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*Research article*

# Dynamics of a stochastic epidemic model with quarantine and non-monotone incidence

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**Abstract:** In this paper, a stochastic SIQR epidemic model with non-monotone incidence is investigated. First of all, we consider the disease-free equilibrium of the deterministic model is globally asymptotically stable by using the Lyapunov method. Secondly, the existence and uniqueness of positive solution to the stochastic model is obtained. Then, the sufficient condition for extinction of the stochastic model is established. Furthermore, a unique stationary distribution to stochastic model will exist by constructing proper Lyapunov function. Finally, numerical examples are carried out to illustrate the theoretical results, with the help of numerical simulations, we can see that the higher intensities of the white noise or the bigger of the quarantine rate can accelerate the extinction of the disease. This theoretically explains the significance of quarantine strength (or isolation measures) when an epidemic erupts.

**Keywords:** stochastic model; non-monotone incidence; Itô's formula; Lyapunov function; basic reproduction number; extinction; stationary distribution
**Mathematics Subject Classification:** 34D30, 60H10, 92D25

## 1. Introduction

Novel coronavirus first emerged in Wuhan in December 2019 [1] and then spread like a fire around the world. The social loss caused by the disease was far beyond what we thought possible. It is crucial to investigate the transmission mechanism and development trend of infectious disease through mathematical modelling so as to take appropriate measures to control the epidemic. The SIR epidemic model was originally proposed by Kermack and McKendrick [2]. Up to now, many researchers have built different types of epidemic models, for instance [3–11]. It is known to all that one of the most direct and effective measures against infectious disease is to quarantine identified contacts through timely contact tracing [12], so that more people can be prevented from becoming infected. Therefore,

epidemic models with quarantine are increasingly being investigated, such as [13–19].

It is widely known that the incidence function is crucial to the transmission dynamics of infectious diseases. In 1978, Capasso and Serio [20] introduced a saturated incidence rate into epidemic models to investigate the spread of cholera in Paris. Ruan and Wang [21] studied an epidemic model with a specific nonlinear incident rate. A non-monotone incidence function was proposed by Xiao and Ruan [22], the incidence rate includes the behavioral change and crowding effect of the infective individuals. Based on the analysis of SARS [23], they discovered that the number of effective contacts between infected and susceptible persons decreases at high levels of infection by the isolation of infected persons or self-protection of susceptible individuals.

Let $S(t), I(t), R(t), Q(t)$ denote the number of susceptible, infective, removed and quarantined. By learning from the experience of predecessors, the deterministic SIQR epidemic model has the following form

$$\begin{cases} S'(t) = A - \mu S(t) - \frac{\beta S(t) I(t)}{1+\alpha I^2(t)}, \\ I'(t) = \frac{\beta S(t) I(t)}{1+\alpha I^2(t)} - (\varepsilon + \mu + \gamma) I(t), \\ Q'(t) = \gamma I(t) - (\mu + \xi) Q(t), \\ R'(t) = \epsilon I(t) + \xi Q(t) - \mu R(t), \end{cases} \tag{1.1}$$

where $A$ is the recruitment rate of $S(t)$, $\mu$ is the natural death rate, $\beta$ is the average number of adequate contacts, $\gamma$ is the removal rate from $I(t)$, $\epsilon$ and $\xi$ represent the recovery rates from $I(t), Q(t)$ to $R(t)$. All parameters in the above are assumed to be nonnegative. The dynamics of (1.1) is completely determined by the basic reproduction number.

In real life, the models of population dynamics of diseases are inevitably affected by random fluctuations. Li et al. [24] considered a stochastic SIRS epidemic model to understand the mechanism of influenza A transmission. Yuan et al. [25] proves that environmental noises can change the qualitative behaviors. In this paper, we assumed that the coefficient of quarantine $\gamma$ is subject to the environment white noise, namely $\gamma \to \gamma + \sigma_5 \dot{B}_5(t)$. In addition, as the research work mentioned above, adding a linear perturbation using Brownian motion (terms $\sigma_1 S(t)\mathrm{d}B_1(t), \sigma_2 I(t)\mathrm{d}B_2(t), \sigma_3 Q(t)\mathrm{d}B_3(t), \sigma_4 R(t)\mathrm{d}B_4(t)$) in model. We then consider the following stochastic model

$$\begin{cases} \mathrm{d}S(t) = (A - \mu S(t) - \frac{\beta S(t) I(t)}{1+\alpha I^2(t)})\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1(t), \\ \mathrm{d}I(t) = (\frac{\beta S(t) I(t)}{1+\alpha I^2(t)} - (\varepsilon + \mu + \gamma) I(t))\mathrm{d}t + \sigma_2 I(t)\mathrm{d}B_2(t) - \sigma_5 I(t)\mathrm{d}B_5(t), \\ \mathrm{d}Q(t) = (\gamma I(t) - (\mu + \xi) Q(t))\mathrm{d}t + \sigma_3 Q(t)\mathrm{d}B_3(t) + \sigma_5 I(t)\mathrm{d}B_5(t), \\ \mathrm{d}R(t) = (\epsilon I(t) + \xi Q(t) - \mu R(t))\mathrm{d}t + \sigma_4 R(t)\mathrm{d}B_4(t), \end{cases} \tag{1.2}$$

where $B_i(t)(i = 1, 2, 3, 4, 5)$ are mutually independent standard Brownian motions with $B_i(0) = 0$, $\sigma_i(i = 1, 2, 3, 4, 5)$ denote the intensities of the white noise $B_i(t)$.

The article is organized as follows: In Section 2, we investigate the disease-free equilibrium of the deterministic model (1.1) is globally asymptotically stable by using the Lyapunov-LaSalle asymptotic theorem. In Section 3, the existence and uniqueness of globally positive solutions to the stochastic model (1.2) are obtained. In Section 4, we establish sufficient conditions for extinction of the model (1.2). In Section 5, we verify that the stationary distribution of stochastic model (1.2) will exist under certain conditions. In Section 6, numerical simulations are given to support our theoretical results. Finally, a brief conclusion is given In Section 7.

## 2. The dynamics of deterministic model (1.1)

In the section, we will investigate the dynamics of the deterministic model (1.1). By summing all the equations of model (1.1) one can obtain that the total population $N(t) = S(t) + I(t) + Q(t) + R(t)$ satisfies $N'(t) = A - \mu N$. Define

$$\Gamma = \{(S, I, Q, R), \frac{A}{\mu + \delta} \leq S + I + Q + R \leq \frac{A}{\mu}, \text{and}, S, I, Q, R \geq 0\}. \tag{2.1}$$

Obviously, $\Gamma$ is positively invariant of model (1.1). For the convenience of analysis, we will have model

$$\begin{cases} N'(t) = A - \mu N, \\ I'(t) = \frac{\beta(N-I-R-Q)I}{1+\alpha I^2} - (\varepsilon + \mu + \gamma)I, \\ Q'(t) = \gamma I - (\mu + \xi)Q, \\ R'(t) = \epsilon I + \xi Q - \mu R. \end{cases} \tag{2.2}$$

The region becomes

$$F = \{(N, I, Q, R), \frac{A}{\mu + \delta} \leq N \leq \frac{A}{\mu}, \text{and}, 0 \leq I, Q, R, I + R, I + Q + R \leq N\}. \tag{2.3}$$

We will analyze model (2.2) in the region (2.3).

Applying the approach of next generation matrix [26] to model (2.2), we obtain

$$\mathfrak{R}_0 = \rho(FV^{-1}) = \frac{\beta A}{\mu(\varepsilon + \mu + \gamma)}. \tag{2.4}$$

**Theorem 1.** *Consider model (1.1). The disease-free equilibrium $E_0 = (\frac{A}{\mu}, 0, 0, 0)$ always exists. It is globally asymptotically stable if $\mathfrak{R}_0 < 1$ .*

*Proof.* Define a Lyapunov function

$$V = \frac{1}{2}(N - \frac{A}{\mu})^2 + n_1 I + R + Q, \tag{2.5}$$

where $n_1 = \frac{\gamma+\varepsilon}{(1-\mathfrak{R}_0)(\varepsilon+\mu+\gamma)} > 0$. If $\mathfrak{R}_0 < 1$, then the total derivative of $V$ along the trajectories of model (2.2) is given by

$$\begin{aligned} \frac{dV}{dt} &= (N - \tfrac{A}{\mu})(A - \mu N) + n_1[\tfrac{\beta(N-I-R-Q)}{1+\alpha I^2} - (\varepsilon + \mu + \gamma)]I \\ &\quad +\gamma I - (\epsilon I + \xi)Q + \epsilon I + \xi Q - \mu R \\ &= 2NA - \mu N^2 - \tfrac{A^2}{\mu} + n_1[\tfrac{\beta(N-I-R-Q)}{1+\alpha I^2} - (\varepsilon + \mu + \gamma)]I \\ &\quad +\gamma I + \epsilon I - \mu(Q + R) \\ &\leq -\mu(N - \tfrac{A}{\mu})^2 + [n_1(\tfrac{\beta A}{\mu} - (\varepsilon + \mu + \gamma)) + \varepsilon + \gamma]I \\ &= -\mu(N - \tfrac{A}{\mu})^2 + (n_1(\varepsilon + \mu + \gamma)(\mathfrak{R}_0 - 1) + \varepsilon + \gamma)I. \end{aligned} \tag{2.6}$$

Thus $\frac{dV}{dt} \leq 0$, and $\frac{dV}{dt} = 0$ if and only if $N = \frac{A}{\mu}, I = 0, R = 0, Q = 0$. Applying the Lyapunov-LaSalle asymptotic theorem, we can obtain that the disease-free equilibrium $E_0 = (\frac{A}{\mu}, 0, 0, 0)$ is globally asymptotically stable if $\mathfrak{R}_0 < 1$. □

## 3. Existence and uniqueness of the global positive solution

Denote $\mathbb{R}^4_+ = \{(S(t), I(t), Q(t), R(t)) | S(t) > 0, I(t) > 0, Q(t) > 0, R(t) > 0\}$, $\mathbb{R}_+ = [0, +\infty)$. Throughout this paper, we let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). We assume that model (1.2) is defined on $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

**Theorem 2.** *For any initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$, there is a unique positive solution $(S(t), I(t), Q(t), R(t))$ of system (1.2). Furthermore, the solution will remain in $\mathbb{R}^4_+$ with probability 1, namely, $(S(t), I(t), Q(t), R(t)) \in \mathbb{R}^4_+$ for all $t \geq 0$ almost surely.*

*Proof.* Since the coefficients of system (1.2) satisfy the local Lipschitz conditions, then for any initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$, there exists a unique local solution $(S(t), I(t), Q(t), R(t))$ on $t \in [0, \tau_e)$, where $\tau_e$ is the explosion time [27]. To show this solution is global, we only need to show that $\tau_e = \infty$ almost surely. Let $k_0 \geq 1$ be sufficiently large such that$S(0), I(0), Q(0)$ and $R(0)$ all lie within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : S(t) \notin (\frac{1}{k}, k) \text{or} I(t) \notin (\frac{1}{k}, k) \text{or} Q(t) \notin (\frac{1}{k}, k) \text{or} R(t) \notin (\frac{1}{k}, k)\}.$$

□

Throughout this paper, we set $\inf \emptyset = \infty$ (as usual $\emptyset$ is the empty set). Clearly $\tau_k$ is increasing when $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$, then $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$, then $\tau_e = \infty$ and $(S(t), I(t), Q(t), R(t)) \in \mathbb{R}^4_+$ a.s. for all $t \geq 0$. If this statement is not true, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon.$$

Hence there exists an integer $k_1 \geq k_0$ such that

$$\mathbb{P}\{\tau_k \leq T\} \geq \varepsilon \tag{3.1}$$

for all $k \geq k_1$.

Define a $C^2-$function $V : \mathbb{R}^4_+ \to \mathbb{R}_+$ by

$$V(S, I, Q, R) = (S(t) - c - c \ln \frac{S(t)}{c}) + (I(t) - 1 - \ln I(t)) + (Q(t) - 1 - \ln Q(t)) + (R(t) - 1 - \ln R(t)), \tag{3.2}$$

where c is a positive constant to be determined later. For any $u > 0$, $u - 1 - \ln u \geq 0$, so the nonnegativity of this function can be seen. Applying Itô's formula to $V$, we have

$$\begin{aligned}
\mathrm{d}V &= [(1 - \tfrac{c}{S(t)})(A - \mu S(t) - \tfrac{\beta S(t) I(t)}{1+\alpha I^2(t)}) + (1 - \tfrac{1}{I(t)})(\tfrac{\beta S(t) I(t)}{1+\alpha I^2(t)} - (\varepsilon + \mu + \gamma) I(t)) \\
&\quad + (1 - \tfrac{1}{Q(t)})(\gamma I(t) - (\mu + \xi) Q(t)) + (1 - \tfrac{1}{R(t)})(\epsilon I(t) + \xi Q(t) - \mu R(t))]\mathrm{d}t \\
&\quad + \tfrac{1}{2}(c\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 + \sigma_5^2 \tfrac{I^2(t)}{Q^2(t)})\mathrm{d}t + (1 - \tfrac{c}{S(t)})\sigma_1 S(t)\mathrm{d}B_1(t) \\
&\quad + (1 - \tfrac{1}{I(t)})\sigma_2 I(t)\mathrm{d}B_2(t) - (1 - \tfrac{1}{I(t)})\sigma_5 I(t)\mathrm{d}B_5(t) \\
&\quad + (1 - \tfrac{1}{Q(t)})\sigma_3 Q(t)\mathrm{d}B_3(t) + (1 - \tfrac{1}{Q(t)})\sigma_5 I(t)\mathrm{d}B_5(t) + (1 - \tfrac{1}{R(t)})\sigma_4 R(t)\mathrm{d}B_4(t) \\
&\leq [A + (c\beta - \mu) I(t) + c\mu + 3\mu + \varepsilon + \gamma + \xi + \tfrac{c}{2}\sigma_1^2 + \tfrac{1}{2}\sigma_2^2 + \tfrac{1}{2}\sigma_3^2 + \tfrac{1}{2}\sigma_4^2 + \tfrac{1}{2}\sigma_5^2 + \tfrac{1}{2}\sigma_5^2 \tfrac{I^2(t)}{Q^2(t)}]\mathrm{d}t \\
&\quad + \sigma_1(S(t) - c)\mathrm{d}B_1(t) + \sigma_2(I(t) - 1)\mathrm{d}B_2(t) + \sigma_3(Q(t) - 1)\mathrm{d}B_3(t) \\
&\quad + \sigma_4(R(t) - 1)\mathrm{d}B_4(t) + \sigma_5(1 - \tfrac{I(t)}{Q(t)})\mathrm{d}B_5(t).
\end{aligned} \tag{3.3}$$

Let $c = \frac{\mu}{\beta}$. Then

$$LV \leq A + (3+c)\mu + \varepsilon + \gamma + \xi + \frac{c}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2 + \frac{1}{2}\sigma_4^2 + \frac{1}{2}\sigma_5^2 + \frac{1}{2}\sigma_5^2\frac{I^2(t)}{Q^2(t)}. \tag{3.4}$$

Similar to the reference [28], the solutions of model (1.2) are stochastically ultimately bounded, Hence there exists a suitable constant $K > 0$ independent of $S, I, Q, R$ and $t$ such that $LV \leq K$. So

$$\begin{aligned} \mathrm{d}V(S(t), I(t), Q(t), R(t)) \quad &\leq K\mathrm{d}t + \sigma_1(S(t)-c)\mathrm{d}B_1(t) + \sigma_2(I(t)-1)\mathrm{d}B_2(t) \\ &+\sigma_3(Q(t)-1)\mathrm{d}B_3(t) + \sigma_4(R(t)-1)\mathrm{d}B_4(t) + \sigma_5(1-\tfrac{I(t)}{Q(t)})\mathrm{d}B_5(t). \end{aligned} \tag{3.5}$$

Integrating both sides of (3.5) from 0 to $\tau_k \wedge T = \min\{\tau_k, T\}$ and then taking the expectation,

$$\mathbb{E}V(S(\tau_k \wedge T), I(\tau_k \wedge T), Q(\tau_k \wedge T), R(\tau_k \wedge T)) \leq V(S(0), I(0), Q(0), R(0)) + K\mathbb{E}(\tau_k \wedge T). \tag{3.6}$$

Thus,

$$\mathbb{E}V(S(\tau_k \wedge T), I(\tau_k \wedge T), Q(\tau_k \wedge T), R(\tau_k \wedge T)) \leq V(S(0), I(0), Q(0), R(0)) + KT. \tag{3.7}$$

Set $\Omega_k = \{\tau_k \leq T\}$ for $k \geq k_1$ and in view of (3.1), we get $\mathbb{P}(\Omega_k) \geq \varepsilon$. Notice that for every $\omega \in \Omega_k$, it exists that $S(\tau_k, \omega)$ or $I(\tau_k, \omega)$ or $Q(\tau_k, \omega)$ or $R(\tau_k, \omega)$ equals either $k$ or $\frac{1}{k}$. Thereby, $V(S(\tau_k, \omega), I(\tau_k, \omega), Q(\tau_k, \omega), R(\tau_k, \omega))$ is no less than either $k - 1 - \ln k$, $k - c - c\ln\frac{k}{c}$, $\frac{1}{k} - 1 - \ln\frac{1}{k}$ or $\frac{1}{k} - c - c\ln\frac{1}{ck}$. Consequently, denote $\tilde{M} = (k - 1 - \ln k) \wedge (k - c - c\ln\frac{k}{c}) \wedge (\frac{1}{k} - 1 - \ln\frac{1}{k}) \wedge (\frac{1}{k} - c - c\ln\frac{1}{ck})$, we have

$$V(S(\tau_k, \omega), I(\tau_k, \omega), Q(\tau_k, \omega), R(\tau_k, \omega)) \geq \tilde{M}. \tag{3.8}$$

It follows from (3.7) that

$$\begin{aligned} V(S(0), I(0), Q(0), R(0)) + KT \geq \quad &\mathbb{E}\left(I_{\Omega_k}(\omega)V(S(\tau_k, \omega), I(\tau_k, \omega), Q(\tau_k, \omega), R(\tau_k, \omega))\right) \\ \geq \quad &\varepsilon\tilde{M}, \end{aligned} \tag{3.9}$$

where $I_{\Omega_k}$ represents the indicator function of $\Omega_k$. Letting $k \to \infty$, then

$$\infty > V(S(0), I(0), Q(0), R(0)) + KT = \infty, \tag{3.10}$$

which leads to the contradiction and hence we obtain $\tau_\infty = +\infty$ a.s., which implies that $(S(t), I(t), Q(t), R(t)) \in \mathbb{R}_+^4$ for all $t \in \mathbb{R}_+$. The proof is completed.

## 4. Extinction of the disease

The following theorem establish a condition for extinction of the disease.

For convenience, let

$$\langle x(t)\rangle = \frac{1}{t}\int_0^t x(r)\mathrm{d}r \tag{4.1}$$

and

$$\mathfrak{R}_0^s = \frac{\beta A}{\mu(\varepsilon + \mu + \gamma + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_5^2)}. \tag{4.2}$$

**Theorem 3.** *For any initial value* $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$, *there is a positive solution* $(S(t), I(t), Q(t), R(t))$ *of system (1.2) which has the following property*

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le (\varepsilon + \mu + \gamma + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_5^2)(\Re_0^s - 1). \tag{4.3}$$

*If* $\Re_0^s < 1$, *then the disease I goes to extinction exponentially with probability 1 a.s.*

$$\lim_{t\to\infty} I(t) = 0. \tag{4.4}$$

*In addition, we have*

$$\lim_{t\to\infty}\langle Q(t)\rangle = 0, \lim_{t\to\infty}\langle R(t)\rangle = 0, \lim_{t\to\infty}\langle S(t)\rangle = \frac{A}{\mu}.$$

*Proof.* Integration of the first three equations of the model (1.2) and then dividing by $t$ gives

$$\begin{cases} \frac{S(t)-S(0)}{t} = A - \mu\langle S(t)\rangle - \frac{\beta\int_0^t \frac{S(u)I(u)}{1+\alpha I^2(u)}\mathrm{d}u}{t} + \frac{\sigma_1\int_0^t S(u)\mathrm{dB}_1(u)}{t}, \\ \frac{I(t)-I(0)}{t} = \frac{\beta\int_0^t \frac{S(u)I(u)}{1+\alpha I^2(u)}\mathrm{d}u}{t} - (\varepsilon+\mu+\gamma)\langle I(t)\rangle + \frac{\sigma_2\int_0^t I(u)\mathrm{dB}_2(u)}{t} - \frac{\sigma_5\int_0^t I(u)\mathrm{dB}_5(u)}{t}, \\ \frac{Q(t)-Q(0)}{t} = \gamma\langle I(t)\rangle - (\mu+\xi)\langle Q(t)\rangle + \frac{\sigma_3\int_0^t Q(u)\mathrm{dB}_3(u)}{t} + \frac{\sigma_5\int_0^t I(u)\mathrm{dB}_5(u)}{t}. \end{cases} \tag{4.5}$$

Summing (4.5), one has

$$\begin{aligned} \frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} + \frac{Q(t)-Q(0)}{t} = {} & A - \mu\langle S(t)\rangle - (\varepsilon+\mu)\langle I(t)\rangle - (\mu+\xi)\langle Q(t)\rangle \\ & + \frac{\sigma_1\int_0^t S(u)\mathrm{dB}_1(u)}{t} + \frac{\sigma_2\int_0^t I(u)\mathrm{dB}_2(u)}{t} + \frac{\sigma_3\int_0^t Q(u)\mathrm{dB}_3(u)}{t}. \end{aligned} \tag{4.6}$$

For convenience, $\phi(t)$ is defined via subsequent equation

$$\begin{aligned} \phi(t) = {} & -\frac{1}{\mu}[\frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} + \frac{Q(t)-Q(0)}{t} \\ & - \frac{\sigma_1\int_0^t S(u)\mathrm{dB}_1(u)}{t} - \frac{\sigma_2\int_0^t I(u)\mathrm{dB}_2(u)}{t} - \frac{\sigma_3\int_0^t Q(u)\mathrm{dB}_3(u)}{t}]. \end{aligned} \tag{4.7}$$

Then

$$\langle S(t)\rangle = \frac{A}{\mu} - \frac{\varepsilon+\mu}{\mu}\langle I(t)\rangle - \frac{\mu+\xi}{\mu}\langle Q(t)\rangle + \phi(t). \tag{4.8}$$

Obviously, applying large number theorem for local martingales [27], we have

$$\lim_{t\to\infty}\phi(t) = 0. \tag{4.9}$$

Using Itô's formula to the 2nd equation of model (1.2), we have

$$\begin{aligned} \mathrm{d}\ln I = {} & [\frac{\beta S(t)}{1+\alpha I^2(t)} - (\varepsilon+\mu+\gamma) - \frac{1}{2}(\sigma_2^2+\sigma_5^2)]\mathrm{d}t + \sigma_2\mathrm{d}B_2(t) - \sigma_5\mathrm{d}B_5(t) \\ & \le [\beta S(t) - (\varepsilon+\mu+\gamma+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_5^2)]\mathrm{d}t + \sigma_2\mathrm{d}B_2(t) - \sigma_5\mathrm{d}B_5(t). \end{aligned} \tag{4.10}$$

Integration of (4.10) and then dividing by $t$ gives

$$\begin{aligned} \frac{\ln I(t)-\ln I(0)}{t} & \le \beta\langle S(t)\rangle - (\varepsilon+\mu+\gamma+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_5^2) + \frac{\sigma_2\int_0^t \mathrm{dB}_2(u)}{t} - \frac{\sigma_5\int_0^t \mathrm{dB}_5(u)}{t} \\ & = \frac{A\beta}{\mu} - \frac{\beta(\varepsilon+\mu)}{\mu}\langle I(t)\rangle - \frac{\beta(\mu+\xi)}{\mu}\langle Q(t)\rangle + \beta\phi(t) \\ & -(\varepsilon+\mu+\gamma+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_5^2) + \frac{\sigma_2\int_0^t \mathrm{dB}_2(u)}{t} - \frac{\sigma_5\int_0^t \mathrm{dB}_5(u)}{t} \\ & \le \frac{A\beta}{\mu} - (\varepsilon+\mu+\gamma+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_5^2) + \frac{\sigma_2\int_0^t \mathrm{dB}_2(u)}{t} - \frac{\sigma_5\int_0^t \mathrm{dB}_5(u)}{t} + \beta\phi(t), \end{aligned} \tag{4.11}$$

namely

$$\begin{aligned}\tfrac{\ln I(t)}{t} &\leq (\varepsilon+\mu+\gamma+\tfrac{1}{2}\sigma_2^2+\tfrac{1}{2}\sigma_5^2)(\mathfrak{R}_0^s-1)\\ &+\tfrac{\sigma_2\int_0^t \mathrm{dB}_2(u)}{t}-\tfrac{\sigma_5\int_0^t \mathrm{dB}_5(u)}{t}+\beta\phi(t)+\tfrac{\ln I(0)}{t}.\end{aligned} \tag{4.12}$$

Obviously,

$$\lim_{t\to\infty}\frac{\ln I(0)}{t}=0,\lim_{t\to\infty}\phi(t)=0, \tag{4.13}$$

$$\lim_{t\to\infty}\frac{\sigma_2\int_0^t \mathrm{dB}_2(u)}{t}=0, \tag{4.14}$$

$$\lim_{t\to\infty}\frac{\sigma_5\int_0^t \mathrm{dB}_5(u)}{t}=0. \tag{4.15}$$

Then for (4.12), if $\mathfrak{R}_0^s<1$, we reach to the following fact that

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}\leq(\varepsilon+\mu+\gamma+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_5^2)(\mathfrak{R}_0^s-1)<0, \tag{4.16}$$

namely,

$$\lim_{t\to\infty}I(t)=0. \tag{4.17}$$

On the other hand

$$\begin{aligned}\mathrm{d}[\tfrac{\xi}{\mu+\xi}Q(t)+R(t)] &= [\tfrac{\xi\gamma}{\mu+\xi}I(t)-\xi Q(t)+\varepsilon I(t)+\xi Q(t)-\mu R(t)]\mathrm{d}t\\ &+\tfrac{\xi\sigma_3}{\mu+\xi}Q(t)\mathrm{d}B_3(t)+\tfrac{\xi\sigma_5}{\mu+\xi}I(t)\mathrm{d}B_5(t)+\sigma_4R(t)\mathrm{d}B_4(t)\\ &= [(\tfrac{\xi\gamma}{\mu+\xi}+\varepsilon)I(t)-\mu R(t)]\mathrm{d}t\\ &+\tfrac{\xi\sigma_3}{\mu+\xi}Q(t)\mathrm{d}B_3(t)+\tfrac{\xi\sigma_5}{\mu+\xi}I(t)\mathrm{d}B_5(t)+\sigma_4R(t)\mathrm{d}B_4(t).\end{aligned} \tag{4.18}$$

Integrating (4.18) from 0 to $t$ and then dividing by $t$ gives

$$\begin{aligned}\tfrac{\xi}{\mu+\xi}\tfrac{Q(t)-Q(0)}{t}+\tfrac{R(t)-R(0)}{t} &= (\tfrac{\xi\gamma}{\mu+\xi}+\varepsilon)\langle I(t)\rangle-\mu\langle R(t)\rangle\\ &+\tfrac{\xi\sigma_3}{\mu+\xi}\tfrac{\int_0^t Q(u)\mathrm{dB}_3(u)}{t}+\tfrac{\xi\sigma_5}{\mu+\xi}\tfrac{\int_0^t I(u)\mathrm{dB}_5(u)}{t}+\tfrac{\sigma_4\int_0^t R(u)\mathrm{dB}_4(u)}{t}.\end{aligned} \tag{4.19}$$

Therefore,

$$\begin{aligned}\langle R(t)\rangle &= -\tfrac{1}{\mu}[\tfrac{\xi}{\mu+\xi}\tfrac{Q(t)-Q(0)}{t}+\tfrac{R(t)-R(0)}{t}-\tfrac{\xi\sigma_3}{\mu+\xi}\tfrac{\int_0^t Q(u)\mathrm{dB}_3(u)}{t}-\tfrac{\xi\sigma_5}{\mu+\xi}\tfrac{\int_0^t I(u)\mathrm{dB}_5(u)}{t}-\tfrac{\sigma_4\int_0^t R(u)\mathrm{dB}_4(u)}{t}]\\ &+\tfrac{1}{\mu}(\tfrac{\xi\gamma}{\mu+\xi}+\varepsilon)\langle I(t)\rangle.\end{aligned} \tag{4.20}$$

Let

$$m(t)=-\tfrac{1}{\mu}[\tfrac{\xi}{\mu+\xi}\tfrac{Q(t)-Q(0)}{t}+\tfrac{R(t)-R(0)}{t}-\tfrac{\xi\sigma_3}{\mu+\xi}\tfrac{\int_0^t Q(u)\mathrm{dB}_3(u)}{t}-\tfrac{\xi\sigma_5}{\mu+\xi}\tfrac{\int_0^t I(u)\mathrm{dB}_5(u)}{t}-\tfrac{\sigma_4\int_0^t R(u)\mathrm{dB}_4(u)}{t}], \tag{4.21}$$

consequently,

$$\langle R(t)\rangle=\frac{1}{\mu}(\frac{\xi\gamma}{\mu+\xi}+\varepsilon)\langle I(t)\rangle+m(t), \tag{4.22}$$

evidently,

$$\lim_{t\to\infty}m(t)=0, \tag{4.23}$$

we get

$$\lim_{t\to\infty}\langle R(t)\rangle = 0. \tag{4.24}$$

In addition,

$$\mathrm{d}Q(t) = [\gamma I(t) - (\mu+\xi)Q(t)]\mathrm{d}t + \sigma_3 Q(t)\mathrm{d}B_3(t) + \sigma_5 I(t)\mathrm{d}B_5(t). \tag{4.25}$$

Integration of (4.25) and then dividing by $t$ gives

$$\frac{Q(t)-Q(0)}{t} = \gamma\langle I(t)\rangle - (\mu+\xi)\langle Q(t)\rangle + \frac{\sigma_3\int_0^t Q(u)\mathrm{d}B_3(u)}{t} + \frac{\sigma_5\int_0^t I(u)\mathrm{d}B_5(u)}{t}, \tag{4.26}$$

by algebraic manipulation,

$$\langle Q(t)\rangle = \frac{\gamma}{\mu+\xi}\langle I(t)\rangle - \frac{1}{\mu+\xi}\frac{Q(t)-Q(0)}{t} + \frac{1}{\mu+\xi}\frac{\sigma_3\int_0^t Q(u)\mathrm{d}B_3(u)}{t} + \frac{1}{\mu+\xi}\frac{\sigma_5\int_0^t I(u)\mathrm{d}B_5(u)}{t}. \tag{4.27}$$

Because,

$$\lim_{t\to\infty}\frac{Q(t)-Q(0)}{t} = 0, \lim_{t\to\infty}\frac{\sigma_3\int_0^t Q(u)\mathrm{d}B_3(u)}{t} = 0, \tag{4.28}$$

hence, we have

$$\lim_{t\to\infty} Q(t) = 0. \tag{4.29}$$

Finally, putting the above Eqs (4.17) and (4.29) into (4.8), we obtain

$$\lim_{t\to\infty} S(t) = \frac{A}{\mu}.$$

Thus we conclude that the disease extinction depends on the value of the parameter $\mathfrak{R}_0^s$, if $\mathfrak{R}_0^s < 1$, the disease will extinct. The proof is completed. □

## 5. Existence of stationary distribution.

In order to prove the existence of a stationary distribution, we recall a famous result from Khasminskii's [29]. Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}_+^d$ described by the following stochastic differential equation

$$dX(t) = b(X)\mathrm{d}t + \sum_{l=1}^{k}\sigma_l(X)\mathrm{d}B_l(t). \tag{5.1}$$

The diffusion matrix is defined by

$$A(X) = (a_{ij}(x)), a_{ij}(x) = \sum_{l=1}^{k}\sigma_l^i(x)\sigma_l^j(x). \tag{5.2}$$

Define the differential operator $L$ associated with Eq (5.1) by

$$L = \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x} + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}. \tag{5.3}$$

**Lemma 1.** *( [29]) Assume that there exists a bounded domain $U \subset \mathbb{R}^d_+$ with regular boundary such that its closure $\overline{U} \subset \mathbb{R}^d_+$, having the following properties:*

*(i) there is a positive number M such that $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j > M|\xi|^2$ for $x \in U$ and $\xi \in \mathbb{R}^d_+$;*

*(ii) there exists a nonnegative $C^2$-function V such that LV is negative for any $x \in \mathbb{R}^d_+ \setminus U$.*

*Then the Markov process X(t) has a unique stationary distribution $\pi(\cdot)$.*

*The existence of a stationary distribution can be regarded as the stochastic weak stability of the model, which means that the diseases will prevail in the long term. Notice that R(t) is independent of S(t), I(t), and Q(t), then system (1.2) has the following form by removing R(t)*

$$\begin{cases} \mathrm{d}S(t) = (A - \mu S(t) - \frac{\beta S(t)I(t)}{1+\alpha I^2(t)})\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1(t), \\ \mathrm{d}I(t) = (\frac{\beta S(t)I(t)}{1+\alpha I^2(t)} - (\varepsilon + \mu + \gamma)I(t))\mathrm{d}t + \sigma_2 I(t)\mathrm{d}B_2(t) - \sigma_5 I(t)\mathrm{d}B_5(t), \\ \mathrm{d}Q(t) = (\gamma I(t) - (\mu + \xi)Q(t))\mathrm{d}t + \sigma_3 Q(t)\mathrm{d}B_3(t) + \sigma_5 I(t)\mathrm{d}B_5(t). \end{cases} \tag{5.4}$$

*Here we only consider the dynamics of system (5.4) and establish the following theorem.*

**Theorem 4.** *Assume that $\mathfrak{R}_0 > 1$ and the following conditions hold*

$$0 < \mathbb{F} < \min\{m_1(S^*)^2, m_2(I^*)^2, m_3(Q^*)^2\}, \tag{5.5}$$

*where*

$$\begin{cases} m_1 = \mu - \sigma_1^2, \\ m_2 = \varepsilon + \mu + \gamma - \frac{\gamma^2}{2} - \sigma_2^2 - 2\sigma_5^2, \\ m_3 = \mu + \xi - \frac{1}{2} - \sigma_3^2, \end{cases} \tag{5.6}$$

$$\mathbb{F} = \sigma_1^2(S^*)^2 + (\sigma_2^2 + 2\sigma_5^2)(I^*)^2 + \sigma_3^2(Q^*)^2 + \frac{(2\mu + \varepsilon + \gamma)(1 + \alpha(I^*)^2)}{2\beta} I^*(\sigma_2^2 + \sigma_5^2) \tag{5.7}$$

*are all positive constants.*

*Then for any initial value $(S(0), I(0), Q(0)) \in R^3_+$, there is a stationary distribution $\pi(\cdot)$ for system (5.4). Especially, we have*

$$\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t [m_1(S(u) - S^*)^2 + m_2(I(u) - I^*)^2 + m_3(Q(u) - Q^*)^2]\mathrm{d}u \le \mathbb{F}, \tag{5.8}$$

*where $E^* = (S^*, I^*, Q^*)$ is the unique endemic equilibrium of (1.1).*

*Proof.* To prove the existence of stationary distribution of model (5.4), we need to verify Lemma 1. We can rewrite system (5.4) as the following form

$$\begin{aligned} \mathrm{d}\begin{pmatrix} S(t) \\ I(t) \\ Q(t) \end{pmatrix} &= \begin{pmatrix} A - \mu S(t) - \frac{\beta S(t)I(t)}{1+\alpha I^2(t)} \\ \frac{\beta S(t)I(t)}{1+\alpha I^2(t)} - (\varepsilon + \mu + \gamma)I(t) \\ \gamma I(t) - (\mu + \xi)Q(t) \end{pmatrix}\mathrm{dt} + \begin{pmatrix} \sigma_1 S(t) \\ 0 \\ 0 \end{pmatrix}\mathrm{dB_1(t)} \\ &+ \begin{pmatrix} 0 \\ \sigma_2 I(t) \\ 0 \end{pmatrix}\mathrm{dB_2(t)} + \begin{pmatrix} 0 \\ 0 \\ \sigma_3 Q(t) \end{pmatrix}\mathrm{dB_3(t)} + \begin{pmatrix} 0 \\ -\sigma_5 I(t) \\ \sigma_5 I(t) \end{pmatrix}\mathrm{dB_5(t)}. \end{aligned} \tag{5.9}$$

Here the diffusion matrix is

$$A(X) = \begin{pmatrix} \sigma_1^2 S^2 & 0 & 0 \\ 0 & (\sigma_2^2 + \sigma_5^2)I^2 & -\sigma_5^2 I^2 \\ 0 & -\sigma_5^2 I^2 & \sigma_3^2 Q^2 + \sigma_5^2 I^2 \end{pmatrix}, \tag{5.10}$$

There is a positive number $M = \min\{\sigma_1^2 S^2, \sigma_2^2 I^2, \sigma_3^2 Q^2, (S, I, Q) \in \overline{U}\}$ such that for all $(S, I, Q) \in \overline{U}$ and $\xi \in R^3$, we have

$$\begin{aligned} \sum_{i,j=1}^{3} a_{ij}(x)\xi_i\xi_j &= \sigma_1^2 S^2\xi_1^2 + (\sigma_2^2 + \sigma_5^2)I^2\xi_2^2 \\ &+(\sigma_3^2 Q^2 + \sigma_5^2 I^2)\xi_3^2 - 2\sigma_5^2 I^2\xi_2\xi_3 \\ &= \sigma_1^2 S^2\xi_1^2 + \sigma_2^2 I^2\xi_2^2 + \sigma_3^2 Q^2\xi_3^2 + \sigma_5^2 I^2(\xi_2 - \xi_3)^2 \\ &\geq \sigma_1^2 S^2\xi_1^2 + \sigma_2^2 I^2\xi_2^2 + \sigma_3^2 Q^2\xi_3^2 \\ &\geq \min\{\sigma_1^2 S^2, \sigma_2^2 I^2, \sigma_3^2 Q^2\}|\xi|^2 = M|\xi|^2, \end{aligned} \tag{5.11}$$

which implies that Lemma 1 (i) is satisfied.

Since $\mathfrak{R}_0 > 1$, then there exists a unique positive equilibrium $E^* = (S^*, I^*, Q^*)$ of system (1.1) and the components satisfy

$$\begin{cases} A = \mu S^* + \frac{\beta S^* I^*}{1+\alpha(I^*)^2}, \\ \frac{\beta S^* I^*}{1+\alpha(I^*)^2} = (\varepsilon + \mu + \gamma)I^*, \\ \gamma I^* = (\mu + \xi)Q^*. \end{cases} \tag{5.12}$$

To verify Lemma 1 (ii), we define a $C^2$-function $V$

$$V(S(t), I(t), Q(t)) = V_1(t) + \frac{(2\mu + \varepsilon + \gamma)(1 + \alpha(I^*)^2)}{\beta}V_2(t) + V_3(t), \tag{5.13}$$

where

$$\begin{cases} V_1(t) = \frac{1}{2}(Q(t) - Q^*)^2 + \frac{1}{2}Q^*, \\ V_2(t) = I(t) - I^* \ln \frac{I(t)}{I^*}, \\ V_3(t) = \frac{1}{2}(S(t) - S^* + I(t) - I^*)^2 + \frac{1}{2}(S^* + I^*). \end{cases} \tag{5.14}$$

Using Itô's formula, we have

$$\mathrm{d}V(S(t), I(t), Q(t)) = \mathrm{d}V_1(t) + \frac{(2\mu + \varepsilon + \gamma)(1 + \alpha(I^*)^2)}{\beta}\mathrm{d}V_2(t) + \mathrm{d}V_3(t). \tag{5.15}$$

Making use of differential operator $L$ on $V_i$ (i=1,2,3), we have

$$\begin{aligned} LV_1(Q) &= (Q - Q^*)[\gamma I - (\mu + \xi)Q] + \tfrac{1}{2}\sigma_3^2 Q^2 + \tfrac{1}{2}\sigma_5^2 I^2 \\ &= (Q - Q^*)[\gamma(I - I^*) - (\mu + \xi)(Q - Q^*)] + \tfrac{1}{2}\sigma_3^2(Q - Q^* + Q^*)^2 + \tfrac{1}{2}\sigma_5^2(I - I^* + I^*)^2 \\ &= -(\mu + \xi)(Q - Q^*)^2 + \gamma(I - I^*)(Q - Q^*) + \tfrac{1}{2}\sigma_3^2(Q - Q^* + Q^*)^2 + \tfrac{1}{2}\sigma_5^2(I - I^* + I^*)^2 \\ &\leq -(\mu + \xi - \tfrac{1}{2} - \sigma_3^2)(Q - Q^*)^2 + (\tfrac{\gamma^2}{2} + \sigma_5^2)(I - I^*)^2 + \sigma_3^2(Q^*)^2 + \sigma_5^2(I^*)^2, \end{aligned} \tag{5.16}$$

where $\frac{1}{2}(a + b)^2 \leq a^2 + b^2$ for any $a, b \in \mathbb{R}$ and $\gamma(I - I^*)(Q - Q^*) \leq \frac{\gamma^2}{2}(I - I^*)^2 + \frac{1}{2}(Q - Q^*)^2$ are used.

Similarly

$$\begin{aligned} LV_3(S, I) &= (S - S^* + I - I^*)[A - \mu S - (\varepsilon + \mu + \gamma)I] + \tfrac{1}{2}(\sigma_1^2 S^2 + \sigma_2^2 I^2 + \sigma_5^2 I^2) \\ &= (S - S^* + I - I^*)[-\mu(S - S^*) - (\varepsilon + \mu + \gamma)(I - I^*)] \\ &+\tfrac{1}{2}\sigma_1^2(S - S^* + S^*)^2 + \tfrac{1}{2}(\sigma_1^2 + \sigma_5^2)(I - I^* + I^*)^2 \\ &\leq -(\mu - \sigma_1^2)(S - S^*)^2 + \sigma_1^2(S^*)^2 - (\varepsilon + \mu + \gamma - \sigma_2^2 - \sigma_5^2)(I - I^*)^2 \\ &-(2\mu + \varepsilon + \gamma)(S - S^*)(I - I^*) + (\sigma_2^2 + \sigma_5^2)(I^*)^2, \end{aligned} \tag{5.17}$$

$$\begin{aligned} LV_2(I) &= (I-I^*)[\tfrac{\beta S}{1+\alpha I^2} - (\varepsilon+\mu+\gamma)] + \tfrac{1}{2}(\sigma_2^2+\sigma_5^2)I^* \\ &= (I-I^*)(\tfrac{\beta S}{1+\alpha I^2} - \tfrac{\beta S^*}{1+\alpha (I^*)^2}) + \tfrac{1}{2}(\sigma_2^2+\sigma_5^2)I^* \\ &= (I-I^*)[-\tfrac{\beta S\alpha(I^2-(I^*)^2)}{(1+\alpha I^2)(1+\alpha(I^*)^2)} + \tfrac{\beta(S-S^*)}{1+\alpha(I^*)^2}] + \tfrac{1}{2}(\sigma_2^2+\sigma_5^2)I^* \\ &\le \tfrac{\beta(S-S^*)(I-I^*)}{1+\alpha(I^*)^2} + \tfrac{1}{2}(\sigma_2^2+\sigma_5^2)I^*, \end{aligned} \tag{5.18}$$

where $\frac{\beta S\alpha(I^2-(I^*)^2)(I-I^*)}{(1+\alpha I^2)(1+\alpha(I^*)^2)} = \frac{\beta S\alpha(I+I^*)(I-I^*)^2}{(1+\alpha I^2)(1+\alpha(I^*)^2)} \ge 0$.

Making use of the differential operator $L$ on (5.13) and substituting (5.16)–(5.18) into it lead to

$$\begin{aligned} LV(S,I,Q) &\le -(\mu+\xi-\tfrac{1}{2}-\sigma_3^2)(Q-Q^*)^2 + (\tfrac{\gamma^2}{2}+\sigma_5^2)(I-I^*)^2 + \sigma_3^2(Q^*)^2 + \sigma_5^2(I^*)^2 \\ &+\tfrac{(2\mu+\varepsilon+\gamma)(1+\alpha(I^*)^2)}{\beta}\tfrac{\beta(S-S^*)(I-I^*)}{1+\alpha(I^*)^2} + \tfrac{(2\mu+\varepsilon+\gamma)(1+\alpha(I^*)^2)}{2\beta}(\sigma_2^2+\sigma_5^2)I^* \\ &-(\mu-\sigma_1^2)(S-S^*)^2 + \sigma_1^2(S^*)^2 - (\varepsilon+\mu+\gamma-\sigma_2^2-\sigma_5^2)(I-I^*)^2 \\ &-(2\mu+\varepsilon+\gamma)(S-S^*)(I-I^*) + (\sigma_2^2+\sigma_5^2)(I^*)^2 \\ &= -(\mu-\sigma_1^2)(S-S^*)^2 - (\varepsilon+\mu+\gamma-\tfrac{\gamma^2}{2}-\sigma_2^2-2\sigma_5^2)(I-I^*)^2 \\ &-(\mu+\xi-\tfrac{1}{2}-\sigma_3^2)(Q-Q^*)^2 + \sigma_1^2(S^*)^2 + \sigma_3^2(Q^*)^2 + (\sigma_2^2+2\sigma_5^2)(I^*)^2 \\ &+\tfrac{(2\mu+\varepsilon+\gamma)(1+\alpha(I^*)^2)}{2\beta}(\sigma_2^2+\sigma_5^2)I^* \\ &\triangleq -m_1(S(t)-S^*)^2 - m_2(I(t)-I^*)^2 - m_3(Q(t)-Q^*)^2 + \mathbb{F} \end{aligned} \tag{5.19}$$

namely

$$LV(S,I,Q) \triangleq -m_1(S(t)-S^*)^2 - m_2(I(t)-I^*)^2 - m_3(Q(t)-Q^*)^2 + \mathbb{F}. \tag{5.20}$$

Therefore

$$\begin{aligned} \mathrm{d}V(t) \le\ & -m_1(S(t)-S^*)^2 - m_2(I(t)-I^*)^2 - m_3(Q(t)-Q^*)^2 + \mathbb{F} \\ & +(S(t)-S^*+I(t)-I^*)(\sigma_1 S(t)\mathrm{d}B_1(t) + \sigma_2 I(t)\mathrm{d}B_2(t) - \sigma_5 I(t)\mathrm{d}B_5(t)) \\ & +(I(t)-I^*)(\sigma_2\mathrm{d}B_2(t) - \sigma_5\mathrm{d}B_5(t)) + (Q(t)-Q^*)(\sigma_3 Q(t)\mathrm{d}B_3(t) + \sigma_5 I(t)\mathrm{d}B_5(t)) \end{aligned} \tag{5.21}$$

where $m_1, m_2, m_3$ and $\mathbb{F}$ are defined in Theorem 4 respectively. Integrating (5.21) from 0 to $t$

$$\begin{aligned} & V(S(t),I(t),Q(t)) - V(S(0),I(0),Q(0)) \\ & \le \int_0^t [-m_1(S(u)-S^*)^2 - m_2(I(u)-I^*)^2 - m_3(Q(u)-Q^*)^2]\mathrm{d}u + \mathbb{F}t + H(t) \end{aligned} \tag{5.22}$$

where $H(t)$ is a local martingale defined by

$$\begin{aligned} H(t) = & \int_0^t (S(u)-S^*+I(u)-I^*)(\sigma_1 S(u)\mathrm{d}B_1(u) + \sigma_2 I(u)\mathrm{d}B_2(u) - \sigma_5 I(u)\mathrm{d}B_5(u)) \\ & + \int_0^t (I(u)-I^*)\sigma_2\mathrm{d}B_2(u) + \int_0^t (I(u)-I^*)\sigma_5\mathrm{d}B_5(u) \\ & + \int_0^t (Q(u)-Q^*)\sigma_3 Q(u)\mathrm{d}B_3(u) + \int_0^t (I(u)-I^*)\sigma_5 I(u)\mathrm{d}B_5(u). \end{aligned} \tag{5.23}$$

Then taking the expectation on both sides of (5.22) yields

$$\begin{aligned} & \mathbb{E}V(S(t),I(t),Q(t)) - \mathbb{E}V(S(0),I(0),Q(0)) \le \\ & \mathbb{E}\int_0^t [-m_1(S(t)-S^*)^2 - m_2(I(t)-I^*)^2 - m_3(Q(t)-Q^*)^2]\mathrm{d}u + \mathbb{F}t. \end{aligned} \tag{5.24}$$

Hence we get

$$\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t [m_1(S(u)-S^*)^2 + m_2(I(u)-I^*)^2 + m_3(Q(u)-Q^*)^2]\mathrm{d}u \le \mathbb{F}. \tag{5.25}$$

Noting that the condition of theorem

$$0 < \mathbb{F} < \min\{m_1(S^*)^2, m_2(I^*)^2, m_3(Q^*)^2\}, \tag{5.26}$$

then the ellipsoid

$$m_1(S(u) - S^*)^2 + m_2(I(u) - I^*)^2 + m_3(Q(u) - Q^*)^2 = \mathbb{F} \tag{5.27}$$

lies entirely in $\mathbb{R}^3_+$. We can choose $U$ as any neighborhood of the ellipsoid such that $\overline{U} \subset \mathbb{R}^3_+$, where $\overline{U}$ is the closure of $U$. Therefore we can derive that $LV(S, I, Q) < 0$ for any $(S, I, Q) \in \mathbb{R}^3_+ \setminus U$, which shows that the condition (ii) in Lemma 1 also holds. The proof is thus completed. □

## 6. Numerical simulations

Numerical simulations are presented for supporting our theoretical findings of model (1.2) through the Milstein method, which can be found in [30].

In deterministic system (1.1), the value of the basic reproduction number $\mathfrak{R}_0$ determines persistence or extinction of the disease. If $\mathfrak{R}_0 < 1$, the disease will die out, while $\mathfrak{R}_0 > 1$, the disease will be persistent. With the help of numerical simulations, it's pretty straightforward to see the property can be changed by stochastic perturbations.

(1) We firstly assume that $A = 2, \beta = 0.08, \mu = 0.1, \epsilon = 0.1, \xi = 0.01, \alpha = 5$, $\sigma_1 = 0.2, \sigma_2 = 1.6, \sigma_3 = 0.1, \sigma_4 = 0.5$; the initial value is $(S(0), I(0), Q(0), R(0)) = (3, 3, 2.5, 2)$. Then choose different parameters $\sigma_5$ and $\gamma$ to observe their influence on the asymptotic behavior of solutions of the system (1.2). From Theorem 4.1, we know for any initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$, there is a positive solution $(S(t), I(t), Q(t), R(t))$ of system (1.2) which has the following property $\lim_{t\to\infty} \sup \frac{\ln I(t)}{t} \leq (\varepsilon + \mu + \gamma + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_5^2)(\mathfrak{R}_0^s - 1)$, where $\mathfrak{R}_0^s = \frac{\beta A}{\mu(\varepsilon+\mu+\gamma+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_5^2)}$. If $\mathfrak{R}_0^s < 1$, then the disease $I(t)$ goes to extinction exponentially with probability 1, (See Figures 1 and 2 ).

Case 1: Let $\sigma_5 = 0.5$. When $\gamma = 0.13$, then $\mathfrak{R}_0 \approx 4.848 > 1$, $\mathfrak{R}_0^s \approx 0.922 < 1$; when $\gamma = 1.38$, then $\mathfrak{R}_0 \approx 1.013 > 1$, $\mathfrak{R}_0^s \approx 0.536 < 1$. With other parameters unchanged, the increase of the quarantine rate $\gamma$ accelerates the velocity of extinction. we can see Figure 1 (a)$\gamma$=0.13, (b)$\gamma$=1.38.

Case 2: Let $\gamma = 0.15$. We presumed the coefficient of quarantine $\gamma$ is subject to the environment white noise, namely $\gamma \to \gamma + \sigma_5 \dot{B}_5(t)$ . With other parameters unchanged, the higher the intensities of the white noise $\sigma_5$, the faster the velocity of disease extinction. We can see Figure 2 (a)$\sigma_5$=0.5, (b)$\sigma_5$=2.

(2) We know that the system (1.2) has a unique stationary distribution under some conditions by Theorem 5.1, which means that the diseases will prevail in the long term. We choose $A = 2, \beta = 0.08, \mu = 0.1, \epsilon = 0.1, \xi = 0.41, \gamma = 0.15, \alpha = 5$, $\sigma_1 = 0.3, \sigma_2 = 0.3, \sigma_3 = 0.09, \sigma_4 = 0.2, \sigma_5 = 0.05$. The system (1.2) has a unique stationary distribution with smaller white noise for different initial values by Figure 3(a) $(S(0), I(0), Q(0), R(0)) = (0.3, 0.3, 0.2, 0.2)$, (b) $(S(0), I(0), Q(0), R(0)) = (0.5, 0.6, 0.4, 0.4)$.

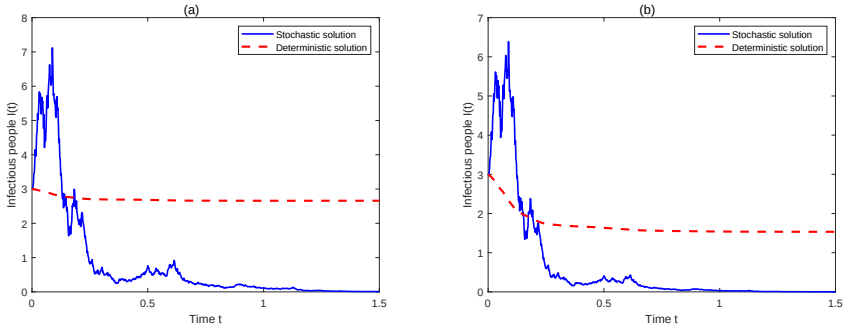


**Figure 1.** The increase of the coefficient of quarantine $\gamma$ accelerates the velocity of extinction. (a)$\gamma$=0.13, (b)$\gamma$=1.38.

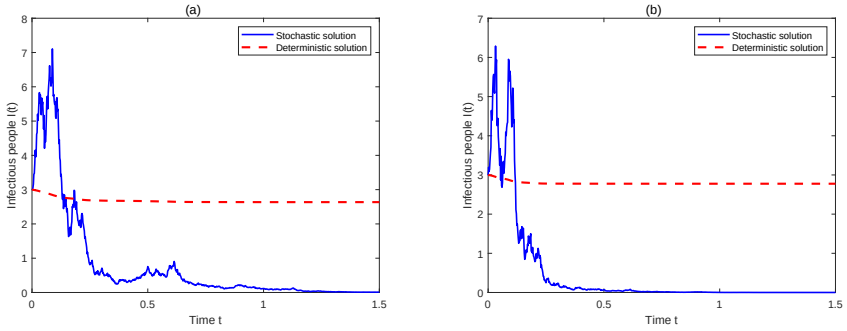


**Figure 2.** The higher intensities of the white noise $\sigma_5$, the faster the velocity of disease extinction. (a)$\sigma_5$=0.5, (b)$\sigma_5$=2.

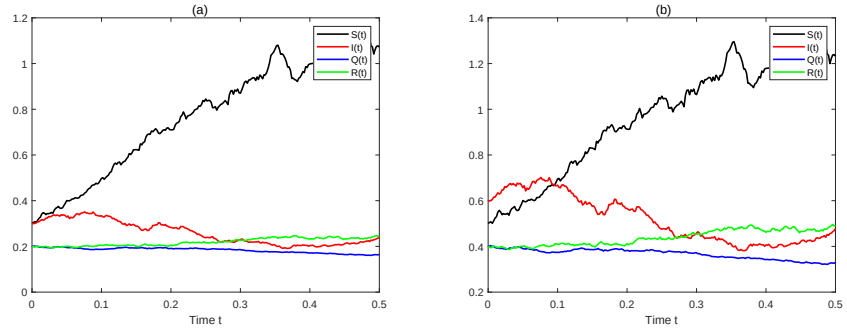


**Figure 3.** The system (1.2) has a unique stationary distribution with smaller white noise for different initial values. (a) $(S(0), I(0), Q(0), R(0)) = (0.3, 0.3, 0.2, 0.2)$, (b) $(S(0), I(0), Q(0), R(0)) = (0.5, 0.6, 0.4, 0.4)$.

## 7. Conclusions

In this paper, we investigated a stochastic epidemic model with quarantine and non-monotone incidence. For the deterministic system (1.1), if $\mathfrak{R}_0 < 1$, the disease will die out, while $\mathfrak{R}_0 > 1$, the

disease will be persistent. We have proved that the property of the solutions of the system can be changed by the stochastic perturbations. Furthermore, we assumed that the coefficient of quarantine $\gamma$ is subject to the environment white noise, namely $\gamma \to \gamma + \sigma_5 \dot{B}_5(t)$. With the help of numerical simulations, we can see that the higher intensities of the white noise $\sigma_5$ or the bigger of the quarantine rate $\gamma$ can accelerate the extinction of the disease. This theoretically explains the significance of quarantine strength (or isolation measures) when an epidemic erupts.

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## Conflict of interest

The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

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