{\Theta\left(\frac{f\frac{2fg}{f+g}}{\varrho f+(1-\varrho)\frac{2fg}{f+g}},.\right)}{\left[h\left(\frac{1}{2}\right)\right]}+\frac{\Theta\left(\frac{f\frac{2fg}{f+g}}{(1-\varrho)f+\varrho\frac{2fg}{f+g}},.\right)}{\left[h\left(\frac{1}{2}\right)\right]}.$$

Integration over $(0,1)$, we have

$$\frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\Theta\left(\frac{4fg}{3g+f},.\right)\preceq_{cr}\frac{fg}{g-f}\int_f^{\frac{2fg}{f+g}}\frac{\Theta(\varepsilon,.)}{\varepsilon^2}d\varepsilon. \tag{3.4}$$

Similarly for interval $\left[\frac{2fg}{f+g},g\right]$, we have

$$\frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\Theta\left(\frac{4fg}{f+3g},.\right)\preceq_{cr}\frac{fg}{g-f}\int_{\frac{2fg}{f+g}}^{g}\frac{\Theta(\varepsilon,.)}{\varepsilon^2}d\varepsilon. \tag{3.5}$$

Adding inequalities (3.4) and (3.5), we get

$$\triangle_1=\frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left[\Theta\left(\frac{4fg}{f+3g},.\right)+\Theta\left(\frac{4fg}{3f+g},.\right)\right]\preceq_{cr}\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)}{\varepsilon^2}d\varepsilon.$$

Now,

$$\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\Theta\left(\frac{2fg}{f+g},.\right)$$
$$=\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\Theta\left(\frac{1}{2}\left(\frac{4fg}{3g+f},.\right)+\frac{1}{2}\left(\frac{4fg}{3f+g},.\right)\right)$$

$$
\begin{aligned}
&\preceq_{cr} \frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\left[\frac{\Theta\left(\frac{4fg}{3f+g},.\right)}{h\left(\frac{1}{2}\right)}+\frac{\Theta\left(\frac{4fg}{3g+f},.\right)}{h\left(\frac{1}{2}\right)}\right]\\
&=\frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left[\Theta\left(\frac{4fg}{f+3g},.\right)+\Theta\left(\frac{4fg}{3f+g},.\right)\right]\\
&=\triangle_1\\
&\preceq_{cr} \frac{fg}{g-f}\int_f^g \frac{\Theta(\varepsilon,.)}{\varepsilon^2}d\varepsilon\\
&\preceq_{cr} \frac{1}{2}\left[\Theta(f,.)+\Theta(g,.)+2\Theta\left(\frac{2fg}{f+g},.\right)\right]\int_0^1 \frac{d\varrho}{h(\varrho)}\\
&=\triangle_2\\
&\preceq_{cr} \left[\frac{\Theta(f,.)+\Theta(g,.)}{2}+\frac{\Theta(f,.)}{h\left(\frac{1}{2}\right)}+\frac{\Theta(g,.)}{h\left(\frac{1}{2}\right)}\right]\int_0^1 \frac{d\varrho}{h(\varrho)}\\
&\preceq_{cr} \left[\frac{\Theta(f,.)+\Theta(g,.)}{2}+\frac{1}{h\left(\frac{1}{2}\right)}\left[\Theta(f,.)+\Theta(g,.)\right]\right]\int_0^1 \frac{d\varrho}{h(\varrho)}\\
&\preceq_{cr} \left\{\left[\Theta(f,.)+\Theta(g,.)\right]\left[\frac{1}{2}+\frac{1}{h\left(\frac{1}{2}\right)}\right]\right\}\int_0^1 \frac{d\varrho}{h(\varrho)}.
\end{aligned}
$$

□

Example 3.2. Let $[f,g]=[1,2]$, $h(\varrho)=\frac{1}{\varrho}$, $\forall\ \varrho\in\ (0,1)$. $\Theta:[f,g]\to {R_I}^+$ is defined as

$$\Theta(\varepsilon,.)=\left[\frac{-1}{\varepsilon^4}+2,\frac{1}{\varepsilon^4}+3\right]$$

where

$$\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\Theta\left(\frac{2fg}{f+g},.\right)=\Theta\left(\frac{4}{3},.\right)=\left[\frac{431}{256},\frac{849}{256}\right],$$

$$\triangle_1=\frac{1}{2}\left[\Theta\left(\frac{8}{5},.\right)+\Theta\left(\frac{8}{7},.\right)\right]=\left[\frac{6679}{4096},\frac{13801}{4096}\right],$$

$$\triangle_2=\left[\frac{\Theta(1,.)+\Theta(2,.)}{2}+\Theta\left(\frac{4}{3},.\right)\right]\int_0^1\frac{d\varrho}{h(\varrho)},$$

$$\triangle_2=\left[\frac{1935}{512},\frac{4465}{512}\right],$$

$$\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)}{\varepsilon^2}d\varepsilon=\left[\frac{258}{160},\frac{542}{160}\right],$$

$$\left\{\left[\Theta(f,.)+\Theta(g,.)\right]\left[\frac{1}{2}+\frac{1}{h(\frac{1}{2})}\right]\right\}\int_0^1\frac{d\varrho}{h(\varrho)}=\left[\frac{47}{8},\frac{113}{8}\right].$$

Thus, we obtain

$$\left[\frac{431}{256},\frac{849}{256}\right]\preceq_{cr}\left[\frac{6679}{4096},\frac{13801}{4096}\right]\preceq_{cr}\left[\frac{258}{160},\frac{542}{160}\right]\preceq_{cr}\left[\frac{1935}{512},\frac{4465}{512}\right]\preceq_{cr}\left[\frac{47}{8},\frac{113}{8}\right].$$

This verify the the Theorem 3.2.

Theorem 3.3. *Let $h_1,h_2:(0,1)\to\mathbf{R}^+$ and $h_1,h_2\neq 0$. A functions $\Theta,\Phi: I\times\Omega\to\mathbf{R}_I{}^+$ are harmonic h-Godunova-Levin stochastic process as well as mean square integrable for cr-$\mathcal{IVFS}$. For every f,g $\in I$, $(f<g)$, if $\Theta\in SGHPX(cr\text{-}h_1,[f,g],\mathbf{R}_I{}^+)$, $\Phi\in SGHPX(cr\text{-}h_2,[f,g],\mathbf{R}_I{}^+)$ and $\Theta,\Phi\in\mathbf{IR}_I$. Almost everywhere, the following inequality is satisfied*

$$\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)\Phi(\varepsilon,.)}{\varepsilon^2}d\varepsilon\preceq_{cr}T(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(\varrho)}d\varrho+U(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(1-\varrho)}d\varrho,\tag{3.6}$$

where

$$T(f,g)=\Theta(f,.)\Phi(f,.)+\Theta(g,.)\Phi(g,.),U(f,g)=\Theta(f,.)\Phi(g,.)+\Theta(g,.)\Phi(f,.).$$

Proof. Conider $\Theta\in SGHPX(cr\text{-}h_1,[f,g],R_I{}^+)$, $\Phi\in SGHPX(cr\text{-}h_2,[f,g],R_I{}^+)$ then, we have

$$\Theta\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)\preceq_{cr}\frac{\Theta(f,.)}{h_1(\varrho)}+\frac{\Theta(g,.)}{h_1(1-\varrho)},$$

$$\Phi\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)\preceq_{cr}\frac{\Phi(f,.)}{h_2(\varrho)}+\frac{\Phi(g,.)}{h_2(1-\varrho)}.$$

Then,

$$\Theta\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)\Phi\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)$$
$$\preceq_{cr}\frac{\Theta(f,.)\Phi(f,.)}{h_1(\varrho)h_2(\varrho)}+\frac{\Theta(f,.)\Phi(g,.)}{h_1(\varrho)h_2(1-\varrho)}+\frac{\Theta(g,.)\Phi(f,.)}{h_1(1-\varrho)h_2(\varrho)}+\frac{\Theta(g,.)\Phi(g,.)}{h_1(1-\varrho)h_2(1-\varrho)}.$$

Integration over (0,1), we have

$$\int_0^1\Theta\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)\Phi\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)d\varrho$$
$$=\left[\int_0^1\underline{\Theta}\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)\underline{\Phi}\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)d\varrho,\right.$$
$$\left.\int_0^1\overline{\Theta}\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)\overline{\Phi}\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)d\varrho\right]$$
$$=\left[\frac{fg}{g-f}\int_f^g\frac{\underline{\Theta}(\varepsilon,.)\underline{\Phi}(\varepsilon,.)}{\varepsilon^2}d\varepsilon,\frac{fg}{g-f}\int_f^g\frac{\overline{\Theta}(\varepsilon,.)\overline{\Phi}(\varepsilon,.)}{\varepsilon^2}d\varepsilon\right]=\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)\Phi(\varepsilon,.)}{\varepsilon^2}d\varepsilon$$
$$\preceq_{cr}\int_0^1\frac{[\Theta(f,.)\Phi(f,.)+\Theta(g,.)\Phi(g,.)]}{h_1(\varrho)h_2(\varrho)}d\varrho+\int_0^1\frac{[\Theta(f,.)\Phi(g,.)+\Theta(g,.)\Phi(f,.)]}{h_1(\varrho)h_2(1-\varrho)}d\varrho.$$

It follows that

$$\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)\Phi(\varepsilon,.)}{\varepsilon^2}d\varepsilon\preceq_{cr}T(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(\varrho)}d\varrho+U(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(1-\varrho)}d\varrho.$$

Theorem is proved. □

Example 3.3. Let $[f,g]=[1,2]$, $h_1(\varrho)=h_2(\varrho)=\frac{1}{\varrho}$, $\forall\,\varrho\in\ (0,1)$. $\Theta,\Phi:[f,g]\to {R_I}^+$ be defined as

$$\Theta(\varepsilon,.)=\left[\frac{-1}{\varepsilon^4}+2,\frac{1}{\varepsilon^4}+3\right],\Phi(\varepsilon)=\left[\frac{-1}{\varepsilon}+1,\frac{1}{\varepsilon}+2\right].$$

Then,

$$\begin{aligned}
&\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)\Phi(\varepsilon,.)}{\varepsilon^2}d\varepsilon=\left[\frac{282}{640},\frac{5986}{640}\right],\\
&T(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(\varrho)}d\varrho=T(1,2)\int_0^1\varrho^2d\varrho=\left[\frac{31}{96},\frac{629}{96}\right],\\
&U(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(1-\varrho)}d\varrho=U(1,2)\int_0^1(\varrho-\varrho^2)d\varrho=\left[\frac{1}{12},\frac{307}{96}\right].
\end{aligned}$$

It follows that

$$\left[\frac{282}{640},\frac{5986}{640}\right]\preceq_{cr}\left[\frac{31}{96},\frac{629}{96}\right]+\left[\frac{1}{12},\frac{307}{96}\right]=\left[\frac{13}{32},\frac{39}{4}\right].$$

This verifies the Theorem 3.3.

Theorem 3.4. *Let $h_1,h_2:(0,1)\to\mathbf{R}^+$ and $h_1,h_2\neq 0$. A functions $\Theta,\Phi: I\times\Omega\to{\mathbf{R}_I}^+$ are harmonic h-Godunova-Levin stochastic process as well as mean square integrable for cr-$\mathcal{IVFS}$. For every f,g $\in I$, ($f<g$), if $\Theta\in SGHPX(cr\text{-}h_1,I,{\mathbf{R}_I}^+)$, $\Phi\in SGHPX(cr\text{-}h_2,I,{\mathbf{R}_I}^+)$ and $\Theta,\Phi\in\mathbf{IR}_I$. Almost everywhere, the following inequality is satisfied*

$$\begin{aligned}
&\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{2}\Theta\left(\frac{2fg}{f+g},.\right)\Phi\left(\frac{2fg}{f+g},.\right)\\
&\preceq_{cr}\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)\Phi(\varepsilon,.)}{\varepsilon^2}d\mu+T(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(1-\varrho)}d\varrho+U(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(\varrho)}d\varrho.
\end{aligned}$$

Proof. Since $\Theta\in SGHPX(cr\text{-}h_1,[f,g],{R_I}^+)$, $\Phi\in SGHPX(cr\text{-}h_2,[f,g],{R_I}^+)$, we have

$$\begin{aligned}
\Theta\left(\frac{2fg}{f+g},.\right)&\preceq_{cr}\frac{\Theta\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)}{h_1\left(\frac{1}{2}\right)}+\frac{\Theta\left(\frac{fg}{f(1-\varrho)+\varrho g},.\right)}{h_1\left(\frac{1}{2}\right)},\\
\Phi\left(\frac{2fg}{f+g},.\right)&\preceq_{cr}\frac{\Phi\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)}{h_2\left(\frac{1}{2}\right)}+\frac{\Phi\left(\frac{fg}{f(1-\varrho)+\varrho g},.\right)}{h_2\left(\frac{1}{2}\right)}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\Theta\left(\frac{2fg}{f+g},.\right)\Phi\left(\frac{2fg}{f+g},.\right)\\
&\preceq_{cr}\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\Theta\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)\Phi\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)+\Theta\left(\frac{fg}{f(1-\varrho)+\varrho g},.\right)\Phi\left(\frac{fg}{f(1-\varrho)+\varrho g},.\right)\right]\\
&+\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\Theta\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)\Phi\left(\frac{fg}{f(1-\varrho)+\varrho g},.\right)+\Theta\left(\frac{fg}{f(1-\varrho)+\varrho g},.\right)\Phi\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)\right]
\end{aligned}$$

$$\preceq_{cr} \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\Theta\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)\Phi\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)+\Theta\left(\frac{fg}{f(1-\varrho)+\varrho g},.\right)\Phi\left(\frac{fg}{f(1-\varrho)+\varrho g},.\right)\right]$$
$$+\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\left(\frac{\Theta(f,.)}{h_1(\varrho)}+\frac{\Theta(g,.)}{h_1(1-\varrho)}\right)\left(\frac{\Phi(g,.)}{h_2(1-\varrho)}+\frac{\Phi(g,.)}{h_2(\varrho)}\right)+\left(\frac{\Theta(f,.)}{h_1(1-\varrho)}+\frac{\Theta(g,.)}{h_1(\varrho)}\right)\left(\frac{\Phi(f,.)}{h_2(\varrho)}+\frac{\Phi(g,.)}{h_2(1-\varrho)}\right)\right]$$
$$\preceq_{cr} \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\Theta\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)\Phi\left(\frac{fg}{f\varrho+(1-\varrho)g},.\right)+\Theta\left(\frac{fg}{f(1-\varrho)+\varrho g},.\right)\Phi\left(\frac{fg}{f(1-\varrho)+\varrho g},.\right)\right]$$
$$+\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\left(\frac{1}{h_1(\varrho)h_2(1-\varrho)}+\frac{1}{h_1(1-\varrho)h_2(\varrho)}\right)T(f,g)+\left(\frac{1}{h_1(\varrho)h_2(\varrho)}+\frac{1}{h_1(1-\varrho)h_2(1-\varrho)}\right)U(f,g)\right].$$

Integration over $(0,1)$, we have

$$\int_0^1 \Theta\left(\frac{2fg}{f+g},.\right)\Phi\left(\frac{2fg}{f+g},.\right)d\varrho=\left[\int_0^1 \underline{\Theta}\left(\frac{2fg}{f+g},.\right)\underline{\Phi}\left(\frac{2fg}{f+g},.\right)d\varrho,\int_0^1 \overline{\Theta}\left(\frac{2fg}{f+g},.\right)\overline{\Phi}\left(\frac{2fg}{f+g},.\right)d\varrho\right]$$
$$=\Theta\left(\frac{2fg}{f+g},.\right)\Phi\left(\frac{2fg}{f+g},.\right)\preceq_{cr}\frac{2}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\left[\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)\Phi(\varepsilon,.)}{\varepsilon^2}d\varepsilon\right]$$
$$+\frac{2}{h\left(\frac{1}{2}\right)h(\frac{1}{2})}\left[T(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(1-\varrho)}d\varrho+U(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(\varrho)}d\varrho\right]$$

Multiply both sides by $\frac{h_1(\frac{1}{2})h_2(\frac{1}{2})}{2}$ above equation, we get required result

$$\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{2}\Theta\left(\frac{2fg}{f+g},.\right)\Phi\left(\frac{2fg}{f+g},.\right)$$
$$\preceq_{cr}\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)\Phi(\varepsilon,.)}{\varepsilon^2}d\mu+T(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(1-\varrho)}d\varrho+U(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(\varrho)}d\varrho.$$

□

Example 3.4. Recall the Example 3.3, we have

$$\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{2}\Theta\left(\frac{2fg}{f+g},.\right)\Phi\left(\frac{2fg}{f+g},.\right)=2\Theta\left(\frac{4}{3}\right)\Phi\left(\frac{4}{3}\right)=\left[\frac{431}{512},\frac{9339}{512}\right],$$
$$\frac{fg}{g-f}\int_f^g\frac{\Theta(\varepsilon,.)\Phi(\varepsilon,.)}{\varepsilon^2}d\varepsilon=\left[\frac{282}{640},\frac{5986}{640}\right],$$
$$T(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(1-\varrho)}d\varrho=T(1,2)\int_0^1(\varrho-\varrho^2)d\varrho=\left[\frac{31}{192},\frac{629}{192}\right],$$
$$U(f,g)\int_0^1\frac{1}{h_1(\varrho)h_2(\varrho)}d\varrho=U(1,2)\int_0^1\varrho^2d\varrho=\left[\frac{1}{6},\frac{307}{48}\right].$$

It follows that

$$\left[\frac{431}{512},\frac{9339}{512}\right]\preceq_{cr}\left[\frac{282}{640},\frac{5986}{640}\right]+\left[\frac{31}{192},\frac{629}{192}\right]+\left[\frac{1}{6},\frac{307}{48}\right]=\left[\frac{123}{160},\frac{761}{40}\right].$$

This verifies the Theorem 3.4.

4. Conclusions

This paper introduces a center-radius order relation for $\mathcal{IVFS}$ by using harmonic-Godunova-Levin stochastic processes in the setting of $\mathcal{IVFS}$. Using these ideas, we created some variants of $\mathcal{H.H}$ inequalities. The fact that inequality terms derived from this order relation produce precise results is one of its distinguishing features. Furthermore, in this article, we generalise the findings of the following authors [34, 49, 51], which is a novel approach for future research. Furthermore, the study provides interesting examples to demonstrate the validity of theorems. These ideas can be used to push convex optimization to new heights. This concept should be useful to scientists working in a variety of fields. Future research may investigate the use of different integral operators for determining equivalent inequalities.

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Conflict of interest

The authors declare that there is no conflict of interest in publishing this paper.

References

1. R. E. Moore, *Methods and applications of interval analysis*, SIAM, Philadelphia, 1966.
2. D. P. Mitchell, Three applications of interval analysis in computer graphics, *Front. Rend. Course Note.*, **14** (1991).
3. M. Ramezanadeh, M. Heidari, O. S. Fard, On the interval differential equation: Novel solution methodology, *Adv. Differ. Equ.*, **1** (2015), 1–23. https://doi.org/10.1186/s13662-015-0671-8
4. E. Rothwell, M. J. Cloud, Automatic error analysis using intervals, *IEEE Trans. Educ.*, **55** (2011), 9–15. https://doi.org/10.1109/TE.2011.2109722
5. S. L. Ho, M. Xie, T. N. Goh, A comparative study of neural network and Box-Jenkins ARIMA modeling in time series prediction, *Comput. Ind. Eng.*, **42** (2002), 371–375. https://doi.org/10.1016/S0360-8352(02)00036-0
6. S. Zheng, C. Ding, F. Nie, H. Huang, Harmonic mean linear discriminant analysis, *IEEE T. Knowl. Data En.*, **31** (2018), 1520–1531. https://doi.org/10.1109/TKDE.2018.2861858
7. D. Zhao, T. An, G. Ye, W. Liu, New Jensen and Hermite-Hadamard type inequalities for h-convex interval-valued functions, *J. Inequal. Appl.*, **2018** (2018), 1–14. https://doi.org/10.1186/s13660-018-1896-3

8. S. Obeidat, M. A. Latif, S. S. Dragomir, Fejér and Hermite-Hadamard type inequalities for differentiable h-convex and quasi convex functions with applications, *Miskolc Math. Notes*, **23** (2022), 401–415. http://doi.org/10.18514/MMN.2022.3065

9. P. O. Mohammed, On new trapezoid type inequalities for h-convex functions via generalized fractional integral, *Turk. J. Anal. Number Theor.*, **6** (2018), 125–128. http://doi.org/10.12691/tjant-6-4-5

10. X. Zhang, K. Shabbir, W. Afzal, H. Xiao, D. Lin, Hermite-Hadamard and Jensen-type inequalities via Riemann integral operator for a generalized class of Godunova-Levin functions, *J. Math.*, **2022** (2022), 3830324. https://doi.org/10.1155/2022/3830324

11. P. Yang, S. Zhang, Mean square integral inequalities for generalized convex stochastic processes via Beta function, *J. Funct. Space.*, **2021** (2021), 4398901. https://doi.org/10.1155/2021/4398901

12. L. Akin, A characterization of boundedness of fractional maximal operator with variable kernel on Herz-Morrey spaces, *Anal. Theory Appl.*, **36** (2020), 60–68. https://doi.org/10.4208/ata.OA-2018-1006

13. D. Zhao, T. An, G. Ye, D. F. M. Torres, On Hermite-Hadamard type inequalities for harmonical *h*-convex interval-valued functions, *Math. Inequal. Appl.*, **23** (2020), 95–105.

14. P. Korus, J. E. N. Valdes, q-Hermite-Hadamrd inequalities for functions with convex or h-convex q-derivarive, *Math. Inequal. Appl.*, **25** (2022), 601–610. http://doi.org/10.7153/mia-2022-25-36

15. M. I. Asjad, S. Z. Majid, W. A. Faridi, S. M. Eldin, Sensitive analysis of soliton solutions of nonlinear Landau-Ginzburg-Higgs equation with generalized projective Riccati method, *AIMS Math.*, **8** (2023), 10210–10227. https://doi.org/10.3934/math.2023517

16. I. Siddique, K. B. Mehdi, S. M. Eldin, A. Zafar, Diverse optical solitons solutions of the fractional complex Ginzburg-Landau equation via two altered methods, *AIMS Math.*, **5** (2023), 11480–11497. https://doi.org/10.3934/math.2023581

17. S. S. Dragomir, Hermite-Hadamard type inequalities for generalized Riemann-Liouville fractional integrals of h-convex functions, *Math. Method. Appl. Sci.*, **44** (2021), 2364–2380. https://doi.org/10.1002/mma.5893

18. G. D. Anderson, M. K. Vamanamurthy, Generalized convexity and inequalities, *J. Math. Anal. Appl.*, **335** (2007), 1294–1308. https://doi.org/10.1016/j.jmaa.2007.02.016

19. M. A. Noor, K. I. Noor, Harmonic variational inequalities, *Appl. Math. Inform. Sci.*, **10** (2016), 1811–1814. https://doi.org/10.1016/j.jmaa.2007.02.016

20. S. I. Butt, S. Rashid, M. Tariq, M. K. Wang, Novel refinements via-polynomial harmonically-type convex functions and application in special functions, *J. Funct. Space.*, **2021** (2021), 6615948. https://doi.org/10.1155/2021/6615948

21. S. I. Butt, A. O. Akdemir, M. Nadeem, N. Mlaiki, I. Iscan, T. Abdeljawad, (*m*-*n*)-Harmonically polynomial convex functions and some Hadamard inequalities on co-ordinates, *AIMS Math.*, **6** (2021), 4677–4690. https://doi.org/10.3934/math.2021275

22. S. I. Butt, S. Yousaf, K. A. Khan, Fejer-Pachpatte-Mercer-type inequalities for harmonically convex functions involving exponential function in kernel, *Math. Probl. Eng.*, **2022** (2022), 7269033. https://doi.org/10.1155/2022/7269033

23. S. I. Butt, P. Agarwal, S. Yousaf, Generalized fractal Jensen and Jensen-Mercer inequalities for harmonic convex function with applications, *J. Inequal. Appl.*, **2022** (2022), 1–18. https://doi.org/10.1186/s13660-021-02735-3

24. M. Tariq, S. I. Butt, Some Ostrowski type integral inequalities via generalized harmonic convex functions, *Open J. Math. Sci.*, **5** (2021), 200–208. https://doi.org/10.30538/oms2021.0157

25. W. Afzal, K. Shabbir, S. Treanţă, K. Nonlaopon, Jensen and Hermite-Hadamard type inclusions for harmonical h-Godunova-Levin functions, *AIMS Math.*, **8** (2022), 3303–3321. https://doi.org/10.3934/math.2023170

26. R. Liu, R. Xu, Hermite-Hadamard type inequalities for harmonical (h_1, h_2)-convex interval-valued functions, *Math. Found. Comput.*, **4** (2021), 89. https://doi.org/10.3934/mfc.2021005

27. W. Afzal, A. A. Lupaş, K. Shabbir, Hermite-Hadamard and Jensen-type inequalities for harmonical $(h_1,\ h_2)$-Godunova Levin interval-valued functions, *Mathematics*, **10** (2022), 2970. https://doi.org/10.3390/math10162970

28. W. Afzal, K. Shabbir, T. Botmart, Generalized version of Jensen and Hermite-Hadamard inequalities for interval-valued $(h_1,\ h_2)$-Godunova-Levin functions, *AIMS Math.*, **7** (2022), 19372–19387. https://doi.org/2010.3934/math.20221064

29. K. Nikodem, On convex stochastic processes, *Aequationes Math.*, **20** (1980), 184–197. https://doi.org/10.1007/BF02190513

30. M. Shaked, J. G. Shanthikumar, Stochastic convexity and its applications, *Adv. Appl. Probab.*, **20** (1988), 427–446. https://doi.org/10.ADA170112

31. A. Skowronski, On some properties ofj-convex stochastic processes, *Aequationes Math.*, **44** (1992), 249–258. https://doi.org/10.1007/BF01830983

32. D. Kotrys, Hermite-Hadamard inequality for convex stochastic processes, *Aequationes Math.*, **83** (2012), 143–151. https://doi.org/10.1007/s00010-011-0090-1

33. S. Varoşanec, On h-convexity, *J. Math. Anal. Appl.*, **326** (2007), 303–311. https://doi.org/10.1016/j.jmaa.2006.02.086

34. D. Barraez, L. Gonzalez, N. Merentes, On h-convex stochastic processes, *Math. Aeterna.*, **5** (2015), 571–581. https://doi.org/10.20180421095428id

35. W. Afzal, E. Y. Prosviryakov, S. M. El-Deeb, Y. Almalki, Some new estimates of Hermite-Hadamard, Ostrowski and Jensen-type inclusions for h-convex stochastic process via interval-valued functions, *Symmetry*, **15** (2023), 831. https://doi.org/10.3390/sym15040831

36. N. Okur, R. Aliyev, Some Hermite-Hadamard type integral inequalities for multidimensional general preinvex stochastic processes, *Commun. Stat.-Theor. M.*, **50** (2021), 3338–3351. https://doi.org/10.1080/03610926.2019.1696976

37. M. J. Vivas Cortez, Some inequalities via strongly p-harmonic log-convex stochastic processes, *Appl. Math. Inform. Sci.*, **12** (2018), 593–600. https://doi.org/10.23000/5745

38. L. Akin, New principles of non-linear integral inequalities on time scales, *Appl. Math. Nonlinear Sci.*, **6** (2021), 387–394. https://doi.org/10.2478/amns.2021.1.00001

39. O. Almutairi, A. Kilicman, Generalized Fejér-Hermite-Hadamard type via generalized $(h-m)$-convexity on fractal sets and applications, *Chaos Soliton. Fract.*, **147** (2021), 110938. https://doi.org/10.1016/j.chaos.2021.110938

40. C. Y. Jung, M. S. Saleem, S. Bilal, W. Nazeer, Some properties of η-convex stochastic processes, *AIMS Math.*, **6** (2021), 726–736. http://doi.org/2010.3934/math.2021044

41. H. Agahi, A. Babakhani, On fractional stochastic inequalities related to Hermite-Hadamard and Jensen types for convex stochastic processes, *Aequationes Math.*, **90** (2016), 1035–1043. http://doi.org/10.1007/s00010-016-0425-z

42. L. Akin, On the fractional maximal delta integral type inequalities on time scales, *Fractal Fract.,* **4** (2020), 26. https://doi.org/10.3390/fractalfract4020026

43. M. Tunc, Ostrowski-type inequalities via h-convex functions with applications to special means, *J. Inequal. Appl.*, **1** (2013), 1–10. https://doi.org/10.1186/1029-242X-2013-326

44. L. Gonzales, J. Materano, M. V. Lopez, Ostrowski-type inequalities via hconvex stochastic processes, *JP J. Math. Sci.*, **16** (2016), 15–29. https://doi.org/10.5f4e522592851c250b850e30

45. A. K. Bhunia, S. S. Samanta, A study of interval metric and its application in multi-objective optimization with interval objectives, *Comput. Ind. Eng.*, **74** (2014), 169–178. https://doi.org/10.1016/j.cie.2014.05.014

46. W. Afzal, W. Nazeer, T. Botmart, S. Treanţă, Some properties and inequalities for generalized class of harmonical Godunova-Levin function via center radius order relation, *AIMS Math.*, **8** (2022), 1696–1712. https://doi.org/10.3934/math.20221064

47. T. Saeed, W. Afzal, K. Shabbir, S. Treanţă, M. D. L. Sen, Some novel estimates of Hermite-Hadamard and Jensen type inequalities for (h_1, h_2)-convex functions pertaining to total order relation, *Mathematics*, **10** (2022), 4770. https://doi.org/10.3390/math10244777

48. W. Afzal, K. Shabbir, T. Botmart, S. Treanţă, Some new estimates of well known inequalities for (h_1, h_2)-Godunova-Levin functions by means of center-radius order relation, *AIMS Math.*, **8** (2022), 3101–3119. https://doi.org/10.3934/math.2023160

49. W. Afzal, T. Botmart, Some novel estimates of Jensen and Hermite-Hadamard inequalities for h-Godunova-Levin stochastic processes, *AIMS Math.*, **8** (2023), 7277–7291. https://doi.org/10.3934/math.2023366

50. T. Saeed, W. Afzal, M. Abbas, S. Treanţă, M. D. L. Sen, Some new generalizations of integral inequalities for harmonical cr-(h_1, h_2)-Godunova-Levin functions and applications, *Mathematics*, **10** (2022), 4540. https://doi.org/10.3390/math10234540

51. W. Afzal, M. Abbas, J. E. Macias-Diaz, S. Treanţă, Some H-Godunova-Levin function inequalities using center radius (cr) order, *Fractal Fract.*, **6** (2022), 518. https://doi.org/10.3390/fractalfract6090518

52. W. C. Qi, Z. P. Qiu, A collocation interval analysis method for interval structural parameters and stochastic excitation, *Phys. Mech. Astron.*, **55** (2012), 66–77. https://doi.org/10.1007/s11433-011-4570-z

53. G. Muscolino, A. Sofi, Stochastic analysis of structures with uncertain-but-bounded parameters via improved interval analysis, *Probabilist. Eng. Mech.*, **28** (2012), 152–163. https://doi.org/10.1016/j.probengmech.2011.08.011
54. I. Dawoud, M. R. Abonazel, F. A. Awwad, S. M. Eldin, A new Tobit Ridge-type estimator of the censored regression model with multicollinearity problem, *Front. Appl. Math. Stat.*, **8** (2022), 952142. https://doi.org/10.3389/fams.2022.952142
55. M. M. A. Aziz, S. M. Eldin, D. K Ibrahim, M. Gilany, A phasor-based double ended fault location scheme for aged power cables, *Electr. Pow. Compo. Syst.*, **34** (2006), 417–432. https://doi.org/10.1109/PES.2005.1489162
56. B. J. Leira, A comparison of stochastic process models for definition of design contours, *Struct. Saf.*, **30** (2008), 493–505. https://doi.org/10.1016/j.strusafe.2007.09.006
57. A. Rakhlin, O. Shamir, K. Sridharan, Making gradient descent optimal for strongly convex stochastic optimization, *arXiv:1109.5647*, **30** (2011). https://doi.org/10.48550/arXiv.1109.5647
58. H. Jin, Z. Q. Xu, X. Y. Zhou, A convex stochastic optimization problem arising from portfolio selection, *Math. Financ.*, **18** (2008), 171–183. https://doi.org/10.1111/j.1467-9965.2007.00327.x

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