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Research article

On some new vector valued sequence spaces $E(X, \lambda, p)$

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Abstract: To define a new sequence space and determine the Köthe-Toeplitz duals of this sequence space, characterizing the matrix transformation classes between the defined sequence spaces and classical sequence spaces has been an important area of work for researchers. Defining and examining a new vector-valued sequence space is also a considerable field of study since it generalizes classical sequence spaces. In this study, new vector-valued sequence spaces $E(X, \lambda, p)$ are introduced. The Köthe-Toeplitz duals of $E(X, \lambda, p)$ spaces are identified. Also, necessary and sufficient conditions are determined for $A = (A_{nk})$ to belong to the matrix classes $(E(X, \lambda, p), c(q))$; where $A_{nk} \in B(X, Y)$, $X \in \{c, \ell_\infty\}$ and Y is any Banach spaces.

Keywords: vector-valued sequence spaces; generalized Köthe-Toeplitz duals; matrix transformations
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1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and $p = (p_k)$ be any bounded sequence of strictly positive real numbers. Let $w(X)$ be the set of all sequences $x = x_k$ with $x_k \in X$. Bounded, convergent and null vector-valued sequence spaces in X are denoted by $\ell_\infty(X)$, $c(X)$, and $c_0(X)$, respectively. These sequence spaces are linear subspace of $w(X)$. When $X = \mathbb{R}$ or $\mathbb{C}$ (the real or complex numbers) it is used the familiar notations ℓ_∞, c and c_0. Here, $\ell_\infty(X)$, $c(X)$, and $c_0(X)$ are Banach spaces with the norm

$$\|\mathrm{x}\|_\infty = \sup_k \|x_k\|,$$

where $x_k \in X$ for $k = 1, 2, ...$ The set

$$S = \{x \in X : \|x\| \leq 1\}$$

is the unit ball. $B(X, Y)$ denotes the set of all bounded operators from a Banach space X into the Banach space Y. If $T \in B(X, Y)$, the operator norm of T is

$$\|T\| = \sup_{x \in S} \|Tx\|.$$

In this paper, to indicate the continuous dual of X we use the notation X^* instead of $B(X, \mathbb{C})$. Also, e and $e(x)$ denote the sequences $(1, 1, 1, ...)$ and $(x, x, x, ...)$, respectively.

In the classical theory of matrix transformations, one of the basic problems is the characterization of matrices that map a sequence space E into a sequence space F. The first step of this characterization is the determination of the Köthe-Toeplitz duals of E. Also, the Köthe-Toeplitz duals of E are called β-dual and α-dual of E.

Let X and Y be Banach spaces and (A_k) be the sequence of operators A_k defined from X to Y, which are linear but not necessarily bounded. Then, the β-dual and α-dual of E are defined as follows:

$$\begin{aligned} E^{\alpha} &= \Big\{A = (A_k) : \sum_{k=1}^{\infty} \|A_k x_k\| < \infty \quad \text{for all } x \in E\Big\}, \\ E^{\beta} &= \Big\{A = (A_k) : \sum_{k=1}^{\infty} A_k x_k \quad \text{converges in the } Y - \text{norm for all } x \in E\Big\}. \end{aligned}$$

Generalized Köthe-Toeplitz duals of X-valued sequence spaces $\ell_\infty(X)$, $c(X)$ and $c_0(X)$ were defined by Maddox [6].

Let $A\mathrm{x} = (A_{nk})$ be an infinite matrix of linear, but not necessarily bounded, operators A_{nk} from X to Y. Suppose E is a nonempty subset of $w(X)$ and suppose F is a nonempty subset of $w(Y)$. We define the matrix classes (E, F) by saying that $A \in (E, F)$ if and only if, for every $\mathrm{x} = (x_k) \in E$,

$$A_n\mathrm{x} = \sum_{k=1}^{\infty} A_{nk} x_k,$$

converges for each n in the norm topology of Y and the sequence

$$A\mathrm{x} = \Big(\sum A_{nk} x_k\Big)_n \text{ belongs to } F.$$

X-valued paranormed sequence spaces $c_0(X, p)$, $c(X, p)$, $\ell_\infty(X, p)$ and $\ell(X, p)$ that are the generalization of Maddox spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ and $\ell(p)$ were first defined by Rath [1]. Rath also determined the generalized Köthe-Toeplitz duals of $c_0(X, p)$, $c(X, p)$, $\ell_\infty(X, p)$ and $\ell(X, p)$. Suantai [8] and Suantai and Sudsukh [7] gave the matrix characterizations $(\xi(X, p), c(q))$ where

$$\xi \in \{c_0, \underline{c_0}, c, \ell, \ell_\infty\ E_r, F_r\}.$$

Srivastava and Srivastava [2] introduced the set $c_0(X, \lambda, p)$ and determined the generalized Köthe-Toeplitz duals of these spaces where $\lambda = (\lambda_k)$ is any sequence of non-zero complex numbers.

Now, we define the new X-valued sequence spaces as follows:

$$\begin{aligned} c(X, \lambda, p) &= \{\mathrm{x} = (x_k) : \exists l \in X, \lim_{k\to\infty} \|\lambda_k x_k - l\|^{p_k} = 0\}, \\ \ell_\infty(X, \lambda, p) &= \{\mathrm{x} = (x_k) : \sup_k \|\lambda_k x_k\|^{p_k} < \infty\}. \end{aligned}$$

The spaces $c(X, \lambda, p)$ and $\ell_\infty(X, \lambda, p)$ are BK spaces with the norm

$$\|\mathrm{x}\|_{\lambda,p} = \sup \|\lambda_k x_k\|^{p_k/M},$$

where

$$p = (p_k) \in \ell_\infty,$$

and

$$M = \max\{1, \sup_k p_k\}.$$

$c(X, \lambda, p)$ and $\ell_\infty(X, \lambda, p)$ are generalization of several sequence spaces. Some of these generalizations are as follows:

If $\lambda_k = 1$ for all k and $X = \mathbb{C}$, then we obtain spaces $c(p)$ and $\ell_\infty(p)$ introduced by Maddox in [5].

If $\lambda_k = 1$ for all k, then we obtain spaces $c(X, p)$ and $\ell_\infty(X, p)$ introduced by Rath in [1] and also introduced by Suantai and Sudsukh [7].

If $\lambda_k = 1$ and $p_k = 1$ for all k, then we obtain spaces $c(X)$ and $\ell_\infty(X)$ introduced by Maddox in [6].

2. Köthe-Toeplitz duals

This section is concerned with the generalized Köthe-Toeplitz duals of the spaces $c(X, \lambda, p)$, $\ell_\infty(X, \lambda, p)$.

Now we define the group norm and consider an inequality to be used to calculate the β- duals of our spaces. If (F_k) is a sequence in $B(X, Y)$, the group norm of (T_k) is

$$\|(F_k)\| = \sup \left\| \sum_{k=1}^{n} F_k x_k \right\|, \tag{2.1}$$

where the supremum is taken over all $n \in \mathbb{N}$ and all choices of $x_k \in S$. Inequality

$$\left\| \sum_{k=n}^{n+p} F_k x_k \right\| \leq \|R_n\|. \max\{\|x_k\| : n \leq k \leq n + p\} \tag{2.2}$$

holds for any x_k and all $n \geq 1$, and all non-negative integers p, where

$$R_n = (F_n, F_{n+1}, F_{n+2}, ...)$$

is called n^{th} tail of (F_n), see [5].

Some results related to Köthe-Toeplitz duals of the space $c_0(X, \lambda, p)$ were obtained by Srivastava and Srivastava [2] as follows:

Lemma 1. *$A = (A_k) \in c_0^\alpha(X, \lambda, p)$ if and only if*

(i) There exist an integer $m \geq 1$ such that $A_k \in B(X, Y)$ for each $k \geq m$.

(ii) There exist an integer $N > 1$ such that $\sum_{k=m}^{\infty} |\lambda_k|^{-1} \|A_k\| N^{-r_k} < \infty$.

Lemma 2. *Let (T_k) is a sequence in $B(X, Y)$. Then, exactly one of the following is true:*

(i) $\|R_k\| = \infty$ for all $k \geq 1$.

(ii) $\|R_k\| < \infty$ for all $k \geq 1$.

Lemma 3. *$A = (A_k) \in c_0^\beta(X, \lambda, p)$ if and only if*

(i) There exist an integer $m \geq 1$ such that $A_k \in B(X, Y)$ for each $k \geq m$.

(ii) $\|R_m(\lambda, N)\| < \infty$ for some $N > 1$, where

$$R_m(\lambda, N) = (\lambda_m^{-1} N^{-r_m} A_m, \lambda_{m+1}^{-1} N^{-r_{m+1}} A_{m+1}, \lambda_{m+2}^{-1} N^{-r_{m+2}} A_{m+2}, ...).$$

Theorem 1. *$A = (A_k) \in c^\beta(X, \lambda, p)$ if and only if*

(i) $A = (A_k) \in c_0^\beta(X, \lambda, p)$.

(ii) $\sum_{k=1}^{\infty} \lambda_k^{-1} A_k x$ converges for each $x \in X$.

Proof. Suppose x $\in c(X, \lambda, p)$ and suppose *(i)* and *(ii)* hold. We can write

$$c(X, \lambda, p) = c_0(X, \lambda, p) + E,$$

where

$$E = \{e(z) : z \in X\}.$$

See Theorem 3.9 [9]. Then, we can write

$$\lambda_k x_k = \lambda_k y_k + z$$

for all $k \in \mathbb{N}$ where the sequence z in X and the sequence (y_k) in $c_0(X, \lambda, p)$.

Therefore, $\sum_{k=1}^{\infty} \lambda_k^{-1} A_k z$ is convergent by condition *(ii)* and $\sum_{k=1}^{\infty} A_k y_k$ is convergent by Lemma 3. Thus we have that $\sum_{k=1}^{\infty} A_k x_k$ converges.

Since $c^\beta(X, \lambda, p) \subset c_0^\beta(X, \lambda, p)$ condition *(i)* follows from Lemma 3. If we take any $x \in c(X, \lambda, p)$ and $y \in c_0(X, \lambda, p)$, then $\sum_{k=1}^{\infty} A_k x_k$ and $\sum_{k=1}^{\infty} A_k y_k$ are converge. Morever, we consider

$$\lambda_k x_k = \lambda_k y_k + z$$

for all $k \in \mathbb{N}$ where the sequence z in X and the sequence (y_k) in $c_0(X, \lambda, p)$. Hence, $\sum_{k=1}^{\infty} \lambda_k^{-1} A_k z$ converges for each $z \in X$. □

If we take $\lambda_k = 1$ and $p_k = 1$ for all k in the Theorem 1, then we have Maddox [6] Proposition 3.2 as follows:

Corollary 1. *$(A_k) \in c^\beta(X)$ if and only if*

(i) There exists $m \in \mathbb{N}$ such that $A_k \in B(X, Y)$ for all $k \geq m$.

(ii) $\sum_{k=1}^{\infty} A_k$ converges in the strong operator topology.

Theorem 2. *$A = (A_k) \in \ell_\infty^\beta(X, \lambda, p)$ if and only if*

(i) There exist an integer $m \geq 1$ such that $A_k \in B(X, Y)$ for each $k \geq m$.

(ii) $\|R_m(\lambda, \mathcal{N})\| \to 0$ for all $N > 1$, where

$$R_m(\lambda, \mathcal{N}) = (\lambda_m^{-1} N^{r_m} A_m, \lambda_{m+1}^{-1} N^{r_{m+1}} A_{m+1}, \lambda_{m+2}^{-1} N^{r_{m+2}} A_{m+2}, ...).$$

Proof. Let *(i)* and *(ii)* hold and $x \in \ell_\infty(X, \lambda, p)$. Then, there exists an integer $N_0 > 1$ such that $\|x_k\| < |\lambda_k|^{-1} N_0^{r_k}$ for all $k \in \mathbb{N}$. If we use inequality (2.2), then we obtain

$$\begin{aligned}\left\| \sum_{k=n}^{n+i} A_k x_x \right\| &= \left\| \sum_{k=n}^{n+i} \lambda_k^{-1} N^{r_k} A_k \lambda_k N^{-r_k} x_k \right\| \\ &\leq \|R_m(\lambda, \mathcal{N})\|. \max_{n \leq k \leq n+i} \{|\lambda_k| N^{-r_k} \|x_k\|\} \\ &\leq \|R_m(\lambda, \mathcal{N})\|.\end{aligned}$$

Hence, $\sum_{k=1}^{\infty} A_k x_k$ is convergent since Y is a Banach space.

Since $\ell_\infty^\beta(X, \lambda, p) \subset c_0^\beta(X, \lambda, p)$ Lemma 3.2 yields condition *(i)*. Now, suppose that *(ii)* fails, say

$$\limsup_n \|R_m(\lambda, \mathcal{N})\| = 3\kappa > 0.$$

Then, there exist integer $0 < m \leq m_1 \leq n_1$ and $z_{m_1}, z_{m_1+1}, ..., z_{n_1}$ in S such that

$$\left\| \sum_{m_1}^{n_1} \lambda_k^{-1} N^{r_k} A_k z_k \right\| > \kappa.$$

Choose $n_1 < m_2$ such that $\|R_{m_2}(\lambda)\| > 2\kappa$. Then, there exist $m_2 \leq n_2$ and $z_{m_2}, z_{m_2+1}, ..., z_{n_2}$ in S such that

$$\left\| \sum_{m_2}^{n_2} \lambda_k^{-1} N^{r_k} A_k z_k \right\| > \kappa.$$

Now, by proceeding in this way, we can define the sequence $x = (x_k)$ as follows

$$x_k = \begin{cases} \lambda_k^{-1} N^{r_k} z_k, & m_i \leq k \leq n_i, \\ \theta, & otherwise, \end{cases}$$

for all natural numbers i. Then, we have that x $\in \ell_\infty(X, \lambda, p)$, since

$$\|\text{x}\| = \sup \|\lambda_k x_k\|^{p_k} \leq N.$$

On the other hand $\sum_{k=0}^{\infty} A_k x_k$ is divergent, that is a contradiction. Hence the proof is completed. □

The Köthe-Toeplitz duals of $c_0(X, \lambda, p)$, $c(X, \lambda, p)$ and $\ell_\infty(X, \lambda, p)$ can be rewritten in the case of $A_k \in B(X, Y)$ for all integers k. For these concept let define the following sets, which are the subspace of $w(B(X, Y))$:

$$M_0(B(X, Y), \lambda, p) = \bigcup_{N>1} \left\{ A = (A_k) : \sum_{k=1}^{\infty} |\lambda_k|^{-1} \|A_k\| N^{-r_k} < \infty \right\}, \tag{2.3}$$

$$M_0(X^*, \lambda, p) = \bigcup_{N>1} \left\{ f = (f_k) : \sum_{k=1}^{\infty} |\lambda_k|^{-1} \|f_k\| N^{-r_k} < \infty \right\}, \tag{2.4}$$

$$M_\infty(B(X, Y), \lambda, p) = \bigcap_{N>1} \left\{ A = (A_k) : \sum_{k=1}^{\infty} |\lambda_k|^{-1} \|A_k\| N^{r_k} < \infty \right\}, \tag{2.5}$$

$$M_\infty(X^*,\lambda,p)=\bigcap_{N>1}\left\{f=(f_k):\sum_{k=1}^{\infty}|\lambda_k|^{-1}\|f_k\|N^{r_k}<\infty\right\}. \tag{2.6}$$

Srivastava and Srivastava [2] showed that in the Theorem 3.1

$$c_0^\alpha(X,\lambda,p)=M_0(B(X,Y),\lambda,p).$$

Moreover, they showed that if $Y=\mathbb{C}$ and $f_k\in X^*$ for $k\geq 1$, then

$$c_0^\alpha(X,\lambda,p)=c_0^\beta(X,\lambda,p)=M_0(X^*,\lambda,p)$$

in the Theorem 3.5.

The following result can now be proved in the same way as Theorem 3.1 in [2] .

Theorem 3. $\ell_\infty^\alpha(X,\lambda,p)=M_\infty(B(X,Y),\lambda,p)$.

Theorem 4. *Let Y be complex field $\mathbb{C}$ and $f=(f_k)$ be a sequence of continuous linear functionals on X. Then,*

$$\ell_\infty^\alpha(X,\lambda,p)=\ell_\infty^\beta(X,\lambda,p)=M_\infty(X^*,\lambda,p).$$

Proof.

$$\ell_\infty^\alpha(X,\lambda,p)=M_\infty(X^*,\lambda,p)$$

from Theorem 3. Now, we show that

$$\ell_\infty^\beta(X,\lambda,p)\subset M_\infty(X^*,\lambda,p).$$

Following Srivastava and Srivastava [2] as in Theorem 3.5; suppose

$$f=(f_k)\notin M_\infty(X^*,\lambda,p),$$

then we obtain

$$\sum_{k=1}^{\infty}|\lambda_k|^{-1}\|f_k\|N^{r_k}>2,$$

for some $N>1$. Also, there exists $z_k\in S$ such that $\|f_k\|<2|f_k(z_k)|$ for each $k\geq 1$. If define the sequence $x=(x_k)$ as

$$x_k=\operatorname{sgn}(f_k(z_k))|\lambda_k|^{-1}N^{r_k}z_k,$$

for $k\in S(N)$, then the sequence $x=(x_k)$ in $\ell_\infty(X,\lambda,p)$ but

$$\sum_{k=1}^{\infty}f_k(x_k)=\infty.$$

These show that $f=(f_k)\notin\ell_\infty^\beta(X,\lambda,p)$, so $\ell_\infty^\beta(X,\lambda,p)\subset M_\infty(X^*,\lambda,p)$.

On the other hand $\ell_\infty^\alpha(X,\lambda,p)\subset\ell_\infty^\beta(X,\lambda,p)$ since $\mathbb{C}$ is complete. □

Taking f $=(f_k)\in B(X,\mathbb{C})$ instead of A $=(A_k)\in B(X,Y)$ in Theorem 1, the following result can be obtained.

Theorem 5. *Let $f=(f_k)$ be a sequence such that $f_k\in X$ for all $k\in\mathbb{N}$. Then, $f=(f_k)\in c^\beta(X,\lambda,p)$ if and only if*

(i) $\sum_{k=1}^{\infty}|\lambda_k|^{-1}\|f_k\|N^{-1/p_k}<\infty$ for some $N\in\mathbb{N}$.

(ii) $\sum_{k=1}^{\infty}\lambda_k^{-1}f_k(x)$ converges for each $x\in X$.

3. Matrix transformations

In this section, some matrix classes will be characterized using similar methods developed by Wu and Liu [11] and Suantai and Sudsukh [7]. Throughout these section, $F = (f_k^n)$ be an infinite matrix and $f_k^n \in X^*$ for all $k, n \in \mathbb{N}$. Moreover, let $u = (u_k)$ and $v = (v_k)$ be scalar sequences and $u_k \neq 0$, $v_k \neq 0$ and we can define two sets as follow:

$$E_u = \{x = (x_k) \in w(X) : (u_k x_k) \in E\},$$

and

$$_u(E, F)_v = \{F = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E, F)\}.$$

An X-valued sequence space E is *solid* when $x = (x_k) \in E$ and $\|y_k\| \leq \|x_k\|$ for all $k \in \mathbb{N}$ together imply that $y = (y_k) \in E$.

Now we give two lemmas, their proof can be found in [7].

Lemma 4. *Let E be a solid X-valued sequence space which is an FK-space and contain $\Phi(X)$. Then*

$$(E, c(q)) = (E, c_0(q)) \oplus (E, \langle \mathrm{e} \rangle).$$

Lemma 5. *Let E and E_n be X-valued sequence spaces, and F and F_n be scalar-valued sequence spaces for all $n \in \mathbb{N}$. Then,*

(i) $(\bigcup_{n=1}^{\infty} E_n, F) = \bigcap_{n=1}^{\infty}(E_n, F)$.

(ii) $(E, \bigcap_{n=1}^{\infty} F_n) = \bigcap_{n=1}^{\infty}(E, F_n)$.

(iii) $(E_1 + E_2, F) = (E_1, F) \bigcap (E_2, F)$.

(iv) $(E_u, F_v) =_v (E, F)_{u^{-1}}$.

Theorem 6. $F = (f_k^n) \in (c_0(X, \lambda, p), c(q))$ *if and only if there is a sequence* $(f_k) \subset X^*$ *such that*

(i) $\sum_{k=1}^{\infty} |\lambda_k|^{-1} \|f_k\| N^{-1/p_k} < \infty$ *for some* $N \in \mathbb{N}$.

(ii) $m^{1/q_n}(f_k^n - f_k) \to 0$ *as* $n \to \infty$ *for every* $k, m \in \mathbb{N}$.

(iii) $m^{1/q_n} \sum_{k=1}^{\infty} \|f_j^n\| r^{1/p_k} \to 0$ *as* $n, r \to \infty$ *for each* $m \in \mathbb{N}$.

Proof. For the sufficiency assume that

$$F = (f_k^n) \in (c_0(X, \lambda, p), c(q)).$$

Since

$$c_q = c_0(q) \oplus \langle \mathrm{e} \rangle$$

from [10] (page-301), then we obtain

$$F = (f_k^n) \in (c_0(X, \lambda, p), c_0(q) \oplus \langle \mathrm{e} \rangle).$$

Now, we can choose

$$D \in (c_0(X, \lambda, p), c_0(q))$$

and

$$E = (h_k^n) \in (c_0(X, \lambda, p), \langle e \rangle)$$

such that $F = D + E$ by Lemma 4. In addition, $(h_k^n(x))_{n=1}^{\infty} \in \langle e \rangle$ for all $x \in X$ and $k \in \mathbb{N}$ since $\Phi(X) \subset c_0(X, \lambda, p)$. These yields that $h_k^n = h_k^{n+1}$ for all $k, n \in \mathbb{N}$. Let choose $f_k = h_k^1$, so we obtain

$$D = (f_k^n - f_k)_{n,k} \in (c_0(X, \lambda, p), c_0(q)).$$

We know that

$$c_0(q) = \bigcap_{m=1}^{\infty} c_{0_{(m^{1/p_k})}}$$

from Theorem 0 in [10], which yields that

$$(m^{1/q_n}(f_k^n - f_k))_{n,k} \in (c_0(X, \lambda, p), c_0)$$

for all $m \in \mathbb{N}$ by Lemma 5*(ii)* and *(iv)*. If we use Theorem 2.4 in [11], then conditions *(ii)* and *(iii)* hold. Further $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all

$$x = (x_k) \in c_0(X, \lambda, p),$$

since

$$E = (f_k)_{n,k} \in (c_0(X, \lambda, p), \langle e \rangle).$$

So (i) holds by Theorem 3.5 in [2].

For the necessity suppose that (f_k) be a sequence such that $f_k \in X^*$ for all $k \in \mathbb{N}$ and conditions *(i)–(iii)* hold. Also, let $F = D + E$ where

$$D = (f_k^n - f_k)_{n,k}$$

and $E = (f_k)_{n,k}$. Hence

$$D \in (c_0(X, \lambda, p), c_0(q)),$$

by conditions *(i)*, *(ii)*, Proposition 3.1 (ii) and Theorem 2.4 in [11]. Moreover, $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all

$$x = (x_k) \in c_0(X, \lambda, p),$$

by proposition 3.3 and the condition *(i)*. Hence $E \in (c_0(X, \lambda, p), \langle e \rangle)$. So Lemma 4 yields

$$F = (f_k^n) \in (c_0(X, \lambda, p), c(q)).$$

□

As an easy consequence of Theorem 4.3 in [7], we have

Lemma 6. $F = (f_k^n) \in (\ell_\infty(X, \lambda), c(q))$ *if and only if there is a sequence* $(f_k) \subset X^*$ *such that*

(i) $\sum_{k=1}^{\infty} |\lambda_k|^{-1} \|f_k\| < \infty$.

(ii) $m^{1/q_n}(f_k^n - f_k) \to 0$ *as* $n \to \infty$ *for every* $k, m \in \mathbb{N}$.

(iii) $\sum_{j=k+1}^{\infty} m^{1/q_n} \|f_j^n - f_j\| r^{1/p_k} \to 0$ *as* $k \to \infty$ *uniformly on* $n \in \mathbb{N}$, *for each* $m, r \in \mathbb{N}$.

Lemma 7. $\ell_\infty(X,\lambda,p) = \bigcup_{m=1}^{\infty} \ell_\infty(X,\lambda)_{(m^{-1/p_k})}$.

Proof. Let x $\in \ell_\infty(X,\lambda,p)$, then there is some $m \in \mathbb{N}$ such that $\sup_k \|\lambda_k x_k\|^{p_k} < m$. Therefore,

$$\|\lambda_k x_k\| m^{-1/p_k} \leq 1$$

for all $k \in \mathbb{N}$. Hence,

$$\mathrm{x} \in \cup_{m=1}^{\infty} \ell_\infty(X,\lambda)_{(m^{-1/p_k})}.$$

Conversely, suppose that

$$\mathrm{x} \in \cup_{m=1}^{\infty} \ell_\infty(X,\lambda)_{(m^{-1/p_k})}.$$

Then, there are some $m \in \mathbb{N}$ and $K > 1$ such that

$$\|\lambda_k x_k\| m^{-1/p_k} \leq K$$

for all $k \in \mathbb{N}$. Hence, this yields

$$\|\lambda_k x_k\|^{p_k} \leq mK^{p_k} \leq mK^t$$

for all $k \in \mathbb{N}$, where $t = \sup_k p_k$, so that x $\in \ell_\infty(X,\lambda,p)$. □

Theorem 7. $F = (f_k^n) \in (\ell_\infty(X,\lambda,p), c(q))$ *if and only if there is a sequence* $(f_k) \subset X^*$ *such that*

(i) $\sum_{k=1}^{\infty} |\lambda_k|^{-1} \|f_k\| N^{1/p_k} < \infty$ *for all* $N \in \mathbb{N}$.

(ii) $r^{1/q_n}(m^{1/p_k} f_k^n - f_k) \to 0$ *as* $n \to \infty$ *for every* $k, m \in \mathbb{N}$.

(iii) $r^{1/q_n} \sum_{j=k+1}^{\infty} \|m^{1/p_j} f_j^n - f_j\| s^{1/p_j} \to 0$ *as* $k \to \infty$ *uniformly on* $n \in \mathbb{N}$, *for each* $m, r, s \in \mathbb{N}$.

Proof. Let consider the Lemmas 5–7. Then, we derive from two-sided implication Fx is in $c(q)$ whenever $x \in \ell_\infty(X,\lambda,p)$ if and only if

$$(m^{1/p_k} f_k^n)_{n,k} \in (\ell_\infty(X,\lambda), c(q)).$$

So we have for the necessary and sufficient *(i)–(iii)* hold. □

Theorem 8. $F = (f_k^n) \in (c(X,\lambda,p), c(q))$ *if and only if there is a sequence* $(f_k) \subset X^*$ *such that*

(i) $\sum_{k=1}^{\infty} |\lambda_k|^{-1} \|f_k\| N^{-1/p_k} < \infty$ *for some* $N \in \mathbb{N}$.

(ii) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for every* $k, m \in \mathbb{N}$.

(iii) $m^{1/q_n} \sum_{k=1}^{\infty} \|f_j^n\| r^{1/p_k} \to 0$ *as* $n, r \to \infty$ *for each* $m \in \mathbb{N}$.

(iv) $(\sum_{k=1}^{\infty} f_k^n(x))_n \in c(q)$ *for all* $x \in X$.

Proof. As in the proof of Theorem 1

$$c(X,\lambda,p) = c_0(X,\lambda,p) + \{\mathrm{e}(x) : x \in X\}.$$

Hence, Lemma 5 show that

$$F \in (c(X,\lambda,p), c(q))$$

if and only if

$$F \in (c_0(X, \lambda, p), c(q))$$

and

$$F \in (\{\mathrm{e}(x) : x \in X\}, c(q)).$$

Then,

$$F \in (c_0(X, \lambda, p), c(q))$$

if and only if the conditions *(i)–(iii)* hold, follows easily from Theorem 6. Also, one can show that

$$F \in (\{\mathrm{e}(x) : x \in X\}, c(q))$$

if and only if *(iv)* holds. □

4. Conclusions

The computation of Köthe-Toeplitz duals of sequence spaces is an important field of study in order to determine the matrix transformation classes between two spaces. Many researchers try to derive new sequence spaces using well-known sequence spaces. They also investigate for matrix transformation classes between the proposed new sequence spaces and classical sequence spaces by identifying the duals of the spaces they have obtained.

Many different methods have been used to obtain new sequence spaces. One of these is paranormed sequence spaces defined by Maddox. Maddox has constructed new sequence spaces from classical sequence spaces ℓ_∞, c_0 and c. These sequence spaces are $\ell_\infty(p)$, $c_0(p)$, $c_0(p)$ where

$$p = (p_k) \in \ell_\infty$$

and

$$M = \max\{1, \sup_k p_k\}.$$

With these sequence spaces defined by Maddox a wide working area has been opened. An other method to obtaining new sequence spaces is "matrix domain" method. For an arbitrary sequence space X, the matrix domain of an infinite matrix A in X is defined by

$$X_A = \{x \in w : Ax \in X\}$$

as in [3,4]. The other method is to generate a new sequence space from classical sequence spaces using any scalar sequence $\lambda = (\lambda_k)$. For example, Srivastava and Srivastava [2] introduced the sequence space

$$c_0(X, \lambda, p) = \{\mathrm{x} = (x_k) : \lim_{k\to\infty} \|\lambda_k x_k\|^{p_k} = 0\}.$$

The researchers can obtain new sequence spaces by using above methods.

In this study, we introduced generalized vector valued sequence spaces $c(X, \lambda, p)$ and $\ell_\infty(X, \lambda, p)$. We determined the Köthe-Toeplitz duals of these spaces and characterized some matrix classes $(E(X, \lambda, p), c(q))$ where $E \in \{\ell_\infty, c\}$. These spaces are the generalization of some sequence spaces given in the references.

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Conflict of interest

Author declares no conflict of interest in this paper.

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