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Research article

Symmetry of positive solutions of a p-Laplace equation with convex nonlinearites

Keqiang Li[1] **and Shangjiu Wang**[2,3,*]

[1] College of Digital Technology and Engineering, Ningbo University of Finance and Economics, Ningbo 315175, China
[2] School of Mathematics and Statistics, Shaoguan University, Shaoguan 512005, China
[3] School of Economics and Statistics, Guangzhou University, Guangzhou 510006, China

* **Correspondence:** Email: wangshangjiu@163.com.

Abstract: In this paper, we consider the symmetry properties of the positive solutions of a p-Laplacian problem of the form

$$\begin{cases} -\Delta_p u = f(x,u), & \text{in} \quad \Omega, \\ \quad u = g(x), & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is an open smooth bounded domain in $R^N, N \geq 2$, and symmetric w.r.t. the hyperplane $T_0^\nu(\nu$ is a direction vector in $R^N, |\nu| = 1)$, $f\colon \Omega \times R^+ \to R^+$ is a continuous function of class C^1 w.r.t. the second variable, $g \geq 0$ is continuous, and both f and g are symmetric w.r.t. T_0^ν, respectively. Introducing some assumptions on nonlinearities, we get that the positive solutions of the problem above are symmetric w.r.t. the direction ν by a new simple idea even if Ω is not convex in the direction ν.

Keywords: symmetry of positive solutions; p-Laplace equation; convex nonlinearities; weak maximum principle; degenerate elliptic operator
Mathematics Subject Classification: 35A21, 35B06

1. Introduction

Let ν be a direction vector in $R^N, N \geq 2, |\nu| = 1$, and $\Omega \subset R^N$ be an open bounded smooth domain which is symmetric with respect to the hyperplane $T_0^\nu := \{x \in R^N | x \cdot \nu = 0\}$. Let us study the symmetry of the positive solutions of the p-Laplace problem of the form

$$\begin{cases} -\Delta_p u = f(x,u), & \text{in} \quad \Omega, \\ \quad u = g(x), & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 1$, $f\colon \Omega \times R^+ \to R^+$ is a continuous function of class C^1 w.r.t. the second variable, $g \geq 0$ is continuous, and both f and g are symmetric w.r.t. T_0^ν.

For the case $p = 2$, the research of symmetry properties for solutions of differential equation was started by Serrin [1], by the method of moving planes (MMP), which was also called Alexandrov reflection method. Since the MMP is essentially a monotonicity method it usually works very well, when $g \equiv 0, u > 0$ in Ω and f has some monotonicity in x. Later, in the celebrated papers [2, 3], by MMP, Gidas, Ni and Nirenberg proved that any positive solution $u \in C^2(\overline{\Omega})$ of the problem $-\Delta u = f(u)$ in Ω with $u = 0$ on $\partial\Omega$ is radially symmetric when $f(s)$ is C^1 and Ω is a ball in R^N or $\Omega = R^N$ (assuming that $u(x) = o(|x|^{2-N})$ at infinity). After that, in a general bounded domain Ω ($\subseteq R^N$), which is convex in the x_1 direction, Berestycki and Nirenberg in [4] proved that monotonicity and symmetry w.r.t. the x_1 direction for positive solutions $u \in W^{2,N}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ of nonlinear elliptic equations.

For the p-Laplace operator, the coefficient ($|\nabla u|^{p-2}$) is vanishing or singular at the critical points of u for $p > 2$ or $1 < p < 2$, respectively. So, many symmetry results of positive solutions for p-Laplace equations were proved under some assumptions on the critical set of u. Symmetry results for positive solutions of p-Laplace equations, without any assumptions on the critical set of u, were obtained in [5, 6] for bounded domains and in [7] for R^N, by MMP. In [8], the authors studied the symmetry of nonnegative C^1 ground states of a class of quasilinear elliptic equations. For $p > 2$, the monotonicity and symmetry properties for nonnegative solutions of $-\Delta_p u = f(u)$ in B with the boundary condition $u = 0$, where Δ_p is the p-Laplace-Beltrami operator and B is a geodesic ball in hyperbolic space H^N, was studied in [9].

In this paper, even if Ω is not convex in the direction ν orthogonal to T_0^ν, f may not have the right monotonicity in x, and u doesn't vanish on the boundary of Ω, we obtain u is symmetric w.r.t. the hyperplane T_0^ν, which is different from the previous results.

To be more precise, let x_0^ν denote the reflection point of x w.r t. T_0^ν, i.e.,

$$x_0^\nu = R_0^\nu(x) := x - 2(x \cdot \nu)\nu,$$

and to guarantee that the p-Laplace equation is uniformly elliptic, we give an hypothesis on the positive solution u of (1.1),

$$\nabla u(x) \neq 0, \quad \forall\, x \in \overline{\Omega}. \tag{1.2}$$

So, by (1.2), it follows that the quasilinear second order operator in (1.1) is nondegenerate elliptic.

Now, we state the main result of the paper as follows.

Theorem 1.1. *Assume $u \in C^1(\overline{\Omega})$ is a positive solution of (1.1) and (1.2) holds. If $f(x, s)$ and $g(x) \geq 0$ are symmetric w.r.t the hyperplane T_0^ν, f is strictly convex in s and the derivative of f with respect to the second variable s is nonpositive, i.e., f_s' satisfies following inequality*

$$f_s'(x, s) \leq 0, \ \ \forall\, (x, s) \in \Omega \times R^+, \tag{1.3}$$

then u is symmetric with respect to T_0^ν, i.e., $u(x) = u(x_0^\nu)$ for any $x \in \Omega$. Furthermore, if $f(x, s)$ is convex in s and $f_s'(x, s) < 0$, for $(x, s) \in \Omega \times R^+$, the same result holds.

This paper is motivated by [10], where the authors studied the symmetry of the solutions of semilinear elliptic equations with convex nonlinearites for the case $p = 2$. In [10], the author proved that nonnegativity of the first eigenvalue of the linearized operator in the caps determined by the symmetry of Ω is a sufficient condition for the symmetry of the solution, when the nonlinearities and the boundary value condition have some symmetric in x. Here, for the p-Laplace equation in (1.1), $p > 1$, we introduce the nonpositivity of the derivative of f w.r.t. the second variable instead of the nonnegativity of the first eigenvalue of linearized operator in [10]. Under our assumptions, we prove that the positive solution u of (1.1) is symmetric w.r.t. the direction ν by a novel simple method instead of the method of moving planes.

2. Preliminaries

In this section, we give some notations. For the direction $\nu \in \mathbf{R}^N$, set Ω^- and Ω^+ for the caps of the left and right of T_0^ν, i.e.,

$$\Omega^- = \{x \in \Omega |\ x \cdot \nu < 0\} \text{ and } \Omega^+ = \{x \in \Omega |\ x \cdot \nu > 0\}.$$

Let us define the reflected functions of u in the domains Ω^- and Ω^+ by v^- and v^+, respectively,

$$v^-(x) = u(x - 2(x \cdot \nu)\, \nu), \quad x \in \Omega^-,$$

$$v^+(x) = u(x - 2(x \cdot \nu)\, \nu), \quad x \in \Omega^+.$$

Hence, by definition, v^- and v^+ are solutions of (1.1) in Ω^- and Ω^+, respectively, and by the condition (1.2), we have

$$\nabla u \neq 0 \text{ in } \Omega, \qquad \nabla v^- \neq 0, \text{ in } \Omega^-, \qquad \nabla v^+ \neq 0 \text{ in } \Omega^+. \tag{2.1}$$

Furthermore, by (2.1) and the definitions of v^- and v^+, we also get

$$\nabla u \neq \nabla v^- \text{ in } \Omega^-, \qquad \nabla u \neq \nabla v^+ \text{ in } \Omega^+. \tag{2.2}$$

What's more, regularity theory for quasilinear elliptic equations [11] give us that any positive solution of the p-Laplacian equation in (1.1) satisfying (1.2) also satisfies $u \in C^2(B_R \setminus Z)(= C^2(B_R))$, where Z is the set of critical points of u.

Now, by assumptions $u \in C^1(\overline{\Omega})$ and (1.2), it follows that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then, we can write $-\Delta_p u = f(x, u)$ in the form (see also (2.5) in [8])

$$-\sum_{i,j}^{N} a_{ij}(x) u_{ij} = f(x, u) \text{ in } \Omega, \tag{2.3}$$

where $a_{ij}(x)$ is a bounded continuous function in Ω, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, ..., N$.

Remark 2.1. *By (1.2) and Lemma 2.1 in [8], the matrix $\{a_{ij}\}$ is a positive definite $\forall\ x \in \Omega$ and the Eq (2.3) is uniformly elliptic in Ω.*

Next, we recall a lemma proved by Simon in [12] and Damascelli in [13], which will be used to prove the main result later.

Lemma 2.1. *Let $p > 1$ and $N(\geq 2) \in \mathbf{N}$. There exists a positive constant c depending on p and N such that for all $\eta, \eta' \in \mathbf{R}^N$ with $|\eta| + |\eta'| > 0$,*

$$(|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta') \cdot (\eta - \eta') \geq c(|\eta| + |\eta'|)^{p-2} |\eta - \eta'|^2. \tag{2.4}$$

3. The proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. We adopt the notations introduced in Section 2. At last, a remark on the proof is given.

In order to have a clear proof, we divide it in two steps.

Firstly, we assume that $f(x,s)$ is the strictly convex in s. In this situation, we obtain

$$f(x, v^-(x)) - f(x, u(x)) \geq f'_u(x, u(x))(v^-(x) - u(x)), \quad \text{in } \Omega^-, \tag{3.1}$$

$$f(x, v^+(x)) - f(x, u(x)) \geq f'_u(x, u(x))(v^+(x) - u(x)), \quad \text{in } \Omega^+, \tag{3.2}$$

with the strict inequality whenever $v^-(x) \neq u$ or $v^+(x) \neq u$. Furthermore, we set

$$w^- = v^- - u \text{ in } \Omega^-, \quad \text{and} \quad w^+ = v^+ - u \text{ in } \Omega^+.$$

Then, since v^- (or v^+) is also a solution of (1.1) in Ω^- (or Ω^+), using the symmetry of f and g with respect to T_0^ν, by (1.1) and (3.1), we have,

$$-\Delta_p v^- - (-\Delta_p u) = f(x, v^-) - f(x,u) \geq f'_u(x,u)(v^- - u), \text{ in } \Omega^-, \tag{3.3}$$

$$-\Delta_p v^+ - (-\Delta_p u) = f(x, v^+) - f(x,u) \geq f'_u(x,u)(v^+ - u), \text{ in } \Omega^+, \tag{3.4}$$

and the strict inequality holds whenever $w^-(x) \neq 0$ or $w^+(x) \neq 0$, and

$$w^- = 0 \quad \text{on } \partial\Omega^-, \quad w^+ = 0 \quad \text{on } \partial\Omega^+. \tag{3.5}$$

Actually, if w^+ and w^- are both nonnegative in domains Ω^+ and Ω^-, respectively, then $w^+ \equiv w^- \equiv 0$, by the definition. Then, we get u is symmetric w.r.t. T_0^ν at once.

Now, we can use two methods to prove w^+ and w^- are nonnegative in Ω^+ and Ω^-, respectively.

3.1. *First proof of nonnegativity of w^+ and w^- in Ω^+ and Ω^-, respectively*

In this proof, we argue by a contradiction. So we assume one of two functions, without loss of generality, w^- is negative somewhere in Ω^-. Then, by (3.3), it follows that

$$-\Delta_p v^- - (-\Delta_p u) - f'_u(x,u)(v^- - u) \geq 0 \quad \text{in} \quad \Omega^-, \tag{3.6}$$

with the strict inequality whenever $w^-(x) \neq 0$. Next, considering a connected component D in Ω^- of the set where $w^- < 0$, multiplying w^- on the both sides of (3.6), integrating and by (3.5), we get

$$\int_D (|\nabla v^-|^{p-2}\nabla v^- - |\nabla u|^{p-2}\nabla u)\cdot \nabla w^- - \int_D f'_u(x,u)(w^-)^2 < 0. \tag{3.7}$$

But by Lemma 2.1, Eqs (1.3), (2.1) and (2.2), we have

$$\begin{aligned}
& \int_D (|\nabla v^-|^{p-2}\nabla v^- - |\nabla u|^{p-2}\nabla u)\cdot \nabla w^- - \int_D f'_u(x,u)(w^-)^2 \\
= & \int_D (|\nabla v^-|^{p-2}\nabla v^- - |\nabla u|^{p-2}\nabla u)\cdot (\nabla v^- - \nabla u) - \int_D f'_u(x,u)(w^-)^2 \\
\geq & \int_D c(|\nabla v^-| + |\nabla u|)^{p-2}|\nabla v^- - \nabla u|^2 - \int_D f'_u(x,u)(w^-)^2 \\
> & \ 0.
\end{aligned} \tag{3.8}$$

So it contradicts to (3.7).

3.2. Second proof of nonnegativity of w^+ and w^- in Ω^+ and Ω^-, respectively

In this proof, we use the weak maximum principle. Now, since f and g are symmetric with respect to the hyperplane T_0^ν, by (1.1), differencing the equation for v^- and u in Ω^-, v^+ and u in Ω^+, respectively, and applying the mean value theorem, by (2.3), for w^- and w^+, we obtain

$$-\sum_{i,j}^{N} a'_{ij}(x)w^-_{ij} + \sum_{i}^{N} b'_i(x)w^-_i = f(x,v^-) - f(x,u), \quad \text{in } \Omega^-, \tag{3.9}$$

$$-\sum_{i,j}^{N} a^*_{ij}(x)w^+_{ij} + \sum_{i}^{N} b^*_i(x)w^+_i = f(x,v^+) - f(x,u), \quad \text{in } \Omega^+, \tag{3.10}$$

where $w^\mp_{ij} = \frac{\partial w^\mp}{\partial x_i \partial x_j}$, $w^\mp_i = \frac{\partial w^\mp}{\partial x_i}$, $a'_{ij}(x)$ and $b'_i(x)$, $a^*_{ij}(x)$ and $b^*_i(x)$ are bounded continuous functions in Ω^- and Ω^+ respectively, $i, j \in 1, ..., N$, the matrix $\{a'_{ij}\}$ is positive definite for $x \in \Omega^-$, and the matrix $\{a^*_{ij}\}$ is positive definite for $x \in \Omega^+$.

Meanwhile, by (1.2) and Remark 2.1 or Lemma 2.1 in [8], we know the Eqs (3.9) and (3.10) are uniformly elliptic in Ω^- and Ω^+, respectively. Actually, since $\overline{\Omega}$ doesn't contain the set Z of critical points of u, the Eqs (3.9) and (3.10) are uniformly elliptic. This is an easy consequence of $Z = \emptyset$ and of the linearized process exposed by Serrin in [1].

So, by (1.3), applying the weak maximum principle in the following problems, respectively,

$$\begin{cases} -\sum\limits_{i,j}^{N} a'_{ij}(x)w^-_{ij} + \sum\limits_{i}^{N} b'_i(x)w^-_i - f'_u(x,u)w^- \geq 0, & \text{in } \Omega^-, \\ w^- = 0, & \text{on } \partial\Omega^-, \end{cases} \tag{3.11}$$

$$\begin{cases} -\sum\limits_{i,j}^{N} a^*_{ij}(x)w^+_{ij} + \sum\limits_{i}^{N} b^*_i(x)w^+_i - f'_u(x,u)w^+ \geq 0, & \text{in } \Omega^+, \\ w^+ = 0, & \text{on } \partial\Omega^+, \end{cases} \tag{3.12}$$

we obtain

$$w^- \geq \min_{x\in\overline{\Omega^-}} w^- = \min_{x\in\partial\Omega^-} w^- = 0, \text{ in } \Omega^-,$$
$$w^+ \geq \min_{x\in\overline{\Omega^+}} w^+ = \min_{x\in\partial\Omega^+} w^+ = 0, \text{ in } \Omega^+.$$

Secondly, we assume that f is convex. Then, we can also get (3.3)–(3.5). To prove the symmetry of u in Ω, the fact that $w^- \geq 0$ and $w^+ \geq 0$ are both nonnegative in the respective domains Ω^- and Ω^+ are useful. In this situation, the arguments to prove that w^+ and w^- are both negative are similar to the situation above. On the one hand, we can argue by a contradiction. In this way, we still assume w^- is negative somewhere in Ω^-. Next, the (3.7) is changed to

$$\int_D (|\nabla v^-|^{p-2}\nabla v^- - |\nabla u|^{p-2}\nabla u)\cdot\nabla w^- - \int_D f'_u(x,u)(w^-)^2 \leq 0, \tag{3.13}$$

and the (3.8) still holds. Then, we get a contradiction by (3.13) and (3.8). On the other hand, we can also get $w^+ \geq 0$ and $w^- \geq 0$ in Ω^+ and Ω^- by the weak maximum principle, respectively. So the symmetry of u w.r.t. T_0^ν is proved.

Remark 3.1. *(1) Since the domain Ω can be not convex in the direction ν in our problem and the method of moving planes can't be applied to the symmetry of the solutions, we don't get the monotonicity of the solutions in Ω^+ or Ω^- by our method.*
(2) The assumption that the critical set $Z = \emptyset$ plays an important role to guarantee the quasilinear second order operator in (1.1) is nondegenerate and the quasilinear Eqs (2.3), (3.9) and (3.10) are uniformly elliptic.

As the critical set $Z = \{x \in \Omega | \nabla u(x) = 0\} \neq \emptyset$, the authors in [5, 14] considered not only the symmetry but also the monotonicity of the positive solution of the problem

$$\begin{cases} -\Delta_p u = f(u), & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.14}$$

where Ω is an open bounded smooth convex domain in $\mathbf{R}^N, N \geq 2, 1 < p < 2$ or $p > 1$.

Exactly, in [5], for $1 < p < 2$, the authors considered the symmetry and monotonicity for the positive solution u of $-\Delta_p u = f(u)$ satisfying an homogenuous Dirichlet boundary condition in Ω by MMP, and in [14], for $p > 1$, under the assumption that the critical set Z has only one point in Ω that is the origin, i.e., $Z \cap \Omega = \{0\}$, the symmetry and monotonicity of the positive solution of (3.14) is gotten by MMP.

4. Conclusions

In this paper, we get the symmetry of the positive solutions of the problem

$$\begin{cases} -\Delta_p u = f(x, u), & \text{in} \quad \Omega, \\ u = g(x), & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 1$, Ω is an open smooth bounded domain in $R^N, N \geq 2$, and symmetric w.r.t. the hyperplane T_0^ν(ν is a direction vector in $R^N, |\nu| = 1$), both f and g are symmetric w.r.t. T_0^ν. Assuming some nonlinearities, we prove that the solutions are symmetric w.r.t. the direction ν by a novel simple idea even if Ω is not convex in the direction ν and there are nonzero boundary values. In this paper, the symmetry is different from that gotten by the method of moving planes. So, by our method, the monotonicity of the solutions is not gotten.

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Conflict of interest

The authors declare no conflicts of interest.

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