AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 13257–13275.
DOI:10.3934/math.2023670
Received: 22 February 2023
Revised: 16 March 2023
Accepted: 28 March 2023
Published: 04 April 2023

Research article

New decision rules under strict uncertainty and a general distance-based approach

Francisco Salas-Molina*, **David Pla-Santamaria, Maria Luisa Vercher-Ferrandiz and Ana Garcia-Bernabeu**

Department of Economics and Social Sciences, Universitat Politècnica de València, Ferrándiz y Carbonell, 03801, Alcoy, Spain

* **Correspondence:** Email: frasamo@upv.es.

Abstract: Strict uncertainty implies a complete lack of knowledge about the probabilities of possible future states of the world. However, there is complete information about the set of alternatives under consideration, the set of future states, and the scalar evaluation of choosing every alternative if a given state occurs. The principle of insufficient reason by Laplace, the maximin rule by Wald, the Hurwicz criterion, or the minimax regret criterion by Savage are examples of decision rules under strict uncertainty. Within the context of strict uncertainty, moderate pessimism implies the existence of a decision-maker who cautiously assumes that the most favorable state will not occur when the action has been taken with no conjecture being made about the other states. The criterion of moderate pessimism proposed by Ballestero implies the use of the inverse of the range of evaluation for each state as a weight system. In this paper, we extend the notion of moderate pessimism under strict uncertainty to solve some of its limitations. First, we propose a new domination analysis that avoids removing dominated alternatives that are still relevant in the final ranking of alternatives. Second, we propose additional score functions using the inverse of the standard deviation and the mean absolute deviation instead of the range of evaluations for each future state to reduce the impact of the possible existence of outliers in the decision table. This partial result is later generalized through the concept of average deviation of a given order. Finally, we show that all the mentioned decision rules are special cases of a general ranking method based on the Minkowski distance function. We illustrate the use of distance-based decision rules through an application in the context of portfolio selection.

Keywords: moderate pessimism; measures of dispersion; outliers; Minkowski distance function; finance
Mathematics Subject Classification: 90B50, 91B06

1. Introduction

Some management and economic problems can be perfectly described by means of deterministic equations. Some others such as investment, budgeting, or ranking of alternatives include uncertainty about the future states of nature. A particular case of this situation happens when there is almost a complete lack of knowledge about the possible outcomes. This situation is called strict uncertainty and is characterized by the absence of knowledge about the probability of any future state [1]. However, there is complete information about the set of alternatives under consideration and the scalar evaluation of choosing every alternative if a given state occurs. The principle of insufficient reason by Laplace [2], the maximin rule by Wald [3], the Hurwicz criterion [4], or the minimax regret criterion by Savage [5] are examples of different decision rules under strict uncertainty. Ballestero [1] proposed a new decision rule to rank alternatives under strict uncertainty. This rule is suitable for moderately pessimistic individuals and social groups, these agents being neither maximax nor maximin decision-makers but people who assume that the best outcome from the action will not occur. The application of Ballestero's criterion to economic problems has been described by Ballestero in textile product selection [6], and portfolio selection [7]. More recently, Bravo y Pla-Santamaria [8] and Pla-Santamaria et al. [9] in the context of credit ranking. Other related works are those by Chambers [10], Rostek [11], and Sadhegi and Moslemi [12].

Decision-making under strict uncertainty requires the use of a decision table with alternatives in rows and future states in columns as shown in Table 1. Furthermore, the moderate pessimism criterion by Ballestero [1] implies the application of a particular domination analysis to remove dominated alternatives from the decision table. An important limitation is that the definition of domination proposed by the author does not guarantee that a non-dominated alternative obtains a score of as least as good as the one obtained by a dominated alternative. As a result, some relevant alternatives may be removed when they are not inferior to any of the non-dominated alternatives in terms of the scoring ranking function. To solve this limitation, we propose a new domination analysis that guarantees that the score of a non-dominated alternative is not inferior to the score of a dominated alternative.

In this paper, we also extend the moderate pessimism criterion by Ballestero [1] by considering novel score functions. On the one hand, we first propose the use of the inverse of standard deviation and the mean absolute deviation instead of the inverse of range to obtain the system of weights attached to each of the future states of nature. The rationale behind this proposal is the possibility that the range, computed as the difference between the maximum and the minimum evaluations for each state in the decision table, may be affected by the presence of outliers. On the other hand, we further elaborate on the notion of duality established between future states of nature and possible multiple-criteria to evaluate the set of alternatives in the decision table. This duality leads us to consider decision rules based on the concept of distance to a reference point as in multiple-criteria decision-making [13–15]. More precisely, we propose a general ranking method based on the Minkowski distance to a reference point. As a result, we show that the principle of insufficient reason by Laplace [2], the maximin rule by Wald [3], the Hurwicz criterion [4], the minimax regret criterion by Savage [5], and the Ballestero moderate pessimism criterion [1] are special cases of a general ranking method based on the Minkowski distance function.

Summarizing, this paper contributes to the development of decision rules under the context of strict uncertainty in three different ways:

1. We propose a new domination analysis to avoid removing relevant alternatives.
2. We extend the moderate pessimism criterion by Ballestero [1] through the use of other measures of dispersion that mitigate the impact of outliers.
3. We present a general distance-based ranking method that subsumes existing decision rules under strict uncertainty.

The previous three contributions are connected by the notion of distance-based decision rules. Indeed, we prove that Ballestero's criterion is a special case of the general distance-based ranking method described in this paper. By proposing alternative ways to derive the weight system of the criterion, we reduce the impact of the possible existence of outliers when deploying our method. Finally, we solve the limitation of removing relevant alternatives of the original moderate pessimism criterion by a new domination analysis as a previous step to implement our distance-based ranking method.

In addition to this introduction, Section 2 provides useful background on the concept of strict uncertainty. Section 3 extends the concept of moderate pessimism under strict uncertainty and presents several theoretical results. Finally, Section 4 concludes and highlights natural extensions of this work.

2. Materials and methods

In this section, we recall some basic concepts related to decision rules under the context of strict uncertainty paying special attention to the formal definition of moderate pessimism.

2.1. Decision rules under strict uncertainty

The notion of strict uncertainty refers to a situation in which the available information is limited to the future states of nature, the set of alternatives under consideration, and the consequence of choosing every alternative if a given state occurs in the form of a valuation function [1]. As a result, it is assumed that the decision-maker can say nothing about the probability of any state. More formally, strict uncertainty is defined as follows:

Definition 1. ***Strict uncertainty****. A decision-maker is said to rank alternatives under strict uncertainty when the available information is limited to:*

1. *A finite set of alternatives* $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$.
2. *A finite set of states* $C = \{r_1, r_2, \ldots, r_n\}$.
3. *A scalar evaluation* $V : \mathcal{A} \times C \to \mathbb{R}$ *of every alternative for each state.*

For convenience, all scalar evaluations V are assumed to be of the type the more the better. The combination of alternative a_i within state r_j results in evaluation v_{ij} as summarized in Table 1.

The Laplace [2] criterion leads to the assumption that knowing nothing about the future state is equivalent to using the same weight for each state when computing the alternative score. As a result, the Laplace score function L_i for each alternative is the following:

$$L_i = \sum_{j=1}^{n} v_{ij}. \tag{2.1}$$

Table 1. Decision table.

Alternatives	States					
	r_1	r_2	...	r_j	...	r_n
a_1	v_{11}	v_{12}	...	v_{1j}	...	v_{1n}
a_2	v_{21}	v_{22}	...	v_{2j}	...	v_{2n}
...	...	...	...	...	...	...
a_i	v_{i1}	v_{i2}	...	v_{ij}	...	v_{in}
...	...	...	...	...	...	...
a_m	v_{m1}	v_{m1}	...	v_{m1}	...	v_{mn}

and the decision-maker should choose the alternative with maximum L_i.

The maximin rule by Wald [3] is an extremely pessimistic rule because assumes that the worst state will occur. Then, the decision-maker considers the evaluations for each alternative in the worst-case state by comparing function W_i:

$$W_i = \min_j(v_{ij}) \tag{2.2}$$

and selects the alternative with maximum W_i.

The Hurwicz [4] criterion implies a balance between optimism and pessimism determined by preference α in function H_i:

$$H_i = \alpha \cdot \max_j(v_{ij}) + (1 - \alpha)\min_j(v_{ij}) \tag{2.3}$$

and the decision-maker should choose the alternative with maximum H_i.

The minimax regret criterion by Savage [5] focuses on the regret that each combination of alternative and state produces in the decision-maker. Regret is defined as the difference between the best evaluation in the state and the particular evaluation of each alternative in this state. Then, the decision-maker considers the regrets for each alternative and state by comparing function S_i:

$$S_i = \max_j(v_{ij}) - v_{ij} \tag{2.4}$$

and selects the alternative with minimum S_i.

The previous rules under strict uncertainty are examples of methods within the field of decision theory [16]. In addition, the definition of the decision table fits well in that of a game and von Neumann and Morgenstern's theory of expected utility [17], and a wide range of subsequent game theory works. More recent works about decision rules and strict uncertainty include applications in sewer network planning [18], the use of the minimax and maximin criteria in a game against nature for the case of a partial a priori uncertainty [19], an axiomatic extension to three-dimensional matrix games [20], and a study about how large-scale railway projects of federal significance should be evaluated in a context of uncertainty [21]. A departure from the traditional approach to decision-making under strict uncertainty is the moderate pessimism criterion proposed by Ballestero [1], which was later applied to different decision-making problems as described in [6–9]. In what follows, we focus on the moderate pessimism criterion to solve some of its limitations.

2.2. Moderate pessimism

Definition 2. ***Moderate pessimism [1].*** *A decision-maker who cautiously assumes that the most favorable state when the action has been taken will not occur (no conjecture being made about the other states) is named a moderately pessimistic decision-maker.*

Moderate pessimism implies that dominated alternatives are removed from the decision table before ranking the alternatives.

Definition 3. ***Dominated alternatives [1].*** *An alternative a_k is dominated by a convex combination of alternatives if the following relationship is satisfied:*

$$\sum_{i=1}^{m} \varphi_j v_{ij} \geq v_{kj}, \quad \forall j = 1, 2, \ldots, n, \tag{2.5}$$

where φ_j is a coefficient for the j-th state bounded to the interval $[0, 1]$.

The principle of moderate pessimism proposed by Ballestero [1] is based on the rationale that the larger the dispersion of evaluations for each state, the higher the distrust of the decision-maker towards the state. As a result, column dispersion is a piece of critical information for the ranking procedure.

Definition 4. ***Column dispersion [1].*** *In the set of non-dominated alternatives in the decision table for the j-th column, the v_{ij} values of this set are ranged over:*

$$\min_i(v_{ij}) \leq v_{ij} \leq \max_i(v_{ij}) \tag{2.6}$$

where $\Delta_j = max(v_{ij}) - min(v_{ij})$ is the column dispersion for the j-th state.

This criterion implies the selection of weights inversely proportional to the range of evaluations leading to the following aggregation weights for the set of criteria:

$$w_j = \frac{1}{\Delta_j K} \tag{2.7}$$

where w_j is the weight for the j-th state, and K is equal to:

$$K = 1 + \sum_{j=1}^{n} \frac{\min_i(v_{ij})}{\Delta_j}. \tag{2.8}$$

Finally, from aggregation weights w_j, we get score function B_i for each alternative:

$$B_i = \sum_{j=1}^{n} w_j v_{ij}. \tag{2.9}$$

The weighting scheme in Ballestero's criterion has the advantage of controlling both optimistic and pessimistic evaluations. The criterion is a consistent weight system in the sense that one and only one weight is attached to each state of the world under plausible conditions of domination. It incorporates the idea of security associated with attitudes of moderate pessimism as opposed to the extreme pessimism by Wald [3]. Finally, Ballestero's criterion uses all the available information in the decision table and satisfies most of the traditional ranking axioms.

3. Results and discussion

In this section, we extend the notion of moderate pessimism under strict uncertainty to solve some of its limitations and propose a general ranking method based on the distance to a reference point. First, we propose a new domination analysis that avoids removing dominated alternatives that are still relevant in the final ranking of alternatives. Second, we propose two additional score functions using the inverse of the standard deviation and the mean absolute deviation instead of the range of evaluations for each future state to reduce the impact of the possible existence of outliers in the decision table. Third, we show that the decision rules described in Section 2 are special cases of a general distance-based approach to rank alternatives through the Minkowski distance function.

Before proceeding with the extension of moderate pessimism, we provide the following result regarding Ballestero's score function B_i that will allow us to simplify notation:

Proposition 1. *Parameter K in equation* (2.7) *is irrelevant for ranking purposes.*

Proof. Note that K is a constant in the score function B_i in equation (2.9). Then, given two arbitrary alternatives i and h, the following holds:

$$B_i \geq B_h \iff KB_i \geq KB_h. \tag{3.1}$$

□

In what follows, we use weights $w_j = 1/\Delta_j$ to extend the moderate pessimism criterion.

3.1. Domination analysis

It is proven elsewhere [1] that if an alternative a_D dominates the $(\varphi_1, \varphi_2, \ldots, \varphi_m)$ convex combination alternatives according to Definition 3, then its ranking value B_D must be greater than (or equal to) the combination of ranking values:

$$B_D \geq \sum_{i=1}^{m} \varphi_i B_i. \tag{3.2}$$

Similarly, if an alternative a_d is dominated by the $(\varphi_1, \varphi_2, \ldots, \varphi_m)$ convex combination alternatives, then its ranking value B_d must be less than (or equal to) the combination of ranking values:

$$B_d \leq \sum_{i=1}^{m} \varphi_i B_i. \tag{3.3}$$

Convex combinations gathered in row vector $\boldsymbol{\varphi}^T = (\varphi_1, \varphi_1, \ldots, \varphi_m)$ are obtained by solving the following linear programming model:

$$\min \ \varphi_i \tag{3.4}$$

subject to:

$$\boldsymbol{\varphi}^T M \geq \boldsymbol{v}_i \tag{3.5}$$

$$\mathbf{1}^T \boldsymbol{\varphi} = 1 \tag{3.6}$$

with non-negativity constraints where M is the decision matrix of size $m \times n$ with elements set to v_{ij}, $\boldsymbol{v_i}$ is the row vector of M with the evaluations for the i-th alternative for all states, $\mathbf{1}$ is a column vector of ones, and superscript T denotes transposition.

Remark 1. *For a non-dominated alternative, denoted by a_D, the solution of the linear program encoded from equations* (3.4) *to* (3.6) *is $\varphi_D = 1$, being the remaining elements of vector $\boldsymbol{\varphi}$ equal to zero.*

Remark 2. *For a dominated alternative, denoted by a_d, the solution of the linear program encoded from equations* (3.4) *to* (3.6) *is $\varphi_d = 0$, being the remaining elements of vector $\boldsymbol{\varphi}$ equal to some value between zero and one.*

However, domination in the sense of Definition 3 is not a sufficient condition for $B_D \geq B_d$, which can reasonably be considered as a desired property for decision-making. The following result provides a sufficient condition derived from dominance analysis.

Theorem 2. *Let a_D be a non-dominated alternative and let a_d be a dominated alternative according to Definition 3. A sufficient condition for $B_D \geq B_d$ is that $\boldsymbol{v_D}^T \boldsymbol{w} \geq \boldsymbol{\varphi_d}^T M \boldsymbol{w}$.*

Proof. If alternative a_d is dominated, then $v_{dj} \leq \sum_{i=1} \phi_i v_{ij}, \forall j = 1, 2, \ldots, n$. By considering vector $\boldsymbol{v_d} = (v_{d1}, v_{d2}, \ldots, v_{dn})$, an equivalent domination relation can be rewritten in vector notation as:

$$\boldsymbol{v_d} \leq \boldsymbol{\varphi_d}^T M \tag{3.7}$$

where $\boldsymbol{\varphi_d}$ is the vector of convex combinations. By multiplying both sides of equation (3.7) by vector of weights $\boldsymbol{w}$, we obtain:

$$B_d = \boldsymbol{v_d}^T \boldsymbol{w} \leq \boldsymbol{\varphi_d}^T M \boldsymbol{w} \tag{3.8}$$

$$B_D = \boldsymbol{v_D}^T \boldsymbol{w} \geq \boldsymbol{\varphi_D}^T M \boldsymbol{w}. \tag{3.9}$$

As a result, if $\boldsymbol{\varphi_D}^T M \boldsymbol{w} \geq \boldsymbol{\varphi_d}^T M \boldsymbol{w}$, then $B_D \geq B_d$. According to Remark (1), the previous sufficient condition reduces to:

$$\boldsymbol{v_D}^T \boldsymbol{w} \geq \boldsymbol{\varphi_d}^T M \boldsymbol{w} \rightarrow B_D \geq B_d. \tag{3.10}$$

□

As a result, some relevant alternatives may be removed from the decision table when they are not inferior to all of the non-dominated alternatives in terms of the scoring ranking function B_i. To illustrate this point, consider the set of alternatives, states, and evaluations described in Table 2 taken from an example described in [1]. Domination analysis derived from Definition 3 results in classifying alternatives a_1^*, a_2^*, and a_5^* as dominated alternatives. However, if we compute the ranking score function B_i using equation (2.9), we find that $B_1 > B_6$ even though alternative a_6 is non-dominated. Consequently, removing a_1^* from the set of available alternatives implies the impossibility of selecting an alternative (assuming that more than one can be selected) that is better than a_6.

Table 2. An illustrative decision table (a_i^* means dominated).

Alternatives	States r_1	r_2	r_3	B_i	Rank
a_1^*	452	26	11	2.904	4
a_2^*	332	73	62	3.525	3
a_3	379	105	86	4.498	1
a_4	393	94	78	4.329	2
a_5^*	267	42	49	2.612	6
a_6	453	26	10	2.896	5

Corollary 1. *A necessary and sufficient condition for $B_D \geq B_d$ is the following:*

$$B_D \geq B_d \iff \boldsymbol{v_D}^T\boldsymbol{w} \geq \boldsymbol{v_d}^T\boldsymbol{w} \iff \boldsymbol{v_D} \geq \boldsymbol{v_d}. \tag{3.11}$$

Proof. It is a direct consequence of equation 2.9. □

We argue that a domination analysis that satisfies the necessary and sufficient condition in Corollary 1 improves the moderate pessimism criterion and decision-making under strict. As a result, we propose a new definition of non-dominated and dominated alternatives based on the notion of Pareto efficiency [14, 15].

Definition 5. ***A substitute non-domination analysis.*** *Let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^n$ be two evaluation vectors for two alternatives such that $\boldsymbol{v}_1 = (v_{1j}, \ldots, v_{1n})$ and $\boldsymbol{v}_2 = (v_{2j}, \ldots, v_{2n})$. Then $\boldsymbol{v}_1$ dominates $\boldsymbol{v}_2$ if and only if $\boldsymbol{v}_1 \geq \boldsymbol{v}_2$ and $\boldsymbol{v}_1 \neq \boldsymbol{v}_2$ (i.e., $v_{1j} \geq v_{2j}$ for all j and $v_{1j} > v_{2j}$ for at least one j). Otherwise, $\boldsymbol{v}_1$ is dominated by $\boldsymbol{v}_2$.*

Corollary 2. *Let a_D be a non-dominated alternative and let a_d be a dominated alternative according to Definition 5. Then, $B_D \geq B_d$.*

Proof. By definition a_D dominates a_d if and only if $\boldsymbol{v}_D \geq \boldsymbol{v}_d$ and $\boldsymbol{v}_D \neq \boldsymbol{v}_d$. Then, the following condition holds.

$$B_D = \boldsymbol{v_D}^T\boldsymbol{w} \geq \boldsymbol{v_d}^T\boldsymbol{w} = B_d. \tag{3.12}$$

□

As a result, the substitute domination analysis from Definition 5, solves the limitation of removing relevant alternatives from the decision table when they are not inferior to some of the non-dominated alternatives in terms of the scoring ranking function. A further advantage of our approach in terms of selecting alternatives is that any dominated alternative is replaced by a better one instead of being replaced by a combination of alternatives. In a wide range of decision-making contexts, alternatives must be selected or non-selected but cannot be replaced by a combination of alternatives. For instance, a company considering the alternatives: 1) build a new production plant; 2) sell a production plant; and 3) do nothing; and three different states of the future economic situation, would select alternative 1, 2, or 3 according to their respective evaluations, but is more difficult to assume that alternative 3 can be replaced by a convex combination of alternatives 1 and 2.

3.2. Reducing the impact of outliers

The use of the range of evaluations for each state, computed as the difference between the maximum and the minimum evaluations for each state in the decision table, may be affected by the presence of outliers. It is likely that an outlier (or an error of measurement) within the decision table is either the minimum or the maximum of evaluations. The system of weights derived from the moderate pessimism criterion by Ballestero [1] will be different if there is an outlier that coincides with a minimum or maximum evaluation for a state. To solve this limitation, we propose the use of the inverse of standard deviation, and the inverse of the mean absolute deviation, instead of the inverse of range to obtain the system of weights attached to each of the future states of nature. Alternatively, the use of winsorized weight systems has been proposed elsewhere [9]. Because of that, we will not cover this option in this paper but it has to be considered as an additional suitable way to reduce the impact of outliers in strict uncertainty.

Given set of evaluations $(v_{1j}, v_{2j}, \ldots, v_{mj})$ for the j-th state, with average value $\overline{v}_j$, the standard deviation σ_j is defined as:

$$\sigma_j = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(v_{ij} - \overline{v}_j)^2}, \tag{3.13}$$

and the mean absolute deviation:

$$MAD_j = \frac{1}{m}\sum_{i=1}^{m}|v_{ij} - \overline{v}_j|. \tag{3.14}$$

Provided that the standard deviation and the mean absolute deviation are not null, their respective values are always smaller than the range of evaluations as the following lemma shows.

Lemma 1. *Let d_i and Δ_j be two real positive values, and let p be a positive integer. Assume that $d_i < \Delta_j$, then it follows that:*

$$|d_i|^p < |\Delta_j|^p \tag{3.15}$$

$$\sum_{i=1}^{m}|d_i|^p < \sum_{i=1}^{m}|\Delta_j|^p = m|\Delta_j|^p \tag{3.16}$$

$$\left[\frac{1}{m}\sum_{i=1}^{m}|d_i|^p\right]^{1/p} < \left[\frac{m}{m}|\Delta_j|^p\right]^{1/p} = \left[|\Delta_j|^p\right]^{1/p} = \Delta_j. \tag{3.17}$$

If we set $p = 2$, and define $d_i = (v_{ij} - \overline{v}_j)$, then $\sigma_j < \Delta_j$. Similarly, if we set $p = 1$, and define $d_i = |v_{ij} - \overline{v}_j|$, then $MAD_j < \Delta_j$.

We next define two new score functions within the framework of moderate pessimism relying on the standard deviation and the mean absolute deviation as measures of dispersion:

$$B_i^{\sigma} = \sum_{j=1}^{n} \frac{v_{ij}}{\sigma_j} \tag{3.18}$$

$$B_i^{MAD} = \sum_{j=1}^{n} \frac{v_{ij}}{MAD_j}. \tag{3.19}$$

Theorem 3. *Given $\sigma_j > 0$ and $MAD_j > 0$, for at least one j in a decision table under strict uncertainty, then $B_i^{\sigma} > B_i$, $B_i^{MAD} > B_i$, and $B_i^{MAD} \geq B_i^{\sigma}$.*

Proof. Inequalities $B_i^{\sigma} > B_i$ and $B_i^{MAD} > B_i$ are a direct consequence of Lemma 1. Inequality $B_i^{MAD} \geq B_{\sigma}$ holds because $MAD_j \leq \sigma_j$ according to the Cauchy-Schwarz inequality $||x||_1 \leq \sqrt{m}||x||_2$, as we next show:

$$MAD_j = \frac{1}{m}||\boldsymbol{x}||_1 \leq \frac{1}{\sqrt{m}}||\boldsymbol{x}||_2 = \sigma_j \tag{3.20}$$

where $\boldsymbol{x} = \boldsymbol{v_j} - \overline{v}_j$, in which $\boldsymbol{v_j}$ is a vector of evaluations for the j-the state, and $||\boldsymbol{x}||_p$ denotes the p-norm of vector $\boldsymbol{x}$ computed as:

$$||\boldsymbol{x}||_p = \left[\sum_{i=1}^{m} |x_i|^p\right]^{1/p}. \tag{3.21}$$

□

As a result, the presence of outliers is mitigated by assigning a larger weight than in the original method to states with low weights due to the presence of an outlier that should not be in the decision table. Indeed, the lower the measure of dispersion, the larger the weight assigned to the states, and the larger the eventual mitigation as in the case mean absolute deviation compared to the standard deviation because $MAD_j \leq \sigma_j$. Then, $B_i^{MAD} \geq B_i^{\sigma}$. This fact leads us to consider a general measure of dispersion, which we call the average deviation of order p, denoted by σ_j^p:

$$\sigma_j^p = \left[\frac{1}{m}\sum_{i=1}^{m} |v_{ij} - \overline{v}_j|^p\right]^{1/p} = \frac{1}{m^{1/p}}||\boldsymbol{x}||_p. \tag{3.22}$$

Next, we generalize the result $B_i^{MAD} \geq B_{\sigma}$, through the concept of average deviation of order p:

Theorem 4. *Let $\sigma_j^p > 0$ be the average deviation of order p from $\overline{v}_j$, for at least one j in a decision table under strict uncertainty, and let p be a positive integer, then $B_i^{\sigma^p} \geq B_i^{\sigma^{p+1}}$.*

Proof. Again, this result derives from the Cauchy-Schwarz inequality for $0 < p < q$:

$$||\boldsymbol{x}||_p \leq m^{1/p-1/q}||\boldsymbol{x}||_q. \tag{3.23}$$

By setting $q = p + 1$, we find that:

$$\sigma_j^p = \frac{1}{m^{1/p}}||\boldsymbol{x}||_p \leq \frac{1}{m^{1/(p+1)}}||\boldsymbol{x}||_{p+1} = \sigma_j^{p+1}. \tag{3.24}$$

Then, weights derived from inverting σ_j^p are always greater or equal to weights derived from σ_j^{p+1}, resulting in $B_i^{\sigma^p} \geq B_i^{\sigma^{p+1}}$. □

This generalization implies the possibility of controlling the degree of mitigation of the eventual existence of outliers in the decision table. Finally, the use of the standard deviation and the mean absolute deviation presents the advantage of using all the information in the decision table to compute the column dispersion. This advantage is in direct contrast to the range used in [1] and the winsorized range used in [9] when there is a loss of information.

3.3. A general distance-based ranking method

In this subsection, we follow the approach of considering states in a decision table as a set of multiple-criteria under consideration for evaluation purposes. Indeed, there is a duality between states and evaluation criteria because an alternative evaluated for different states can be viewed as a multiple-criteria decision-making problem. Furthermore, multiple-criteria evaluations and multiple states can also be assimilated into a multiple-criteria decision-making problem. As a result, each evaluation v_{ij} becomes the degree of achievement of the i-th alternative in terms of the j-th criterion. For instance, alternative investments can be evaluated in terms of expected return, risk, and liquidity.

The concept of duality and the possibility to consider a decision table under strict uncertainty as a multiple-criteria decision-making problem leads us to propose a general ranking method based on the concept of distance to a reference point. In what follows, we consider two reference points: the ideal and the anti-ideal (or nadir) points. The ideal point is usually infeasible but it plays a key role in selecting the best alternatives. Given alternative $a_i \in \mathcal{A}$, with $i = 1, 2, \ldots, m$, and evaluations v_{ij} in terms of the j-the criterion, with $j = 1, 2, \ldots, n$, in a maximization context, the ideal and anti-ideal (or nadir) points are defined as follows [22]:

Definition 6. *Point* $\boldsymbol{v}_I = (v_1^I, \ldots, v_j^I, \ldots, v_n^I)$ *given by* $v_j^I := \underset{a_i \in \mathcal{A}}{max}\, v_j(a_i) = \underset{i}{max}\, v_{ij}$ *is called the ideal point of the multicriteria problem* $\underset{a_i \in \mathcal{A}}{max}\, (v_1(a_i), \ldots, v_j(a_i), \ldots, v_n(a_i))$.

Definition 7. *Point* $\boldsymbol{v}_N = (v_1^N, \ldots, v_j^N, \ldots, v_n^N)$ *given by* $v_j^N := \underset{a_i \in \mathcal{A}}{min}\, v_j(a_i) = \underset{i}{min}\, v_{ij}$ *is called the anti-ideal (nadir) point of the multicriteria problem* $\underset{a_i \in \mathcal{A}}{max}\, (v_1(a_i), \ldots, v_j(a_i), \ldots, v_n(a_i))$.

Using either $\boldsymbol{v}_I$ or $\boldsymbol{v}_N$ as reference points, we here propose to evaluate alternatives employing the Minkowski distance function. For instance, the Minkowski distance between evaluations v_{ij} for alternative a_i and the nadir point $\boldsymbol{v}_N$ is computed as follows:

$$\mathcal{D}(a_i, \boldsymbol{v}_N, p) = \left[\sum_{j=1}^{n} w_j^p |v_{ij} - v_{jN}|^p \right]^{1/p}. \tag{3.25}$$

Note that parameter p in equation (3.25) is a topological metric belonging to the closed interval $[-\infty, \infty]$. There are some values of p that are relevant to this work which we next summarize in a group of remarks.

Remark 3. *Setting* $p = -\infty$ *leads to computing the minimum deviation over* j:

$$\mathcal{D}(a_i, \boldsymbol{v}_N, -\infty) = \underset{j}{min}(w_j |v_{ij} - v_{jN}|). \tag{3.26}$$

Remark 4. *Setting $p = \infty$ leads to computing the maximum deviation over j:*

$$\mathcal{D}(a_i, \boldsymbol{v}_N, \infty) = \max_j(w_j|v_{ij} - v_{jN}|). \tag{3.27}$$

Remark 5. *Setting $p = 1$ leads to computing the Manhattan distance over j:*

$$\mathcal{D}(a_i, \boldsymbol{v}_N, 1) = \sum_{j=1}^{n} w_j|v_{ij} - v_{jN}|. \tag{3.28}$$

Remark 6. *Setting $p = 2$ leads to computing the Euclidean distance over j:*

$$\mathcal{D}(a_i, \boldsymbol{v}_N, 2) = \sqrt{\sum_{j=1}^{n} w_j^2(v_{ij} - v_{jN})^2}. \tag{3.29}$$

As a result, decision-makers can select the best alternative by finding out which alternative presents either the minimum distance to the ideal point or the maximum distance from the nadir point. Similarly, decision-makers can produce a ranking of alternatives by computing their respective distances and sorting the values. Taking advantage of the previous remarks, the following results establish a link between the decision rules by Laplace [2], by Wald [3], by Hurwicz [4], by Savage [5], and by Ballestero [1] described in Section 2. To this end, and without loss of generality, we consider a fictitious nadir point $\boldsymbol{v}_N = \mathbf{0}$ of size n with all its elements set to zero.

Theorem 5. *Setting $w_j = 1, \forall j = 1, 2, \ldots, n$, and $\boldsymbol{v}_N = \mathbf{0}$, the Laplace rule $\max_i(L_i)$ is equivalent to:*

$$\max_i(1/n \cdot \mathcal{D}(a_i, \boldsymbol{v}_N, 1)). \tag{3.30}$$

Proof. It is a direct consequence of Remark 5. □

Theorem 6. *Setting $w_j = 1, \forall j = 1, 2, \ldots, n$, and $\boldsymbol{v}_N = \mathbf{0}$, the Wald rule $\max_i(W_i)$ is equivalent to:*

$$\max_i(\mathcal{D}(a_i, \boldsymbol{v}_N, -\infty)). \tag{3.31}$$

Proof. It is a direct consequence of Remark 3. □

Theorem 7. *Setting $w_j = 1, \forall j = 1, 2, \ldots, n$, and $\boldsymbol{v}_N = \mathbf{0}$, the Hurwicz rule $\max_i(H_i)$ is equivalent to:*

$$\max_i(\alpha\mathcal{D}(a_i, \boldsymbol{v}_N, \infty) + (1 - \alpha)\mathcal{D}(a_i, \boldsymbol{v}_N, -\infty)). \tag{3.32}$$

Proof. It is a direct consequence of Remarks 3 and 4. □

Theorem 8. *Setting $w_j = 1, \forall j = 1, 2, \ldots, n$, and $\boldsymbol{v}_N = \mathbf{0}$, the Savage rule $\underset{i}{min}(S_i)$ is equivalent to:*

$$\underset{i}{min}(\mathcal{D}(a_i, \boldsymbol{v}_N, \infty) - v_{ij}), \forall j = 1, 2, \ldots, n. \tag{3.33}$$

Proof. It is a direct consequence of Remark 4. □

Theorem 9. *Setting $w_j = 1/\Delta_j, \forall j = 1, 2, \ldots, n$, and $\boldsymbol{v}_N = \mathbf{0}$, the Ballestero rule $\underset{i}{max}(B_i)$ is equivalent to:*

$$\underset{i}{max}(\mathcal{D}(a_i, \boldsymbol{v}_N, 1)). \tag{3.34}$$

Proof. It is a direct consequence of Remark 5. □

By changing the reference point, we can design further decision rules based on the concept of distance to the ideal point and Zeleny's axiom of choice [13], which states that alternatives that are closer to the ideal point are preferred to those that are further. To this end, we first reformulate the Minkowski distance function in equation (3.25) to consider ideal point $\boldsymbol{v}_I$ as a reference point for decision-making:

$$\mathcal{D}(a_i, \boldsymbol{v}_I, p) = \left[\sum_{j=1}^{n} w_j^p |v_{jI} - v_{ij}|^p \right]^{1/p}. \tag{3.35}$$

Next, we establish the following preference relations for two arbitrary alternatives $a_1, a_2 \in \mathcal{A}$:

$$a_1 \succ a_2 \iff \mathcal{D}(a_1, \boldsymbol{v}_I, p) < \mathcal{D}(a_2, \boldsymbol{v}_I, p) \tag{3.36}$$

$$a_1 \sim a_2 \iff \mathcal{D}(a_1, \boldsymbol{v}_I, p) = \mathcal{D}(a_2, \boldsymbol{v}_I, p) \tag{3.37}$$

where $\succ$ means "is preferred to", and $\sim$ means "is indifferent to".

When using distances considering multiple-criteria (states), measurements must be comparable to avoid meaningless comparisons due to problems of scale. For instance, in Table 2, we observe that the first criterion (state) presents significantly higher values than the remaining criteria (states). If we directly apply a distance function to evaluate alternatives, the results will be biased toward the best alternative in the first column because of the higher scale. To solve this limitation, we normalize the evaluation using the following change of variable [15]:

$$\theta_{ij} = \frac{v_{ij} - \underset{j}{min}(v_{ij})}{\underset{j}{max}(v_{ij}) - \underset{j}{min}(v_{ij})} \tag{3.38}$$

where normalized evaluation θ_{ij} ranges in the interval $[0, 1]$.

As a result, the Minkowski distance function of the alternatives to the ideal point becomes:

$$\mathcal{D}(a_i, \boldsymbol{v}_I, p) = \left[\sum_{j=1}^{n} w_j^p |1 - \theta_{ij}|^p \right]^{1/p} \tag{3.39}$$

and we can evaluate all alternatives in a decision table according to some decision-making principles as suggested by Romero [23] and Gonzalez-Pachon and Romero [24]. For instance, $p = 1$ corresponds to the idea of maximum freedom of the individuals derived from the theory of utilitarianism by Bentham [25], whereas $p = \infty$ corresponds to the principle of maximum fairness, from the Rawlsian idea of considering only the welfare of the worst-off group [26]. Indeed, by increasing p, we are increasing the balance of solutions, meaning that the best solutions tend to be those with less inequality in the evaluations for different criteria (states). Note also that weights w_j attached to each of the criteria (states) are not necessarily equal. They may be equal when the criteria (states) are equally important for decision-making purposes as in the case of the Laplace rule, but they may be different when some criteria (states) are considered more important than others as in the case of Ballestero's rule. Then, these weights must be set by the decision-maker according to the rule used. In the following illustrative example, we consider that all criteria (states) are equally important. Let us extend Table 2 with equally-weighted distances to the ideal point $(1, 1, 1)$ after normalization for $p = 1$ (Benthamite principle), $p = 2$ (principle of the smallest Euclidean deviation), and $p = \infty$ (Rawlsian principle) as shown in Table 3.

Table 3. A normalized decision table with three different distances to the ideal point.

Alternatives	States			$\mathcal{D}(a_i, \boldsymbol{v}_I, 1)$	Rank	$\mathcal{D}(a_i, \boldsymbol{v}_I, 2)$	Rank	$\mathcal{D}(a_i, \boldsymbol{v}_I, \infty)$	Rank
	r_1	r_2	r_3						
a_1	0.995	0.000	0.013	1.992	4	1.405	4	1.000	4
a_2	0.349	0.595	0.684	1.371	3	0.829	3	0.651	3
a_3	0.602	1.000	1.000	0.398	1	0.398	2	0.398	2
a_4	0.677	0.861	0.895	0.567	2	0.367	1	0.323	1
a_5	0.000	0.203	0.513	2.284	6	1.369	6	1.000	4
a_6	1.000	0.000	0.000	2.000	5	1.414	5	1.000	4

Note that in the case of considering distances to the ideal point, the best alternative is the one with the minimum distance value. By comparing Tables 2 and 3, we observe that the final ranking of alternatives derived from the application of the distance-based method for $p = 1$ (Benthamite principle) is equivalent to the Ballestero moderate pessimism criterion. However, we find differences when the value of p is increased to 2 (principle of the smallest Euclidean deviation). In this case, the best option is alternative a_4. Finally, by setting $p = \infty$ (Rawlsian principle) the first three best alternatives are the same as in case $p = 2$ and the remaining alternatives obtain the same score because all of them present the worst evaluation for at least one of the criteria (states).

3.4. Application of distance-based decision rules in portfolio selection

In this section, we extend the results reported by Ballestero et al. [7] in the context of portfolio selection. More precisely, we evaluate the implications in the selection of the best portfolio when considering 15 scenarios under strict uncertainty by varying the weights and the topological metric p when using the distance-based decision rules proposed in this paper. Table 4 shows the simulation-based multicriteria performance indices described by Ballestero et al. [7] for non-dominated pre-selected efficient portfolios (PEP) for a profitability seeker investor. Although the authors also reported results for a neutral and a conservative investor, we restrict the following analysis to a profitability seeker because our findings are very similar in the three cases. Summarizing, these values were obtained by: 1) computing the efficient frontier for portfolios based on historical data; 2) defining a number of uncertain scenarios from the range of past returns of the market index; 3) simulating market returns; and 4) deriving from the market simulation a return-risk performance index including a parameter describing the risk preferences of the investor. To facilitate the analysis, we extend the table of results with the inverse of the range, the inverse of standard deviation, and the inverse of the mean absolute deviation that will be later used in the computation of the scores for the alternatives.

Table 4. Performance indices for non-dominated pre-selected efficient portfolios (PEP) for a profitability seeker investor.

	Scenarios														
PEP	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
13	0.729	2.396	1.375	1.157	1.570	1.329	1.223	1.754	2.404	2.707	1.676	1.891	2.036	2.074	2.019
12	0.966	2.175	1.427	1.271	1.622	1.356	1.305	1.664	2.175	2.385	1.589	1.754	1.859	1.889	1.85
14	0.429	2.581	1.275	0.991	1.439	1.278	1.099	1.829	2.603	3.009	1.758	2.020	2.208	2.253	2.18
11	1.140	1.920	1.432	1.332	1.595	1.360	1.344	1.558	1.915	2.044	1.497	1.610	1.678	1.698	1.674
10	1.199	1.741	1.400	1.331	1.529	1.338	1.335	1.476	1.736	1.823	1.428	1.510	1.556	1.572	1.555
$1/\Delta_j$	1.299	1.190	6.369	2.933	5.464	12.195	4.082	2.833	1.153	0.843	3.030	1.961	1.534	1.468	1.600
$1/\sigma_j$	3.526	3.275	17.489	7.724	15.669	34.018	10.912	7.828	3.176	2.325	8.421	5.427	4.246	4.060	4.425
$1/MAD_j$	3.986	3.764	22.007	8.778	18.657	43.554	12.475	8.980	3.665	2.692	9.812	6.297	4.910	4.694	5.125

The results in Table 4 represent an example of a decision table under strict uncertainty which is the basic input information to apply the set of decision rules described in this paper. Along the lines of Ballestero [1], we consider that decision rules that do not exhaustively use all the information in the decision table are less desirable than others using all the information. As a result, we dismiss the rules proposed by Wald, Hurwicz, and Savage in the next analysis. Following the same reasoning, we also discard the use of distance function $\mathcal{D}(a_i, \boldsymbol{v}_N, \infty)$, because this rule only focuses on the maximum performance for each alternative. To analyze the impact of selecting different weight systems and different topological metrics required in the distance-based ranking method described in Section 3.3, we consider the following variants to compute the distance to the nadir point: 1) weights set to the inverse of the range of performance indices for each scenario ($w_j = 1/\Delta_j$) and $p = 1$ (Ballestero's criterion = $\mathcal{D}_1$); 2) equal weights for each scenario ($w_j = 1/15$) and $p = 1$ (Laplace criterion = $\mathcal{D}_2$); 3) equal weights for each scenario ($w_j = 1$) and $p = 2$ ($\mathcal{D}_3$); 4) weights set to the inverse of the standard deviation of performance indices for each scenario ($w_j = 1/\sigma_j$) and $p = 1$ ($\mathcal{D}_4$); 5) weights set to the inverse of the standard deviation of performance indices for each scenario ($w_j = 1/\sigma_j$) and $p = 2$ ($\mathcal{D}_5$); 6) weights set to the inverse of the mean absolute deviation of performance indices for each scenario ($w_j = 1/MAD_j$) and $p = 1$ ($\mathcal{D}_6$); and 7) weights set to the inverse of the mean absolute deviation of performance indices for each scenario ($w_j = 1/MAD_j$) and $p = 2$ ($\mathcal{D}_7$). The results of the previous distance-based scores for the alternatives in Table 4 are summarized in Table 5.

We observe that the best alternative for $\mathcal{D}_1$ (Ballestero) is different to $\mathcal{D}_2$ (Laplace) and $\mathcal{D}_3$. This is not surprising because the weight system of $\mathcal{D}_1$ is obtained from the inverse of the range, and $\mathcal{D}_2$ and $\mathcal{D}_3$ are equally-weighted distances. In addition, we find that the use of the inverse of the standard deviation in $\mathcal{D}_4$ and the mean absolute deviation in $\mathcal{D}_6$ as weight systems do not produce any change in the best alternative despite the different weight system. However, we observe a change in the selection

Table 5. Distance-based scores for alternatives in Table 4. Bold values mean best values.

PEP	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_6$	$\mathcal{D}_7$
13	**73.938**	1.756	7.094	**204.591**	66.105	**245.049**	81.502
12	73.460	1.686	6.688	203.214	**66.813**	243.642	**82.500**
14	72.783	**1.797**	**7.444**	201.449	64.016	241.022	78.785
11	71.365	1.586	6.214	197.362	66.072	236.859	81.711
10	68.763	1.502	5.853	190.132	64.333	228.312	79.636
w_j	$1/\Delta_j$	1/15	1	$1/\sigma_j$	$1/\sigma_j$	$1/MAD_j$	$1/MAD_j$
p	1	1	2	1	2	1	2

of the best alternative when these weights are combined with metric $p = 2$ in $\mathcal{D}_5$ and $\mathcal{D}_7$. This behavior is caused by the fact that both weights and performance indexes are raised to exponent 2, hence overweighing those scenarios with less variation that may lead to a change in the final scores. As a result, we find that the impact of the use of the inverse of measures of dispersion such as the range, the standard deviation, or the mean absolute deviation is increased when the topological metric is increased. From this observation, we conclude that metric p can be used as a key variable to control the degree of confidence (importance) in the scenarios with less dispersion. This fact provides a higher degree of flexibility than the moderate pessimism criterion by Ballestero. This feature may be useful to accommodate well to the needs or beliefs of different decision-makers with respect to either the distrust of high dispersion scenarios or the confidence in low dispersion scenarios.

4. Conclusions

Within the context of decision rules under strict uncertainty, this paper has extended the moderate pessimism criterion by Ballestero [1] to solve two important limitations. On the one hand, we solve the problem of removing alternatives that are relevant in the final ranking by proposing a new definition of dominated and non-dominated alternatives based on the concept of Pareto optimality. On the other hand, we propose the use of the inverse of the standard deviation and the inverse of the mean absolute deviation instead of the inverse of the range of evaluations for each future state to reduce the impact of the possible existence of outliers in the decision table. To motivate this proposal, we show in Theorem 3 that the presence of outliers is mitigated by using the standard deviation and the mean absolute deviations as measures of dispersion by attaching a larger weight than in the original method to states with low weights due to the presence of an outlier that should not be in the decision table. In addition, we find that this mitigation is larger in the case of the mean absolute deviation than in the case of the standard deviation due to well-known properties of these measures. We generalize the use of additional measures of dispersion through the concept of average deviation of order p and prove in Theorem 4 that this generalization implies the possibility of controlling the degree of mitigation of the eventual existence of outliers in the decision table. Finally, the use of the standard deviation and the mean absolute deviation presents the advantage of using all the information in the decision table to compute the column dispersion.

We also propose a general distance-based ranking method that subsumes existing decision rules under strict uncertainty. This approach can be applied from two different points of view. The first one implies the maximization of distances to an anti-ideal point. Following this first approach, we show that all the decision rules under strict uncertainty considered in this paper are indeed particular cases of maximizing distances to an anti-ideal point with all elements set to zero. The second approach implies the minimization of distances to an ideal point with maximum achievements for each of the criteria (states) considered. In this case, the use of a parameter in the Minkowski distance function introduces the possibility of ranking alternatives according to some general decision-making principles by varying the balance of solutions. We also observe that metric p can be used as a key variable to accommodate to the needs and beliefs of different decision-makers with respect to either the distrust of high dispersion scenarios or the confidence in low dispersion scenarios. From this observation, we conclude that the distance-based ranking methods proposed in this paper not only generalize previous approaches but also provide a higher degree of flexibility for decision-making.

Finally, we consider that a natural extension of this work is to extend the analysis of the duality established between states and criteria and strengthen the link between decision-making under strict uncertainty and the whole range of tools and methodologies of multiple-criteria decision-making such as mathematical programming or combinatorial optimization.

Conflict of interest

The authors declare no conflict of interest.

References

1. E. Ballestero, Strict uncertainty: A criterion for moderately pessimistic decision makers, *Decision Sci.,* **33** (2002), 87–108. https://doi.org/10.1111/j.1540-5915.2002.tb01637.x
2. P. S. Laplace, *A philosophical essay on probabilities*, New York: Dover, 1952.
3. A. Wald, *Statistical decision functions*, New York: John Wiley, 1950.
4. L. Hurwicz, Optimality criteria for decision making under ignorance, *Cowles Commission Discussion Paper, Statistics,* **370** (1951).
5. L. J. Savage, The theory of statistical decision, *J. Am. Stat. Assoc.,* **46** (1951), 55–67.
6. E. Ballestero, Selecting textile products by manufacturing companies under uncertainty, *Asia Pac. J. Oper. Res.,* **21** (2004), 141–161. https://doi.org/10.1142/s0217595904000102
7. E. Ballestero, M. Günther, D. Pla-Santamaria, C. Stummer, Portfolio selection under strict uncertainty: A multi-criteria methodology and its application to the Frankfurt and Vienna Stock Exchanges, *Eur. J. Oper. Res.,* **181** (2007), 1476–1487. https://doi.org/10.1016/j.ejor.2005.11.050
8. M. Bravo, D. Pla-Santamaria, Evaluating loan performance for bank offices: a multicriteria decision-making approach, *Inform. Syst. Oper. Res.,* **50** (2012), 127–133. https://doi.org/10.3138/infor.50.3.127

9. D. Pla-Santamaria, M. Bravo, J. Reig-Mullor, F. Salas-Molina, A multicriteria approach to manage credit risk under strict uncertainty, *Top,* **29** (2021), 494–523. https://doi.org/10.1007/s11750-020-00571-0

10. C. P. Chambers, Ordinal aggregation and quantiles, *J. Econ. Theory,* **137** (2007), 416–431. https://doi.org/10.1016/j.jet.2006.12.002

11. M. Rostek, Quantile maximization in decision theory, *Rev. Econ. Stud,* **77** (2010), 339–371. https://doi.org/10.1111/j.1467-937x.2009.00564.x

12. H. Sadeghi, F. Moslemi, A multiple objective programming approach to linear bilevel multi-follower programming, *AIMS Math.,* **4** (2019), 763–778. https://doi.org/10.3934/math.2019.3.763

13. M. Zeleny, Compromise programming, *Multiple criteria decision making*, University of South Carolina Press, Columbia, (1973), 262–301.

14. R. E. Steuer, *Multiple criteria optimization. Theory, Computation, and Application*, New York: John Wiley, 1986.

15. E. Ballestero, C. Romero, *Multiple criteria decision making and its applications to economic problems*, Dordrecht: Kluwer Academic Publishers, 1998.

16. S. French, *Decision theory: an introduction to the mathematics of rationality*, New York: John Wiley and Sons, 1986.

17. J. Von Neumann, O. Morgenstern, *Theory of games and economic behavior*, Princeton: Princeton University Press, 1944.

18. Z. Wu, D. L. St-Pierre, G. Abdul-Nour, Decision making under strict uncertainty: case study in sewer network planning, *Int. J. Comput. Infor. Eng.,* **11** (2017), 815–823. https://doi.org/10.5281/zenodo.1131355

19. V. Ulansky, A. Raza, Generalization of minimax and maximin criteria in a game against nature for the case of a partial a priori uncertainty, *Heliyon,* **7** (2021), e07498. https://doi.org/10.1016/j.heliyon.2021.e07498

20. M. Özkaya, B. İzgi, M. Perc, Axioms of Decision Criteria for 3D Matrix Games and Their Applications, *Math.,* **10** (2022), 4524. https://doi.org/10.3390/math10234524

21. G. Balakina, E. Kibalov, M. Pyataev, System assessment of the Kyzyl-Kuragino railroad project as an element of Russia-Mongolia-China transport corridors, *Transpor. Res. Proc.,* **63** (2022), 1817–1825. https://doi.org/10.1016/j.trpro.2022.06.199

22. M. Ehrgott, *Multicriteria optimization*, Berlin: Springer, 2005.

23. C. Romero, A note on distributive equity and social efficiency, *J. Agr. Econ.,* **52** (2001), 110–112. https://doi.org/10.1111/j.1477-9552.2001.tb00928.x

24. J. González-Pachón, C. Romero, Bentham, Marx and Rawls ethical principles: In search for a compromise, *Omega,* **62** (2016), 47–51. https://doi.org/10.1016/j.omega.2015.08.008

25. J. Bentham, *An introduction to the principles of morals and legislation*, London: Payne and Son, 1789.

26. J. Rawls, *A theory of justice*, Oxford: Oxford University Press, 1973.

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