AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 13219–13240.
DOI:10.3934/math.2023668
Received: 14 February 2023
Revised: 17 March 2023
Accepted: 22 March 2023
Published: 03 April 2023

Research article

Results on monochromatic vertex disconnection of graphs

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Abstract: Let G be a vertex-colored graph. A vertex cut S of G is called a *monochromatic vertex cut* if the vertices of S are colored with the same color. A graph G is *monochromatically vertex-disconnected* if any two nonadjacent vertices of G have a monochromatic vertex cut separating them. The *monochromatic vertex disconnection number* of G, denoted by $mvd(G)$, is the maximum number of colors that are used to make G monochromatically vertex-disconnected. In this paper, the connection between the graph parameters are studied: $mvd(G)$, connectivity and block decomposition. We determine the value of $mvd(G)$ for some well-known graphs, and then characterize G when $n-5 \leq mvd(G) \leq n$ and all blocks of G are minimally 2-connected triangle-free graphs. We obtain the maximum size of a graph G with $mvd(G) = k$ for any k. Finally, we study the Erdős-Gallai-type results for $mvd(G)$, and completely solve them.

Keywords: monochromatic vertex cut; monochromatic vertex disconnection number; connectivity; block; Erdős-Gallai-type problems
Mathematics Subject Classification: 05C15, 05C35, 05C40

1. Introduction

Connectivity is perhaps the most fundamental graph theoretic subject, in both the combinatorial and network science senses. To expand its application area, connectivity is strengthened through tasks such as requiring graph coloring, Hamiltonicity [13] and conditional connectivity [11].

In 2008, Chartrand et al. [5, 6] introduced an interesting way, i.e., the rainbow k-connection, to strengthen conventional connectivity. An edge-colored graph G is called rainbow k-connected if any pair of vertices are connected by k internally vertex-disjoint rainbow paths, where a rainbow path is a path with edges that all carry different colors. This concept comes from the communication of information between agencies of the government, and it is also applied to communication networks. More developments for variants of rainbow k-connectivity of different graph families can be found, e.g., in the survey [18].

Moving away from deterministic graphs, similar results have also been investigated for random graph models (see [7, 22]). In particular, a multiplex approach to cope with rainbow-related concepts has recently attracted increasing research attention in random settings. For example, Shang [23] approached rainbow k-connectivity by considering an alternative random network with multiple layers. Meanwhile, many colored versions of connectivity parameters have been introduced in recent years. For example, the monochromatic connection introduced by Caro and Yuster [3] in 2011 is defined from the monochromatic version; the monochromatic vertex connection introduced by Cai, Li and Wu [4] in 2018 is defined from the vertex version. For more results, we refer the reader to [10, 14, 19, 23].

There are two ways to study the connectivity, one using paths and the other using vertex cuts. These concepts mentioned above use paths, so it is natural to consider monochromatic vertex cuts. Let G be a vertex-colored graph. A vertex cut S is called a *monochromatic vertex cut* if the vertices of S are colored with the same color, and a *monochromatic x-y vertex cut* is a monochromatic vertex cut that separates x and y. Obviously, if x is adjacent to y, there is no x-y vertex cut, so we only need to consider nonadjacent vertices in the sequel. Then, G is called *monochromatically vertex-disconnected* if any two nonadjacent vertices of G have a monochromatic vertex cut separating them; the corresponding coloring is called *monochromatic vertex-disconnection coloring* (*MVD-coloring* for short). The *monochromatic vertex disconnection number* of G, denoted by $mvd(G)$, is the maximum number of colors that are used to make G monochromatically vertex-disconnected. An MVD-coloring with $mvd(G)$ colors is called an *mvd-coloring* of G.

In addition to being a natural combinatorial measure, our parameter can also be applied to communication networks. Suppose that G represents a network (e.g., a cellular network) where messages can be transmitted between any two vertices. To intercept messages (e.g., to prevent the transmission of error messages), each vertex is equipped with an interceptor that requires a fixed password (color) to be turned on. There is a fixed interception passphrase between any two different vertices. Entering this passphrase at the vertex cut where the password matches will turn on these interceptors and intercept the message between the two vertices. To enhance system security, the number of passwords should be as large as possible. This number is precisely $mvd(G)$.

In this paper, the connection between graph parameters are studied: $mvd(G)$ and connectivity $\kappa(G)$. We obtain the following results in Section 2:

Theorem 1.1. *If G is a connected non-complete graph, then $1 \le mvd(G) \le n-\kappa^+(G)+1 \le n-\kappa(G)+1$. The upper bound is tight.*

Further, for the minimally 2-connected graph of order $n \ge 4$, we give a new upper bound:

Theorem 1.2. *If G is a minimally 2-connected graph of order $n \ge 4$, then $mvd(G) \le \left\lfloor \frac{n}{2} \right\rfloor$. The bound is tight.*

For the graph G with $\kappa(G) = 1$, mvd-coloring, a global property of G, is transformed into a local property of each block, which greatly simplifies the original problem:

Theorem 1.3. *If $\kappa(G) = 1$ and G has r blocks $B_1, \ldots, B_r$, then $mvd(G) = (\sum_{i=1}^{r} mvd(B_i)) - r + 1$.*

In Section 3, we focus on the value of $mvd(G)$ for some well-known graphs with $\kappa(G) \ge 2$ and obtain several classes of graphs with $mvd(G) = k$, where $k \in \{1, 2, n\}$. Moreover, we completely characterize G when $n - 5 \le mvd(G) \le n$ and all blocks of G are minimally 2-connected triangle-free graphs.

The Erdős-Gallai theorem [8] originated in 1959 and is an interesting problem in extremal graph theory, where the Erdős-Gallai-type result aims at determining the maximum or minimum value of a graph parameter with some given properties. For a graph parameter, it is always interesting and challenging to get the Erdős-Gallai-type results; see [1, 9, 15, 16, 20] for more such results on various kinds of graph parameters. In Section 4, we study two kinds of Erdős-Gallai-type problems for our parameter $mvd(G)$.

Problem I. Given two positive integers n and k with $1 \leq k \leq n$, compute the minimum integer $f_v(n,k)$ such that, for any graph G of order n, if $|E(G)| \geq f_v(n,k)$, then $mvd(G) \geq k$.

Problem II. Given two positive integers n and k with $1 \leq k \leq n$, compute the maximum integer $g_v(n,k)$ such that, for any graph G of order n, if $|E(G)| \leq g_v(n,k)$, then $mvd(G) \leq k$.

By Theorem 1.3, for any tree G of order n, we have $mvd(G) = n$, so $g_v(n,k)$ does not exist for $1 \leq k \leq n-1$. Since $mvd(K_n) = n$, where K_n is a complete graph, $g_v(n,n) = \frac{n(n-1)}{2}$. The result of Problem I is shown below.

Theorem 1.4. *Given two positive integers n and k with $n \geq 5$ and $1 \leq k \leq n$,*

$$f_v(n,k) = \begin{cases} n-1, & k = 1, \\ \frac{n(n-1)}{2} - 1, & 2 \leq k \leq 3, \\ \frac{n(n-1)}{2}, & 4 \leq k \leq n. \end{cases}$$

Moreover, we obtain the maximum size of G with $mvd(G) = k$ for any k.

Theorem 1.5. *Given two positive integers n and k with $n > 4$ and $1 \leq k \leq n$, the maximum size of a connected graph G of order n with $mvd(G) = k$ is*

$$|E(G)|_{max} = \begin{cases} \frac{n(n-1)}{2} - 2, & k = 1 \text{ and } n \geq 5, \\ 7, & k = 2 \text{ and } n = 5, \\ \frac{n(n-1)}{2} - 4, & k = 2 \text{ and } n \geq 6, \\ \frac{n(n-1)}{2} - k + 2, & 3 \leq k \leq n-1, \\ \frac{n(n-1)}{2}, & k = n. \end{cases}$$

2. Some basic results

All graphs considered in this paper are simple, connected, finite and undirected. We follow the terminology and notation of Bondy and Murty [2]. Let $n = |G|$ be the order of G, and let $|E(G)|$ be the size of G. For $D \subseteq E(G)$, $G - D$ is the graph obtained by removing D from G. For $S \subseteq V(G)$, $G - S$ is the graph obtained by removing S and the edges incident to the vertices of S from G. We use $[r]$ to denote the set $\{1, 2, \ldots, r\}$. For a vertex-coloring τ of G, $\tau(v)$ is the color of vertex v, $\tau(G)$ is the set of colors used in G and $|\tau(G)|$ is the number of colors in $\tau(G)$. If H is a subgraph of G, then the part of the coloring of τ on H is called *τ-restricted* on H. To show the connection between $mvd(G)$ and connectivity $\kappa(G)$, we need the following lemma:

Lemma 2.1. *If τ is an MVD-coloring of G, then τ restricted on $G[S]$ is also an MVD-coloring of $G[S]$.*

Proof. Let the coloring of τ restricted on $G[S]$ be denoted as τ', and let x and y be two nonadjacent vertices of $G[S]$. If D is a monochromatic x-y vertex cut of G, then $D' = D \cap V(G[S])$ is a monochromatic x-y vertex cut in $G[S]$. Otherwise, if there is an x, y-path P in $G[S] - D'$, then P is also in $G - D$, which is a contradiction. Thus, τ' is an MVD-coloring of $G[S]$. □

Proof of Theorem 1.1. G is a non-complete graph and x and y are two nonadjacent vertices of G. Let $\kappa(x, y)$ be the minimum size of an x-y vertex cut, and let $\kappa^+(G)$ be the upper bound of the function $\kappa(x, y)$. Obviously, $mvd(G) \geq 1$. For the upper bound, assume that S is a monochromatic x-y vertex cut. Therefore, $mvd(G) \leq 1+(n-|S|) \leq n-\kappa(x, y)+1$. Thus, $mvd(G) \leq n-\kappa^+(G)+1 \leq n-\kappa(G)+1$. □

Tight example 1: Let G be a graph obtained by adding k edges to K_{n-1} from a vertex v outside K_{n-1}, where $k \in [n-2]$. Clearly, $\kappa(G) = k$. Define a vertex-coloring τ of G: $V(G) \to [n-k+1]$ such that $\tau(G-N(v)) \to [n-k]$ and $\tau(N(v)) = n-k+1$. If x and y are two nonadjacent vertices in G, then either x or y is v, i.e., $v = x$. Since $\tau(N(v)) = n-k+1$, $N(v)$ is a monochromatic v-y vertex cut and τ is a MVD-coloring of G. So, $mvd(G) \geq n-k+1$. It follows that $mvd(G) = n-\kappa(G)+1$. The upper bound is tight.

A graph G is *minimally* 2*-connected* (*minimal block*) if G is 2-connected but $G - \{e\}$ is not 2-connected for every $e \in E(G)$. Before proving Theorem 1.2, we need more preparation: a nest sequence of graphs is a sequence $G_0, G_1, \ldots, G_t$ of graphs such that $G_i \subset G_{i+1}, 0 \leq i < t$; an *ear decomposition* of a 2-connected graph G is a nest sequence $G_0, G_1, \ldots, G_t$ of 2-connected subgraphs of G satisfying the following conditions: (i) G_0 is a cycle of G; (ii) $G_{i+1} = G_i \cup P_i$, where P_i is an ear of G_i, $0 \leq i < t$; (iii) $G_t = G$.

Lemma 2.2. *[17] Let G be a minimally* 2*-connected graph, and G is not a cycle. Then, G has an ear decomposition* $G_0, G_1, \ldots, G_t$ $(t \geq 1)$ *satisfying the following conditions:*

(i) $G_{i+1} = G_i \cup P_i$ $(0 \leq i < t)$*, where* P_i *is an ear of* G_i *in G and at least one vertex of* P_i *has degree two in G;*

(ii) each of the two internally disjoint paths in G_0 *between the endpoints of* P_0 *has at least one vertex with degree two in G.*

Lemma 2.3. *[21] If G is a minimally* 2*-connected graph of order* $n \geq 4$*, then G contains no triangles.*

Lemma 2.4. *If G is a cycle of order* $n \geq 4$*, then* $mvd(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Let $G = v_1e_1 \ldots v_ne_nv_1$. Define a vertex-coloring $\tau : V(G) \to [\left\lfloor \frac{n}{2} \right\rfloor]$ such that, if $j \equiv i$ (*mod* $\left\lfloor \frac{n}{2} \right\rfloor$), then $\tau(v_j) = i$, where $i \in [\left\lfloor \frac{n}{2} \right\rfloor]$ and $j \in [n]$. It can be shown that, for any two nonadjacent vertices x and y, there is a monochromatic x-y vertex cut. So, τ is an MVD-coloring and $mvd(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$.

If, on the contrary, for $n \geq 4$, $mvd(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$ and τ is an mvd-coloring of G with $|\tau(G)| = mvd(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$, then there must be a color that colors only one vertex v_i; otherwise, we have $V(G) \geq 2|\tau(G)| \geq 2\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \geq n+1$. Since $\kappa(G) = 2$, the monochromatic v_{i-1}-v_{i+1} vertex cut must contain v_i and some vertex v_j in $G - \{v_{i-1}, v_i, v_{i+1}\}$. However, $\tau(v_i) \neq \tau(v_j)$, which contradicts the fact that τ is an mvd-coloring. Thus, $mvd(G) = \left\lfloor \frac{n}{2} \right\rfloor$. □

Proof of Theorem 1.2. We make the following claim:

Claim 1: If G is a 2-connected triangle-free graph and every ear P_i ($0 \le i < t$) has internal vertices, then $mvd(G) \le \left\lfloor \frac{n}{2} \right\rfloor$.

Let $F = \{G_0, G_1, \ldots, G_t\}$ be an ear decomposition of G. We use induction on $|F|$. By Lemma 2.4, the claim holds for $|F| = 1$. If $|F| = t + 1 > 1$, let τ be an mvd-coloring of G. Since $|P_{t-1}| \ge 3$, G_{t-1} is a connected vertex-induced subgraph of G. By Lemma 2.1, τ restricted on G_{t-1} is an MVD-coloring of G_{t-1}. By induction, we have

$$|\tau(G_{t-1})| \le mvd(G_{t-1}) \le \left\lfloor \frac{|G_{t-1}|}{2} \right\rfloor = \left\lfloor \frac{n - |P_{t-1}| + 2}{2} \right\rfloor.$$

Suppose that the endpoints of P_{t-1} are a and b and L is the shortest a, b-path in G_{t-1}. Since P_{t-1} is the last ear, cycle $C = L \cup P_{t-1}$ is a connected vertex-induced subgraph of G and τ restricted on C is an MVD-coloring of C. Then, there are at most $|P_{t-1}|-2$ vertices that are assigned colors in $\tau(G)-\tau(G_{t-1})$. Since $|C| \ge 4$, each color in $\tau(G) - \tau(G_{t-1})$ colors at least two internal vertices of P_{t-1}. Otherwise, if $j \in \tau(G) - \tau(G_{t-1})$ and only colors one internal vertex of P_{t-1}, say x_j, then x_{j-1} and x_{j+1} are two nonadjacent vertices of G and the monochromatic x_{j-1}-x_{j+1} vertex cut must contain x_j and another vertex, which is a contradiction. Then, $|\tau(G) - \tau(G_{t-1})| \le \left\lfloor \frac{|P_{t-1}|-2}{2} \right\rfloor$. So,

$$mvd(G) = |\tau(G)| = |\tau(G_{t-1})| + |\tau(G) - \tau(G_{t-1})| \le \left\lfloor \frac{n - |P_{t-1}| + 2}{2} \right\rfloor + \left\lfloor \frac{|P_{t-1}| - 2}{2} \right\rfloor \le \left\lfloor \frac{n}{2} \right\rfloor.$$

Above all, $mvd(G) \le \left\lfloor \frac{n}{2} \right\rfloor$.

By Lemmas 2.2 and 2.3, if G is not a cycle, then there is an ear decomposition satisfying the conditions in Claim 1. Thus, $mvd(G) \le \left\lfloor \frac{n}{2} \right\rfloor$. □

Tight example 2: If G is a cycle, then, according to Lemma 2.4, we have $mvd(G) = \left\lfloor \frac{n}{2} \right\rfloor$. The bound is tight.

A *block* is a maximal connected subgraph of G that has no cut-vertex. Every block of a nontrivial connected graph is either K_2 or a 2-connected subgraph, called *trivial* and *nontrivial*, respectively. To show the connection between $mvd(G)$ and block decomposition, we need the following result:

Theorem 2.1. *Let G be a connected graph with r blocks. Then, τ is an mvd-coloring of G if and only if τ restricted on each block is an mvd-coloring of each block and the colors of different blocks are different except at the cut vertices.*

Proof. Let $\{B_1, \ldots, B_r\}$ be the block decomposition of G. Let τ be a vertex-coloring of G, and let τ_i be the coloring of τ restricted on B_i, $i \in [r]$. If G has no cut vertex, then $G = B_1$ and the result follows. Now, we assume that G has at least one cut vertex:

Claim 1: τ is an MVD-coloring of G if and only if τ restricted on each block is an MVD-coloring of each block.

Since each block is a vertex-induced subgraph of G, the necessity is obvious by Lemma 2.1. Now, let τ_i be an MVD-coloring of B_i, where $i \in [r]$. For any two nonadjacent vertices x and y in G, if there is a block, say B_1, which contains both x and y, then any monochromatic x-y vertex cut in B_1 is also a monochromatic x-y vertex cut in G. If x and y are in different blocks, then there is exactly one internally disjoint x, y-path containing at least one cut vertex v. The vertex v is a monochromatic x-y vertex cut in G. Thus, τ is an MVD-coloring of G.

Next, let τ_i be an *mvd*-coloring of B_i satisfying that, if $B_i \cap B_j = v$, then $\tau_i(B_i) \cap \tau_j(B_j) = \tau_i(v) = \tau_j(v)$, and if $B_i \cap B_j = \emptyset$, then $\tau_i(B_i) \cap \tau_j(B_j) = \emptyset$, where $i, j \in [r]$ and $i \neq j$. Then, τ is an *MVD*-coloring of G by Claim 1. We claim that τ is an *mvd*-coloring of G. Otherwise, there is an *mvd*-coloring τ' of G satisfying $|\tau'(G)| > |\tau(G)|$. Let the coloring of τ' restricted on B_i be τ'_i, which is an *MVD*-coloring of B_i by Claim 1. Then, for any $i \in [r]$, $|\tau'_i(B_i)| \leq |\tau_i(B_i)|$, which contradicts $|\tau'(G)| > |\tau(G)|$. Thus, τ is an *mvd*-coloring of G.

Now, we prove the necessity of the theorem. Let τ be an *mvd*-coloring of G. According to Claim 1, τ_i is an *MVD*-coloring of B_i, $i \in [r]$. Similar to the proof of Claim 1, it can be shown that, if $B_i \cap B_j = v$, then $\tau_i(B_i) \cap \tau_j(B_j) = \tau_i(v) = \tau_j(v)$, and if $B_i \cap B_j = \emptyset$, then $\tau_i(B_i) \cap \tau_j(B_j) = \emptyset$, where $i, j \in [r]$ and $i \neq j$. We only need to prove that τ_i is an *mvd*-coloring of B_i, $i \in [r]$. If τ_1 is not an *mvd*-coloring of B_1, then there must be an *mvd*-coloring τ'_1 of B_1 satisfying that all colors in $\tau'_1(B_1)$ are unused colors, except for those owned by cut vertices. It follows that $|\tau'_1(B_1)| > |\tau_1(B_1)|$ and $|\tau'(G)| > |\tau(G)|$. According to the sufficiency of Claim 1, τ' is an *MVD*-coloring of G, which contradicts the maximality of τ. □

Proof of Theorem 1.3. Let G be a connected graph with blocks $B_1, B_2, \ldots, B_r$, and let τ be an *mvd*-coloring of G. We use induction on r. The result holds for $r = 1$. If $r > 1$, then G is not 2-connected; we know that there is a block, say B_r, containing only one cut vertex, say v. Let $G' = G - (V(B_r) - \{v\})$. Then, G' is a connected graph with blocks $B_1, B_2, \ldots, B_{r-1}$. By Theorem 2.1, τ restricted on G' is an *mvd*-coloring of G', and, combined with the induction hypothesis, we have $|\tau(G')| = mvd(G') = (\sum_{i=1}^{r-1} mvd(B_i)) - (r-1) + 1$. According to Theorem 2.1, τ restricted on B_r is an *mvd*-coloring of B_r and $\tau(B_r) \cap \tau(G') = \tau(v)$. So, we deleted $|mvd(B_r)| - 1$ colors from $\tau(G)$ to obtain $\tau(G')$; $mvd(G)$ is as desired. □

3. Results for special graphs

If G and H are vertex-disjoint, then let $G \vee H$ denote the *join* of G and H, which is obtained from G and H by adding edges $\{xy : x \in V(G), y \in V(H)\}$. If C_{n-1} is a cycle of order $n-1$, then $W_n = C_{n-1} \vee K_1$ is called a *wheel graph*. We first show several classes of graphs with $mvd(G) = k$, where $k \in \{1, 2, n\}$.

Theorem 3.1. *If G is one of the following graphs, then $mvd(G) = 1$.*
(i) G is a wheel graph other than W_4;
(ii) $G = K_{n_1,\ldots,n_k}$ is a complete k-partite graph with n_k, $n_{k-1} \geq 2$ and $k > 2$.

(1) Let $G = W_n = C_{n-1} \vee K_1$, where cycle $C_{n-1} = v_1v_2 \ldots v_{n-1}v_1$ and $K_1 = v$. It is known that $mvd(W_4) = mvd(K_4) = 4$. We claim that $mvd(W_n) = 1$ for $n > 4$.

Let τ be an *mvd*-coloring of W_n, say $\tau(v) = 1$. Since $n > 4$, v_1 and v_3 are two nonadjacent vertices with three internally disjoint v_1, v_3-paths, namely, $v_1v_2v_3$, v_1vv_3 and $v_1v_{n-1}v_{n-2} \cdots v_4v_3$. Since $\kappa(W_n) = 3$, any monochromatic v_1-v_3 vertex cut must contain the vertex set $\{v_2, v, v_i\}$, where $v_i \in \{v_{n-1}, v_{n-2}, \ldots, v_4\}$, so $\tau(v_2) = 1$. Similarly, v_2 and v_4 are two nonadjacent vertices, and we get $\tau(v_3) = 1$. Repeat operations above until all vertices of C_{n-1} are colored, and we get $\tau(v) = \tau(v_1) = \tau(v_2) = \cdots = \tau(v_{n-1}) = 1$. Therefore, $mvd(W_n) = 1$ for $n > 4$.

(2) Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph of order n, where $k \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$. Let $V_1, V_2, \ldots, V_k$ be the vertex-partition sets of G with $|V_i| = n_i$, where $i \in [k]$. There are four cases below.

Case 1. $n_i = 1$ for $i \in [k]$; then, $mvd(G) = mvd(K_n) = n$.
Case 2. $n_i = 1$ for $i \in [k-1]$, and $n_k \geq 2$:

Define a vertex-coloring $\tau : V(G) \to [n-k+2]$ such that $\tau(V_k) \to [n-k+1]$ and $\tau(V_i) = n-k+2$ for $i \in [k-1]$. If x and y are two nonadjacent vertices in G, then $x,\ y \in V_k$ and $V(G) - V_k$ is a monochromatic x-y vertex cut in G. Thus, τ is an MVD-coloring of G and $mvd(G) \geq n-k+2$. On the other hand, since any two vertices in V_k have $k-1$ internally disjoint paths, according to Theorem 1.1, $mvd(G) \leq n - \kappa^+(G) + 1 = n-k+2$.
Case 3. $k > 2$ and $n_k \geq n_{k-1} \geq 2$:

x and y are nonadjacent. If $x,\ y \in V_{k-1}$, then any x-y vertex cut must contain $V(G) - V_{k-1}$. So, $V(G) - V_{k-1}$ are assigned the same color. Similarly, if $x,\ y \in V_k$, then $V(G) - V_k$ are assigned the same color. Since $k > 2$, the sets $V(G) - V_{k-1}$ and $V(G) - V_{k-1}$ intersect. Then, $mvd(G) = 1$.
Case 4. $k = 2$, $n_2 \geq n_1 \geq 2$:

Similarly, since $k = 2$, the sets $V(G) - V_1$ and $V(G) - V_2$ are disjoint; then, $mvd(K_{n_1,n_2}) = 2$.

The *Cartesian product* of G and H, written as $G \square H$, is the graph with the vertex set $V(G) \times V(H)$, specified by putting (u, v) adjacent to (u', v') if and only if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. If P_n is a path with order n, then $P_m \square P_n$ is called the *m-by-n grid*.

Theorem 3.2. *If G is one of the following graphs, then* $mvd(G) = 2$.

(i) $G = P_m \square P_n$ *is a nontrivial grid other than* $P_1 \square P_n$ *with* $n > 2$;

(ii) G is a Petersen graph.

Proof. **(1)** Let $G = P_m \square P_n$ and define $x_{i,j}$ to be the vertex in the i-th row and j-th column, where $i \in [m]$ and $j \in [n]$. It is known that $mvd(P_1 \square P_n) = mvd(P_n) = n$. Then, $mvd(P_1 \square P_2) = 2$ and $mvd(P_1 \square P_n) > 2$ for $n > 2$. We claim that $mvd(P_m \square P_n) = 2$ for $m, n \geq 2$.

Define a vertex-coloring τ of G: $V(G) \to [2]$ such that $\tau(x_{i,j}) = 1$ if $i + j$ is even and $\tau(x_{i,j}) = 2$ if $i + j$ is odd. For any vertex x in G, the set $N(x)$ is monochromatic. Thus, for any two nonadjacent vertices x and y in G, $N(x)$ is a monochromatic x-y vertex cut. So, $mvd(G) \geq 2$.

Now, we prove that $mvd(G) = 2$. Any MVD-coloring of a 4-cycle can have only two cases, where one is trivial and the other is to assign colors 1 and 2 to the four vertices of the 4-cycle alternately. Suppose that $mvd(G) > 2$ and τ is an mvd-coloring of G. By Lemma 2.1, τ restricted on each 4-cycle $G[x_{i,j}, x_{i,j+1}, x_{i+1,j+1}, x_{i+1,j}]$ is an MVD-coloring, given $1 \leq i < m, 1 \leq j < n$, which contradicts that $mvd(G) > 2$. Therefore, $mvd(P_m \square P_n) = 2$ for $m, n \geq 2$.

(2) Define a vertex-coloring τ of G: $V(G) \to [2]$ as shown in Figure 1(2). For any two nonadjacent vertices x and y, there is only one common neighbor, say z. Suppose that $\tau(z) = t$; it can be shown that the set of all vertices colored by t, except x and y, is a monochromatic x-y vertex cut. Thus, τ is an MVD-coloring of G and $mvd(G) \geq 2$. We prove that $mvd(G) = 2$ below.

Any MVD-coloring of C_5 can only be two cases, where one is trivial and the other is to assign colors 1 and 2 to the five vertices of C_5 alternately. At least two adjacent vertices in C_5 have the same color. Suppose that $mvd(G) > 2$ and τ is an mvd-coloring of G. Let the four 5-cycles of G be $G_1 = G[a, b, c, d, e], G_2 = G[c, d, i, f, h], G_3 = G[a, b, c, h, f]$ and $G_4 = G[f, h, j, g, i]$. Since τ restricted on G_1 is an MVD-coloring, there are two cases.

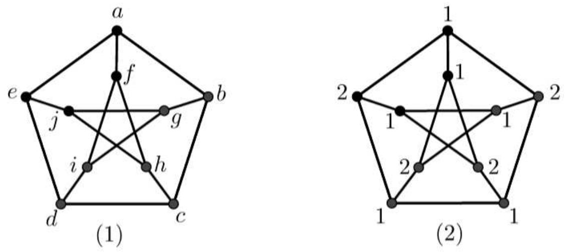

Figure 1. Vertex-coloring of Peterson graph.

Case 1. G_1 is colored nontrivially.

Suppose that $\tau(a) = \tau(c) = \tau(d) = 1$ and $\tau(b) = \tau(e) = 2$. Then, τ restricted on G_2 is an MVD-coloring. If G_2 is colored trivially, i.e., $\tau(f) = \tau(h) = \tau(i) = 1$, it is obvious that τ is not an MVD-coloring restricted on G_3. By Lemma 2.1, τ is not an MVD-coloring of G, which contradicts that τ is an *mvd*-coloring of G. If G_2 is colored nontrivially, i.e., $\tau(f) = 1$ and $\tau(h) = \tau(i) = 2$, then τ is a nontrivial MVD-coloring restricted on G_4 with $\tau(g) = \tau(j) = 1$, which contradicts that $mvd(G) > 2$.

Case 2. G_1 is colored trivially.

Suppose that $\tau(a) = \tau(b) = \tau(c) = \tau(d) = \tau(e) = 1$. Then, τ is a trivial MVD-coloring restricted on G_3 with $\tau(f) = \tau(h) = 1$. Since τ is an MVD-coloring restricted on G_4, $|\tau(G_4)| \leq 2$, which contradicts that $mvd(G) > 2$.

Above all, $mvd(G) = 2$. □

Theorem 3.3. *Let G be a connected graph of order n. Then, $mvd(G) = n$ if and only if each block of G is complete.*

Proof. Let $\{B_1, B_2, \ldots, B_r\}$ be a block decomposition of G. If B_i ($i \in [r]$) is complete, we define a coloring $\tau : V(G) \to [n]$ such that all vertices of G have different colors. By Theorem 2.1, τ is an *mvd*-coloring of G and $mvd(G) = n$. On the contrary, if $mvd(G) = n$, we define a coloring $\tau : V(G) \to [n]$ such that all vertices of G have different colors. By Theorem 2.1, $\tau(B_i)$ is an *mvd*-coloring of B_i. Then, B_i is complete. Otherwise, since B_i is 2-connected, by Theorem 1.1, $mvd(B_i) \leq |B_i| - \kappa(B_i) + 1 \leq |B_i| - 1$, which is a contradiction. □

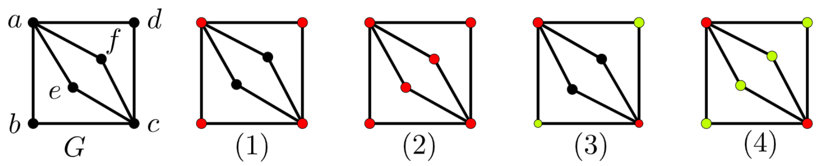

Figure 2. An example.

Now, we focus on minimally 2-connected graphs [12] of order 10 or less. As an example, see Figure 2; let the three 4-cycles of G be $G_1 = G[a, b, c, d], G_2 = G[a, b, c, e]$ and $G_3 = G[a, f, c, d]$. According to Lemma 2.1, if τ is an *mvd*-coloring of G, then $\tau(G_1)$ may be (1) or (3), i.e., an MVD-coloring of G_1. For Case (1), $\tau(G_2)$ and $\tau(G_3)$ must be (2). For Case (3), $\tau(G_2)$ and $\tau(G_3)$ must be (4).

It can be shown that both (2) and (4) are MVD-colorings of G and (4) is an mvd-coloring of G. Using the same method, after tedious calculations, the $mvd(G)$ of minimally 2-connected graph G is shown in *Table 1, and the mvd-coloring of G is shown in Appendix A (In fact, this method is applicable to many graphs with small order).

Table 1. $mvd(G)$ for minimally 2-connected graph G.

$mvd(G)$	$n \le 5$	n=6	n=7	n=8	n=9	n=10
5	-	-	-	-	-	$P(4,4)$
4	-	-	-	$P(3,3)$	$P(4,3)$ $P(5,1,1)$ $P(3,3,1)$	㊽-㊿ $P(5,1,1,1)$ $P(5,2,1)$ $P(4,2,2)$ $P(4,3,1)$ $P(3,3,2)$ $P(6,1,1)$ $P(3,3,1,1)$
3	K_3	$P(2,2)$	$P(3,2)$ $P(3,1,1)$	④ $P(3,2,1)$ $P(2,2,2)$ $P(4,1,1)$ $P(3,1,1,1)$	⑫-⑯ $P(3,2,1,1)$ $P(4,2,1)$ $P(3,4*1)$ $P(3,2,2)$ $P(4,1,1,1)$	㉘-㊼ $P(4,2,1,1)$ $P(3,2,2,1)$ $P(4*2)$ $P(4,4*1)$ $P(3,5*1)$ $P(3,2,3*1)$
2	C_4 K_2 $P(2,1)$ $P(1,1,1)$	$P(2,1,1)$ $P(1,1,1,1)$	① $P(5*1)$ $P(2,2,1)$ $P(2,1,1,1)$	①-③ $P(2,2,1,1)$ $P(2,4*1)$ $P(6*1)$	①-⑪ $P(7*1)$ $P(2,5*1)$ $P(2,2,2,1)$ $P(2,2,1,1,1)$	①-㉗ $P(2,2,4*1)$ $P(8*1)$ $P(2,6*1)$ $P(2,2,2,1,1)$

Finally, when $n-5 \le mvd(G) \le n$ and all blocks of the graph G are minimally 2-connected triangle-free graphs, we characterize G. We need the following lemma:

Lemma 3.1. *G is a connected graph of order n with r blocks, where t blocks are trivial. If all blocks are minimally 2-connected triangle-free graphs, then* $mvd(G) \le \left\lfloor \frac{n+2t-r+1}{2} \right\rfloor$.

Proof. Let $\{B_1, B_2, \ldots, B_r\}$ be the block decomposition of G. We claim that

$$n = (\sum_{i=1}^{r} |B_i|) - r + 1. \tag{3.1}$$

The proof proceeds by induction on r. The result holds for $r = 1$. If $r > 1$, then G is not 2-connected, we know that there is a block, say B_r, containing only one cut vertex, say v. Let $G' = G - (V(B_r) - \{v\})$. Then, G' is a connected graph with blocks $B_1, B_2, \ldots, B_{r-1}$. By the induction hypothesis, $|G'| = (\sum_{i=1}^{r-1} |B_i|) - (r-1) + 1$. Since we deleted $|B_r| - 1$ vertices from G to obtain G', the number of vertices in G is as desired.

Without loss of generality, let the trivial blocks be $B_1, \ldots, B_t$, and let the nontrivial blocks be $B_{t+1}, \ldots, B_r$. By Theorems 1.2 and 1.3, we have

$$\begin{aligned} mvd(G) &= (\sum_{i=1}^{t} mvd(B_i)) + (\sum_{i=t+1}^{r} mvd(B_i)) - r + 1 \le 2t + \left\lfloor \frac{|B_{t+1}|}{2} \right\rfloor + \ldots + \left\lfloor \frac{|B_r|}{2} \right\rfloor - r + 1 \\ &= \left\lfloor t + \frac{|B_{t+1}|}{2} \right\rfloor + \left\lfloor \frac{|B_{t+2}|}{2} \right\rfloor + \ldots + \left\lfloor \frac{|B_r|}{2} \right\rfloor + t - r + 1 \le \left\lfloor \frac{2t + |B_{t+1}| + \ldots + |B_r|}{2} \right\rfloor + t - r + 1 = I. \end{aligned}$$

*$P_{n_1}, \ldots, P_{n_k}$ are k disjoint paths with $|P_{n_i}| = n_i$. The first and the last vertices of p_i are denoted by $f(P_{n_i})$ and $l(P_{n_i})$. $P(n_1, \ldots, n_k)$ is the graph with the vertex set $\{\cup_{i\in[k]} V(P_{n_i})\} \cup \{u, v\}$ and edge set $\cup_{i\in[k]}[E(P_{n_i}) \cup \{f(P_{n_i})u, l(P_{n_i})v\}]$, where $u, v \notin \cup_{i\in[k]} V(P_{n_i})$. If $n_{i+1} = \cdots = n_{i+j}$, then $P(n_1, \ldots, n_k) = P(n_1, \ldots, n_i, j * n_{i+1}, n_{i+j+1}, \ldots, n_k)$.

Combined with Eq (3.1), we have

$$I = \left\lfloor \frac{n+r-1}{2} \right\rfloor + t - r + 1 = \left\lfloor \frac{n+2t-r+1}{2} \right\rfloor.$$

□

The *nontrivial block-induced subgraph* of G is the subgraph induced by all nontrivial blocks of G. Let $\mathcal{A}$ be a set of connected graphs whose nontrivial block-induced graph is exactly one of the graphs shown in Figure 3(a). Similarly, we define $\mathcal{B}$ and $\mathcal{C}$ according to Figure 3(b) and 3(c), respectively.

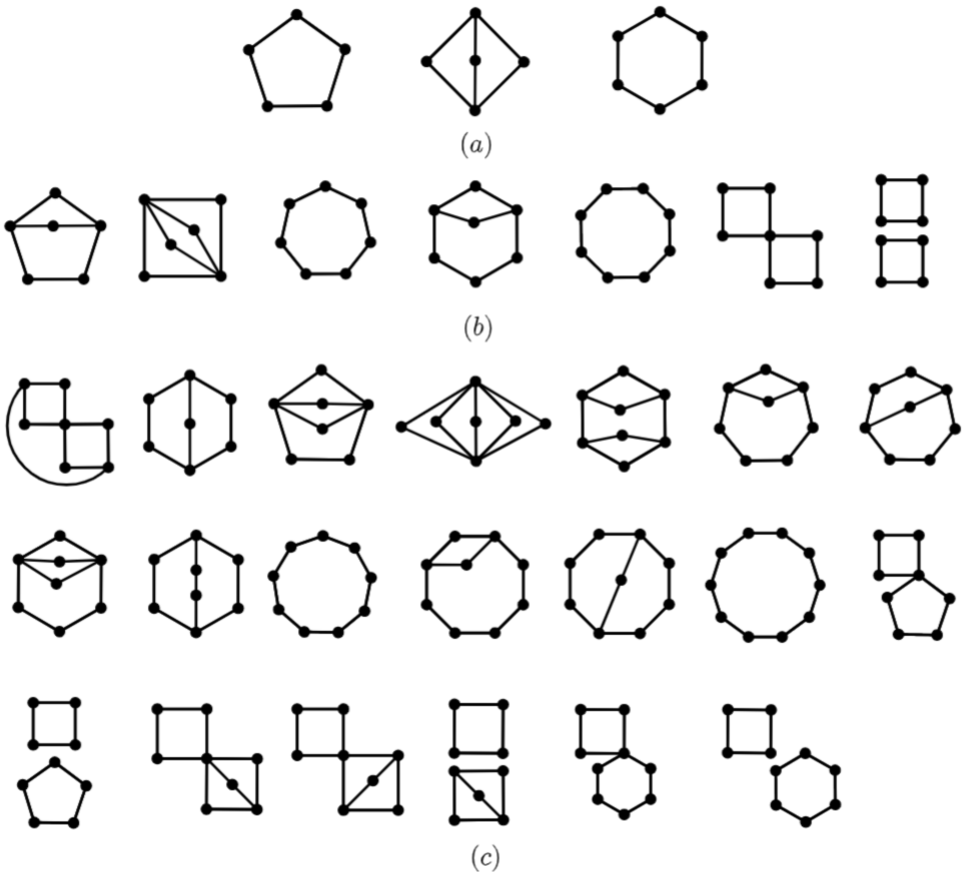

Figure 3. Results for Theorem 3.4.

Theorem 3.4. *For a connected graph G, if blocks in G are all minimally* 2*-connected triangle-free graphs, then*

$$mvd(G) = \begin{cases} n, & \Leftrightarrow G \text{ is a tree}, \\ n-1, & \text{no graph}, \\ n-2, & \Leftrightarrow G \text{ is a unicycle graph with cycle } C_4, \\ n-3, & \Leftrightarrow G \in \mathcal{A}, \\ n-4, & \Leftrightarrow G \in \mathcal{B}, \\ n-5, & \Leftrightarrow G \in \mathcal{C}. \end{cases}$$

Proof. By Table 1 and Theorems 1.3 and 2.1, it is easy to verify the sufficiency. Now, we prove the necessity. If $mvd(G) = n$, then B_i is complete by Theorem 3.3. Since G is triangle-free, G is a tree.

Now, suppose that $mvd(G) \geq n - 5$ and G has r blocks, of which t are trivial. By Lemma 3.1, $n - 5 \leq mvd(G) \leq \left\lfloor \frac{n+2t-r+1}{2} \right\rfloor$. There are two cases below.

Case 1. $n - r$ is even.

So, $2t \geq n + r - 10$. Since $t \leq r$, $r \geq n - 10$, then r may be $n - 2,\ n - 4,\ n - 6,\ n - 8$ or $n - 10$. We discuss them below.

(I) For $r = n - 2$, combined with $2t \geq n + r - 10$ and the fact that $r = t$ if and only if G is a tree (i.e., $r = n - 1$), we have that $n - 6 \leq t \leq n - 3$. Since G is triangle-free, when $t = n - 3, n - 4, n - 5$ or $n - 6$, respectively, we have $\sum_{i=1}^{r} |B_i| \geq 4 + 2(n - 3), 8 + 2(n - 4), 12 + 2(n - 5)$ or $16 + 2(n - 6)$, contradicting Eq (3.1). Note that, in other cases, we first use this method to determine the number of nontrivial blocks in G.

(II) For $r = n - 4$, similarly, we have that $n - 7 \leq t \leq n - 5$, and there is only one nontrivial block in G. By Eq (3.1), this nontrivial block is of order 5, i.e., $P(2, 1)$ or $P(1, 1, 1)$ (see Appendix A. 5 VERTEX). According to Table 1 and Theorems 1.3 and 2.1, $mvd(G) = n - 3$ in both cases.

(III) For $r = n - 6$, similarly, we have that $n - 8 \leq t \leq n - 7$. Since G is triangle-free, combined with Eq (3.1), we obtain the following.

If $t = n - 7$, then the order of this nontrivial block is 7, i.e., one of the graphs in Appendix A. 7 VERTEX.

If $t = n - 8$, then there are two nontrivial blocks B_i and B_j with $|B_i| + |B_j| = 9$, and the nontrivial block-induced subgraph of G is one of the graphs in Figure 4. Note that B_i and B_j may or may not be adjacent, and the figure only shows the case when B_i and B_j are not adjacent.

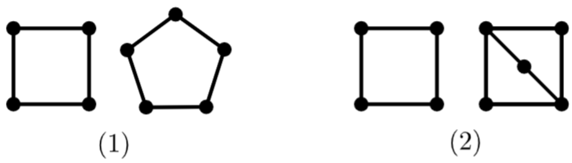

Figure 4. Case when $r = n - 6$, $t = n - 8$.

According to Table 1 and Theorems 1.3 and 2.1, $mvd(G) = n - 4$ when the nontrivial block-induced subgraph of G is $P(3, 2)$ or $P(3, 1, 1)$, and $mvd(G) = n - 5$ for the remaining cases.

(IV) For $r = n - 8$, similarly, we have that $t = n - 9$. By Eq (3.1), this nontrivial block is of order 9, i.e., one of the graphs in Appendix A. 9 VERTEX. According to Table 1 and Theorems 1.3 and 2.1, $mvd(G) = n - 5$ when the nontrivial block-induced subgraph of G is one among $\{P(4, 3), P(5, 1, 1), P(3, 3, 1)\}$, and $mvd(G) < n - 5$ for the remaining cases.

(V) For $r = n - 10$, similarly, we have that $n - 10 \leq t \leq n - 11$, which is a contradiction.

Case 2. $n - r$ is odd.

So, $2t \geq n + r - 11$. Since $t \leq r$, $r \geq n - 11$, then r may be $n - 1,\ n - 3,\ n - 5,\ n - 7,\ n - 9$ or $n - 11$. We discuss them below.

(I) For $r = n - 1$, G is a tree and $mvd(G) = n$.

(II) For $r = n - 3$, combined with $2t \geq n + r - 11$ and the fact that $r = t$ if and only if G is a tree (i.e., $r = n - 1$), we have that $n - 7 \leq t \leq n - 4$. Since G is triangle-free, when $t = n - 5, n - 6$ or

$n-7$, respectively, we have that $\sum_{i=1}^{r}|B_i| \geq 8+2(n-5), 12+2(n-6)$ or $16+2(n-7)$, contradicting Eq (3.1). Then, G has exactly one nontrivial block. Note that, in other cases, we first use this method to determine the number of nontrivial blocks in G. By Eq (3.1), this nontrivial block is of order 4, i.e., C_4. According to Table 1 and Theorems 1.3 and 2.1, $mvd(G) = n-2$.

(III) For $r = n-5$, similarly, we have that $n-8 \leq t \leq n-6$, and there are at most two nontrivial blocks in G. Since G is triangle-free, combined with Eq (3.1), we obtain the following.

If $t = n-6$, then the order of this nontrivial block is 6, i.e., one of the graphs in Appendix A. 6 VERTEX.

If $t = n-7$, then there are two nontrivial blocks B_i and B_j with $|B_i| + |B_j| = 8$, i.e., both B_i and B_j are C_4.

According to Table 1 and Theorems 1.3 and 2.1, $mvd(G) = n-3$ when the nontrivial block-induced subgraph of G is $P(2,2)$, and $mvd(G) = n-4$ for the remaining cases.

(IV) For $r = n-7$, similarly, we have that $n-9 \leq t \leq n-8$. Since G is triangle-free, combined with Eq (3.1), we obtain the following.

If $t = n-8$, then the order of this nontrivial block is 8, i.e., one of the graphs in Appendix A. 8 VERTEX.

If $t = n-9$, then there are two nontrivial blocks B_i and B_j with $|B_i| + |B_j| = 10$, and the nontrivial block-induced subgraph of G is one of the graphs in Figure 5. Note that B_i and B_j may or may not be adjacent, and the figure only shows the case when B_i and B_j are not adjacent.

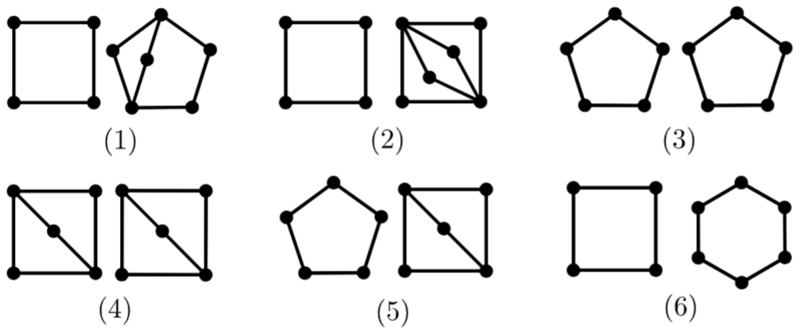

Figure 5. Case when $r = n-7$, $t = n-9$.

According to Table 1 and Theorems 1.3 and 2.1, $mvd(G) = n-4$ when the nontrivial block-induced subgraph of G is $P(3,3)$, $mvd(G) = n-5$ when it is one among $\{P(4,1,1), P(3,2,1), P(3,1,1,1), P(2,2,2)$, Appendix A. 8VERTEX④, Figure 5(6)$\}$ and $mvd(G) = n-6$ for the remaining cases.

(V) For $r = n-9$, similarly, we have that $t = n-10$. By Eq (3.1), this nontrivial block is of order 10, i.e., one of the graphs in Appendix A. 10 VERTEX. According to Table 1 and Theorems 1.3 and 2.1, $mvd(G) = n-5$ when the nontrivial block-induced subgraph of G is $P(4,4)$, and $mvd(G) < n-5$ for the remaining cases.

(VI) For $r = n-11$, similarly, we have that $n-11 \leq t \leq n-12$, which is a contradiction.

For easy reading, when $mvd(G) = n-3, n-4$ or $n-5$, the possible configurations of the nontrivial block-induced subgraph of G are as summarized in Figure 3. The theorem is proved. □

4. Erdős-Gallai-type problems

In this section, we first study the following extremal problem and obtain Theorem 1.5. To solve this problem, we show some lemmas.

For integers k and n with $1 \leq k \leq n$, what is the maximum possible size of a connected graph G of order n with $mvd(G) = k$?

Lemma 4.1. *If G is obtained by removing any edge e from K_n, where $n \geq 3$, then $mvd(G) = 3$; if G is obtained by removing any two edges e_1 and e_2 from K_n, where $n \geq 4$, then $mvd(G) \leq 4$.*

Proof. Let v be one of the endpoints of e; then, G is obtained by adding $n-2$ edges from v to K_{n-1}. From (1), $mvd(G) = 3$. If e_1 and e_2 are adjacent edges incident with a vertex v of G, then G is obtained by adding $n-3$ edges from v to K_{n-1}. So, $mvd(G) = 4$. If e_1 and e_2 are nonadjacent, then $G = K_{2,2}$ when $n = 4$, or $G = K_{2,2,1,\ldots,1}$ when $n > 4$. According to Theorem 3.1(2), $mvd(G) = 2$ or $mvd(G) = 1$. □

Lemma 4.2. *Let G be a connected graph of order $n \geq 2$. Then, the maximum size of G with $mvd(G) = 2$ is*

$$|E(G)|_{max} = \begin{cases} 1, & n = 2, \\ -\infty, & n = 3, \\ 4, & n = 4, \\ 7, & n = 5, \\ \frac{n(n-1)}{2} - 4, & n \geq 6. \end{cases}$$

Proof. Since $mvd(K_n) = n$, the maximum size of G with $mvd(G) = n = 2$ is 1. There is no graph G with $n = 3$ and $mvd(G) = 2$. According to Lemma 4.1, if $n \geq 4$ and $mvd(G) = 2$, then $|E(G)| \leq |E(K_n)| - 2$ and the equation holds only when $n = 4$. Now, let $n \geq 5$.

Claim 1: If G is a graph of order n that is obtained by removing any three edges from K_n, where $n \geq 5$, then $mvd(G) = 2$ if and only if $n = 5$, and the three removed edges are shown as G_4 in Figure 6.

There are five cases to consider for the three removed edges (see $G_i, i \in [5]$ in Figure 6), and if $n = 5$, G_5 is excluded. Note that, for the case G_i, $i \in [4]$, G is connected since $n \geq 5$, and for the case G_5, G is connected only when $n > 5$. For the case G_1, G is obtained by adding $n-4$ edges from v_1 to K_{n-1}. It follows by Tight example 1 that $mvd(G) = 5$. For the case of G_2, define a vertex-coloring $\tau : V(G) \rightarrow [4]$ such that $\tau(N(v_1)) = \tau(N(v_2)) = \tau(N(v_3)) = 1$, $\tau(v_1) = 2$, $\tau(v_2) = 3$ and $\tau(v_3) = 4$. For any two nonadjacent vertices x and y, $N(v_1)$ is a monochromatic x-y vertex cut since $N(v_1) = N(v_2) = N(v_3)$. Then, $mvd(G) \geq 4$. For the case of G_3, define a vertex-coloring $\tau : V(G) \rightarrow [3]$ such that $\tau(N(v_2)) = \tau(N(v_3)) = 1$, $\tau(v_2) = 2$ and $\tau(v_3) = 3$. For any two nonadjacent vertices x and y, $N(v_2)$ or $N(v_3)$ is a monochromatic x-y vertex cut; then, $mvd(G) \geq 3$.

For the case of G_4, we claim that, if $n = 5$, then $mvd(G) = 2$; if $n \geq 6$, then $mvd(G) = 1$. If $n = 5$, define a vertex-coloring $\tau : V(G) \rightarrow [2]$ such that $\tau(v_1) = \tau(v_2) = \tau(v_3) = 1$ and $\tau(v_4) = \tau(v_5) = 2$. For any two nonadjacent vertices x and y, if $\{x, y\} = \{v_4, v_5\}$, then $\{v_1, v_2, v_3\}$ is a monochromatic x-y vertex cut; otherwise, $\{v_4, v_5\}$ is a monochromatic x-y vertex cut. Then, $mvd(G) \geq 2$. Suppose that $mvd(G) > 2$ and τ' is an mvd-coloring of G. For nonadjacent vertices v_4 and v_5, it must be that $\tau'(v_1) = \tau'(v_2) = \tau'(v_3)$ since $N(v_4) = N(v_5) = \{v_1, v_2, v_3\}$. Thus, $\tau'(v_4) \neq \tau'(v_5)$, and both are different from $\tau'(v_1)$, which contradicts the existence of a monochromatic v_1-v_2 vertex cut since $N(v_1) \cap N(v_2) = \{v_4, v_5\}$. If $n \geq 6$ and τ' is an mvd-coloring of G, then $\tau'(V(G) - \{v_4, v_5\})$ is monochromatic since $N(v_4) = N(v_5) =$

$V(G) - \{v_4, v_5\}$. For nonadjacent vertices v_1 and v_2, since $N(v_1) \cap N(v_2) = N(v_1) = V(G) - \{v_1, v_2, v_3\}$ and $(V(G) - \{v_4, v_5\}) \cap (V(G) - \{v_1, v_2, v_3\}) \neq \emptyset$ for $n \geq 6$, $\tau'(V(G) - \{v_1, v_2, v_3\}) = \tau'(V(G) - \{v_4, v_5\})$. Therefore, $mvd(G) = 1$. At last, $G = K_{2,2,2,1,\ldots,1}$ for the case of G_5; then, it follows by Theorem 3.1(2) that $mvd(G) = 1$.

Therefore, the maximum size of G with $n = 5$ and $mvd(G) = 2$ is 7. Furthermore, if $n \geq 6$ and $mvd(G) = 2$, then $|E(G)| \leq |E(K_n)| - 4$. Thus, we only need to prove the following claim to complete the proof of Lemma 4.2.

Claim 2: G is a graph of order n, $n \geq 6$, and if G is obtained by removing any four edges from K_n, where the four removed edges are shown as G_6 in Figure 6, then $mvd(G) = 2$.

G is connected since $n \geq 6$. Define a vertex-coloring $\tau : V(G) \to [2]$ such that $\tau(V(G) - \{v_3\}) = 1$ and $\tau(v_3) = 2$. For any two nonadjacent vertices x and y, $V(G) - \{x, y, v_3\}$ is a monochromatic x-y vertex cut. Then, $mvd(G) \geq 2$. Let τ' be an mvd-coloring of G. For nonadjacent vertices v_4 and v_5, $\tau'(V(G) - \{v_3, v_4, v_5\})$ is monochromatic since $N(v_4) \cap N(v_5) = N(v_4) = V(G) - \{v_3, v_4, v_5\}$. For nonadjacent vertices v_1 and v_2, since $N(v_1) \cap N(v_2) = N(v_2) = V(G) - \{v_1, v_2, v_3\}$ and $(V(G) - \{v_3, v_4, v_5\}) \cap (V(G) - \{v_1, v_2, v_3\}) \neq \emptyset$ for $n \geq 6$, $\tau'(V(G) - \{v_1, v_2, v_3\}) = 1$. Uncolored vertex v_3 adds a new color at most. Therefore, $mvd(G) = 2$. □

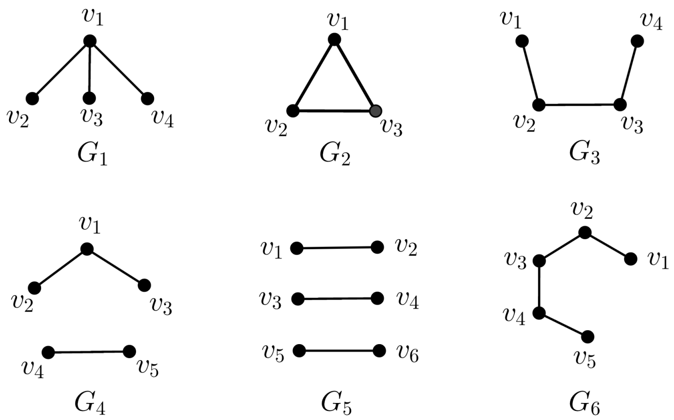

Figure 6. Proof process for Lemma 4.2.

Proof of Theorem 1.5. If $mvd(G) = n$, then the size is maximum when $G = K_n$. By Lemma 4.1, if G is a graph of order n that is obtained by removing any edge from K_n, where $n \geq 3$, then $mvd(G) = 3$. So, if $mvd(G) = 3$ and $n \geq 4$, then the maximum size of G of order n is $\frac{n(n-1)}{2} - 1$. There is no graph G with $1 < n \leq 4$ and $mvd(G) = 1$. By Lemma 4.1, if $mvd(G) = 1$ and $n \geq 5$, then the maximum size of G is $\frac{n(n-1)}{2} - 2$. If $mvd(G) = 2$, we refer to Lemma 4.2. Now, we consider the case that $4 \leq k \leq n - 1$. Let t denote the number of vertices with degree $n - 1$. We first claim that, if the size of a connected graph G of order n is at least $\frac{n(n-1)}{2} - k + 3$, then $mvd(G) \leq k - 1$.

Case 1. $n - k + 2 \leq t \leq n - 1$.

For any two nonadjacent vertices x and y in G, since x and y are adjacent to each vertex of degree $n - 1$ in G, $|N(x) \cap N(y)| \geq n - k + 2$. Therefore, $mvd(G) \leq k - 1$ by Theorem 1.1.

Case 2. $0 \leq t \leq n - k + 2$.

Claim 1: There is at least one vertex with degree $n - 2$.

Otherwise, with the exception of t vertices with degree $n-1$, the maximum degree of the remaining vertices in G is $n-3$ at most. Then, the size is

$$|E(G)| \leq \frac{t(n-1)+(n-t)(n-3)}{2} = \frac{n^2-3n+2t}{2} \leq \frac{n^2-3n+2(n-k+2)}{2} < \frac{n(n-1)}{2} - k + 3,$$

which is a contradiction. Thus, G has at least one vertex with degree $n-2$.

Claim 2: $\delta(G) \geq n-k+2$.

Otherwise, there is a vertex with degree at most $n-k+1$; then, the size is

$$|E(G)| \leq \frac{(n-k+2)(n-1)+(n-k+1)+(k-3)(n-2)}{2} = \frac{n^2-n-2k+5}{2} < \frac{n(n-1)}{2} - k + 3,$$

which is a contradiction. Thus, $\delta(G) \geq n-k+2$.

Let x be a vertex of G with degree $n-2$. There is a vertex y in G which is nonadjacent to x. By Claim 2, $d(y) \geq n-k+2$; then, $|N(x) \cap N(y)| \geq n-k+2$. In other words, there are at least $n-k+2$ internal disjoint x, y-paths. Therefore, $mvd(G) \leq k-1$ by Theorem 1.1.

Above all, if $mvd(G) \geq k$, then the size of G of order n is at most $\frac{n(n-1)}{2} - k + 2$. It remains to show that, for any integers k and n, where $4 \leq k \leq n-1$, there must be a connected graph G of order n and size $\frac{n(n-1)}{2} - k + 2$ such that $mvd(G) = k$. Suppose that G is a graph of order n and size $\frac{n(n-1)}{2} - k + 2$ that is obtained by adding $n-k+1$ edges to K_{n-1} from a vertex v outside K_{n-1}. By Tight example 1, $mvd(G) = k$. □

Proof of Theorem 1.4. It is worth mentioning that the parameter $f_v(n,k)$ is equivalent to another parameter. Let $s_v(n,k) = max\{|E(G)| : |G| = n, mvd(G) \leq k\}$. It is easy to see that $f_v(n,k) = s_v(n,k-1)+1$. Let $n \geq 5$. There are three cases, as follows.

Case 1. $k = 1$.

Since $mvd(G) \geq 1$ holds for any graph G and the tree has minimum size, $f_v(n,1) = n-1$.

Case 2. $2 \leq k \leq 3$.

According to Theorem 1.5, $|E(G)|_{max} = \frac{n(n-1)}{2} - 2$ if $mvd(G) = 1$, $|E(G)|_{max} = \frac{n(n-1)}{2} - 3$ if $mvd(G) = 2$ and $n = 5$ and $|E(G)|_{max} = \frac{n(n-1)}{2} - 4$ if $mvd(G) = 2$ and $n \geq 6$. So, we have that $s_v(n,1) = s_v(n,2) = \frac{n(n-1)}{2} - 2$. Since $f_v(n,k) = s_v(n,k-1)+1$, $f_v(n,2) = f_v(n,3) = \frac{n(n-1)}{2} - 1$.

Case 3. $4 \leq k \leq n$.

According to Theorem 1.5, $|E(G)|_{max} = \frac{n(n-1)}{2} - 1$ if $mvd(G) = 3$, and $|E(G)|_{max} < \frac{n(n-1)}{2} - 1$ if $3 < mvd(G) \leq n-1$. So, we have that $s_v(n,3) = s_v(n,4) = \cdots = s_v(n,n-1) = \frac{n(n-1)}{2} - 1$. Since $f_v(n,k) = s_v(n,k-1)+1$, $f_v(n,4) = f_v(n,5) = \cdots = f_v(n,n) = \frac{n(n-1)}{2}$. □

Remark 1. *For positive integers n and k with $1 \leq k \leq n \leq 4$, the results are shown in Table 2. Moreover, to further understand Theorems 1.4 and 1.5, when $n = 5, 6, 7$, the evolution between k and $f_v(n,k)$ are shown in Figure 7(1), and the evolution between k and $|E(G)|_{max}$ are shown in Figure 7(2).*

Table 2. Values of $|E(G)|_{max}$ and $f_v(n,k)$ when $1 \leq k \leq n \leq 4$.

n	1				2				3				4			
k	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
$\|E(G)\|_{max}$	0	-	-	-	-	1	-	-	-	-	3	-	-	4	5	6
$f_v(n,k)$	0	-	-	-	1	1	-	-	2	2	2	-	3	3	5	6

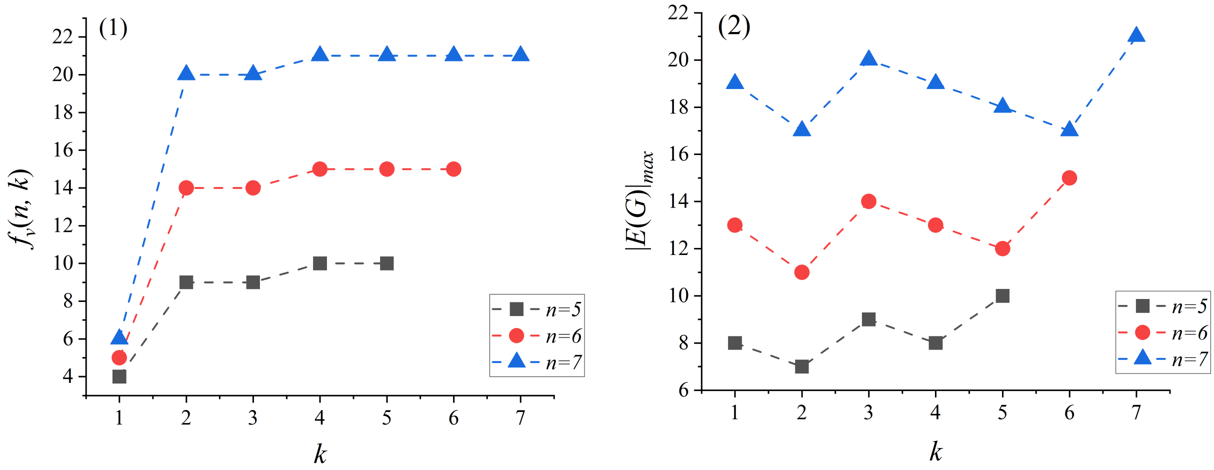

Figure 7. For $n = 5, 6, 7$, the evolution of $f_v(n, k)$ and $|E(G)|_{max}$.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (NSFC11801410 and NSFC11971346) and the China Scholarship Council.

Conflict of interest

The authors declare no conflict of interest.

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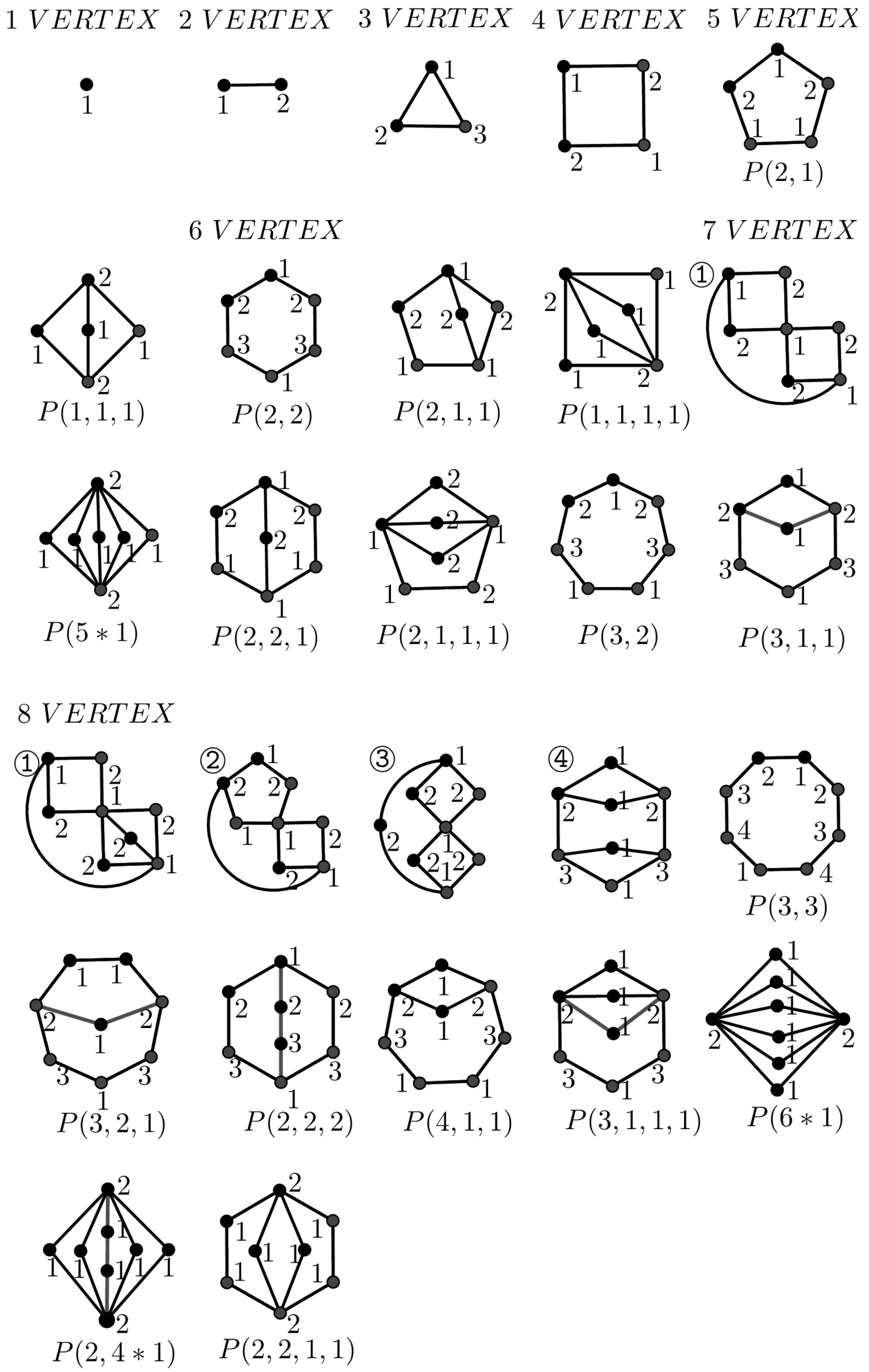
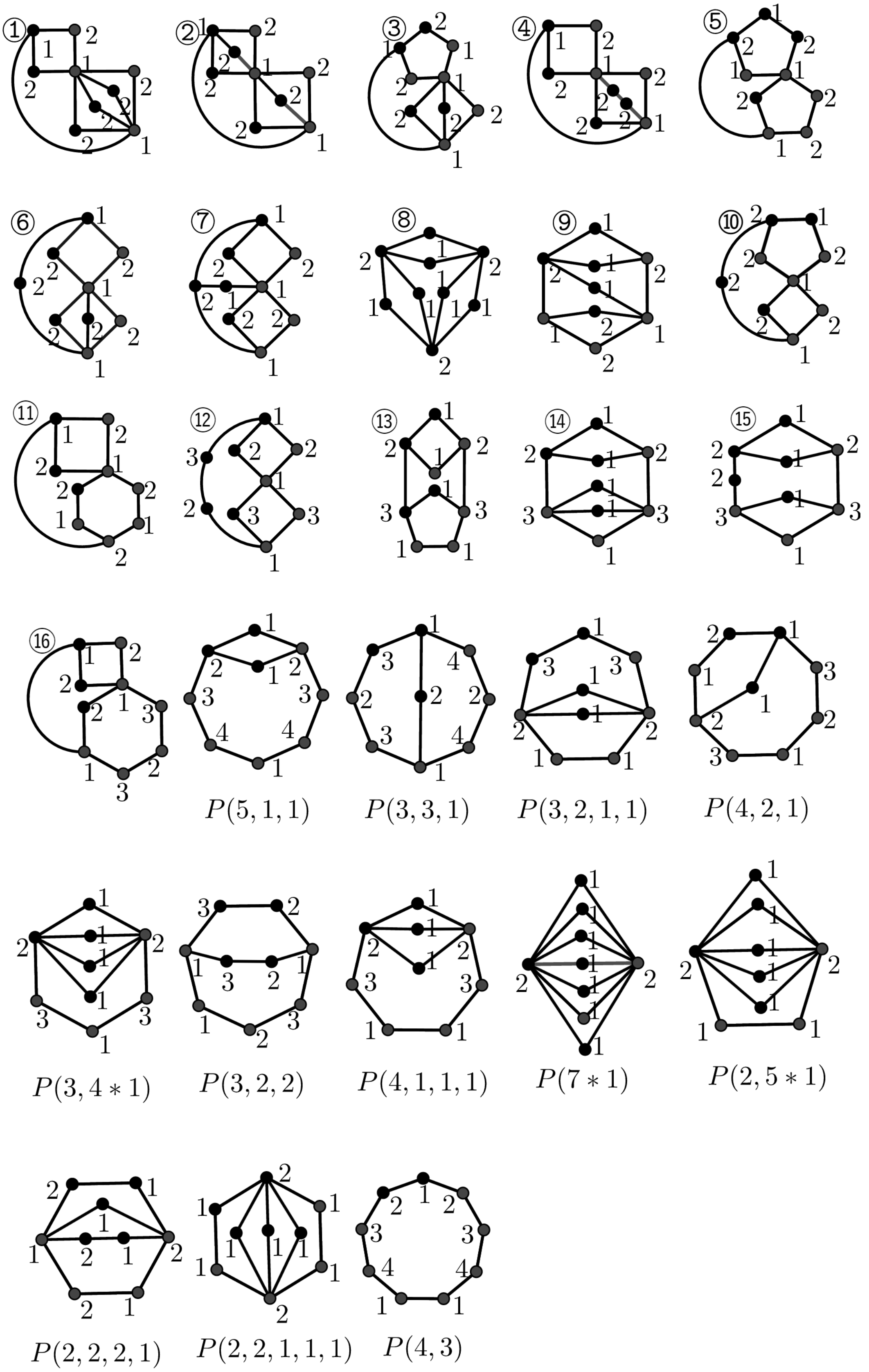
A. Appendix

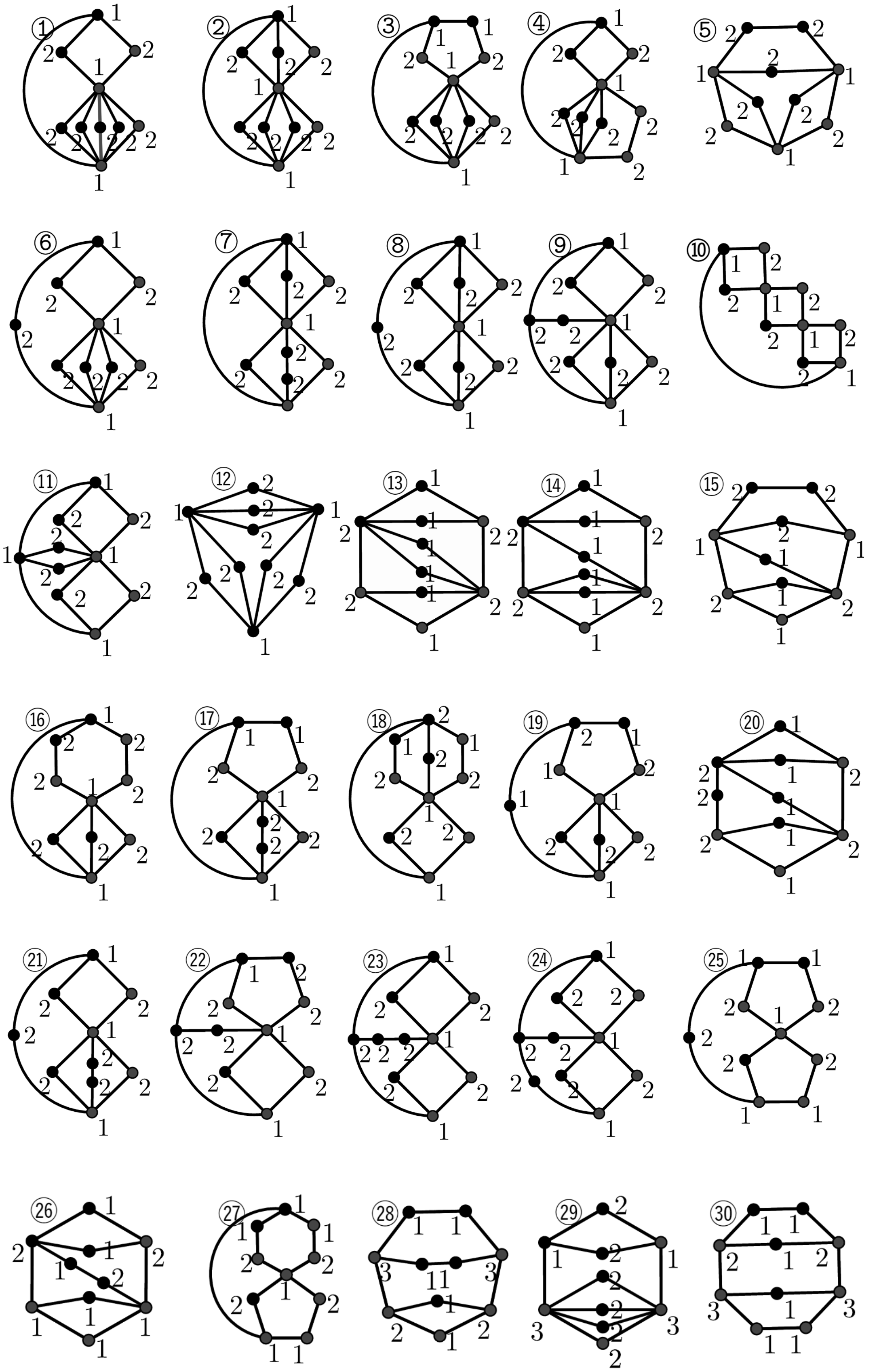
mvd-colorings of minimal blocks with small order.

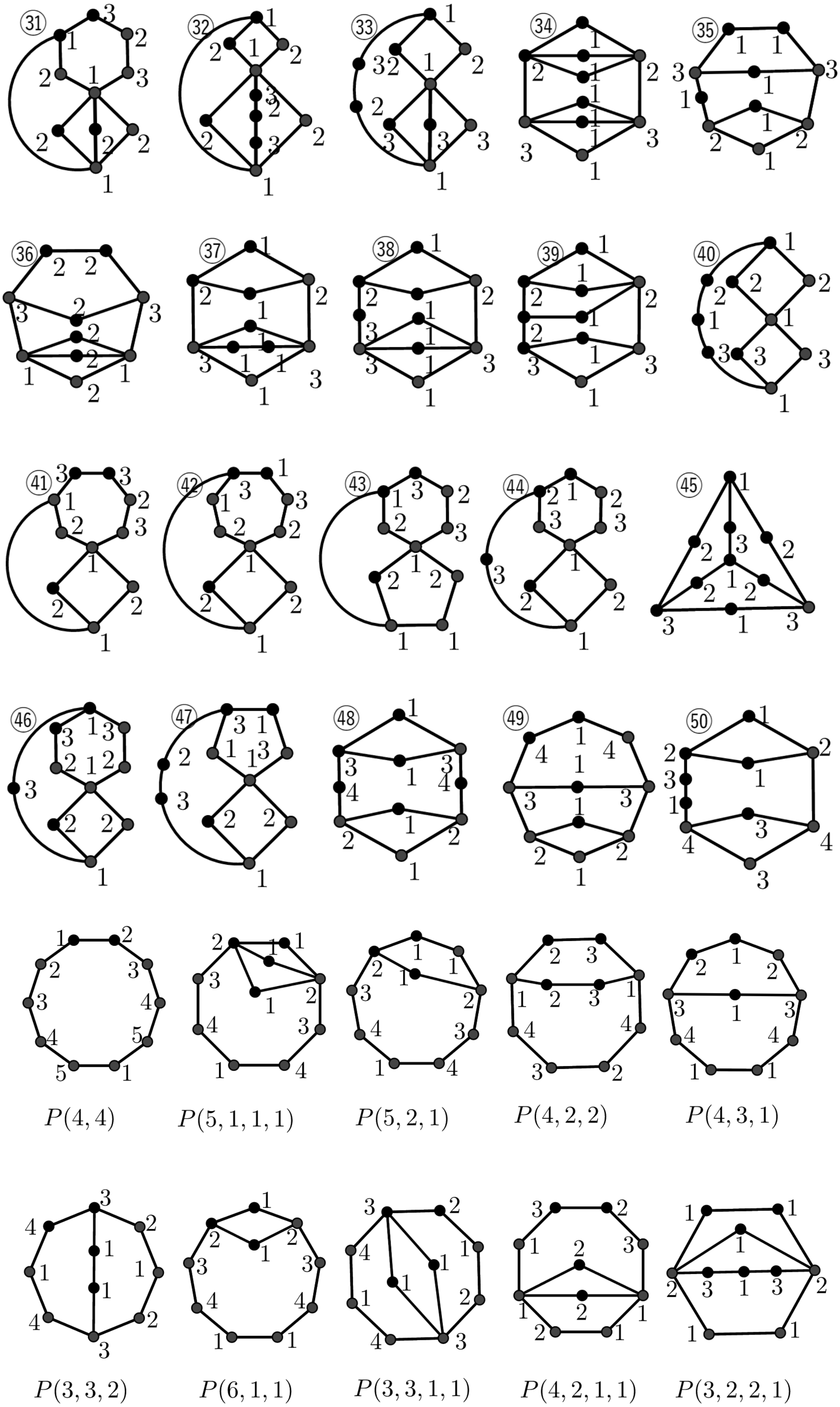

1 VERTEX
2 VERTEX
3 VERTEX
4 VERTEX
5 VERTEX
P(2, 1)
6 VERTEX
7 VERTEX
P(1, 1, 1)
P(2, 2)
P(2, 1, 1)
P(1, 1, 1, 1)
P(5 ∗ 1)
P(2, 2, 1)
P(2, 1, 1, 1)
P(3, 2)
P(3, 1, 1)
8 VERTEX
P(3, 3)
P(3, 2, 1)
P(2, 2, 2)
P(4, 1, 1)
P(3, 1, 1, 1)
P(6 ∗ 1)
P(2, 4 ∗ 1)
P(2, 2, 1, 1)

$9\ VERTEX$

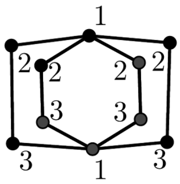

P(5, 1, 1)
P(3, 3, 1)
P(3, 2, 1, 1)
P(4, 2, 1)
P(3, 4 ∗ 1)
P(3, 2, 2)
P(4, 1, 1, 1)
P(7 ∗ 1)
P(2, 5 ∗ 1)
P(2, 2, 2, 1)
P(2, 2, 1, 1, 1)
P(4, 3)

10 $VERTEX$

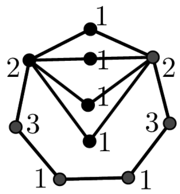




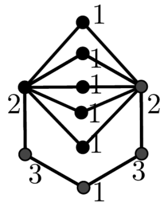
$P(4,4)$
$P(5,1,1,1)$
$P(5,2,1)$
$P(4,2,2)$
$P(4,3,1)$
$P(3,3,2)$
$P(6,1,1)$
$P(3,3,1,1)$
$P(4,2,1,1)$
$P(3,2,2,1)$



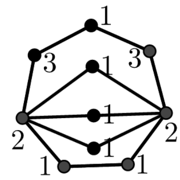
$P(4*2)$



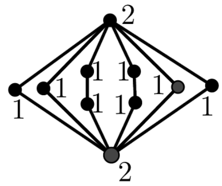
$P(4,4*1)$



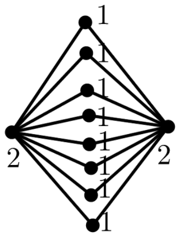
$P(3,5*1)$



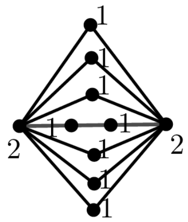
$P(3,2,3*1)$



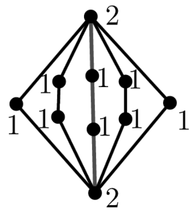
$P(2,2,4*1)$



$P(8*1)$



$P(2,6*1)$



$P(2,2,2,1,1)$