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*Research article*

## Set-valued fractional programming problems with $\sigma$ -arcwisely connectivity

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**Abstract:** In this paper, we determine the sufficient Karush-Kuhn-Tucker (KKT) conditions of optimality of a set-valued fractional programming problem (in short, SVFP) (FP) under the suppositions of contingent epidifferentiation and  $\sigma$ -arcwisely connectivity. We additionally explore the results of duality of parametric (PD), Mond-Weir (MWD), Wolfe (WD), and mixed (MD) kinds for the problem (FP).

**Keywords:** set-valued map; convex cone; contingent epidifferentiation; duality

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### 1. Introduction

Since the last decade researchers have been working in the theory of set-valued optimization problems (in short, SVOPs), a new branch of optimization theory. SVOPs include SVMs as constraints and objective functions. It has applications in viability theory, image processing, mathematical economics and engineering. Borwein [8] proposed the idea of cone convexity of SVMs. It has a significant role to obtain conditions of optimality of SVOPs. Different kinds of differentiability of SVMs have been presented. Jahn and Rauh [23] proposed the idea of contingent epidifferentiation of SVMs. They [6] accordingly proposed the idea of cone preinvexity for SVMs. SVFP is a special class of SVOPs. In the year of 1997, Bhatia and Mehra [6] constituted the Lagrangian theorems of duality for the SVFPs. They [7] additionally formulated the results of duality for Geoffrion solutions of efficiency of the SVFPs under cone convexity supposition. In 2013, Gadhi and Jawhar [22] constituted

the necessary conditions of optimality of the SVFPs. Bhatia and Garg [5], Kaul and Lyall [26], Suneja and Lalitha [44] Suneja and Gupta [43], and Lee and Ho [32] constituted the conditions of optimality and studied the theorems of duality for fractional programming using extended convexity. Das and Nahak [10–15], Das [9], Das and Treanță [16, 17], Das et al. [18], and Treanță and Das [46] proposed the idea of  $\sigma$  convex SVMs. They consequently constituted the KKT conditions of sufficiency and derived the results of duality of various kinds of SVOPs under contingent epidifferentiation and  $\sigma$  convexity suppositions.

In 1976, Avriel [4] proposed the idea of arcwisely connectivity in optimization theory. By substituting a continuous arc for the line segment that connects two points, it essentially generalizes convexity. In 2003, Fu and Wang [21] and Lalitha et al. [31] proposed the idea of arcwisely connected SVMs which is an elongation of of convex SVMs. Lalitha et al. [31] provided the sufficient condition of optimality for SVOPs via contingent epidifferentiation and cone arcwisely connectivity suppositions. Qiu and Yang [35] derived the connectivity of the set of Henig weakly efficiency of SVOPs in 2012. In 2013, Yu [49] provided the sufficient as well as necessary conditions of optimality for global proper efficiency in vector optimization problem (in short, VOP) under arcwisely connected SVMs. In the year of 2016, Yihong and Min [48] proposed the idea of nearly arcwisely connected SVMs of  $\alpha$ -order. They additionally provided the sufficient as well as necessary conditions of optimality of SVOPs. Yu [50] provided the sufficient as well as necessary conditions of optimality for global proper efficiency of VOPs involving arcwisely connected SVMs. In 2018, Peng and Xu [34] proposed the idea of cone subarcwisely connected SVMs. They accordingly constituted the necessary conditions of optimality of second-order for local global proper efficient elements of SVOPs.

Van Su and Hang [47] examined a non-smooth multiobjective fractional programming problem with set, generalized inequality, and equality constraints. For the local weak minimizers, several primary and dual necessary optimality requirements are given in terms of contingent derivatives. An uncertain nonsmooth multiobjective fractional semi-infinite programming problem is established along with certain robust optimality requirements of the Karush-Kuhn-Tucker type in the article by Thuy and Su [45] in the year of 2022. Su and Hang [42] established second-order optimality requirements for a locally Lipschitz multiobjective fractional programming problem with inequality constraints for a second-order strict and weak local Pareto minimum. In this scenario, second-order differentiability is not a requirement at all. But using a contingent epidifferentiation premise, we establish the KKT conditions of sufficiency for the SVFP in this study. Therefore, under the assumption of cone arcwise connection, we give the duals of parametric, Mond-Weir, Wolfe and mixed kinds and prove the accompanying strong, weak, and converse theorems of duality.

In the study by Agarwal et al. [1] in 2023, fuzzy-valued fractional optimization problems were given KKT optimality requirements. The Dinkelbach algorithm, Lagrange multipliers, and  $\alpha$ -cuts were used to build the solution notion. In order to solve the stochastic fuzzy multi-level multi-objective fractional decision making problem (ML-MOFDM), El Sayed et al. [38] introduced a novel modified technique for order preference by similarity to ideal solution (M-TOPSIS). In the year of 2022, El Sayed et al. [39] used a bi-level multi-objective supply chain model that is interactive (BL-MOSCM). A modified Hungarian method-based algorithm for identifying the best fuzzy AP solution was presented by Elsisy et al. [20]. The solution to the fully intuitionistic fuzzy multi-objective fractional transportation problem was illustrated in the work (FIF-MOFTP) by El Sayed and Abo-Sinna [37] in 2021. In the work by Elsisy et al. [19], a new algorithm for the bi-level multi-

objective rough nonlinear programming problem was discussed (BL-MRNPP). For other connected ideas, see [27–30].

This paper is structured as follows. We clarify a few terms and introduce some basic ideas in Section 2 of the set-valued optimization theory. In Section 3, we determine the sufficient conditions of optimality for weak efficiency of the SVFPs under extended cone arcwisely connectivity supposition. We additionally formulate the results of duality of parametric (PD), Mond-Weir (MWD), Wolfe (WD), and mixed (MD) kinds in this section.

## 2. Definitions and preliminaries

Let  $\beta$  be a real normed space (in short, RNS) and  $Q$  be a nonvoid subset of  $\beta$ . Then  $Q$  is referred to as a cone if  $\xi q \in Q$ , for every  $q \in Q$  and  $\xi \geq 0$ . Furthermore, the cone  $Q$  is referred to as pointed if  $Q \cap (-Q) = \{\mathbf{0}_\beta\}$ , solid if  $\text{int}(Q) \neq \emptyset$ , closed if  $\bar{Q} = Q$ , and convex if

$$\xi Q + (1 - \xi)Q \subseteq Q, \forall \xi \in [0, 1],$$

where  $\text{int}(Q)$  and  $\bar{Q}$  represent the interior and closure of  $Q$ , respectively and  $\mathbf{0}_\beta$  is the zero element of  $\beta$ .

The positive orthant  $\mathbb{R}_+^j$  of  $\mathbb{R}^j$ , specified by

$$\mathbb{R}_+^j = \{q = (q_1, \dots, q_j) \in \mathbb{R}^j : q_i \geq 0, \forall i = 1, \dots, j\},$$

is a pointed solid closed convex cone of  $\mathbb{R}^j$ .

Let  $Q$  be a solid pointed convex cone in  $\beta$ . There are two types of cone-orderings in  $\beta$  w.r.t.  $Q$ . For any two elements  $q_1, q_2 \in \beta$ , we have

$$q_1 \leq q_2 \text{ if } q_2 - q_1 \in Q$$

and

$$q_1 < q_2 \text{ if } q_2 - q_1 \in \text{int}(Q).$$

The following notions of minimality are mainly used w.r.t. a solid pointed convex cone  $Q$  in a RNS  $\beta$ .

**Definition 1.** Let  $\tilde{\beta}$  be a nonempty subset of a RNS  $\beta$ . Then ideal minimal, minimal, and weakly minimal points of  $\tilde{\beta}$  are defined as

- (i)  $q' \in \tilde{\beta}$  is an ideal minimal point of  $\tilde{\beta}$  if  $q' \leq q$ , for all  $q \in \tilde{\beta}$ .
- (ii)  $q' \in \tilde{\beta}$  is a minimal point of  $\tilde{\beta}$  if there is no  $q \in \tilde{\beta} \setminus \{q'\}$ , such that  $q \leq q'$ .
- (iii)  $q' \in \tilde{\beta}$  is a weakly minimal point of  $\tilde{\beta}$  if there is no  $q \in \tilde{\beta}$ , such that  $q < q'$ .

The sets of ideal minimal points, minimal points, and weakly minimal points of  $\tilde{\beta}$  are denoted by  $\text{I-min}(\tilde{\beta})$ ,  $\text{min}(\tilde{\beta})$ , and  $\text{w-min}(\tilde{\beta})$  respectively.

The following theorem characterizes contingent epiderivative of set-valued maps.

**Theorem 1.** [36] Let  $\pi : \alpha \rightarrow 2^\beta$  be a set-valued map and  $(p', q') \in \text{grp}(\pi)$ . Then the contingent epiderivative  $\vec{\Delta}\pi(p', q')$  of  $\pi$  at  $(p', q')$  exists if and only if the ideal minimal point of the set

$$\{q \in \beta : (p, q) \in \eta(\text{epigrp}(\pi), (p', q'))\}$$

exists, for all  $p \in \Pi$ , where  $\Pi$  is the projection of  $\eta(\text{epigrp}(\pi), (p', q'))$  onto  $\alpha$ . Since  $Q$  is a pointed cone, the ideal minimal point of the set

$$\{q \in \beta : (u, v) \in \eta(\text{epigrp}(\pi), (p', q'))\},$$

if it exists, is unique, for all  $p \in \Pi$ . In this case, the contingent epiderivative  $\vec{\Delta}\pi(p', q')$  is given by

$$\vec{\Delta}\pi(p', q')(p) = \text{I-min}\{q \in \beta : (p, q) \in \eta(\text{epigrp}(\pi), (p', q'))\}, \forall u \in \Pi.$$

Aubin [2, 3] proposed the idea of contingent cone in RNSs.

**Definition 2.** [2, 3] Let  $N$  be a nonvoid subset of a RNS  $\beta$  and  $q' \in \bar{N}$ . Then, the contingent cone to  $N$  at  $q'$ , denoted by  $\eta(N, q')$ , is interpreted as:

$q \in \eta(N, q')$  if there exist sequences  $\{\xi_n\}$  in  $\mathbb{R}$ , together with  $\xi_n \rightarrow 0^+$  and  $\{q_n\}$  in  $\beta$ , together with  $q_n \rightarrow q$ , satisfying

$$q' + \xi_n q_n \in N, \forall n \in \mathbb{N},$$

or, there exist sequences  $\{v_n\}$  in  $\mathbb{R}$ , together with  $v_n > 0$  and  $\{q'_n\}$  in  $N$ , together with  $q'_n \rightarrow q'$ , satisfying

$$v_n(q'_n - q') \rightarrow q, \text{ as } n \rightarrow \infty.$$

Let  $\alpha, \beta$  be RNSs,  $2^\beta$  be the set of all subsets of  $\beta$ , and  $Q$  be a pointed solid convex cone in  $\beta$ . Let  $\pi : \alpha \rightarrow 2^\beta$  be a SVM from  $\alpha$  to  $\beta$ , i.e.,  $\pi(p) \subseteq \beta$ , for every  $p \in \alpha$ . The domain, graph, and epigraph of  $\pi$  are interpreted by

$$\text{domain}(\pi) = \{p \in \alpha : \pi(p) \neq \emptyset\},$$

$$\text{grp}(\pi) = \{(p, q) \in \alpha \times \beta : q \in \pi(p)\},$$

and

$$\text{epigrp}(\pi) = \{(p, q) \in \alpha \times \beta : q \in \pi(p) + Q\}.$$

In 1997, Jahn and Rauh [23] proposed the idea of contingent epidifferentiation of SVMs.

**Definition 3.** [23] A function  $\vec{\Delta}\pi(p', q') : \alpha \rightarrow \beta$  whose epigraph is identical with the contingent cone to the epigraph of  $\pi$  at  $(p', q')$ , i.e.,

$$\text{epigrp}(\vec{\Delta}\pi(p', q')) = \eta(\text{epigrp}(\pi), (p', q')),$$

is presumed to be the contingent epidifferentiation of  $\pi$  at  $(p', q')$ .

Borwein [8] proposed the idea of cone convexity of SVMs.

**Definition 4.** [8] Let  $M$  be a nonvoid convex subset of a RNS  $\alpha$ . A SVM  $\pi : \alpha \rightarrow 2^\beta$ , together with  $M \subseteq \text{domain}(\pi)$ , is referred to as  $Q$ -convex on  $M$  if  $\forall p_1, p_2 \in M$  and  $\xi \in [0, 1]$ ,

$$\xi\pi(p_1) + (1 - \xi)\pi(p_2) \subseteq \pi(\xi p_1 + (1 - \xi)p_2) + Q.$$

Let  $\alpha$  be a RNS and  $M$  be a nonvoid subset of  $\alpha$ . Let  $\pi : \alpha \rightarrow 2^{\mathbb{R}^j}$ ,  $\omega : \alpha \rightarrow 2^{\mathbb{R}^j}$ , and  $\psi : \alpha \rightarrow 2^{\mathbb{R}^l}$  be SVMs, together with

$$M \subseteq \text{domain}(\pi) \cap \text{domain}(\omega) \cap \text{domain}(\psi).$$

Everywhere in the paper, we represent

$$\mathbf{0}_{\mathbb{R}^j} = (0, \dots, 0) \in \mathbb{R}^j$$

and

$$\mathbf{1}_{\mathbb{R}^j} = (1, \dots, 1) \in \mathbb{R}^j.$$

Let  $\pi = (\pi_1, \pi_2, \dots, \pi_j)$ ,  $\omega = (\omega_1, \omega_2, \dots, \omega_j)$ , and  $\psi = (\psi_1, \psi_2, \dots, \psi_l)$ , where the SVMs  $\pi_i : \alpha \rightarrow 2^{\mathbb{R}}$ ,  $\omega_i : \alpha \rightarrow 2^{\mathbb{R}}$ ,  $i = 1, 2, \dots, j$ , and  $\psi_n : \alpha \rightarrow 2^{\mathbb{R}}$ ,  $n = 1, 2, \dots, l$ , are interpreted by:

$$\text{domain}(\pi_i) = \text{domain}(\pi), \text{domain}(\omega_i) = \text{domain}(\omega) \text{ and } \text{domain}(\psi_n) = \text{domain}(\psi),$$

$$p \in M, y = (q_1, q_2, \dots, q_j) \in \pi(p) \iff q_i \in \pi_i(p), \forall i = 1, 2, \dots, j,$$

$$r = (r_1, r_2, \dots, r_j) \in \omega(p) \iff r_i \in \omega_i(p), \forall i = 1, 2, \dots, j,$$

and

$$w = (w_1, w_2, \dots, w_l) \in \psi(p) \iff w_n \in \psi_n(p), \forall n = 1, 2, \dots, l.$$

Take into account that  $\pi_i(p) \subseteq \mathbb{R}_+$  and  $\omega_i(p) \subseteq \text{int}(\mathbb{R}_+)$ ,  $\forall i = 1, 2, \dots, j$  and  $p \in M$ . Let  $\xi' = (\xi'_1, \xi'_2, \dots, \xi'_j) \in \mathbb{R}_+^j$ . Define  $\frac{q}{r} \in \mathbb{R}^j$  and  $\xi' r \in \mathbb{R}^j$  by:

$$\frac{q}{r} = \left( \frac{q_1}{r_1}, \frac{q_2}{r_2}, \dots, \frac{q_j}{r_j} \right)$$

and

$$\xi' r = (\xi'_1 r_1, \xi'_1 r_2, \dots, \xi'_j r_j).$$

For  $p \in M$ , clarify the subset  $\frac{\pi(p)}{\omega(p)}$  of  $\mathbb{R}^j$  by:

$$\frac{\pi(p)}{\omega(p)} = \left\{ \frac{q}{r} = \left( \frac{q_1}{r_1}, \frac{q_2}{r_2}, \dots, \frac{q_j}{r_j} \right) : q_i \in \pi_i(p), r_i \in \omega_i(p), i = 1, 2, \dots, j \right\}.$$

We assume the SVFP (FP).

$$\begin{aligned} & \underset{p \in M}{\text{minimize}} && \frac{\pi(p)}{\omega(p)} = \left( \frac{\pi_1(p)}{\omega_1(p)}, \frac{\pi_2(p)}{\omega_2(p)}, \dots, \frac{\pi_j(p)}{\omega_j(p)} \right) \\ & \text{s. t.,} && \psi(p) \cap (-\mathbb{R}_+^l) \neq \emptyset. \end{aligned} \tag{FP}$$

The set of feasibility of (FP) can be categorized as

$$S = \{p \in M : \psi(p) \cap (-\mathbb{R}_+^l) \neq \emptyset\}.$$

**Definition 5.** A point  $(p', \frac{q'}{r'}) \in \alpha \times \mathbb{R}^j$ , together with  $p' \in S$ ,  $q' \in \pi(p')$ , and  $r' \in \omega(p')$ , is referred to as to minimize the problem (FP) if there exist no  $p \in S$ ,  $q \in \pi(p)$ , and  $r \in \omega(p)$  satisfying

$$\frac{q}{r} - \frac{q'}{r'} \in -\mathbb{R}_+^j \setminus \{0_{\mathbb{R}^j}\}.$$

**Definition 6.** A point  $(p', \frac{q'}{r'}) \in \alpha \times \mathbb{R}^j$ , together with  $p' \in S$ ,  $q' \in \pi(p')$ , and  $r' \in \omega(p')$ , is referred to as to minimize weakly the problem (FP) if there exist no  $p \in S$ ,  $q \in \pi(p)$ , and  $r \in \omega(p)$  satisfying

$$\frac{q}{r} - \frac{q'}{r'} \in -\text{int}(\mathbb{R}_+^j).$$

We assume the parametric problem  $(FP_{\xi'})$  associated with the SVFP (FP).

$$\begin{aligned} & \underset{p \in M}{\text{minimize}} && \pi(p) - \xi' \omega(p) \\ & \text{s. t.,} && \psi(p) \cap (-\mathbb{R}_+^l) \neq \emptyset. \end{aligned} \tag{FP_{\xi'}}$$

**Definition 7.** A point  $(p', q' - \xi' r') \in \alpha \times \mathbb{R}^j$ , together with  $p' \in S$ ,  $q' \in \pi(p')$ , and  $r' \in \omega(p')$ , is referred to as to minimize the problem  $(FP_{\xi'})$ , if there exist no  $p \in S$ ,  $q \in \pi(p)$ , and  $r \in \omega(p)$  satisfying

$$(q - \xi' r) - (q' - \xi' r') \in -\mathbb{R}_+^j \setminus \{0_{\mathbb{R}^j}\}.$$

**Definition 8.** A point  $(p', q' - \xi' r') \in \alpha \times \mathbb{R}^j$ , together with  $p' \in S$ ,  $q' \in \pi(p')$ , and  $r' \in \omega(p')$ , is referred to as to minimize weakly the problem  $(FP_{\xi'})$ , if there exist no  $p \in S$ ,  $q \in \pi(p)$ , and  $r \in \omega(p)$  satisfying

$$(q - \xi' r) - (q' - \xi' r') \in -\text{int}(\mathbb{R}_+^j).$$

Gadhi and Jawhar [22] demonstrated how the solutions of the problems (FP) and  $(FP_{\xi'})$  relate to one another.

**Lemma 1.** [22] A point  $(p', \frac{q'}{r'}) \in \alpha \times \mathbb{R}^j$  minimizes weakly the problem (FP) if and only if  $(p', 0_{\mathbb{R}^j})$  minimizes weakly the problem  $(FP_{\xi'})$ , where  $\xi' = \frac{q'}{r'}$ .

Avriel [4] proposed the idea of arcwisely connectivity. It is mainly a generalization of convexity.

**Definition 9.** A subset  $M$  of a RNS  $\alpha$  is presumed to be an arcwisely connected set if for every  $p_1, p_2 \in M$  there exists a continuous arc  $\chi_{p_1, p_2}(\xi)$  defined on  $[0, 1]$  with a value in  $M$  satisfying  $\chi_{p_1, p_2}(0) = p_1$  and  $\chi_{p_1, p_2}(1) = p_2$ .

Fu and Wang [21] and Lalitha et al. [31] proposed the idea of arcwisely connected SVMs which is an elongation of the sort of cone convex SVMs.

**Definition 10.** [21, 31] Let  $M$  be an arcwisely connected subset of a RNS  $\alpha$  and  $\pi : \alpha \rightarrow 2^\beta$  be a SVM, together with  $M \subseteq \text{domain}(\pi)$ . Then  $\pi$  is presumed to be  $Q$ -arcwisely connected on  $M$  if

$$(1 - \xi)\pi(p_1) + \xi\pi(p_2) \subseteq \pi(\chi_{p_1, p_2}(\xi)) + Q, \quad \forall p_1, p_2 \in M \text{ and } \forall \xi \in [0, 1].$$

Peng and Xu [34] proposed the idea of cone subarcwisely connected SVMs.

**Definition 11.** [34] Let  $M$  be an arcwisely connected subset of a RNS  $\alpha$ ,  $e \in \text{int}(Q)$ , and  $\pi : \alpha \rightarrow 2^\beta$  be a SVM, together with  $M \subseteq \text{domain}(\pi)$ . Then  $\pi$  is presumed to be  $Q$ -subarcwisely connected on  $M$  if

$$\begin{aligned} (1 - \xi)\pi(p_1) + \xi\pi(p_2) + \epsilon e &\subseteq \pi(\chi_{p_1, p_2}(\xi)) + Q, \\ \forall p_1, p_2 \in M, \forall \epsilon > 0, \text{ and } \forall \xi \in [0, 1]. \end{aligned}$$

### 3. Main results

In this section, we present the notion of  $\sigma$ -arcwisely connectivity of SVMs in the broader sense of arcwisely connected SVMs.

**Definition 12.** Let  $M$  be an arcwisely connected subset of a RNS  $\alpha$ ,  $e \in \text{int}(Q)$ , and  $\pi : \alpha \rightarrow 2^\beta$  be a SVM, together with  $M \subseteq \text{domain}(\pi)$ . Then  $\pi$  is presumed to be  $\sigma$ - $Q$ -arcwisely connected w.r.t.  $e$  on  $M$  if there exists  $\sigma \in \mathbb{R}$ , satisfying

$$(1 - \xi)\pi(p_1) + \xi\pi(p_2) \subseteq \pi(\chi_{p_1, p_2}(\xi)) + \sigma\xi(1 - \xi)\|p_1 - p_2\|^2 e + Q, \\ \forall p_1, p_2 \in M \text{ and } \forall \xi \in [0, 1].$$

**Remark 1.** If  $\sigma > 0$ , then  $\pi$  is presumed to be strongly  $\sigma$ - $Q$ -arcwisely connected, if  $\sigma = 0$ , we get the common concept of  $Q$ -arcwisely connectivity, and if  $\sigma < 0$ , then  $\pi$  is presumed to be weakly  $\sigma$ - $Q$ -arcwisely connected. Undoubtedly, strongly  $\sigma$ - $Q$ -arcwisely connectivity  $\Rightarrow$   $Q$ -arcwisely connectivity  $\Rightarrow$  weakly  $\sigma$ - $Q$ -arcwisely connectivity.

We create a case study of  $\sigma$ -arcwisely connected SVM, which is not necessarily arcwisely connected.

**Example 1.** Let  $\alpha = \mathbb{R}^2$ ,  $\beta = \mathbb{R}$ ,  $Q = \mathbb{R}_+$  and

$$M = \left\{ p = (p_1, p_2) \mid p_1 + p_2 \geq \frac{1}{2}, p_1 \geq 0, p_2 \geq 0 \right\} \subseteq \alpha.$$

Define  $\chi_{p,s}(\xi) = (1 - \xi)u + \xi s$ , where  $p = (p_1, p_2)$ ,  $s = (s_1, s_2)$  and  $\xi \in [0, 1]$ . Evidently,  $M$  is an arcwisely connected set. For the SVM  $\pi : \alpha \rightarrow 2^\beta$ , defined as follows:  $\pi(p) = [0, 2]$ ,  $p_1 + p_2 \geq \frac{1}{2}$ ,  $p_1 \neq p_2$ , and  $\pi(p) = [3, 5]$  for  $\{p_1 + p_2 < \frac{1}{2}\} \cup \{p_1 + p_2 \geq \frac{1}{2}, p_1 = p_2\}$ , we find that  $\pi$  is not  $Q$ -arcwisely connected for  $p = (1, 0)$ ,  $s = (0, 1)$  and  $\xi = \frac{1}{2}$ . However, by taking into account  $\sigma = -2$  and  $e = [3, 3] = \{3\}$ , we comprehend that  $\pi$  is a  $\sigma$ - $Q$ -arcwisely connected SVM for  $p = (1, 0)$ ,  $s = (0, 1)$ .

In the following theorem, we characterize  $\sigma$ -arcwisely connectivity of SVMs in relation to contingent epidifferentiation.

**Theorem 2.** Let  $M$  be an arcwisely connected subset of a RNS  $\alpha$ ,  $e \in \text{int}(Q)$ , and  $\pi : \alpha \rightarrow 2^\beta$  be  $\sigma$ - $Q$ -arcwisely connected w.r.t.  $e$  on  $M$ . Let  $p' \in M$  and  $q' \in \pi(p')$ . Then,

$$\pi(p) - q' \subseteq \overrightarrow{\Delta}\pi(p', q')(\chi'_{p', p}(0+)) + \sigma\|p - p'\|^2 e + Q, \quad \forall p \in M,$$

where

$$\chi'_{p', p}(0+) = \lim_{\xi \rightarrow 0+} \frac{\chi_{p', p}(\xi) - \chi_{p', p}(0)}{\xi},$$

presuming that  $\chi'_{p', p}(0+)$  appears for every  $p, p' \in M$ .

*Proof.* Let  $p \in M$ . As  $\pi$  is  $\sigma$ - $Q$ -arcwisely connected w.r.t.  $e$  on  $M$ ,

$$(1 - \xi)\pi(p') + \xi\pi(p) \subseteq \pi(\chi_{p', p}(\xi)) + \sigma\xi(1 - \xi)\|p - p'\|^2 e + Q, \\ \forall \xi \in [0, 1].$$

Let  $q \in \pi(p)$ . Choose a sequence  $\{\xi_n\}$ , together with  $\xi_n \in (0, 1)$ ,  $n \in \mathbb{N}$ , satisfying  $\xi_n \rightarrow 0+$  when  $n \rightarrow \infty$ . Suppose

$$p_n = \chi_{p',p}(\xi_n)$$

and

$$q_n = (1 - \xi_n)q' + \xi_n q - \sigma \xi_n (1 - \xi_n) \|p - p'\|^2 e.$$

So,

$$q_n \in \pi(p_n) + Q.$$

It is undeniable that

$$p_n = \chi_{p',p}(\xi_n) \rightarrow \chi_{p',p}(0) = p', q_n \rightarrow q', \text{ whenever } n \rightarrow \infty,$$

$$\frac{p_n - p'}{\xi_n} = \frac{\chi_{p',p}(\xi_n) - \chi_{p',p}(0)}{\xi_n} \rightarrow \chi'_{p',p}(0+), \text{ whenever } n \rightarrow \infty,$$

and

$$\frac{q_n - q'}{\xi_n} = q - q' - \sigma(1 - \xi_n) \|p - p'\|^2 e \rightarrow q - q' - \sigma \|p - p'\|^2 e, \text{ whenever } n \rightarrow \infty.$$

So,

$$(\chi'_{p',p}(0+), q - q' - \sigma \|p - p'\|^2 e) \in \eta(\text{epigrp}(\pi), (p', q')) = \text{epigrp}(\overrightarrow{\Delta\pi}(p', q')).$$

Accordingly,

$$q - q' - \sigma \|p - p'\|^2 e \in \overrightarrow{\Delta\pi}(p', q')(\chi'_{p',p}(0+)) + Q,$$

that is accurate, for every  $q \in \pi(p)$ . Hence,

$$\pi(p) - q' \subseteq \overrightarrow{\Delta\pi}(p', q')(\chi'_{p',p}(0+)) + \sigma \|p - p'\|^2 e + Q, \quad \forall p \in M.$$

Thus, the theorem is implied. □

### 3.1. Conditions of optimality

We determine the sufficient conditions of optimality for the problem (FP) under  $\sigma$ -arcwisely connectivity and contingent epidifferentiation suppositions.

**Theorem 3. (Sufficient conditions of optimality)** Let  $M$  be an arcwisely connected subset of  $\alpha$ ,  $p'$  be an element of the set of feasibility  $S$  of (FP),  $q' \in \pi(p')$ ,  $r' \in \omega(p')$ ,  $\xi' = \frac{q'}{r'}$ , and  $s' \in \psi(p') \cap (-\mathbb{R}_+^l)$ . Take into account that  $\pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{I}_{\mathbb{R}^j}$ ,  $-\xi' \omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{I}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{I}_{\mathbb{R}^l}$ , on  $M$ . Let  $\pi$  be contingent epidifferentiable at  $(p', q')$ ,  $-\xi' \omega$  be contingent epidifferentiable at  $(p', -\xi' r')$ , and  $\psi$  be contingent epidifferentiable at  $(p', s')$ . Presume that there exists  $(q^*, r^*) \in \mathbb{R}_+^j \times \mathbb{R}_+^l$ , together with  $q^* \neq \mathbf{0}_{\mathbb{R}^j}$ , and

$$(\sigma_1 + \sigma_2) \langle q^*, \mathbf{I}_{\mathbb{R}^j} \rangle + \sigma_3 \langle r^*, \mathbf{I}_{\mathbb{R}^l} \rangle \geq 0, \quad (3.1)$$

satisfying

$$\begin{aligned} & \left\langle q^*, \overrightarrow{\Delta\pi}(p', q')(\chi'_{p',p}(0+)) + \overrightarrow{\Delta}(-\xi' \omega)(p', -\xi' r')(\chi'_{p',p}(0+)) \right\rangle \\ & + \left\langle r^*, \overrightarrow{\Delta\psi}(p', s')(\chi'_{p',p}(0+)) \right\rangle \geq 0, \forall p \in M, \end{aligned} \quad (3.2)$$



$$q' - \xi' r' = 0, \quad (3.3)$$

and

$$\langle r^*, s' \rangle = 0. \quad (3.4)$$

Then  $(p', \frac{q'}{r'})$  minimizes weakly the problem (FP).

*Proof.* Presume that  $(p', \frac{q'}{r'})$  does not minimize the problem (FP). Then there exist  $p \in S$ ,  $q \in \pi(p)$  and  $r \in \omega(p)$  satisfying

$$\frac{q}{r} < \frac{q'}{r'}.$$

As  $q' - \xi' r' = 0$ , we have

$$\frac{q}{r} < \xi'.$$

So,

$$q - \xi' r < 0.$$

Hence,

$$\langle q^*, q - \xi' r \rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^j} \neq q^* \in \mathbb{R}_+^j.$$

Again, as  $q' - \xi' r' = 0$ , we have

$$\langle q^*, q' - \xi' r' \rangle = 0.$$

Since  $p \in S$ , there exists an element  $r \in \psi(p) \cap (-\mathbb{R}_+^l)$ .

Therefore,

$$\langle r^*, r \rangle \leq 0.$$

So,

$$\langle r^*, s - s' \rangle \leq 0, \text{ as } \langle r^*, s' \rangle = 0.$$

Hence,

$$\langle q^*, q - \xi' r - (q' - \xi' r') \rangle + \langle r^*, s - s' \rangle < 0. \quad (3.5)$$

As  $\pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ ,  $-\xi' \omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^l}$ , on  $M$ ,

$$\pi(p) - q' \subseteq \overrightarrow{\Delta} \pi(p', q')(\chi'_{p',p}(0+)) + \sigma_1 \|p - p'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

$$(-\xi' \omega)(p) + \xi' r \subseteq \overrightarrow{\Delta} (-\xi' \omega)(p', -\xi' r')(\chi'_{p',p}(0+)) + \sigma_2 \|p - p'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

and

$$\psi(p) - s' \subseteq \overrightarrow{\Delta} \psi(p', s')(\chi'_{p',p}(0+)) + \sigma_3 \|p - p'\|^2 \mathbf{1}_{\mathbb{R}^l} + \mathbb{R}_+^l.$$

Hence,

$$\begin{aligned} q - q' &\in \overrightarrow{\Delta} \pi(p', q')(\chi'_{p',p}(0+)) + \sigma_1 \|p - p'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j, \\ -\xi' r + \xi' r' &\in \overrightarrow{\Delta} (-\xi' \omega)(p', -\xi' r')(\chi'_{p',p}(0+)) + \sigma_2 \|p - p'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j, \end{aligned}$$

and

$$s - s' \in \overrightarrow{\Delta} \psi(p', s')(\chi'_{p', p}(0+)) + \sigma_3 \|p - p'\|^2 \mathbf{1}_{\mathbb{R}^l} + \mathbb{R}_+^l.$$

So, (3.1) and (3.2) imply that

$$\langle q^*, q - \xi' r - (q' - \xi' r') \rangle + \langle r^*, s - s' \rangle \geq 0,$$

which is in conflict with (3.5). Accordingly,  $(p', q')$  minimizes weakly the problem (FP).  $\square$

We can accordingly prove the following theorem by the same approach.

**Theorem 4. (Sufficient conditions of optimality)** Let  $M$  be an arcwisely connected subset of  $\alpha$ ,  $p'$  be an element of the set of feasibility  $S$  of the problem (FP),  $q' \in \pi(p')$ ,  $r' \in \omega(p')$ , and  $s' \in \psi(p') \cap (-\mathbb{R}_+^l)$ . Take into account that  $r' \pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{I}_{\mathbb{R}^j}$ ,  $-q' \omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{I}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{I}_{\mathbb{R}^l}$ , on  $M$ . Presume that there exists  $(q^*, r^*) \in \mathbb{R}_+^j \times \mathbb{R}_+^l$ , together with  $q^* \neq \mathbf{0}_{\mathbb{R}^j}$ , and (3.1) and (3.4) are fulfilled, together with

$$\begin{aligned} & \langle q^*, \overrightarrow{\Delta}(r' \pi)(p', q' r')(\chi'_{p', p}(0+)) + \overrightarrow{\Delta}(-q' \omega)(p', -q' r')(\chi'_{p', p}(0+)) \rangle \\ & + \langle r^*, \overrightarrow{\Delta} \psi(p', s')(\chi'_{p', p}(0+)) \rangle \geq 0, \forall p \in M. \end{aligned} \quad (3.6)$$

Then  $(p', \frac{q'}{r'})$  minimizes weakly the problem (FP).

We construct the duals of parametric (PD), Mond-Weir (MWD), Wolfe (WD), and mixed (MD) kinds associated with (FP). We consequently explore the related theorems of duality.

### 3.2. Parametric kind dual

We assume the parametric kind dual (PD) connected with the problem (FP).

$$\begin{aligned} & \text{maximize } \xi', \\ & \text{s. t.,} \\ & \langle q^*, \overrightarrow{\Delta} \pi(p', q')(\chi'_{p', p}(0+)) + \overrightarrow{\Delta}(-\xi' \omega)(p', -\xi' r')(\chi'_{p', p}(0+)) \rangle \\ & + \langle r^*, \overrightarrow{\Delta} \psi(p', s')(\chi'_{p', p}(0+)) \rangle \geq 0, \forall p \in M, \\ & q'_i - \xi'_i r'_i \geq 0, \forall i = 1, \dots, j, \\ & p' \in M, q' \in \pi(p'), r' \in \omega(p'), \xi' \in \frac{\pi(p)}{\omega(p)}, s' \in \psi(p'), \\ & q^* \in \mathbb{R}_+^j, r^* \in \mathbb{R}_+^l, \langle r^*, s' \rangle \geq 0 \text{ and } \langle q^*, \mathbf{1}_{\mathbb{R}^j} \rangle = 1. \end{aligned} \quad (\text{PD})$$

A point  $(p', q', r', \xi', s', q^*, r^*)$  meeting all the requirements of the problem (PD) is referred to as feasible to (PD).

**Definition 13.** A point  $(p', q', r', \xi', s', q^*, r^*)$  in the set of feasibility of the problem (PD) is referred to as a weak maximizer of (PD) if there exists no point  $(p, q, r, \xi, s, q_1^*, r_1^*)$  in the set of feasibility of (PD) satisfying

$$\xi - \xi' \in \text{int}(\mathbb{R}_+^j).$$

**Theorem 5. (Weak Duality)** Let  $M$  be an arcwisely connected subset of  $\alpha$ ,  $\bar{p}$  be an element of the set of feasibility  $S$  of the problem (FP), and  $(p', q', r', \xi', s', q^*, r^*)$  be feasible to the problem (PD). Take into account that  $\pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{I}_{\mathbb{R}^j}$ ,  $-\xi' \omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{I}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{I}_{\mathbb{R}^l}$ , on  $M$ , satisfying

$$(\sigma_1 + \sigma_2) + \sigma_3 \langle r^*, \mathbf{I}_{\mathbb{R}^l} \rangle \geq 0. \quad (3.7)$$

Then,

$$\frac{\pi(\bar{p})}{\omega(\bar{p})} - \xi' \subseteq \mathbb{R}^j \setminus -\text{int}(\mathbb{R}_+^j).$$

*Proof.* Presume that for some  $\bar{q} \in \pi(\bar{p})$  and  $\bar{r} \in \omega(\bar{p})$ ,

$$\frac{\bar{q}}{\bar{r}} - \xi' \in -\text{int}(\mathbb{R}_+^j).$$

So,

$$\frac{\bar{q}}{\bar{r}} < \xi'.$$

Therefore,

$$\frac{\bar{q}_i}{\bar{r}_i} < \xi'_i, \forall i = 1, \dots, j.$$

So,

$$\bar{q}_i - \xi'_i \bar{r}_i < 0, \forall i = 1, \dots, j.$$

Therefore,

$$\langle q^*, \bar{q} - \xi' \bar{r} \rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^j} \neq q^* \in \mathbb{R}_+^j.$$

From the requirements of (PD),

$$q'_i - \xi'_i r'_i \geq 0, \forall i = 1, \dots, j.$$

So,

$$\langle q^*, q' - \xi' r' \rangle \geq 0.$$

As  $\bar{p} \in S$ , we have

$$\psi(\bar{p}) \cap (-\mathbb{R}_+^l) \neq \emptyset.$$

We select  $\bar{s} \in \psi(\bar{p}) \cap (-\mathbb{R}_+^l)$ .

So,

$$\langle r^*, \bar{s} \rangle \leq 0.$$

From the requirements of (PD),

$$\langle r^*, s' \rangle \geq 0.$$

So,

$$\langle r^*, \bar{s} - s' \rangle = \langle r^*, \bar{s} \rangle - \langle r^*, s' \rangle \leq 0.$$

Hence,

$$\langle q^*, \bar{q} - \xi' \bar{r} - (q' - \xi' r') \rangle + \langle r^*, \bar{s} - s' \rangle < 0. \quad (3.8)$$

As  $\pi$  is  $\sigma_1\text{-}\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ ,  $-\xi'\omega$  is  $\sigma_2\text{-}\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3\text{-}\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^l}$ , on  $M$ ,

$$\pi(\bar{p}) - q' \subseteq \overrightarrow{\Delta}\pi(p', q')(\chi'_{p', \bar{p}}(0+)) + \sigma_1\|\bar{p} - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

$$(-\xi'\omega)(\bar{p}) + \xi'r' \subseteq \overrightarrow{\Delta}(-\xi'\omega)(p', -\xi'r')(\chi'_{p', \bar{p}}(0+)) + \sigma_2\|\bar{p} - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

and

$$\psi(\bar{p}) - s' \subseteq \overrightarrow{\Delta}\psi(p', s')(\chi'_{p', \bar{p}}(0+)) + \sigma_3\|\bar{p} - p'\|^2\mathbf{1}_{\mathbb{R}^l} + \mathbb{R}_+^l.$$

Hence,

$$\bar{q} - q' \in \overrightarrow{\Delta}\pi(p', q')(\chi'_{p', \bar{p}}(0+)) + \sigma_1\|\bar{p} - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

$$-\xi'\bar{r} + \xi'r' \in \overrightarrow{\Delta}(-\xi'\omega)(p', -\xi'r')(\chi'_{p', \bar{p}}(0+)) + \sigma_2\|\bar{p} - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

and

$$\bar{s} - s' \in \overrightarrow{\Delta}\psi(p', s')(\chi'_{p', \bar{p}}(0+)) + \sigma_3\|\bar{p} - p'\|^2\mathbf{1}_{\mathbb{R}^l} + \mathbb{R}_+^l.$$

From the requirements of (PD) and (3.7), we have

$$\langle q^*, \bar{q} - \xi'\bar{r} - (q' - \xi'r') \rangle + \langle r^*, \bar{s} - s' \rangle \geq 0,$$

which is in conflict with (3.8).

So,

$$\frac{\bar{q}}{r} - \xi' \notin -\text{int}(\mathbb{R}_+^j).$$

Since  $\bar{q} \in F(\bar{p})$  is chosen arbitrarily,

$$\frac{\pi(\bar{p})}{\omega(\bar{p})} - \xi' \subseteq \mathbb{R}^j \setminus -\text{int}(\mathbb{R}_+^j).$$

□

**Theorem 6. (Strong Duality)** Let  $(p', \frac{q'}{r'})$  minimize weakly the problem (FP) and  $s' \in \psi(p') \cap (-\mathbb{R}_+^l)$ . Take into account that for some  $(q^*, r^*) \in \mathbb{R}_+^j \times \mathbb{R}_+^l$ , together with  $\langle q^*, \mathbf{1}_{\mathbb{R}^j} \rangle = 1$  and  $\xi' \in \mathbb{R}^j$ , (3.2)–(3.4) are fulfilled at  $(p', q', r', \xi', s', q^*, r^*)$ . Then  $(p', q', r', \xi', s', q^*, r^*)$  is feasible to the problem (PD). Furthermore, If the Theorem 5 between (FP) and (PD) remains, then  $(p', q', r', \xi', s', q^*, r^*)$  maximizes weakly (PD).

*Proof.* As the (3.2)–(3.4) are fulfilled at  $(p', q', r', \xi', s', q^*, r^*)$ , we have

$$\begin{aligned} & \langle q^*, \overrightarrow{\Delta}\pi(p', q')(\chi'_{p', p}(0+)) + \overrightarrow{\Delta}(-\xi'\omega)(p', -\xi'r')(\chi'_{p', p}(0+)) \rangle \\ & + \langle r^*, \overrightarrow{\Delta}\psi(p', s')(\chi'_{p', p}(0+)) \rangle \geq 0, \forall p \in M, \end{aligned}$$

$$q' - \xi'r' = 0,$$

and

$$\langle r^*, s' \rangle = 0.$$

So  $(p', q', r', \xi', s', q^*, r^*)$  is feasible to (PD). Presume that the Theorem 5 between (FP) and (PD) stays and  $(p', q', r', \xi', s', q^*, r^*)$  does not maximize weakly (PD). Then there exists a point  $(p, q, r, \xi, s, q_1^*, r_1^*)$  in the set of feasibility of (PD) satisfying

$$\xi - \xi' \in \text{int}(\mathbb{R}_+^j).$$

As  $q' - \xi' r' = 0$ ,

$$\xi - \frac{q'}{r'} \in \text{int}(\mathbb{R}_+^j).$$

which is in conflict with the Theorem 5 between (FP) and (PD). Accordingly,  $(p', q', r', \xi', s', q^*, r^*)$  maximizes weakly (PD).  $\square$

**Theorem 7. (Converse Duality)** Let  $M$  be an arcwisely connected subset of  $\alpha$  and  $(p', q', r', \xi', s', q^*, r^*)$  be feasible to (PD), where  $\xi' = \frac{q'}{r'}$ . Take into account that  $\pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{I}_{\mathbb{R}^j}$ ,  $-\xi' \omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{I}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{I}_{\mathbb{R}^l}$ , on  $M$ , satisfying (3.7). If  $p'$  is an element of the set of feasibility  $S$  of (FP), then  $(p', \frac{q'}{r'})$  minimizes weakly the problem (FP).

*Proof.* Suppose  $(p', \frac{q'}{r'})$  does not minimize the problem (FP). Therefore there exist  $p \in S$ ,  $q \in \pi(p)$  and  $r \in \omega(p)$  satisfying

$$\frac{q}{r} < \frac{q'}{r'}.$$

Since  $\xi' = \frac{q'}{r'}$ ,

$$\frac{q}{r} < \xi'.$$

So,

$$q - \xi' r < 0.$$

Hence,

$$\langle q^*, q - \xi' r \rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^j} \neq q^* \in \mathbb{R}_+^j.$$

From the requirements of (PD),

$$q_i - \xi'_i r'_i \geq 0, \forall i = 1, \dots, j.$$

Therefore,

$$\langle q^*, q' - \xi' r' \rangle \geq 0.$$

As  $p \in S$ , there exists an element

$$r \in \psi(p) \cap (-\mathbb{R}_+^l).$$

Therefore,

$$\langle r^*, r \rangle \leq 0.$$

We have

$$\langle r^*, s - s' \rangle \leq 0, \text{ as } \langle r^*, s' \rangle = 0.$$

Hence,

$$\langle q^*, q - \xi' r - (q' - \xi' r') \rangle + \langle r^*, s - s' \rangle < 0. \quad (3.9)$$

As  $\pi$  is  $\sigma_1\text{-}\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ ,  $-\xi'\omega$  is  $\sigma_2\text{-}\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3\text{-}\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^l}$ , on  $M$ ,

$$\pi(p) - q' \subseteq \overrightarrow{\Delta}\pi(p', q')(\chi'_{p',p}(0+)) + \sigma_1\|p - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

$$(-\xi'\omega)(p) + \xi'r \subseteq \overrightarrow{\Delta}(-\xi'\omega)(p', -\xi'r')(\chi'_{p',p}(0+)) + \sigma_2\|p - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

and

$$\psi(p) - s' \subseteq \overrightarrow{\Delta}\psi(p', s')(\chi'_{p',p}(0+)) + \sigma_3\|p - p'\|^2\mathbf{1}_{\mathbb{R}^l} + \mathbb{R}_+^l.$$

Hence,

$$q - q' \in \overrightarrow{\Delta}\pi(p', q')(\chi'_{p',p}(0+)) + \sigma_1\|p - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

$$-\xi'r + \xi'r' \in \overrightarrow{\Delta}(-\xi'\omega)(p', -\xi'r')(\chi'_{p',p}(0+)) + \sigma_2\|p - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

and

$$s - s' \in \overrightarrow{\Delta}\psi(p', s')(\chi'_{p',p}(0+)) + \sigma_3\|p - p'\|^2\mathbf{1}_{\mathbb{R}^l} + \mathbb{R}_+^l.$$

From the requirements of (PD) and (3.7), we have

$$\langle q^*, q - \xi'r - (q' - \xi'r') \rangle + \langle r^*, s - s' \rangle \geq 0,$$

which is in conflict with (3.9). So,  $(p', \frac{q'}{r'})$  minimizes weakly the problem (FP).  $\square$

### 3.3. Mond-Weir kind dual

We assume the Mond-Weir kind dual (MWD) connected with the problem (FP).

$$\text{Maximize } \frac{q'}{r'},$$

s. t.,

$$\left\langle q^*, \overrightarrow{\Delta}(r'\pi)(p', q'r')(\chi'_{p',p}(0+)) + \overrightarrow{\Delta}(-q'\omega)(p', -q'r')(\chi'_{p',p}(0+)) \right\rangle \quad (\text{MWD})$$

$$+ \left\langle r^*, \overrightarrow{\Delta}\psi(p', s')(\chi'_{p',p}(0+)) \right\rangle \geq 0, \forall p \in M,$$

$$\langle r^*, s' \rangle \geq 0,$$

$$p' \in M, q' \in \pi(p'), r' \in \omega(p'), s' \in \psi(p'), q^* \in \mathbb{R}_+^j, r^* \in \mathbb{R}_+^l \text{ and } \langle q^*, \mathbf{1}_{\mathbb{R}^j} \rangle = 1.$$

A point  $(p', q', r', s', q^*, r^*)$  which fulfills all the constraints of (MWD) is referred to as feasible to (MWD).

**Definition 14.** A point  $(p', q', r', s', q^*, r^*)$  in the set of feasibility of the problem (MWD) is referred to as a weak maximizer of (MWD) if there exists no point  $(p, q, r, s, q_1^*, r_1^*)$  in the set of feasibility of (MWD) satisfying

$$\frac{q}{r} - \frac{q'}{r'} \in \text{int}(\mathbb{R}_+^j).$$

**Theorem 8. (Weak Duality)** Let  $M$  be an arcwisely connected subset of  $\alpha$ ,  $\bar{p}$  be an element of the set of feasibility  $S$  of the problem (FP), and  $(p', q', r', s', q^*, r^*)$  be feasible to the problem (MWD). Take into account that  $r'\pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ ,  $-q'\omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^l}$ , on  $M$ , satisfying (3.7).

Then,

$$\frac{\pi(\bar{p})}{\omega(\bar{p})} - \frac{q'}{r'} \subseteq \mathbb{R}^j \setminus -\text{int}(\mathbb{R}_+^j).$$

*Proof.* Presume that for some  $\bar{q} \in \pi(\bar{p})$  and  $\bar{r} \in \omega(\bar{p})$ ,

$$\frac{\bar{q}}{\bar{r}} - \frac{q'}{r'} \in -\text{int}(\mathbb{R}_+^j).$$

Therefore,

$$\frac{\bar{q}}{\bar{r}} < \frac{q'}{r'}.$$

So,

$$\bar{q}r' - q'\bar{r} < 0.$$

Hence,

$$\langle q^*, \bar{q}r' - q'\bar{r} \rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^j} \neq q^* \in \mathbb{R}_+^j.$$

As  $\bar{p} \in S$ , we have

$$\psi(\bar{p}) \cap (-\mathbb{R}_+^l) \neq \emptyset.$$

We select  $\bar{s} \in \psi(\bar{p}) \cap (-\mathbb{R}_+^l)$ .

So,

$$\langle r^*, \bar{s} \rangle \leq 0.$$

From the requirements of (MWD), we have

$$\langle r^*, s' \rangle \geq 0.$$

So,

$$\langle r^*, \bar{s} - s' \rangle = \langle r^*, \bar{s} \rangle - \langle r^*, s' \rangle \leq 0.$$

Hence,

$$\langle q^*, \bar{q}r' - q'\bar{r} \rangle + \langle r^*, \bar{s} - s' \rangle < 0. \quad (3.10)$$

As  $r'\pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ ,  $-q'\omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^l}$ , on  $M$ ,

$$r'\pi(\bar{p}) - q'r' \subseteq \vec{\Delta}\pi(p', q')(X'_{p', \bar{p}}(0+)) + \sigma_1\|\bar{p} - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

$$(-q'\omega)(\bar{p}) + q'r' \subseteq \vec{\Delta}(-q'\omega)(p', -q'r')(X'_{p', \bar{p}}(0+)) + \sigma_2\|\bar{p} - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

and

$$\psi(\bar{p}) - s' \subseteq \vec{\Delta}\psi(p', s')(X'_{p', \bar{p}}(0+)) + \sigma_3\|\bar{p} - p'\|^2\mathbf{1}_{\mathbb{R}^l} + \mathbb{R}_+^l.$$

Hence,

$$\begin{aligned}\bar{q}r' - q'r' &\in \vec{\Delta}(r'\pi)(p', q'r')(\chi'_{p', \bar{p}}(0+)) + \sigma_1 \|\bar{p} - p'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j, \\ -r'\bar{r} + q'r' &\in \vec{\Delta}(-q'\omega)(p', -q'r')(\chi'_{p', \bar{p}}(0+)) + \sigma_2 \|\bar{p} - p'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,\end{aligned}$$

and

$$\bar{s} - s' \in \vec{\Delta}\psi(p', s')(\chi'_{p', \bar{p}}(0+)) + \sigma_3 \|\bar{p} - p'\|^2 \mathbf{1}_{\mathbb{R}^l} + \mathbb{R}_+^l.$$

From the requirements of (MWD) and (3.7),

$$\langle q^*, \bar{q}r' - q'r' \rangle + \langle r^*, \bar{s} - s' \rangle \geq 0,$$

which is in conflict with (3.10).

Therefore,

$$\frac{\bar{q}}{r} - \frac{q'}{r'} \notin -\text{int}(\mathbb{R}_+^j).$$

Since  $\bar{q} \in F(\bar{p})$  is chosen arbitrarily,

$$\frac{\pi(\bar{p})}{\omega(\bar{p})} - \frac{q'}{r'} \subseteq \mathbb{R}^j \setminus -\text{int}(\mathbb{R}_+^j).$$

□

**Theorem 9. (Strong Duality)** Let  $(p', \frac{q'}{r'})$  minimize weakly the problem (FP) and  $s' \in \psi(p') \cap (-\mathbb{R}_+^l)$ . Take into account that for some  $(q^*, r^*) \in \mathbb{R}_+^j \times \mathbb{R}_+^l$ , together with  $\langle q^*, \mathbf{1}_{\mathbb{R}^j} \rangle = 1$ , (3.4) and (3.6) are fulfilled at  $(p', q', r', s', q^*, r^*)$ . Then  $(p', q', r', s', q^*, r^*)$  is feasible to the problem (MWD). Furthermore, If the Theorem 8 between the problems (FP) and (MWD) remains, then  $(p', q', r', s', q^*, r^*)$  maximizes weakly (MWD).

*Proof.* As (3.4) and (3.6) are fulfilled at  $(p', q', r', s', q^*, r^*)$ , we have

$$\begin{aligned}\langle q^*, \vec{\Delta}(r'\pi)(p', q'r')(\chi'_{p', p}(0+)) + \vec{\Delta}(-q'\omega)(p', -q'r')(\chi'_{p', p}(0+)) \rangle \\ + \langle r^*, \vec{\Delta}\psi(p', s')(\chi'_{p', p}(0+)) \rangle \geq 0, \forall p \in M,\end{aligned}$$

and

$$\langle r^*, s' \rangle = 0.$$

So,  $(p', q', r', s', q^*, r^*)$  is feasible to (MWD). Presume that the Theorem 8 between (FP) and (MWD) stays and  $(p', q', r', s', q^*, r^*)$  does not maximize weakly (MWD). Then there exists a point  $(p, q, r, s, q_1^*, r_1^*)$  in the set of feasibility of (MWD) satisfying

$$\frac{q'}{r'} < \frac{q}{r},$$

which is in conflict with the Theorem 8 between (FP) and (MWD). Accordingly,  $(p', q', r', s', q^*, r^*)$  maximizes weakly (MWD). □



**Theorem 10. (Converse Duality)** Let  $M$  be an arcwisely connected subset of  $\alpha$  and  $(p', q', r', s', q^*, r^*)$  be feasible to the problem (MWD). Take into account that  $r'\pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ ,  $-q'\omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^l}$ , on  $M$ , satisfying (3.7). If  $p'$  is an element of the set of feasibility  $S$  of the problem (FP), then  $(p', \frac{q'}{r'})$  minimizes weakly the problem (FP).

*Proof.* Suppose  $(p', \frac{q'}{r'})$  does not minimize the problem (FP). Therefore there exist  $p \in S$ ,  $q \in \pi(p)$  and  $r \in \omega(p)$  satisfying

$$\frac{q}{r} < \frac{q'}{r'}.$$

So,

$$qr' - q'r < 0.$$

Therefore,

$$\langle q^*, qr' - q'r \rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^j} \neq q^* \in \mathbb{R}_+^j.$$

As  $p \in S$ ,

$$\psi(p) \cap (-\mathbb{R}_+^l) \neq \emptyset.$$

We select  $r \in \psi(p) \cap (-\mathbb{R}_+^l)$ .

So,

$$\langle r^*, r \rangle \leq 0.$$

From the requirements of (WD),

$$\langle r^*, s' \rangle \geq 0.$$

So,

$$\langle r^*, s - s' \rangle = \langle r^*, r \rangle - \langle r^*, s' \rangle \leq 0.$$

Hence,

$$\langle q^*, qr' - q'r \rangle + \langle r^*, s - s' \rangle < 0. \quad (3.11)$$

As  $r'\pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ ,  $-q'\omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^l}$ , on  $M$ ,

$$r'\pi(p) - q'r' \subseteq \overrightarrow{\Delta}\pi(p', q')(x'_{p',p}(0+)) + \sigma_1\|p - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

$$(-q'\omega)(p) + q'r' \subseteq \overrightarrow{\Delta}(-q'\omega)(p', -q'r')(x'_{p',p}(0+)) + \sigma_2\|p - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

and

$$\psi(p) - s' \subseteq \overrightarrow{\Delta}\psi(p', s')(x'_{p',p}(0+)) + \sigma_3\|p - p'\|^2\mathbf{1}_{\mathbb{R}^l} + \mathbb{R}_+^l.$$

Hence,

$$yz' - q'r' \in \overrightarrow{\Delta}(r'\pi)(p', q'r')(x'_{p',p}(0+)) + \sigma_1\|p - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

$$-r'z + q'r' \in \overrightarrow{\Delta}(-q'\omega)(p', -q'r')(x'_{p',p}(0+)) + \sigma_2\|p - p'\|^2\mathbf{1}_{\mathbb{R}^j} + \mathbb{R}_+^j,$$

and

$$s - s' \in \overrightarrow{\Delta}\psi(p', s')(x'_{p',p}(0+)) + \sigma_3\|p - p'\|^2\mathbf{1}_{\mathbb{R}^l} + \mathbb{R}_+^l.$$

So, from the requirements of (WD) and (3.7),

$$\langle q^*, qr' - q'r \rangle + \langle r^*, s - s' \rangle \geq 0,$$

which is in conflict with (3.11). Therefore  $(p', \frac{q'}{r'})$  minimizes weakly (FP).  $\square$

### 3.4. Wolfe kind dual

We assume the Wolfe kind dual (WD) connected with the problem (FP).

$$\begin{aligned} & \text{Maximize } \frac{q' + \langle r^*, s' \rangle \mathbf{1}_{\mathbb{R}^j}}{r'}, \\ & \text{s. t.,} \\ & \left\langle q^*, \overrightarrow{\Delta}(r'\pi)(p', q'r')(\chi'_{p',p}(0+)) + \overrightarrow{\Delta}(-q'\omega)(p', -q'r')(\chi'_{p',p}(0+)) \right\rangle \\ & + \left\langle r^*, \overrightarrow{\Delta}\psi(p', s')(\chi'_{p',p}(0+)) \right\rangle \geq 0, \forall p \in M, \\ & p' \in M, q' \in \pi(p'), r' \in \omega(p'), s' \in \psi(p'), q^* \in \mathbb{R}_+^j, r^* \in \mathbb{R}_+^l \text{ and } \langle q^*, \mathbf{1}_{\mathbb{R}^j} \rangle = 1. \end{aligned} \quad (\text{WD})$$

A point  $(p', q', r', s', q^*, r^*)$  which fulfills all the constraints of (WD) is referred to as feasible to (WD).

**Definition 15.** A point  $(p', q', r', s', q^*, r^*)$  in the set of feasibility of the problem (WD) is referred to as a weak maximizer of (WD) if there exists no point  $(p, q, r, s, q_1^*, r_1^*)$  in the set of feasibility of (WD) satisfying

$$\frac{q + \langle r_1^*, s \rangle \mathbf{1}_{\mathbb{R}^j}}{r} - \frac{q' + \langle r^*, s' \rangle \mathbf{1}_{\mathbb{R}^j}}{r'} \in \text{int}(\mathbb{R}_+^j).$$

**Theorem 11. (Weak Duality)** Let  $M$  be an arcwisely connected subset of  $\alpha$ ,  $\bar{p}$  be an element of the set of feasibility  $S$  of the problem (FP) and  $(p', q', r', s', q^*, r^*)$  be feasible to the problem (WD). Take into account that  $r'\pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ ,  $-q'\omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^l}$ , on  $M$ , satisfying (3.7).

Then,

$$\frac{\pi(\bar{p})}{\omega(\bar{p})} - \frac{q' + \langle r^*, s' \rangle \mathbf{1}_{\mathbb{R}^j}}{r'} \subseteq \mathbb{R}^j \setminus -\text{int}(\mathbb{R}_+^j).$$

*Proof.* The proof is comparable to that of Theorems 5 and 8. It is therefore omitted.  $\square$

**Theorem 12. (Strong Duality)** Let  $(p', \frac{q'}{r'})$  minimize weakly the problem (FP) and  $s' \in \psi(p') \cap (-\mathbb{R}_+^l)$ . Take into account that for some  $(q^*, r^*) \in \mathbb{R}_+^j \times \mathbb{R}_+^l$ , together with  $\langle q^*, \mathbf{1}_{\mathbb{R}^j} \rangle = 1$ , (3.4) and (3.6) are fulfilled at  $(p', q', r', s', q^*, r^*)$ . Then  $(p', q', r', s', q^*, r^*)$  is feasible to the problem (WD). Furthermore, If the Theorem 11 between (FP) and (WD) remains, then  $(p', q', r', s', q^*, r^*)$  maximizes weakly (WD).

*Proof.* The proof is comparable to that of Theorems 6 and 9. It is therefore omitted.  $\square$

**Theorem 13. (Converse Duality)** Let  $M$  be an arcwisely connected subset of  $\alpha$ ,  $(p', q', r', s', q^*, r^*)$  be feasible to the problem (WD) and  $\langle r^*, s' \rangle \geq 0$ . Take into account that  $r'\pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ ,  $-q'\omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^l}$ , on  $M$ , satisfying (3.7). If  $p'$  is an element of the set of feasibility  $S$  of the problem (FP), then  $(p', \frac{q'}{r'})$  minimizes weakly the problem (FP).

*Proof.* The proof is comparable to that of Theorems 7 and 10. It is therefore omitted.  $\square$

### 3.5. Mixed kind dual

We assume the mixed kind dual (MD) connected with the problem (FP).

$$\begin{aligned}
 & \text{maximize } \frac{q' + \langle r^*, s' \rangle \mathbf{1}_{\mathbb{R}^j}}{r'}, \\
 & \text{s. t.,} \\
 & \left\langle q^*, \vec{\Delta}(r'\pi)(p', q'r')(\chi'_{p',p}(0+)) + \vec{\Delta}(-q'\omega)(p', -q'r')(\chi'_{p',p}(0+)) \right\rangle \\
 & + \left\langle r^*, \vec{\Delta}\psi(p', s')(\chi'_{p',p}(0+)) \right\rangle \geq 0, \forall p \in M, \\
 & \langle r^*, s' \rangle \geq 0, \\
 & p' \in M, q' \in \pi(p'), r' \in \omega(p'), s' \in \psi(p'), q^* \in \mathbb{R}_+^j, r^* \in \mathbb{R}_+^l \text{ and } \langle q^*, \mathbf{1}_{\mathbb{R}^j} \rangle = 1.
 \end{aligned} \tag{MD}$$

A point  $(p', q', r', s', q^*, r^*)$  meeting all the requirements of the problem (MD) is referred to as feasible to (MD).

**Definition 16.** A point  $(p', q', r', s', q^*, r^*)$  in the set of feasibility of the problem (MD) is referred to as a weak maximizer of (MD) if there exists no point  $(p, q, r, s, q_1^*, r_1^*)$  in the set of feasibility of (MD) satisfying

$$\frac{q + \langle r_1^*, s \rangle \mathbf{1}_{\mathbb{R}^j}}{r} - \frac{q' + \langle r^*, s' \rangle \mathbf{1}_{\mathbb{R}^j}}{r'} \in \text{int}(\mathbb{R}_+^j).$$

**Theorem 14. (Weak Duality)** Let  $M$  be an arcwisely connected subset of  $\alpha$ ,  $\bar{p}$  be an element of the set of feasibility  $S$  of the problem (FP) and  $(p', q', r', s', q^*, r^*)$  be feasible to the problem (MD). Take into account that  $r'\pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ ,  $-q'\omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^l}$ , on  $M$ , satisfying (3.7).

Then,

$$\frac{\pi(\bar{p})}{\omega(\bar{p})} - \frac{q' + \langle r^*, s' \rangle \mathbf{1}_{\mathbb{R}^j}}{r'} \subseteq \mathbb{R}^j \setminus -\text{int}(\mathbb{R}_+^j).$$

*Proof.* The proof is comparable to that of Theorems 5 and 8. It is therefore omitted.  $\square$

**Theorem 15. (Strong Duality)** Let  $(p', \frac{q'}{r'})$  minimize weakly the problem (FP) and  $s' \in \psi(p') \cap (-\mathbb{R}_+^l)$ . Take into account that for some  $(q^*, r^*) \in \mathbb{R}_+^j \times \mathbb{R}_+^l$ , together with  $\langle q^*, \mathbf{1}_{\mathbb{R}^j} \rangle = 1$ , (3.4) and (3.6) are fulfilled at  $(p', q', r', s', q^*, r^*)$ . Then  $(p', q', r', s', q^*, r^*)$  is feasible to the problem (MD). Furthermore, If the Theorem 14 between (FP) and (MD) remains, then  $(p', q', r', s', q^*, r^*)$  maximizes weakly (MD).

*Proof.* The proof is comparable to that of Theorems 6 and 9. It is therefore omitted.  $\square$

**Theorem 16. (Converse Duality)** Let  $M$  be an arcwisely connected subset of  $\alpha$  and  $(p', q', r', s', q^*, r^*)$  be feasible to the problem (MD). Take into account that  $r'\pi$  is  $\sigma_1$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ ,  $-q'\omega$  is  $\sigma_2$ - $\mathbb{R}_+^j$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^j}$ , and  $\psi$  is  $\sigma_3$ - $\mathbb{R}_+^l$ -arcwisely connected w.r.t.  $\mathbf{1}_{\mathbb{R}^l}$ , on  $M$ , satisfying (3.7). If  $p'$  is an element of the set of feasibility  $S$  of the problem (FP), then  $(p', \frac{q'}{r'})$  minimizes weakly the problem (FP).

*Proof.* The proof is comparable to that of Theorems 7 and 10. It is therefore omitted.  $\square$

To clarify the conclusions, we have included the following example.

**Example 2.** Let  $\alpha = \mathbb{R}^2$  and  $M = \{(t, 0) : t \in [-1, 1]\} \subseteq \mathbb{R}^2$ . Let  $p = (p_1, p_2) \in \mathbb{R}^2$  and  $p' = (p'_1, p'_2) \in \mathbb{R}^2$ . Let us consider a continuous arc  $\chi_{p',p} : [0, 1] \rightarrow \mathbb{R}^2$  defined by

$$\chi_{p',p}(\xi) = (1 - \xi^2)p' + \xi^2 p, \forall \xi \in [0, 1].$$

Clearly,  $M$  is an arcwisely connected subset of  $\alpha$ . Now,

$$\chi'_{p',p}(0+) = \lim_{\xi \rightarrow 0+} \frac{\chi_{p',p}(\xi) - \chi_{p',p}(0)}{\xi} = (0, 0).$$

We consider three set-valued maps  $\pi : M \subseteq \mathbb{R}^2 \rightarrow 2^{\mathbb{R}^2}$ ,  $\omega : M \subseteq \mathbb{R}^2 \rightarrow 2^{\mathbb{R}^2}$ , and  $\psi : M \subseteq \mathbb{R}^2 \rightarrow 2^{\mathbb{R}^2}$ , with  $\text{domain}(\pi) = \text{domain}(\omega) = \text{domain}(\psi) = M$ , defined by

$$\pi(t, 0) = \begin{cases} \{(x - 10t^2, x^2 - 10t^2) : x \geq 0\}, & \text{if } 0 \leq t \leq 1, \\ \{(x - 10t^2, x - 10t^2) : x < 0\}, & \text{if } -1 \leq t < 0, \end{cases}$$

$$\omega(t, 0) = (1, 1), \forall t \in [-1, 1],$$

and

$$\psi(t, 0) = \begin{cases} \{x^2 + 11t^2 : x \geq 0\}, & \text{if } 0 \leq t \leq 1, \\ \{x + 11t^2 : x > 0\}, & \text{if } -1 \leq t < 0. \end{cases}$$

We can show that  $\pi$  is  $(-10)\text{-}\mathbb{R}_+^2$ -arcwisely connected w.r.t.  $(1, 1)$ ,  $-\xi'\omega$  is  $0\text{-}\mathbb{R}_+^2$ -arcwisely connected w.r.t.  $(1, 1)$  for any  $\xi' \in \mathbb{R}^2$ , and  $\psi$  is  $11\text{-}\mathbb{R}_+$ -arcwisely connected w.r.t.  $1$  on  $A$ . So  $\sigma_1 = -10$ ,  $\sigma_2 = 0$ , and  $\sigma_3 = 11$ . Now, we have

$$\begin{aligned} \text{epigrp}(\pi) &= \{(u, v) \in \mathbb{R}^2 \times \mathbb{R}^2 : u \in A, v \in \pi(u) + \mathbb{R}_+^2\} \\ &= \{((t, 0), (v_1, v_2)) : u = (t, 0), v = (v_1, v_2), t \in [-1, 1], v \in \pi(u) + \mathbb{R}_+^2\} \\ &= \{((t, 0), (v_1, v_2)) : 0 \leq t \leq 1, v_1 \geq x - 10t^2, v_2 \geq x^2 - 10t^2, x \geq 0\} \\ &\cup \{((t, 0), (v_1, v_2)) : -1 \leq t < 0, v_1 \geq x - 10t^2, v_2 \geq x - 10t^2, x < 0\}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\eta(\text{epigrp}(\pi), ((t', 0), (v'_1, v'_2))) \\ &= \begin{cases} \mathbb{R}_+ \times \{0\} \times \mathbb{R}_+ \times \mathbb{R}_+, & \text{if } t' = 0, v'_1 = v'_2 = 0, \\ \mathbb{R} \times \{0\} \times \mathbb{R} \times \mathbb{R}, & \text{otherwise.} \end{cases} \end{aligned}$$

It is obvious that  $\text{I-min}\{(v_1, v_2) : ((t, 0), (v_1, v_2)) \in \eta(\text{epigrp}(\pi), ((t', 0), (v'_1, v'_2)))\}$  exists for  $0 \leq t \leq 1$  if and only if  $t' = 0, v'_1 = 0$ , and  $v'_2 = 0$ . Clearly,

$$\text{I-min}\{(v_1, v_2) : ((t, 0), (v_1, v_2)) \in \eta(\text{epigrp}(\pi), ((0, 0), (0, 0)))\} = \{(0, 0)\}, \text{ for } 0 \leq t \leq 1.$$

Now, we have

$$\overrightarrow{\Delta}\pi((0, 0), (0, 0))(t, 0) = \text{I-min}\{(v_1, v_2) : ((t, 0), (v_1, v_2)) \in \eta(\text{epigrp}(\pi), ((0, 0), (0, 0)))\}.$$

So,  $\pi$  is contingent epidifferentiable only at  $((0, 0), (0, 0))$  with

$$\overrightarrow{\Delta}\pi((0, 0), (0, 0))(t, 0) = \{(0, 0)\}, \forall t, \text{ with } 0 \leq t \leq 1.$$

Similarly, we can show that  $\omega$  is contingent epidifferentiable everywhere on  $M$  and  $\psi$  is contingent epidifferentiable only at  $((0, 0), 0)$  with

$$\vec{\Delta}\psi((0, 0), 0)(t, 0) = \{0\}, \forall t, \text{ with } 0 \leq t \leq 1.$$

Let  $p' = (0, 0)$ ,  $q' = (0, 0) \in \pi(p')$ ,  $r' = (1, 1) \in \omega(p')$ , and  $s' = 0 \in \psi(p') \cap (-\mathbb{R}_+)$ . So,  $\xi' = \frac{q'}{r'} = (0, 0)$ . The feasible set of the problem (FP) is  $S = \{(t, 0) : \psi(t, 0) \cap (-\mathbb{R}_+) \neq \emptyset\} = \{(0, 0)\}$ . Therefore,  $(p', \frac{q'}{r'}) = ((0, 0), (0, 0))$  minimizes weakly the problem (FP).

Now, we have

$$\vec{\Delta}\pi(p', q')(\chi'_{p', p}(0+)) = \vec{\Delta}\pi(p', q')(0, 0) = (0, 0),$$

$$\vec{\Delta}(-\xi' \omega)(p', -\xi' r')(\chi'_{p', p}(0+)) = \vec{\Delta}(-\xi' \omega)(p', -\xi' r')(0, 0) = (0, 0),$$

and

$$\vec{\Delta}\psi(p', s')(\chi'_{p', p}(0+)) = \vec{\Delta}\psi(p', s')(0, 0) = 0.$$

It is clear that for  $q^* = (\frac{1}{2}, \frac{1}{2})$  and  $r^* = 1$ , the sufficient optimality conditions (3.1)–(3.4) are satisfied and  $\langle q^*, (1, 1) \rangle = 1$ . So  $(p', q', r', s', q^*, r^*)$  is feasible to the (MWD). Again,  $(\sigma_1 + \sigma_2)\langle q^*, (1, 1) \rangle + \sigma_3\langle r^*, 1 \rangle = 1 \geq 0$ . Hence the weak duality Theorem 7 holds between (FP) and (MWD). We prove that  $(p', q', r', s', q^*, r^*)$  maximizes weakly (MWD). Let any point feasible to the Mond-Weir type dual (MWD) be  $(p, q, r, s, q_1^*, r_1^*)$  where  $p \in M, q \in \pi(p), r \in \psi(p), s \in \psi(p) \cap (-\mathbb{R}_+), q_1^* \in \mathbb{R}_+^2, r_1^* \in \mathbb{R}_+$ , and  $\langle q_1^*, (1, 1) \rangle = 1$ . Since  $\pi$  and  $\psi$  are contingent epidifferentiable only at  $((0, 0), (0, 0))$  and the conditions of the Mond-Weir type dual (MWD) are satisfied at the point  $(p, q, r, s, q_1^*, r_1^*)$ , we have  $p = (0, 0), q = (0, 0), r = (1, 1)$ , and  $s = 0$ . Hence  $(p', q', r', s', q^*, r^*)$ , with  $p' = (0, 0), q' = (0, 0), r' = (1, 1)$ , and  $s' = 0$ , maximizes weakly the problem (MWD). Hence Theorem 8 is satisfied. Similarly, we can verify the weak, strong, and converse duality results for parametric (PD), Wolfe (WD), and mixed (MD) kinds for the problem (FP).  $\square$

#### 4. Conclusions

In this paper, we determine the KKT conditions of sufficiency for the SVFP (FP) via contingent epidifferentiation supposition. We accordingly present the duals of parametric (PD), Mond-Weir (MWD), Wolfe (WD), and mixed (MD) kinds and derive the associated strong, weak, and converse theorems of duality under cone arcwisely connectivity supposition.

#### Conflict of interest

The authors declare no conflict of interest.

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