Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 13368–13389.
DOI:10.3934/math.2023677
Received: 26 January 2023
Revised: 13 March 2023
Accepted: 15 March 2023
Published: 06 April 2023

Research article

A new relaxed acceleration two-sweep modulus-based matrix splitting iteration method for solving linear complementarity problems

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Abstract: A new relaxed acceleration two-sweep modulus-based matrix splitting (NRATMMS) iteration method is developed to solve linear complementarity problems. The convergence of the NRATMMS method is established with the system matrix A being an H_+-matrix. Numerical experiments show that the proposed method is superior to some existing algorithms under appropriate conditions.

Keywords: linear complementarity problem; relaxation; acceleration; modulus-based matrix splitting iteration method; convergence analysis
Mathematics Subject Classification: 65F10, 65H10, 90C30

1. Introduction

Over the past decades, owing to a broad variety of applications in engineering, sciences and economics, the linear complementarity problem (LCP) has been an active topic in the optimization community and has garnered a flurry of interest. The LCP is a powerful mathematical model which is intimately related to many significant scientific problems, such as the well-known primal-dual linear programming, bimatrix game, convex quadratic programming, American option pricing problem and others, see e.g., [1–3] for more details. The LCP consists in determining a vector $z \in \mathbb{R}^n$ such that

$$z \geq 0, \quad v = Az + q \geq 0 \quad \text{and} \quad z \perp v, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ are given. We hereafter abbreviate the problem (1.1) by LCP(A, q).

The LCP(A, q) of form (1.1) together with its extensions are extensively investigated in recent years, and designing efficient numerical algorithms to fast and economically obtain the solution of the LCP(A, q) (1.1) is of great significance. Some numerical iterative algorithms have been developed for solving the LCP(A, q) (1.1) over the past decades, such as the pivot algorithms [1, 2, 4], the projected iterative methods [5–8], the multisplitting methods [9–14], the Newton-type iteration methods [15, 16] and others, see e.g., [17–19] and the references therein. The modulus-based matrix splitting (MMS) iteration method, which was first introduced in [20], is particularly attractive for solving the LCP(A, q) (1.1). Based on the general variable transformation, by setting $z = \frac{|x|+x}{\gamma}$ and $v = \frac{\Omega}{\gamma}(|x| - x)$, and let $A = M - N$, Bai reformulated the LCP(A, q) (1.1) as the following equivalent form [20]

$$(\Omega + M)x = Nx + (\Omega - A)|x| - \gamma q,$$

where $\gamma > 0$ and $\Omega \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix. Then he skillfully designed a general framework of MMS iteration method for solving the large-scale sparse LCP(A, q) (1.1), which exhibits the following formal formulation.

Algorithm 1.1. *([20])* (**The MMS method**) *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$. Assume that $x^0 \in \mathbb{R}^n$ is an arbitrary initial guess. For $k = 0, 1, 2, \cdots$, compute $\{x^{k+1}\}$ by solving the linear system*

$$(\Omega + M)x^{k+1} = Nx^k + (\Omega - A)|x^k| - \gamma q,$$

and then set

$$z^{k+1} = \frac{1}{\gamma}(|x^{k+1}| + x^{k+1})$$

until the iterative sequence $\{z^k\}$ is convergent. Here, $\Omega \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix and γ is a positive constant.

The MMS iteration method not only covers some presented iteration methods, such as the nonstationary extrapolated modulus method [21] and the modified modulus method [22] as its special cases, but also yields a series of modulus-based relaxation methods, such as the modulus-based Jacobi (MJ), the modulus-based Gauss-Seidel (MGS), the modulus-based successive overrelaxation (MSOR) and the modulus-based accelerated overrelaxation (MAOR) methods. Thereafter, since the promising behaviors and elegant mathematical properties of the MMS iterative scheme, it immediately received considerable attention and diverse versions of the MMS method occurred. For instance, Zheng and Yin [23] established a new class of accelerated MMS (AMMS) iteration methods for solving the large-scale sparse LCP(A, q) (1.1), and the convergence analyses of the AMMS method with the system matrix A being a positive definite matrix or an H_+-matrix were explored. In order to further accelerate the MMS method, Zheng et al. [24] combined the relaxation strategy with the matrix splitting technique in the modulus equation of [25] and presented a relaxation MMS (RMMS) iteration method for solving the LCP(A, q) (1.1). The parametric selection strategies of the RMMS method were discussed in depth [24]. In addition, the RMMS method covers the general MMS (GMMS) method [25] as a special case. In the sequel, by extending the two-sweep iteration methods [26, 27], Wu and Li [28] developed a general framework of two-sweep MMS (TMMS) iteration method to solve the LCP(A, q) (1.1), and the convergences of the TMMS method were established with the system matrix A being either an H_+-matrix or a positive-definite matrix. Ren et al. [29] proposed a class of general two-sweep MMS

(GTMMS) iteration methods to solve the LCP(A, q) (1.1) which encompasses the TMMS method by selecting appropriate parameter matrices. Peng et al. [30] presented a relaxation two-sweep MMS (RTMMS) iteration method for solving the LCP(A, q) (1.1) and gave its convergence theories with the system matrix A being an H_+-matrix or a positive-definite matrix. Huang et al. [31] combined the parametric strategy, the relaxation technique and the acceleration technique to construct an accelerated relaxation MMS (ARMMS) iteration method for solving the LCP(A, q) (1.1). The ARMMS method can be regarded as a generalization of some existing methods, such as the MMS [20], the GMMS [25] and the RMMS [24]. For more modulus-based matrix splitting type iteration methods, see [32–41] and the references therein.

On the other hand, Bai and Tong [42] equivalently transformed the LCP(A, q) (1.1) into a nonlinear equation without using variable transformation and proposed an efficient iterative algorithm by using the matrix splittings and extrapolation acceleration techniques. Then some relaxed versions of the method proposed in [42] were constructed by Bai and Huang [43] and the convergence theories were established under some mild conditions. Recently, Wu and Li [44] recasted the LCP(A, q) (1.1) into an implicit fixed-point equation

$$(\Omega + M)z = Nz + |(A - \Omega)z + q| - q, \tag{1.2}$$

where $A = M - N$. In fact, if $M = A$ and $\Omega = I$, then (1.2) reduces to the fixed-point equation proposed in [42]. Based on (1.2), the new MMS (NMMS) method for solving the LCP(A, q) (1.1) was constructed in [44].

Algorithm 1.2. *([44])* (**The NMMS method**) *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$ and the matrix $\Omega + M$ be nonsingular, where $\Omega \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix. Given a nonnegative initial vector $z^0 \in \mathbb{R}^n$, for $k = 0, 1, 2, \cdots$ until the iteration sequence $\{z^k\}$ is convergent, compute $z^{k+1} \in \mathbb{R}^n$ by solving the linear system*

$$(\Omega + M)z^{k+1} = Nz^k + |(A - \Omega)z^k + q| - q.$$

It is obvious that the NMMS method does not need any variable transformations, which is different from the above mentioned MMS type iteration methods. However, the NMMS method still inherits the merits of the MMS type iteration methods and some relaxation versions of it are studied.

Remark 1.1. *Let $A = D_A - L_A - U_A$, where D_A, $-L_A$ and $-U_A$ are the diagonal, strictly lower-triangular and strictly upper-triangular parts of A, respectively. It has been mentioned in [44] that the Algorithm 1.2 can reduce to the following methods.*

(i) If $M = A$, $\Omega = I$ and $N = 0$, then the Algorithm 1.2 becomes the new modulus method:

$$(I + A)z^{k+1} = |(A - I)z^k + q| - q.$$

(ii) If $M = A$, $N = 0$ and $\Omega = \alpha I$, then Algorithm 1.2 turns into the new modified modulus iteration method:

$$(\alpha I + A)z^{k+1} = |(A - \alpha I)z^k + q| - q.$$

(iii) Let $M = \frac{1}{\alpha}(D_A - \beta L_A)$ and $N = \frac{1}{\alpha}((1 - \alpha)D_A + (\alpha - \beta)L_A + \alpha U_A)$, then Algorithm 1.2 reduces to the new MAOR iteration method:

$$(\alpha\Omega + D_A - \beta L_A)z^{k+1} = [(1 - \alpha)D_A + (\alpha - \beta)L_A + \alpha U_A]z^k + \alpha(|(A - \Omega)z^k + q| - q). \tag{1.3}$$

Evidently, based on (1.3)*, when* (α,β) *is equal to* (α,α), $(1,1)$ *and* $(1,0)$*, respectively, we can obtain the new MSOR (NMSOR), the new MGS (NMGS) and the new MJ (NMJ) iteration methods, respectively.*

The goal of this paper is to further improve the computing efficiency of the Algorithm 1.2 for solving the LCP(A, q) (1.1). To this end, we utilize the two-sweep matrix splitting iteration technique in [28, 29] and the relaxation technique, and construct a new class of relaxed acceleration two-sweep MMS (NRATMMS) iteration method for solving the LCP(A, q) (1.1). Convergence analysis of the NRATMMS iteration method is studied in detail. By choosing suitable parameter matrices, the NRATMMS iteration method can generate some relaxation versions. Numerical results are reported to demonstrate the efficiency of the NRATMMS iteration method.

The remainder of this paper is organized as follows. In Section 2, we present some notations and definitions used hereinafter. Section 3 is devoted to establishing the NRATMMS iteration method for solving the LCP(A, q) (1.1) and the global linear convergence of the proposed method is explored. Section 4 reports the numerical results. Finally, some concluding remarks are given in Section 5.

2. Preliminaries

In this section, we collect some notations, classical definitions and some auxiliary results which lay the foundation of our developments.

$\mathbb{R}^{n\times n}$ denotes the set of all $n \times n$ real matrices and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. I is the identity matrix with suitable dimension. $|\cdot|$ denotes absolute value for real scalar or modulus for complex scalar. For $x \in \mathbb{R}^n$, x_i refers to its i-th entry, $|x| = (|x_1|, |x_2|, \cdots, |x_n|) \in \mathbb{R}^n$ represents the componentwise absolute value of a vector x. $\mathbf{tridiag}(a, b, c)$ denotes a tridiagonal matrix that has a, b, c as the subdiagonal, main diagonal and superdiagonal entries in the matrix, respectively. $\mathbf{Tridiag}(A, B, C)$ denotes a block tridiagonal matrix that has A, B, C as the subdiagonal, main diagonal and superdiagonal block entries in the matrix, respectively.

Let two matrices $P = (p_{ij}) \in \mathbb{R}^{m\times n}$ and $Q = (q_{ij}) \in \mathbb{R}^{m\times n}$, we write $P \geq Q\,(P > Q)$ if $p_{ij} \geq q_{ij}\,(p_{ij} > q_{ij})$ holds for any i and j. For $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, $A^\top$ and $|A|$ represent the transpose of A and the absolute value of A ($|A| = (|a_{ij}|) \in \mathbb{R}^{m\times n}$), respectively. For $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $\rho(A)$ represents its spectral radius. Moreover, the comparison matrix $\langle A\rangle$ is defined by

$$\langle a_{ij}\rangle = \begin{cases} |a_{ij}|, & \text{if } i = j, \\ -|a_{ij}|, & \text{if } i \neq j, \end{cases} \quad i, j = 1, 2, \cdots, n.$$

A matrix $A \in \mathbb{R}^{n\times n}$ is called a Z-matrix if all of its off-diagonal entries are nonpositive, and it is a P-matrix if all of its principal minors are positive; we call a real matrix as an M-matrix if it is a Z-matrix with $A^{-1} \geq 0$, and it is called an H-matrix if its comparison matrix $\langle A\rangle$ is an M-matrix. In particular, an H-matrix with positive diagonals is called an H_+-matrix [9]. In addition, a sufficient condition for the matrix A to be a P-matrix is that A is an H_+-matrix. $A \in \mathbb{R}^{n\times n}$ is called a strictly diagonal dominant matrix if $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ for all $1 \leq i \leq n$.

Let M be nonsingular, then $A = M - N$ is called an M-splitting if M is an M-matrix and $N \geq 0$, an H-splitting if $\langle M\rangle - |N|$ is an M-matrix and an H-compatible splitting if $\langle A\rangle = \langle M\rangle - |N|$ [45]. Finally, the following lemmas are needed in the convergence analysis of the proposed method.

Lemma 1. ([46]) Let $A \in \mathbb{R}^{n\times n}$ be an H_+-matrix, then the LCP(A, q) (1.1) has a unique solution for any $q \in \mathbb{R}^n$.

Lemma 2. ([47]) Let $B \in \mathbb{R}^{n\times n}$ be a strictly diagonal dominant matrix. Then for all $C \in \mathbb{R}^{n\times n}$,

$$\|B^{-1}C\|_\infty \leq \max_{1\leq i\leq n} \frac{(|C|e)_i}{(\langle B\rangle e)_i}$$

holds, where $e = (1, 1, \cdots, 1)^\top$.

Lemma 3. ([48]) Let A be an H-matrix, then $|A^{-1}| \leq \langle A\rangle^{-1}$.

Lemma 4. ([49]) If A is an M-matrix, there exists a positive diagonal matrix V such that AV is a strictly diagonal dominant matrix with positive diagonal entries.

Lemma 5. ([49]) Let A, B be two Z-matrices, A be an M-matrix, and $B \geq A$. Then B is an M-matrix.

Lemma 6. ([26]) Let

$$A = \begin{pmatrix} B & C \\ I & 0 \end{pmatrix} \geq 0 \text{ and } \rho(B + C) < 1,$$

then $\rho(A) < 1$.

Lemma 7. ([45]) If $A = M - N$ is an M-splitting of A, then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.

3. The method and convergence

In this section, the NRATMMS iteration method for solving the LCP(A, q) (1.1) is developed, and the general convergence analysis of the NRATMMS iteration method will be explored.

Let $A = M_1 - N_1 = M_2 - N_2$ be two splittings of A and $\Omega = \Omega_1 - \Omega_2 = \Omega_3 - \Omega_4$ with Ω_i $(i = 1, 2, 3, 4)$ being all nonnegative diagonal matrices, then (1.2) can be reformulated to the following fixed point format:

$$(\Omega_1 + M_1)z = (N_1 + \Omega_2) [\theta z + (1 - \theta)z] + |(M_2 - \Omega_3)z + (\Omega_4 - N_2)z + q| - q, \tag{3.1}$$

where $\theta \geq 0$ is a relaxation parameter. Based on (3.1), the NRATMMS iteration method is established as in the following Algorithm 3.1.

Algorithm 3.1. (**The NRATMMS iteration method**) *Let $A = M_1 - N_1 = M_2 - N_2$ be two splittings of A and $\Omega = \Omega_1 - \Omega_2 = \Omega_3 - \Omega_4$ with Ω_i $(i = 1, 2, 3, 4)$ being all nonnegative diagonal matrices such that $M_1 + \Omega_1$ is nonsingular. Given two initial guesses $z^0, z^1 \in \mathbb{R}^n$ and a nonnegative relaxation parameter θ, the iteration sequence $\{z^k\}$ is generated by*

$$(\Omega_1 + M_1)z^{k+1} = (N_1 + \Omega_2)\left[\theta z^k + (1 - \theta)z^{k-1}\right] + \left|(M_2 - \Omega_3)z^k + (\Omega_4 - N_2)z^{k-1} + q\right| - q \tag{3.2}$$

for $k = 1, 2, \cdots$ until convergence.

The Algorithm 3.1 provides a general framework of NMMS iteration methods for solving the LCP(A, q) (1.1), and it can yield a series of NMMS type iteration methods with suitable choices of the matrix splittings and the relaxation parameter. For instance, when $\theta = 1$ and $\Omega_i = 0$ $(i = 1, 2, 3, 4)$, the Algorithm 3.1 reduces to the new accelerated two-sweep MMS (NATMMS) iteration method

$$M_1 z^{k+1} = N_1 z^k + \left|M_2 z^k - N_2 z^{k-1} + q\right| - q.$$

When $\theta = 1$, $\Omega_1 = \Omega_3 = \Omega$, $\Omega_2 = \Omega_4 = 0$, $M_2 = A$ and $N_2 = 0$, the Algorithm 3.1 reduces to the Algorithm 1.2. When $M_1 = \frac{1}{\alpha}(D_A - \beta L_A)$, $N_1 = \frac{1}{\alpha}[(1-\alpha)D_A + (\alpha-\beta)L_A + \alpha U_A]$, $M_2 = D_A - U_A$, $N_2 = L_A$ with $\alpha,\ \beta > 0$, the Algorithm 3.1 gives the new relaxed acceleration two-sweep MAOR (NRATMAOR) iteration method. If $(\alpha,\ \beta)$ is equal to $(\alpha,\ \alpha), (1, 1)$, and $(1, 0)$, the NRATMAOR iteration method reduces to the new relaxed acceleration two-sweep MSOR (NRATMSOR) iteration method, the new relaxed acceleration two-sweep MGS (NRATMGS) iteration method and the new relaxed acceleration two-sweep MJ (NRATMJ) iteration method, respectively.

The convergence analysis for Algorithm 3.1 is investigated with the system matrix A of the LCP(A, q) (1.1) being an H_+-matrix.

Lemma 8. *Assume that $A \in \mathbb{R}^{n\times n}$ is an H_+-matrix. Let $A = M_1 - N_1$ and $A = M_2 - N_2$ be an H-splitting and a general splitting of A, respectively, and Ω_i $(i = 1, 2, 3, 4)$ be four nonnegative diagonal matrices such that $M_1 + \Omega_1$ is nonsingular. Denote*

$$\tilde{\mathcal{L}} = (\Omega_1 + \langle M_1\rangle)^{-1}\left[(\theta + |1-\theta|)|N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2|\right],$$

then the iteration sequence $\{z^k\}$ generated by the Algorithm 3.1 converges to the unique solution z^* for arbitrary two initial vectors if $\rho(\tilde{\mathcal{L}}) < 1$.

Proof. Let z^* be the exact solution of the LCP(A, q) (1.1), then it satisfies

$$(\Omega_1 + M_1)z^* = (N_1 + \Omega_2)\left[\theta z^* + (1-\theta)z^*\right] + |(M_2 - \Omega_3)z^* + (\Omega_4 - N_2)z^* + q| - q. \tag{3.3}$$

Subtracting (3.3) from (3.2), we have

$$\begin{aligned}
|z^{k+1} - z^*| &= |(\Omega_1 + M_1)^{-1}(N_1 + \Omega_2)[\theta(z^k - z^*) + (1-\theta)(z^{k-1} - z^*)] \\
&\quad + (\Omega_1 + M_1)^{-1}|(M_2 - \Omega_3)z^k + (\Omega_4 - N_2)z^{k-1} + q| \\
&\quad - (\Omega_1 + M_1)^{-1}|(M_2 - \Omega_3)z^* + (\Omega_4 - N_2)z^* + q|| \\
&\le |(\Omega_1 + M_1)^{-1}||N_1 + \Omega_2||\theta(z^k - z^*) + (1-\theta)(z^{k-1} - z^*)| \\
&\quad + |(\Omega_1 + M_1)^{-1}|\big||(M_2 - \Omega_3)z^k + (\Omega_4 - N_2)z^{k-1} + q| \\
&\quad - |(M_2 - \Omega_3)z^* + (\Omega_4 - N_2)z^* + q|\big| \\
&\le |(\Omega_1 + M_1)^{-1}||N_1 + \Omega_2|[\theta|z^k - z^*| + |1-\theta||z^{k-1} - z^*|] \\
&\quad + |(\Omega_1 + M_1)^{-1}||(M_2 - \Omega_3)(z^k - z^*) + (\Omega_4 - N_2)(z^{k-1} - z^*)| \\
&\le |(\Omega_1 + M_1)^{-1}||N_1 + \Omega_2|[\theta|z^k - z^*| + |1-\theta||z^{k-1} - z^*|] \\
&\quad + |(\Omega_1 + M_1)^{-1}|[|M_2 - \Omega_3||z^k - z^*| + |\Omega_4 - N_2||z^{k-1} - z^*|] \\
&= |(\Omega_1 + M_1)^{-1}|[\theta|N_1 + \Omega_2| + |M_2 - \Omega_3|]|z^k - z^*| \\
&\quad + |(\Omega_1 + M_1)^{-1}|[|1-\theta||N_1 + \Omega_2| + |\Omega_4 - N_2|]|z^{k-1} - z^*|.
\end{aligned}$$

For simplicity, let

$$F = |(\Omega_1 + M_1)^{-1}|[\theta|N_1 + \Omega_2| + |M_2 - \Omega_3|], \tag{3.4}$$

and

$$G = |(\Omega_1 + M_1)^{-1}|[|1-\theta||N_1 + \Omega_2| + |\Omega_4 - N_2|]. \tag{3.5}$$

Then we have

$$\left|\begin{pmatrix} z^{k+1} - z^* \\ z^k - z^* \end{pmatrix}\right| \le \begin{pmatrix} F & G \\ I & 0 \end{pmatrix} \left|\begin{pmatrix} z^k - z^* \\ z^{k-1} - z^* \end{pmatrix}\right|.$$

Let

$$\mathcal{L} = \begin{pmatrix} F & G \\ I & 0 \end{pmatrix},$$

then the iteration sequence $\{z^k\}$ converges to the unique solution z^* if $\rho(\mathcal{L}) < 1$. Since $\mathcal{L} \ge 0$, according to Lemma 6, $\rho(F+G) < 1$ implies $\rho(\mathcal{L}) < 1$. To prove the convergence of the Algorithm 3.1, it is sufficient to prove $\rho(F+G) < 1$.

Under the conditions that A is an H_+-matrix and $A = M_1 - N_1$ is an H-splitting of A, i.e., $\langle M_1 \rangle - |N_1|$ is an M-matrix, then by Lemma 5, $\langle M_1 \rangle \ge \langle M_1 \rangle - |N_1|$ implies that M_1 is an H-matrix, and $\Omega_1 + M_1$ is also an H-matrix. In the light of Lemma 3, it follows that

$$0 \le |(\Omega_1 + M_1)^{-1}| \le (\Omega_1 + \langle M_1 \rangle)^{-1}.$$

Recall (3.4) and (3.5), we obtain

$$F + G = |(\Omega_1 + M_1)^{-1}|[(\theta + |1-\theta|)|N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2|],$$

which yields that

$$0 \le F + G \le (\Omega_1 + \langle M_1 \rangle)^{-1}[(\theta + |1-\theta|)|N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2|] := \tilde{\mathcal{L}}.$$

As a consequence, based on the monotone property of the spectral radius, the iteration sequence $\{z^k\}$ generated by Algorithm 3.1 converges to the unique solution z^* of the LCP(A, q) (1.1) if $\rho(\tilde{\mathcal{L}}) < 1$. The proof is completed. □

Theorem 3.1. *Assume that $A \in \mathbb{R}^{n \times n}$ is an H_+-matrix. Let $A = M_1 - N_1$ be an H-compatible splitting and $A = M_2 - N_2$ be an M-splitting of A, and Ω_i $(i = 1, 2, 3, 4)$ be four nonnegative diagonal matrices such that $M_1 + \Omega_1$ is nonsingular. Denote*

$$\tilde{\mathcal{L}} = (\Omega_1 + \langle M_1 \rangle)^{-1}[(\theta + |1-\theta|)|N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2|],$$

then the iteration sequence $\{z^k\}$ generated by the Algorithm 3.1 converges to the unique solution z^* of the LCP(A, q) (1.1) for arbitrary two initial vectors if one of the following two conditions holds.

(i) $0 < \theta \le 1$ *and* Ω_i $(i = 1, 2, 3, 4)$ *satisfy*

$$\begin{cases} \langle A \rangle Ve > \Omega_4 Ve, & \text{if } \Omega_3 \ge D_{M_2}, \\ (\langle A \rangle + \Omega)Ve > D_{M_2} Ve, & \text{if } \Omega_3 < D_{M_2}. \end{cases} \tag{3.6}$$

(ii) $\theta > 1$ *and* Ω_i $(i = 1, 2, 3, 4)$ *satisfy*

$$\begin{cases} \theta < 1 + \min\limits_{1 \le i \le n} \frac{[(\langle A \rangle - \Omega_4)Ve]_i}{[(|N_1| + \Omega_2)Ve]_i} \text{ and } \frac{[(\langle A \rangle - \Omega_4)Ve]_i}{[(|N_1| + \Omega_2)Ve]_i} > 0, & \text{if } \Omega_3 \ge D_{M_2}, \\ \theta < 1 + \min\limits_{1 \le i \le n} \frac{[(\langle A \rangle + \Omega - D_{M_2})Ve]_i}{[(|N_1| + \Omega_2)Ve]_i} \text{ and } \frac{[(\langle A \rangle + \Omega - D_{M_2})Ve]_i}{[(|N_1| + \Omega_2)Ve]_i} > 0, & \text{if } \Omega_3 < D_{M_2}. \end{cases} \tag{3.7}$$

Here, $\Omega = \Omega_1 - \Omega_2 = \Omega_3 - \Omega_4$ *and* V *is an arbitrary positive diagonal matrix such that* $(\Omega_1 + \langle M_1 \rangle)V$ *is a strictly diagonal dominant matrix.*

Proof. According to Lemma 8, we only need to demonstrate $\rho(\tilde{\mathcal{L}}) < 1$. Then, on the basis of Lemma 2 and Lemma 4, it follows that

$$\begin{aligned}\rho(\tilde{\mathcal{L}}) =& \rho(V^{-1}\tilde{\mathcal{L}}V) \le \|V^{-1}\tilde{\mathcal{L}}V\|_\infty\\ =& \|[(\Omega_1 + \langle M_1 \rangle)V]^{-1}[(\theta + |1-\theta|)|N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2|]V\|_\infty\\ \le& \max_{1\le i\le n} \frac{\{[(\theta + |1-\theta|)|N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2|]Ve\}_i}{[(\Omega_1 + \langle M_1 \rangle)Ve]_i}.\end{aligned}$$

When $0 < \theta \le 1$, it holds that

$$\rho(\tilde{\mathcal{L}}) \le \max_{1\le i\le n} \frac{\{[|N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2|]Ve\}_i}{[(\Omega_1 + \langle M_1 \rangle)Ve]_i}. \tag{3.8}$$

Since $A = M_2 - N_2$ is an M-splitting of A, M_2 is an M-matrix. Let $M_2 = D_{M_2} - B_{M_2}$ be a splitting of M_2, where D_{M_2} is the positive diagonal matrix of M_2.

If $\Omega_3 \ge D_{M_2}$, it can be concluded that

$$\begin{aligned}&(\Omega_1 + \langle M_1 \rangle)Ve - [|N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2|]Ve\\ &= (\Omega_1 + \langle M_1 \rangle - |N_1 + \Omega_2| - |\Omega_3 - M_2| - |\Omega_4 - N_2|)Ve\\ &= (\Omega_1 + \langle M_1 \rangle - |N_1 + \Omega_2| - |\Omega_3 - D_{M_2} + B_{M_2}| - |\Omega_4 - N_2|)Ve\\ &\ge (\Omega_1 + \langle M_1 \rangle - |N_1 + \Omega_2| - |\Omega_3 - D_{M_2}| - |B_{M_2}| - |\Omega_4 - N_2|)Ve\\ &\ge (\Omega_1 + \langle M_1 \rangle - |N_1| - \Omega_2 - \Omega_3 + D_{M_2} - |B_{M_2}| - \Omega_4 - |N_2|)Ve\\ &= (\Omega_1 + \langle M_1 \rangle - |N_1| - \Omega_2 - \Omega_3 + \langle M_2 \rangle - \Omega_4 - |N_2|)Ve\\ &= (2\langle A \rangle + \Omega_1 - \Omega_2 - \Omega_3 - \Omega_4)Ve\\ &= (2\langle A \rangle - 2\Omega_4)Ve.\end{aligned} \tag{3.9}$$

If $\Omega_3 < D_{M_2}$, we get

$$\begin{aligned}&(\Omega_1 + \langle M_1 \rangle)Ve - [|N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2|]Ve\\ &= (\Omega_1 + \langle M_1 \rangle - |N_1 + \Omega_2| - |M_2 - \Omega_3| - |\Omega_4 - N_2|)Ve\\ &= (\Omega_1 + \langle M_1 \rangle - |N_1 + \Omega_2| - |D_{M_2} - \Omega_3 - B_{M_2}| - |\Omega_4 - N_2|)Ve\\ &\ge (\Omega_1 + \langle M_1 \rangle - |N_1 + \Omega_2| - |D_{M_2} - \Omega_3| - |B_{M_2}| - |\Omega_4 - N_2|)Ve\\ &\ge (\Omega_1 + \langle M_1 \rangle - |N_1| - \Omega_2 + \Omega_3 - 2D_{M_2} + D_{M_2} - |B_{M_2}| - \Omega_4 - |N_2|)Ve\\ &= (\Omega_1 + \langle M_1 \rangle - |N_1| - \Omega_2 + \Omega_3 - 2D_{M_2} + \langle M_2 \rangle - \Omega_4 - |N_2|)Ve\\ &= (2\langle A \rangle - 2D_{M_2} + \Omega_1 - \Omega_2 + \Omega_3 - \Omega_4)Ve\\ &= (2\langle A \rangle - 2D_{M_2} + 2\Omega)Ve.\end{aligned} \tag{3.10}$$

According to (3.8), (3.9) and (3.10), we have $\rho(\tilde{\mathcal{L}}) < 1$ if (3.6) holds.

When $\theta > 1$, it follows that

$$\rho(\tilde{\mathcal{L}}) \le \max_{1\le i\le n} \frac{\{[(2\theta - 1)|N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2|]Ve\}_i}{[(\Omega_1 + \langle M_1 \rangle)Ve]_i}. \tag{3.11}$$

If $\Omega_3 \geq D_{M_2}$, it can be derived that

$$\begin{aligned}
&(\Omega_1 + \langle M_1\rangle)Ve - [(2\theta-1)|N_1+\Omega_2| + |M_2-\Omega_3| + |\Omega_4 - N_2|]Ve \\
&\quad = (\Omega_1 + \langle M_1\rangle - (2\theta-1)|N_1+\Omega_2| - |\Omega_3 - M_2| - |\Omega_4 - N_2|)Ve \\
&\quad = (\Omega_1 + \langle M_1\rangle - (2\theta-1)|N_1+\Omega_2| - |\Omega_3 - D_{M_2} + B_{M_2}| - |\Omega_4 - N_2|)Ve \\
&\quad \geq (\Omega_1 + \langle M_1\rangle - (2\theta-1)|N_1| - (2\theta-1)\Omega_2 - |\Omega_3 - D_{M_2}| - |B_{M_2}| - \Omega_4 - |N_2|)Ve \\
&\quad = (\Omega_1 + \langle M_1\rangle - (2\theta-1)|N_1| - (2\theta-1)\Omega_2 - \Omega_3 + D_{M_2} - |B_{M_2}| - \Omega_4 - |N_2|)Ve \\
&\quad = (\Omega_1 + \langle M_1\rangle - |N_1| - 2(\theta-1)|N_1| - 2(\theta-1)\Omega_2 - \Omega_2 - \Omega_3 + \langle M_2\rangle - \Omega_4 - |N_2|)Ve \\
&\quad = (2\langle A\rangle - 2\Omega_4 - 2(\theta-1)(|N_1| + \Omega_2))Ve,
\end{aligned}$$

from which we have

$$[2\langle A\rangle - 2\Omega_4 - 2(\theta-1)(|N_1|+\Omega_2)]Ve > 0 \tag{3.12}$$

provided that $1 < \theta < \min\limits_{1\leq i\leq n} 1 + \frac{[(\langle A\rangle - \Omega_4)Ve]_i}{[(|N_1|+\Omega_2)Ve]_i}$ and $\frac{[(\langle A\rangle - \Omega_4)Ve]_i}{[(|N_1|+\Omega_2)Ve]_i} > 0$ $(i = 1, 2, \cdots, n)$.

If $\Omega_3 < D_{M_2}$, it is implied that

$$\begin{aligned}
&(\Omega_1 + \langle M_1\rangle)Ve - [(2\theta-1)|N_1+\Omega_2| + |M_2-\Omega_3| + |\Omega_4 - N_2|]Ve \\
&\quad = (\Omega_1 + \langle M_1\rangle - (2\theta-1)|N_1+\Omega_2| - |M_2 - \Omega_3| - |\Omega_4 - N_2|)Ve \\
&\quad = (\Omega_1 + \langle M_1\rangle - (2\theta-1)|N_1+\Omega_2| - |D_{M_2} - \Omega_3 - B_{M_2}| - |\Omega_4 - N_2|)Ve \\
&\quad \geq (\Omega_1 + \langle M_1\rangle - (2\theta-1)|N_1| - (2\theta-1)\Omega_2 - |D_{M_2} - \Omega_3| - |B_{M_2}| - \Omega_4 - |N_2|)Ve \\
&\quad = (\Omega_1 + \langle M_1\rangle - (2\theta-1)|N_1| - (2\theta-1)\Omega_2 - D_{M_2} + \Omega_3 - |B_{M_2}| - \Omega_4 - |N_2|)Ve \\
&\quad = (\Omega_1 + \langle M_1\rangle - |N_1| - 2(\theta-1)|N_1| - 2(\theta-1)\Omega_2 - 2D_{M_2} - \Omega_2 + \Omega_3 + \langle M_2\rangle - \Omega_4 - |N_2|)Ve \\
&\quad = (2\langle A\rangle + 2\Omega - 2D_{M_2} - 2(\theta-1)(|N_1| + \Omega_2))Ve,
\end{aligned}$$

from which we have

$$[2\langle A\rangle + 2\Omega - 2D_{M_2} - 2(\theta-1)(|N_1|+\Omega_2)]Ve > 0 \tag{3.13}$$

provided that $1 < \theta < \min\limits_{1\leq i\leq n} 1 + \frac{[(\langle A\rangle + \Omega - D_{M_2})Ve]_i}{[(|N_1|+\Omega_2)Ve]_i}$ and $\frac{[(\langle A\rangle + \Omega - D_{M_2})Ve]_i}{[(|N_1|+\Omega_2)Ve]_i} > 0$ $(i = 1, 2, \cdots, n)$.

According to (3.11), (3.12) and (3.13), we have $\rho(\tilde{\mathcal{L}}) < 1$ if (3.7) holds. □

Theorem 3.2. *Assume that $A \in \mathbb{R}^{n\times n}$ is an H_+-matrix. Let $\varrho \doteq \rho(D_A^{-1}|B_A|)$. Assume that the choices of the four nonnegative diagonal matrices Ω_i $(i = 1, 2, 3, 4)$ and the three positive parameters α, β, θ such that $M_1 + \Omega_1$ is nonsingular and either $\Omega_3 \leq D_A \leq \min\{2\Omega, 2\Omega - 2(\theta-1)\Omega_2\}$ or $\max\{2\Omega_4, 2\Omega_4 + 2(\theta-1)\Omega_2\} \leq D_A < \Omega_3$. Then the NRATMAOR iteration method is convergent for arbitrary two initial vectors if one of the following eight conditions holds:*

(i) $0 < \theta \leq 1, 0 < \alpha \leq 1, 0 < \beta \leq \alpha, \varrho < \frac{1}{2}$;
(ii) $0 < \theta \leq 1, 1 < \alpha < 2, 0 < \beta \leq \alpha, \varrho < \frac{2-\alpha}{2\alpha}$;
(iii) $0 < \theta \leq 1, 0 < \alpha \leq 1, 0 < \alpha \leq \beta, \varrho < \frac{\alpha}{2\beta}$;
(iv) $0 < \theta \leq 1, 1 < \alpha < 2, 0 < \alpha \leq \beta, \varrho < \frac{2-\alpha}{2\beta}$;
(v) $1 < \theta < \frac{2-\alpha}{2\alpha\varrho - 2\alpha + 2}, \frac{2(\theta-1)}{2\theta-1} < \alpha \leq 1, 0 < \beta \leq \alpha, \varrho < \frac{1}{2}$;
(vi) $1 < \theta < \frac{\alpha}{2\alpha\varrho + 2\alpha - 2}, 1 < \alpha < \frac{2\theta}{2\theta-1}, 0 < \beta \leq \alpha, \varrho < \frac{2-\alpha}{2\alpha}$;
(vii) $1 < \theta < \frac{2-\alpha}{2\beta\varrho - 2\alpha + 2}, \frac{2(\theta-1)}{2\theta-1} < \alpha \leq 1, 0 < \alpha \leq \beta, \varrho < \frac{\alpha}{2\beta}$;

(viii) $1<\theta<\frac{\alpha}{2\beta\varrho+2\alpha-2}, 1<\alpha<\frac{2\theta}{2\theta-1}, 0<\alpha\leq\beta, \varrho<\frac{2-\alpha}{2\beta}$.

Proof. For the NRATMAOR iteration method, we have $A = M_1 - N_1 = M_2 - N_2$ with

$$M_1 = \frac{1}{\alpha}(D_A - \beta L_A), \quad N_1 = \frac{1}{\alpha}\left[(1-\alpha)D_A + (\alpha-\beta)L_A + \alpha U_A\right] \tag{3.14}$$

and

$$M_2 = D_A - U_A, \quad N_2 = L_A,$$

where $\alpha, \beta > 0$ are parameters. In order to use the result of Lemma 8, we need $A = M_1 - N_1$ to be an H-splitting of A. Since A is an H_+-matrix, we have $D_A > 0$. It follows from (3.14) that

$$\begin{aligned}
\langle M_1\rangle - |N_1| &= \left\langle \frac{1}{\alpha}(D_A - \beta L_A)\right\rangle - \left|\frac{1}{\alpha}[(1-\alpha)D_A + (\alpha-\beta)L_A + \alpha U_A]\right| \\
&= \frac{1}{\alpha}(D_A - \beta|L_A|) - \frac{1}{\alpha}\,|[(1-\alpha)D_A + (\alpha-\beta)L_A + \alpha U_A]| \\
&= \frac{1}{\alpha}D_A - \frac{\beta}{\alpha}|L_A| - \frac{|1-\alpha|}{\alpha}D_A - \frac{|\alpha-\beta|}{\alpha}|L_A| - |U_A| \\
&= \frac{1-|1-\alpha|}{\alpha}D_A - \frac{\beta+|\alpha-\beta|}{\alpha}|L_A| - |U_A| \doteq S
\end{aligned}$$

If $0<\beta\leq\alpha$, then

$$S = \frac{1-|1-\alpha|}{\alpha}D_A - |L_A| - |U_A| = \frac{1-|1-\alpha|}{\alpha}D_A - |B_A|,$$

and it follows from Lemma 7 that S is an M-matrix if

$$1-|1-\alpha|>0 \quad \text{and} \quad \varrho < \frac{1-|1-\alpha|}{\alpha},$$

which is satisfied if

$$0<\alpha\leq 1 \quad \text{and} \quad \varrho<1$$

or

$$1<\alpha<2 \quad \text{and} \quad \varrho<\frac{2-\alpha}{\alpha}.$$

If $0<\alpha\leq\beta$, then

$$S \geq \frac{1-|1-\alpha|}{\alpha}D_A - \frac{2\beta}{\alpha}|L_A| - |U_A| \geq \frac{1-|1-\alpha|}{\alpha}D_A - \frac{2\beta}{\alpha}|B_A| \doteq \bar{S}. \tag{3.15}$$

It follows from Lemma 7 that $\bar{S}$ is an M-matrix if

$$1-|1-\alpha|>0 \quad \text{and} \quad \varrho<\frac{1-|1-\alpha|}{2\beta},$$

which is satisfied if

$$0<\alpha\leq 1 \quad \text{and} \quad \varrho<\frac{\alpha}{2\beta} \tag{3.16}$$

or

$$1 < \alpha < 2 \quad \text{and} \quad \varrho < \frac{2-\alpha}{2\beta}. \tag{3.17}$$

In this case, since S is a Z-matrix, it follows from Lemma 5 and (3.15) that S is an M-matrix if (3.16) or (3.17) holds.

In conclusion, $A = M_1 - N_1$ is an H-splitting of A (or, equivalently, S is an M-matrix) if one of the following four conditions holds:

$$0 < \alpha \leq 1, \quad 0 < \beta \leq \alpha, \quad \varrho < 1, \tag{3.18}$$

$$1 < \alpha < 2, \quad 0 < \beta \leq \alpha, \quad \varrho < \frac{2-\alpha}{\alpha}, \tag{3.19}$$

$$0 < \alpha \leq 1, \quad 0 < \alpha \leq \beta, \quad \varrho < \frac{\alpha}{2\beta} \tag{3.20}$$

or

$$1 < \alpha < 2, \quad 0 < \alpha \leq \beta, \quad \varrho < \frac{2-\alpha}{2\beta}. \tag{3.21}$$

In the following, let $\hat{A} = \hat{M} - \hat{N}$ with $\hat{M} = \Omega_1 + \langle M_1 \rangle$ and $\hat{N} = (\theta + |1-\theta|)|N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2|$, then $\tilde{\mathcal{L}} = \hat{M}^{-1}\hat{N}$. In order to prove the convergence of the NRATMAOR iteration method, based on Lemma 8, it suffices to prove $\rho(\tilde{\mathcal{L}}) < 1$ provided that $A = M_1 - N_1$ is an H-splitting of A.

Since

$$\hat{M} = \Omega_1 + \left\langle \frac{1}{\alpha}(D_A - \beta L_A) \right\rangle = \Omega_1 + \frac{1}{\alpha}(D_A - \beta |L_A|)$$

is a lower triangular matrix with positive diagonal entries and non-positive off-diagonal entries, it is an M-matrix. In addition, $\hat{N} \geq 0$. According to Lemma 7, $\hat{A}$ is an M-matrix implies $\rho(\tilde{\mathcal{L}}) < 1$. Thus, we will prove that the Z-matrix $\hat{A}$ is an M-matrix in the following.

Case I: $0 < \theta \leq 1$. In this case, we have

$$\begin{aligned}
\hat{N} =& |N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2| \\
=& \left| \frac{1}{\alpha}[(1-\alpha)D_A + (\alpha - \beta)L_A + \alpha U_A] + \Omega_2 \right| + |D_A - \Omega_3 - U_A| + |\Omega_4 - L_A| \\
\leq& \frac{|1-\alpha|}{\alpha} D_A + \frac{\alpha + |\alpha - \beta|}{\alpha}|L_A| + 2|U_A| + \Omega_2 + \Omega_4 + |D_A - \Omega_3| \doteq \tilde{P},
\end{aligned}$$

from which we have

$$\begin{aligned}
\hat{A} =& \hat{M} - \hat{N} \geq \hat{M} - \tilde{P} \\
=& \Omega_1 + \frac{1}{\alpha}(D_A - \beta|L_A|) - \frac{|1-\alpha|}{\alpha} D_A - \frac{\alpha + |\alpha - \beta|}{\alpha}|L_A| - 2|U_A| - \Omega_2 - \Omega_4 - |D_A - \Omega_3| \\
=& (\Omega_3 - 2\Omega_4 - |D_A - \Omega_3|) + \frac{1 - |1-\alpha|}{\alpha} D_A - \frac{\alpha + \beta + |\alpha - \beta|}{\alpha}|L_A| - 2|U_A|.
\end{aligned} \tag{3.22}$$

It can be easy to prove that the first term of (3.22) is nonnegative if

$$\Omega_3 \leq D_A \leq 2\Omega \tag{3.23}$$

or

$$2\Omega_4 \le D_A < \Omega_3. \tag{3.24}$$

Then it follows from (3.22) that

$$\hat{A} \ge \frac{1-|1-\alpha|}{\alpha} D_A - \frac{\alpha+\beta+|\alpha-\beta|}{\alpha}|L_A| - 2|U_A|. \tag{3.25}$$

(i) If $0 < \beta \le \alpha$, then it can be deduced from (3.25) that

$$\hat{A} \ge \frac{1-|1-\alpha|}{\alpha} D_A - 2|L_A| - 2|U_A| = \frac{1-|1-\alpha|}{\alpha} D_A - 2|B_A| \doteq T,$$

from which and Lemma 5, we obtain that $\hat{A}$ is an M-matrix whenever T is. It follows from Lemma 7 that T is an M-matrix if

$$1-|1-\alpha| > 0 \quad \text{and} \quad \varrho < \frac{1-|1-\alpha|}{2\alpha},$$

which is satisfied if

$$0 < \alpha \le 1 \quad \text{and} \quad \varrho < \frac{1}{2}$$

or

$$1 < \alpha < 2 \quad \text{and} \quad \varrho < \frac{2-\alpha}{2\alpha}.$$

(ii) If $0 < \alpha \le \beta$, it can be deduced from (3.25) that

$$\begin{aligned}\hat{A} \ge& \frac{1-|1-\alpha|}{\alpha} D_A - \frac{2\beta}{\alpha}|L_A| - 2|U_A| \\ \ge& \frac{1-|1-\alpha|}{\alpha} D_A - \frac{2\beta}{\alpha}|B_A| = \bar{S},\end{aligned}$$

which is an M-matrix if (3.16) or (3.17) holds.

In Case I, it can be concluded from (i) and (ii) that $\hat{A}$ is an M-matrix if one of the following four conditions holds:

$$0 < \theta \le 1, \quad 0 < \alpha \le 1, \quad 0 < \beta \le \alpha, \quad \varrho < \frac{1}{2}, \tag{3.26}$$

$$0 < \theta \le 1, \quad 1 < \alpha < 2, \quad 0 < \beta \le \alpha, \quad \varrho < \frac{2-\alpha}{2\alpha}, \tag{3.27}$$

$$0 < \theta \le 1, \quad ,0 < \alpha \le 1, \quad 0 < \alpha \le \beta, \quad \varrho < \frac{\alpha}{2\beta} \tag{3.28}$$

or

$$0 < \theta \le 1, \quad 1 < \alpha < 2, \quad 0 < \alpha \le \beta, \quad \varrho < \frac{2-\alpha}{2\beta}. \tag{3.29}$$

Case II: $\theta > 1$. In this case, we have

$$\begin{aligned}\hat{N} =& (2\theta-1)|N_1 + \Omega_2| + |M_2 - \Omega_3| + |\Omega_4 - N_2| \\ =& (2\theta-1)|\frac{1}{\alpha}[(1-\alpha)D_A + (\alpha-\beta)L_A + \alpha U_A] + \Omega_2| + |D_A - \Omega_3 - U_A| + |\Omega_4 - L_A|\end{aligned}$$

$$\le\frac{(2\theta-1)|1-\alpha|}{\alpha}D_A+\frac{\alpha+(2\theta-1)|\alpha-\beta|}{\alpha}|L_A|+2\theta|U_A|+(2\theta-1)\Omega_2+\Omega_4+|D_A-\Omega_3|\doteq\tilde{N},$$

from which we obtain

$$\begin{aligned}\hat{A}=&\hat{M}-\hat{N}\ge\hat{M}-\tilde{N}\\=&\Omega_1+\frac{1}{\alpha}(D_A-\beta|L_A|)-\frac{(2\theta-1)|1-\alpha|}{\alpha}D_A-\frac{\alpha+(2\theta-1)|\alpha-\beta|}{\alpha}|L_A|\\&-2\theta|U_A|-(2\theta-1)\Omega_2-\Omega_4-|D_A-\Omega_3|\\=&(\Omega_3-2\Omega_4-2(\theta-1)\Omega_2-|D_A-\Omega_3|)\\&+\frac{1-(2\theta-1)|1-\alpha|}{\alpha}D_A-\frac{\alpha+\beta+(2\theta-1)|\alpha-\beta|}{\alpha}|L_A|-2\theta|U_A|.\end{aligned}\tag{3.30}$$

The first term of (3.30) is nonnegative if

$$\Omega_3\le D_A\le 2\Omega-2(\theta-1)\Omega_2<2\Omega\tag{3.31}$$

or

$$2\Omega_4\le 2\Omega_4+2(\theta-1)\Omega_2\le D_A<\Omega_3.\tag{3.32}$$

Then it follows from (3.30) that

$$\hat{A}\ge\frac{1-(2\theta-1)|1-\alpha|}{\alpha}D_A-\frac{\alpha+\beta+(2\theta-1)|\alpha-\beta|}{\alpha}|L_A|-2\theta|U_A|.\tag{3.33}$$

(a) If $0<\beta\le\alpha$, then it follows from (3.33) that

$$\hat{A}\ge\frac{1-(2\theta-1)|1-\alpha|}{\alpha}D_A-2\theta|B_A|\doteq R,$$

from which and Lemma 5, we find that $\hat{A}$ is an M-matrix whenever R is. It follows from Lemma 7 that R is an M-matrix if

$$1-(2\theta-1)|1-\alpha|>0\quad\text{and}\quad\varrho<\frac{1-(2\theta-1)|1-\alpha|}{2\theta\alpha},$$

which is satisfied if

$$\frac{2(\theta-1)}{2\theta-1}<\alpha\le 1,\quad 1<\theta<\frac{2-\alpha}{2\alpha\varrho-2\alpha+2},\quad\varrho<\frac{1}{2}$$

or

$$1<\alpha<\frac{2\theta}{2\theta-1},\quad 1<\theta<\frac{\alpha}{2\alpha\varrho+2\alpha-2},\quad\varrho<\frac{2-\alpha}{2\alpha}.$$

(b) If $0<\alpha\le\beta$, then

$$\begin{aligned}\hat{A}\ge&\frac{1-(2\theta-1)|1-\alpha|}{\alpha}D_A-\frac{2\theta\beta-2\alpha(\theta-1)}{\alpha}|L_A|-2\theta|U_A|\\=&\frac{1-(2\theta-1)|1-\alpha|}{\alpha}D_A-\left(\frac{2\theta\beta}{\alpha}|L_A|+2\theta|U_A|\right)+2(\theta-1)|L_A|\end{aligned}$$

$$\geq \frac{1-(2\theta-1)|1-\alpha|}{\alpha} D_A - 2\theta\left(\frac{\beta}{\alpha}|L_A| + |U_A|\right)$$
$$\geq \frac{1-(2\theta-1)|1-\alpha|}{\alpha} D_A - \frac{2\theta\beta}{\alpha}|B_A| \doteq \tilde{R},$$

from which and Lemma 5, we find that $\hat{A}$ is an M-matrix whenever $\tilde{R}$ is. It follows from Lemma 7 that $\tilde{R}$ is an M-matrix if

$$1-(2\theta-1)|1-\alpha| > 0 \quad \text{and} \quad \varrho < \frac{1-(2\theta-1)|1-\alpha|}{2\theta\beta},$$

which is satisfied if

$$\frac{2(\theta-1)}{2\theta-1} < \alpha \leq 1, \quad 1 < \theta < \frac{2-\alpha}{2\beta\varrho - 2\alpha + 2}, \quad \varrho < \frac{\alpha}{2\beta}$$

or

$$1 < \alpha < \frac{2\theta}{2\theta-1}, \quad 1 < \theta < \frac{\alpha}{2\beta\varrho + 2\alpha - 2}, \quad \varrho < \frac{2-\alpha}{2\beta}.$$

In Case II, it can be concluded from (a) and (b) that $\hat{A}$ is an M-matrix if one of the following four conditions holds:

$$\theta > 1, \quad \frac{2(\theta-1)}{2\theta-1} < \alpha \leq 1, \quad 0 < \beta \leq \alpha, \quad 1 < \theta < \frac{2-\alpha}{2\alpha\varrho - 2\alpha + 2}, \quad \varrho < \frac{1}{2}, \tag{3.34}$$

$$\theta > 1, \quad 1 < \alpha < \frac{2\theta}{2\theta-1}, \quad 0 < \beta \leq \alpha, \quad 1 < \theta < \frac{\alpha}{2\alpha\varrho + 2\alpha - 2}, \quad \varrho < \frac{2-\alpha}{2\alpha}, \tag{3.35}$$

$$\theta > 1, \quad \frac{2(\theta-1)}{2\theta-1} < \alpha \leq 1, \quad 0 < \alpha \leq \beta, \quad 1 < \theta < \frac{2-\alpha}{2\beta\varrho - 2\alpha + 2}, \quad \varrho < \frac{\alpha}{2\beta} \tag{3.36}$$

or

$$\theta > 1, \quad 1 < \alpha < \frac{2\theta}{2\theta-1}, \quad 0 < \alpha \leq \beta, \quad 1 < \theta < \frac{\alpha}{2\beta\varrho + 2\alpha - 2}, \quad \varrho < \frac{2-\alpha}{2\beta}. \tag{3.37}$$

The proof is completed by combining (3.18)–(3.21), (3.23), (3.24), (3.26)–(3.29), (3.31), (3.32) and (3.34)–(3.37). □

4. Numerical results

In this section, three numerical examples are performed to validate the effectiveness of the NRATMMS iteration method.

All test problems are conducted in MATLAB R2016a on a personal computer with 1.19 GHz central processing unit (Intel (R) Core (TM) i5-1035U), 8.00 GB memory and Windows 10 operating system. In the numerical results, we report the number of iteration steps (denoted by "IT"), the elapsed CPU time in seconds (denoted as "CPU") and the norm of the absolute residual vector (denoted by "RES"). Here, RES is defined by

$$\text{RES}(z^k) \doteq \left\|\min\{Az^k + q, z^k\}\right\|_2.$$

As [44], the following three examples are used.

Example 4.1. ([20]) Consider the LCP(A, q), where the matrix $A = \hat{A} + \mu I_{m^2} (\mu \geq 0)$ with

$$\hat{A} = \textbf{Tridiag}(-I_m, S_m, -I_m) \in \mathbb{R}^{m^2 \times m^2}, \quad S_m = \textbf{tridiag}(-1, 4, -1) \in \mathbb{R}^{m \times m},$$

and $q = -Az^* \in \mathbb{R}^{m^2}$ with $z^* = (1, 2, 1, 2, \cdots, 1, 2, \cdots)^\top$ being the unique solution of the LCP(A, q) (1.1).

Example 4.2. ([20]) Consider the LCP(A, q), where the matrix $A = \hat{A} + \mu I_{m^2} (\mu \geq 0)$ with

$$\hat{A} = \textbf{Tridiag}(-1.5I_m,\ S_m, -0.5I_m) \in \mathbb{R}^{m^2 \times m^2}, \quad S_m = \textbf{tridiag}(-1.5, 4, -0.5) \in \mathbb{R}^{m \times m},$$

and $q = -Az^* \in \mathbb{R}^{m^2}$ with $z^* = (1, 2, 1, 2, \cdots, 1, 2, \cdots)^\top$ being the unique solution of the LCP(A, q) (1.1).

Example 4.3. *(Black-Scholes American option pricing). In [50], the original Black-Scholes-Merton model changes to the following partial differential complementarity system*

$$(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2})(y(x, \tau) - g(x, \tau)) = 0,\ \frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0,\ y(x, \tau) - g(x, \tau) \geq 0. \tag{4.1}$$

The initial and boundary conditions of the American put option price $y(x, \tau)$ become $y(x, 0) = g(x, 0)$ and $\lim_{x \to \pm\infty} y(x, \tau) = \lim_{x \to \pm\infty} g(x, \tau)$, where $g(x, \tau)$ is the transformed payoff function. For the price $x \in [a, b]$, (a, b) is equal to $(-1.5, 1.5)$, $(-2, 2)$ or $(-4, 4)$. Let ϑ, η be a number of time steps and a number of x-nodes, σ, T be fixed volatility and expiry time. According to [50], by discretizing (4.1), we obtain the LCP:

$$w := Au - d \geq 0,\ u - g \geq 0 \text{ and } w^\top (u - g) = 0, \tag{4.2}$$

with $A = tridiag(-\tau, 1 + 2\tau, -\tau)$ and $\tau = \frac{\Delta t}{(\Delta x)^2}$, where $\Delta t = \frac{0.5\sigma^2 T}{\vartheta}$, $\Delta x = \frac{b-a}{\eta}$ denote the time step and the price step, respectively. And then, if we employ a transformation $z = u - g$ and $q = Ag - d$ to formula (4.2), the American option pricing problem can be rewritten as LCP (1.1). In our numerical computations, we take $g = 0.5z^*$, and $z^* = (1, 0, 1, 0, \cdots, 1, 0, \cdots)^\top$. The vector d is adjusted such that $d = Az^* - w^*$, where $w^* = (0, 1, 0, 1, \cdots, 0, 1, \cdots)^\top$. The involved parameters are detailed in Table 3.

As shown in [44], the NMGS method can be top-priority when the six tested methods there are used to solve the LCP(A, q) in the three examples. Therefore, in this paper, we focus on comparing the performance of the NMGS method in [44] with the NRATMGS method. For the NMGS iteration method, $\Omega = D_A$ is used [44]. For the NRATMGS method, we take $\Omega_1 = \Omega_3 = D_A$, $\Omega_2 = \Omega_4 = 0$, $M_2 = A$, $N_2 = 0$ and $\alpha = \beta = 1$. In addition, the optimal parameter $\theta_{\exp}$ in the NRATMGS iteration method is obtained experimentally (ranging from 0 to 2 with step size 0.1 for Example 4.1 and Example 4.2, and with step size 0.01 for Example 4.3) by minimizing the corresponding iteration step number. For the sake of fairness, each methods are run 10 times and we take the average value of computing times as the reported CPU. Both methods are started from the initial vectors $z^0 = z^1 = (1, 0, 1, 0, \cdots, 1, 0, \cdots)^\top$ and stopped if RES$(z^k) < 10^{-5}$ or the prescribed maximal iteration number $k_{\max} = 500$ is exceeded. The involved linear systems are solved by " \ ". Numerical results for Examples 4.1–4.3 are reported in Tables 1–3. It follows from Tables 1–3 the NRATMGS method is better than the NMGS method (and the other methods tested in [44]) in terms of the iteration step number and CPU time when the parameter $\theta_{\exp}$ is selected appropriately.

Table 1. Numerical results of Example 4.1.

	Method		*m*			
			16	32	64	128
$\mu = 2$	NMGS	IT	28	31	32	34
		CPU	0.0009	0.0018	0.0055	0.0364
		RES	9.6887×10^{-6}	6.1988×10^{-6}	8.7866×10^{-6}	6.8116×10^{-6}
	NRATMGS	$\theta_{\exp}$	1.4	1.4	1.4	1.4
		IT	24	25	26	28
		CPU	**0.0005**	**0.0012**	**0.0048**	**0.0318**
		RES	6.6171×10^{-6}	8.5424×10^{-6}	9.5986×10^{-6}	5.5387×10^{-6}
$\mu = 4$	NMGS	IT	18	20	21	21
		CPU	0.0003	0.0009	0.0036	0.0224
		RES	9.3072×10^{-6}	4.4327×10^{-6}	4.2816×10^{-6}	9.0760×10^{-6}
	NRATMGS	$\theta_{\exp}$	1.2	1.2	1.3	1.3
		IT	16	17	17	18
		CPU	**0.0002**	**0.0009**	**0.0032**	**0.0161**
		RES	9.3811×10^{-6}	9.4009×10^{-6}	8.9594×10^{-6}	5.9780×10^{-6}
$\mu = 6$	NMGS	IT	15	16	16	17
		CPU	0.0002	0.0011	0.0030	0.0194
		RES	4.3687×10^{-6}	3.6549×10^{-6}	8.1157×10^{-6}	5.6681×10^{-6}
	NRATMGS	$\theta_{\exp}$	1.2	1.2	1.2	1.2
		IT	13	14	14	15
		CPU	**0.0002**	**0.0009**	**0.0025**	**0.0138**
		RES	5.5869×10^{-6}	3.4746×10^{-6}	6.9767×10^{-6}	3.9164×10^{-6}
$\mu = 8$	NMGS	IT	13	13	14	15
		CPU	0.0003	0.0006	0.0027	0.0173
		RES	4.0397×10^{-6}	9.9549×10^{-6}	5.9143×10^{-6}	3.3650×10^{-6}
	NRATMGS	$\theta_{\exp}$	1.2	1.2	1.2	1.2
		IT	11	12	12	13
		CPU	**0.0003**	**0.0006**	**0.0022**	**0.0118**
		RES	8.9972×10^{-6}	4.0405×10^{-6}	6.4052×10^{-6}	2.6779×10^{-6}

Table 2. Numerical results of Example 4.2.

	Method		m			
			16	32	64	128
$\mu = 2$	NMGS	IT	24	26	28	29
		CPU	0.0007	0.0011	0.0051	0.0324
		RES	8.9826×10^{-6}	9.8366×10^{-6}	7.5002×10^{-6}	9.2033×10^{-6}
	NRATMGS	θ_{exp}	1.8	1.9	1.9	1.9
		IT	18	19	20	21
		CPU	**0.0003**	**0.0010**	**0.0037**	**0.0232**
		RES	8.1939×10^{-6}	9.1146×10^{-6}	7.2571×10^{-6}	5.7367×10^{-6}
$\mu = 4$	NMGS	IT	16	17	18	19
		CPU	0.0002	0.0007	0.0030	0.0238
		RES	8.3608×10^{-6}	8.5767×10^{-6}	7.6698×10^{-6}	6.2746×10^{-6}
	NRATMGS	θ_{exp}	1.5	1.5	1.6	1.6
		IT	13	14	17	14
		CPU	**0.0002**	**0.0007**	**0.0026**	**0.0145**
		RES	6.5082×10^{-6}	4.9487×10^{-6}	6.5055×10^{-6}	9.6328×10^{-6}
$\mu = 6$	NMGS	IT	13	14	15	15
		CPU	0.0003	0.0008	0.0028	0.0224
		RES	7.5880×10^{-6}	5.6490×10^{-6}	3.6901×10^{-6}	7.7841×10^{-6}
	NRATMGS	θ_{exp}	1.4	1.4	1.4	1.4
		IT	11	11	12	12
		CPU	**0.0002**	**0.0008**	**0.0022**	**0.0121**
		RES	4.7675×10^{-6}	8.9849×10^{-6}	3.8629×10^{-6}	7.2580×10^{-6}
$\mu = 8$	NMGS	IT	12	12	13	13
		CPU	0.0003	0.0005	0.0024	0.0163
		RES	2.8368×10^{-6}	7.0877×10^{-6}	3.6923×10^{-6}	7.7385×10^{-6}
	NRATMGS	θ_{exp}	1.3	1.3	1.3	1.3
		IT	10	10	11	11
		CPU	**0.0002**	**0.0005**	**0.0021**	**0.0112**
		RES	3.3256×10^{-6}	7.2249×10^{-6}	2.6219×10^{-6}	5.2755×10^{-6}

Table 3. Numerical results of Example 4.3.

Case	Grid(η,ϑ)	τ	NMGS			NRATMGS			
			IT	CPU	RES	$\theta_{\exp}$	IT	CPU	RES
$\sigma = 0.2$ $T = 0.5$ $a = -1.5$ $b = 1.5$	$(4000, 2000)$	8.8889	23	0.0034	3.4764×10^{-6}	0.95	20	**0.0028**	3.2943×10^{-6}
	$(6000, 3000)$	13.3333	26	0.0062	7.7964×10^{-6}	0.93	22	**0.0046**	5.1432×10^{-6}
	$(8000, 5000)$	14.2222	25	0.0075	1.5296×10^{-6}	1.03	23	**0.0062**	7.9327×10^{-6}
	$(8000, 8000)$	8.8889	23	0.0065	4.9204×10^{-6}	0.95	20	**0.0056**	4.6315×10^{-6}
	$(10000, 10000)$	11.1111	23	0.0077	9.1134×10^{-7}	1.04	21	**0.0073**	5.2285×10^{-6}
$\sigma = 0.2$ $T = 0.25$ $a = -1.5$ $b = 1.5$	$(6000, 3000)$	6.6667	21	0.0043	6.0408×10^{-6}	0.93	18	**0.0038**	7.5730×10^{-6}
	$(8000, 4000)$	8.8889	23	0.0064	4.9204×10^{-6}	0.95	20	**0.0055**	4.6315×10^{-6}
	$(10000, 5000)$	11.1111	23	0.0078	9.1134×10^{-7}	1.04	21	**0.0073**	5.2285×10^{-6}
	$(15000, 15000)$	8.3333	22	0.0111	8.1100×10^{-6}	0.82	19	**0.0101**	7.3832×10^{-6}
	$(20000, 20000)$	11.1111	23	0.0177	1.2828×10^{-6}	1.04	21	**0.0152**	7.2461×10^{-6}
$\sigma = 0.3$ $T = 0.5$ $a = -2$ $b = 2$	$(4000, 2500)$	9	23	0.0034	3.8926×10^{-6}	0.95	20	**0.0029**	3.1298×10^{-6}
	$(6000, 3000)$	16.875	27	0.0056	7.5378×10^{-6}	1.1	25	**0.0052**	7.0849×10^{-6}
	$(8000, 4000)$	22.5	32	0.0088	5.2859×10^{-6}	0.99	28	**0.0076**	4.8398×10^{-6}
	$(8000, 6000)$	15	26	0.0071	4.9459×10^{-6}	0.86	23	**0.0062**	7.8720×10^{-6}
	$(10000, 10000)$	14.0625	25	0.0087	2.5353×10^{-6}	1.05	23	**0.0081**	6.5500×10^{-6}
$\sigma = 0.3$ $T = 0.25$ $a = -4$ $b = 4$	$(8000, 4000)$	2.8125	18	0.0050	6.2985×10^{-6}	0.92	16	**0.0045**	8.3277×10^{-6}
	$(16000, 8000)$	5.625	21	0.0112	5.5724×10^{-6}	0.94	18	**0.0101**	7.3356×10^{-6}
	$(20000, 10000)$	7.0313	23	0.0177	6.2544×10^{-7}	0.91	20	**0.0144**	6.1941×10^{-6}
	$(24000, 15000)$	6.75	23	0.0202	6.6919×10^{-7}	0.91	20	**0.0179**	6.9170×10^{-6}
	$(30000, 24000)$	6.5918	23	0.0275	7.3343×10^{-7}	0.91	20	**0.0243**	7.7511×10^{-6}

5. Conclusions

In this paper, by applying the matrix splitting, relaxation technique and two-sweep iteration form to the new modulus-based matrix splitting formula, we propose a new relaxed acceleration two-sweep modulus-based matrix splitting (NRATMMS) iteration method for solving the LCP(A, q) (1.1). We investigate the convergence properties of the NRATMMS iteration method with the system matrix A of the LCP(A, q) (1.1) being an H_+-matrix. Numerical experiments illustrate that the NRATMMS iteration method is effective, and it can be superior to some existing methods. However, the choices of the optimal parameters in theory require further investigation.

Acknowledgments

The authors are grateful to the five reviewers and the editor for their helpful comments and suggestions that have helped to improve the paper. This research is supported by the National Natural Science Foundation of China (12201275, 12131004), the Ministry of Education in China of Humanities and Social Science Project (21YJCZH204), the Project of Liaoning Provincial Federation Social Science Circles (2023lslqnkt-044, 2022lslwtkt-069), the Natural Science Foundation of Fujian Province (2021J01661) and the Ministry of Science and Technology of China (2021YFA1003600).

Conflict of interest

The authors confirm that there has no conflict of interest.

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