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A higher-order numerical scheme for system of two-dimensional nonlinear fractional Volterra integral equations with uniform accuracy

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Abstract: We give a modified block-by-block method for the nonlinear fractional order Volterra integral equation system by using quadratic Lagrangian interpolation based on the classical block-by-block method. The core of the method is that we divide its domain into a series of subdomains, that is, block it, and use piecewise quadratic Lagrangian interpolation on each subdomain to approximate $\boldsymbol{\kappa}(x, y, s, r, u(s, r))$. Our proposed method has uniform accuracy and its convergence order is $O(h_x^{4-\alpha} + h_y^{4-\beta})$. We give a strict proof for the error analysis of the method, and give several numerical examples to verify the correctness of the theoretical analysis.

Keywords: nonlinear Volterra integral equation system; higher-order numerical scheme; convergence analysis
Mathematics Subject Classification: 65R20, 65D30, 65L12

1. Introduction

The concepts of integral equations have attracted much interest in recent years for analytical and numerical treatments. Nonlinear phenomena, which appear in many scientific fields such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics, can also be modelled by integral equations. The Volterra integral equations arise from the mathematical modelling of the spatiotemporal development of an epidemic and various physical and biological models. We consider the following system of nonlinear Volterra integral equations (VIEs):

$$\mathbf{u}(x, y) = \mathbf{g}(x, y) + \int_a^x \int_c^y \frac{\boldsymbol{\kappa}(x, y, s, r, u(s, r))}{(x - s)^\alpha (y - r)^\beta} drds, (x, y) \in U, 0 < \alpha, \beta < 1, \tag{1.1}$$

where $\mathbf{u}(x, y) = (u_1(x, y), u_2(x, y), \ldots, u_m(x, y))^{\mathrm{T}}$, $\mathbf{g}(x, y) = (g_1(x, y), g_2(x, y), \ldots, g_m(x, y))^{\mathrm{T}}$, $\boldsymbol{\kappa}(x, y, s, r, u(s, r)) = (\kappa_1(x, y, s, r, u(s, r)), \kappa_2(x, y, s, r, u(s, r)), \ldots, \kappa_m(x, y, s, r, u(s, r)))^{\mathrm{T}}$,and $u_i(x, y)$,

$i = 1, 2, \ldots, m$ is an unknown function, which is defined on $U = [a, b] \times [c, d]$, $g_i(x, y)$ and $k_{i,j}(x, y, s, r, u_j(s, r)), i, j = 1, 2 \ldots, m$ are known continuous functions that are defined on U and $U \times U \times R$, respectively.

Ahmad et al. [1] present the operational matrices of fractional integration of Haar wavelets to solve systems of two-dimensional fractional partial Volterra integral equations (VIEs) with the convergence analysis. Karimi et al. [2] present numerical solution for a system of two-dimensional VIEs by Legendre wavelets with convergence analysis. Liu et al. [3] report a highly accurate meshfree approach based on the barycentric Lagrange basis functions to solve the linear and nonlinear multi-dimensional systems of Fredholm integral equations of the second kind with the convergence analysis and error estimation. Xie et al. [4] propose a numerical scheme for approximating the solutions of nonlinear system of fractional-order Volterra-Fredholm integral-differential equations (VFIDEs) based on the orthogonal functions. Pedro et al. [5] propose an approximation method for solving second kind VIE systems by radial basis functions with convergence analysis. Yaghoobnia et al. [6] use the Bernstein multi-scaling polynomials to solve numerical solution for system of nonlinear integral equations. Jafarian et al. [7] use an artificial neural network to construct the approximate solution of second kind linear VIEs system. Cao et al. [8] present a general technique to construct high order schemes for the numerical solution of the fractional differential equations based on the so-called block-by-block approach with the stability and convergence. Katani et al. [9] give numerical methods for solving system of VIEs with convergence analysis. Sorkun et al. [10] present an approximate method to solve the linear VIE systems by transforming the integral system into the matrix equation with the help of Taylor series. Berenguer et al. [11] the approximating solution for a system of nonlinear VIEs of the second kind with the aid of biorthogonal systems in adequate Banach spaces. Maleknejad et al. [12] use operational matrices of piecewise constant orthogonal functions to solve singular VIEs system of convolution type based on Laplace transform. Tahmasbi et al. [13] present a method for numerical solution of linear VIEs system based on the power series method. Rabbani et al. [14] reduce the system of integral equations to a linear system of ordinary differential equations based on expansion method. Yalçinbaş et al. [15] implement a new approximate method for solving system of nonlinear VIEs based on Taylor series. Researchers can read more references in this area, such as: spectral collocation method [16],hat basis functions [17, 18], decomposition method [19], Runge-Kutta method [20], reproducing kernel method [21], Chebyshev wavelets [22], collocation methods [23], fractional Lanczos vector polynomials [24], two-variable Vieta-Fibonacci polynomials [25], finite difference method [26] and so on.

Block-by-block method is a classical numerical method to solving the VIEs, we will construct a high order uniform accuracy scheme for VIEs systems based on the idea of [8]. The numerical scheme is uniform accuracy high order convergence scheme which is easy to implement given the fact that it is explicit. For the convergence of the numerical scheme, we creatively analyze it by introducing reasonable dimensionality reduction techniques. The method of analyzing the convergence of numerical schemes in this paper is general, and it can provide a good reference method for dealing with similar problems. The present method and the domain decomposition method similarly divide the domain into subdomains. However, the domain decomposition method can solve all subdomains in parallel. The present method must solve all subdomains in a specific order. Therefore, the domain decomposition method can efficiently solve complex domains, and the present method can only solve some regular domains.

The rest of the paper is organized as follows. In Section 2, the higher-order numeric scheme and details of its construction are given. For the higher-order numerical scheme introduced in Section 2, its truncation error estimates and convergence analysis are given in Sections 3 and 4, respectively. Some numerical results are given in Section 5. Final section is for some conclusions.

2. Construction of higher order numerical scheme

We carefully observe the structure of Eq (1.1), which can be written as a nonlinear Volterra integral equation system in component form as follows:

$$u_i(x,y) = g_i(x,y) + \int_a^x \int_c^y \frac{\kappa_i\,(x,y,s,r,u_1(s,r),\ldots,u_m(s,r))}{(x-s)^{\alpha}(y-r)^{\beta}} drds, i = 1,2,\ldots,m, \tag{2.1}$$

where $(x,y) \in U, 0 < \alpha, \beta < 1$. The $\kappa_i\,(x,y,s,r,u_1(s,r),\ldots,u_m(s,r))$ in (2.1) is defined by the following:

$$\kappa_i\,(x,y,s,r,u_1(s,r),\ldots,u_m(s,r)) = \sum_{j=1}^{m} \kappa_{i,j}\left(x,y,s,r,u_j(s,r)\right), \tag{2.2}$$

and $\kappa_{i,j}(x,y,s,r,v)$ satisfies the Lipschitz condition with respect to the fifth variable, that is

$$|\kappa_{i,j}(x,y,s,r,v_1) - \kappa_{i,j}(x,y,s,r,v_2)| \leq L|v_1 - v_2|, \quad L > 0. \tag{2.3}$$

In this part, we introduce the piecewise quadratic Lagrangian interpolation to approximate $\kappa_{i,j}(x,y,s,r,u_j(s,r)), i, j = 1,\ldots,m$, and then obtain the higher-order numerical scheme of (2.1). First, divide D into $2N \times 2L$ equal blocks, and set $x_k = a + kh_x, y_l = c + lh_y, k = 0,\ldots,2N; l = 0,\ldots,2L$, with $h_x = (b-a)/(2N), h_y = (d-c)/(2L)$. For simplicity, we let the approximate solution of Eq (2.1) at point (x_v,y_n) be $u^i_{v,n}$, $g_i(x_v,y_n) = g^i_{v,n}$, $\kappa_{i,j}(x_v,y_n,s,r,u_j(s,r)) = \kappa^{i,j}_{v,n}(s,r,u_j(s,r))$ and $u^i_{v,0} = g^i_{v,0}, u^i_{0,n} = g^i_{0,n}$. Next, we discretize the integration interval of (2.1) and use piecewise quadratic Lagrangian interpolation on the discrete integral interval, where quadratic Lagrangian interpolation is defined as:

$$J^{[y_l,y_{l+2}]}_{[x_k,x_{k+2}]}\kappa^{i,j}_{v,n}(s,r,u_j(s,r)) = \sum_{e=0}^{2}\sum_{f=0}^{2} \psi_{e,k}(s)\gamma_{f,l}(r)\kappa^{i,j}_{v,n}(x_{k+e},y_{l+f},u^j_{k+e,l+f}), k,l \in \mathbb{N},$$

where

$$\psi_{0,k}(s) = \frac{(s-x_{k+1})(s-x_{k+2})}{2h_x^2}, \psi_{1,k}(s) = \frac{(s-x_k)(s-x_{k+2})}{-h_x^2}, \psi_{2,k}(s) = \frac{(s-x_k)(s-x_{k+1})}{2h_x^2},$$
$$\gamma_{0,l}(r) = \frac{(r-y_{l+1})(r-y_{l+2})}{2h_y^2}, \gamma_{1,l}(r) = \frac{(r-y_l)(r-y_{l+2})}{-h_y^2}, \gamma_{2,l}(r) = \frac{(r-y_l)(r-y_{l+1})}{2h_y^2}.$$

Computing the approximate solution for the first two steps requires us to first determine the value of $u_i(x,y), i = 1,\ldots,m$, at points $(x_v,y_n), v,n = 1,2$. Combining (2.1) and (2.2), when $v,n = 1$, we have

$$
\begin{aligned}
u_i(x_1,y_1) &= g^i_{1,1} + \sum_{j=1}^{m}\int_a^{x_1}\int_c^{y_1}\frac{(x_1-s)^{-\alpha}}{(y_1-r)^{\beta}}\kappa^{i,j}_{1,1}(s,r,u_j(s,r))drds\\
&\approx g^i_{1,1} + \sum_{j=1}^{m}\int_a^{x_1}\int_c^{y_1}\frac{(x_1-s)^{-\alpha}}{(y_1-r)^{\beta}}J^{[y_0,y_2]}_{[x_0,x_2]}\kappa^{i,j}_{1,1}(s,r,u_j(s,r))drds\\
&= g^i_{1,1} + \sum_{j=1}^{m}\sum_{e=0}^{2}\sum_{f=0}^{2}\varpi^{e,0}_1\hat{\varpi}^{f,0}_1\kappa^{i,j}_{1,1}(x_e,y_f,u^j_{e,f}),
\end{aligned}
\tag{2.4}
$$

where

$$
\varpi^{e,0}_1 = \int_a^{x_1}\frac{\psi_{e,0}(s)}{(x_1-s)^{\alpha}}ds,\ \ \hat{\varpi}^{f,0}_1 = \int_c^{y_1}\frac{\gamma_{f,0}(r)}{(y_1-r)^{\beta}}dr, e,f=0,1,2. \tag{2.5}
$$

Using the same method as (2.4) to approximate the values of $u_i(x,y)$ at (x_2,y_1) and $(x_v,y_2), v=1,2,$ and we obtain

$$
u_i(x_2,y_1) \approx g^i_{2,1} + \sum_{j=1}^{m}\sum_{e=0}^{2}\sum_{f=0}^{2}\pi^{e,0}_2\hat{\varpi}^{f,0}_1\kappa^{i,j}_{2,1}(x_e,y_f,u^j_{e,f}), \tag{2.6}
$$

$$
u_i(x_1,y_2) \approx g^i_{1,2} + \sum_{j=1}^{m}\sum_{e=0}^{2}\sum_{f=0}^{2}\varpi^{e,0}_1\hat{\pi}^{f,0}_2\kappa^{i,j}_{1,2}(x_e,y_f,u^j_{e,f}), \tag{2.7}
$$

$$
u_i(x_2,y_2) \approx g^i_{2,2} + \sum_{j=1}^{m}\sum_{e=0}^{2}\sum_{f=0}^{2}\pi^{e,0}_2\hat{\pi}^{f,0}_2\kappa^{i,j}_{2,2}(x_e,y_f,u^j_{e,f}), \tag{2.8}
$$

where $\varpi^{e,0}_1$ and $\hat{\varpi}^{f,0}_1$ are in (2.5), and

$$
\pi^{e,0}_2 = \int_a^{x_2}\frac{\psi_{e,0}(s)}{(x_2-s)^{\alpha}}ds,\ \hat{\pi}^{f,0}_2 = \int_c^{y_2}\frac{\gamma_{f,0}(r)}{(y_2-r)^{\beta}}dr, e,f=0,1,2. \tag{2.9}
$$

As we described above, $u^i_{v,n}, v,n = 1,2$ cannot be obtained by only one of the above equations, that is, they are coupled, and it is necessary to use the four Eqs (2.4) and (2.6)–(2.8) to solve them simultaneously.

Next we estimate the case of $v \geq 3, n = 1,2$ or $v = 1,2, n \geq 3$, that is, estimates $u_i(x_{2p+k_1}, y_{l_1}), p = 1,2,\ldots,N-1$ and $u_i(x_{k_1}, y_{2q+l_1}), k_1, l_1 = 1,2; q = 1,2,\ldots,L-1; i = 1,2,\ldots,m$. Before it is estimated, it is assumed that the values of $u^i_{k,l_1}, k = 0,1,\ldots,2p, u^i_{k_1,l}, l = 0,1,\ldots,2q$, has been known by us. For $u_i(x_{2p+1}, y_1)$, we obtain that

$$
\begin{aligned}
u_i(x_{2p+1},y_1) = g^i_{2p+1,1} &+ \sum_{j=1}^{m}\int_a^{x_1}\int_c^{y_1}\frac{\kappa^{i,j}_{2p+1,1}(s,r,u_j(s,r))}{(x_{2p+1}-s)^{\alpha}(y_1-r)^{\beta}}drds\\
&+ \sum_{j=1}^{m}\sum_{k=1}^{p}\int_{x_{2k-1}}^{x_{2k+1}}\int_c^{y_1}\frac{\kappa^{i,j}_{2p+1,1}(s,r,u_j(s,r))}{(x_{2p+1}-s)^{\alpha}(y_1-r)^{\beta}}drds
\end{aligned}
$$

$$\approx g^i_{2p+1,1} + \sum_{j=1}^{m} \int_a^{x_1} \int_c^{y_1} \frac{(x_{2p+1}-s)^{-\alpha}}{(y_1-r)^{\beta}} J^{[y_0,y_2]}_{[x_0,x_2]} \kappa^{i,j}_{2p+1,1}(s,r,u_j(s,r))drds$$

$$+ \sum_{j=1}^{m}\sum_{k=1}^{p} \int_{x_{2k-1}}^{x_{2k+1}} \int_c^{y_1} \frac{(x_{2p+1}-s)^{-\alpha}}{(y_1-r)^{\beta}} J^{[y_0,y_2]}_{[x_{2k-1},x_{2k+1}]} \kappa^{i,j}_{2p+1,1}(s,r,u_j(s,r))drds$$

$$= g^i_{2p+1,1} + \sum_{j=1}^{m}\sum_{e=0}^{2}\sum_{f=0}^{2} \varpi^{e,0}_{2p+1}\hat{\varpi}^{f,0}_{1}\kappa^{i,j}_{2p+1,1}(x_e,y_f,u^j_{e,f})$$

$$+ \sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{e=0}^{2}\sum_{f=0}^{2} \pi^{e,k}_{2p+1}\hat{\varpi}^{f,0}_{1}\kappa^{i,j}_{2p+1,1}(x_{2k-1+e},y_f,u^j_{2k-1+e,f}), \tag{2.10}$$

where $\hat{\varpi}^{f,0}_1$ is defined in (2.5), the definition of $\varpi^{e,0}_{2p+1}$ and $\pi^{e,k}_{2p+1}$ are as follows

$$\varpi^{e,0}_{2p+1} = \int_a^{x_1} \frac{\psi_{e,0}(s)}{(x_{2p+1}-s)^{\alpha}}ds,\ \pi^{e,k}_{2p+1} = \int_{x_{2k-1}}^{x_{2k+1}} \frac{\psi_{e,2k-1}(s)}{(x_{2p+1}-s)^{\alpha}}ds, e=0,1,2; k=1,\ldots,p. \tag{2.11}$$

Use the same technique to process $u_i(x_{2p+2},y_1)$ and $u_i(x_{2p+k_1},y_2), k_1 = 1,2$, we get the following results:

$$u_i(x_{2p+2},y_1) \approx g^i_{2p+2,1} + \sum_{j=1}^{m}\sum_{k=0}^{p}\sum_{e=0}^{2}\sum_{f=0}^{2} \pi^{e,k}_{2p+2}\hat{\varpi}^{f,0}_{1}\kappa^{i,j}_{2p+2,1}(x_{2k+e},y_f,u^j_{2k+e,f}), \tag{2.12}$$

$$u_i(x_{2p+1},y_2) \approx g^i_{2p+1,2} + \sum_{j=1}^{m}\sum_{e=0}^{2}\sum_{f=0}^{2} \varpi^{e,0}_{2p+1}\hat{\pi}^{f,0}_{2}\kappa^{i,j}_{2p+1,2}(x_e,y_f,u^j_{e,f}) \tag{2.13}$$

$$+ \sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{e=0}^{2}\sum_{f=0}^{2} \pi^{e,k}_{2p+1}\hat{\pi}^{f,0}_{2}\kappa^{i,j}_{2p+1,2}(x_{2k-1+e},y_f,u^j_{2k-1+e,f}),$$

$$u_i(x_{2p+2},y_2) \approx g^i_{2p+2,2} + \sum_{j=1}^{m}\sum_{k=0}^{p}\sum_{e=0}^{2}\sum_{f=0}^{2} \pi^{e,k}_{2p+2}\hat{\pi}^{f,0}_{2}\kappa^{i,j}_{2p+2,2}(x_{2k+e},y_f,u^j_{2k+e,f}), \tag{2.14}$$

with $\hat{\varpi}^{f,0}_1$, $\hat{\pi}^{f,0}_2$, $\varpi^{e,0}_{2p+1}$ and $\pi^{e,k}_{2p+1}$ are in (2.5), (2.9) and (2.11), respectively. $\pi^{e,k}_{2p+2}$ is defined by:

$$\pi^{e,k}_{2p+2} = \int_{x_{2k}}^{x_{2k+2}} \frac{\psi_{e,2k}(s)}{(x_{2p+2}-s)^{\alpha}}ds, e=0,1,2; k=0,1,\ldots,p. \tag{2.15}$$

The same method is applied to $u_i(x_{k_1},y_{2q+l_1}), k_1, l_1 = 1,2; q = 1,2,\ldots,L-1$, we can directly obtain

$$u_i(x_1,y_{2q+1}) = g^i_{1,2q+1} + \sum_{j=1}^{m} \int_a^{x_1}\int_c^{y_1} \frac{\kappa^{i,j}_{1,2q+1}(s,r,u_j(s,r))}{(x_1-s)^{\alpha}(y_{2q+1}-r)^{\beta}}drds$$

$$+ \sum_{j=1}^{m}\sum_{l=1}^{q} \int_a^{x_1}\int_{y_{2l-1}}^{y_{2l+1}} \frac{\kappa^{i,j}_{1,2q+1}(s,r,u_j(s,r))}{(x_1-s)^{\alpha}(y_{2q+1}-r)^{\beta}}drds$$

$$\approx g^i_{1,2q+1} + \sum_{j=1}^{m}\sum_{e=0}^{2}\sum_{f=0}^{2} \varpi^{e,0}_{1}\hat{\varpi}^{f,0}_{2q+1}\kappa^{i,j}_{1,2q+1}(x_e,y_f,u^j_{e,f}) \tag{2.16}$$

$$+\sum_{j=1}^{m}\sum_{l=1}^{q}\sum_{e=0}^{2}\sum_{f=0}^{2}\varpi_1^{e,0}\hat{\pi}_{2q+1}^{f,l}\kappa_{1,2q+1}^{i,j}(x_e,y_{2l-1+f},u_{e,2l-1+f}^j),$$

$$u_i(x_2,y_{2q+1})\approx g_{2,2q+1}^i+\sum_{j=1}^{m}\sum_{e=0}^{2}\sum_{f=0}^{2}\pi_2^{e,0}\hat{\varpi}_{2q+1}^{f,0}\kappa_{2,2q+1}^{i,j}(x_e,y_f,u_{e,f}^j) \tag{2.17}$$

$$+\sum_{j=1}^{m}\sum_{l=1}^{q}\sum_{e=0}^{2}\sum_{f=0}^{2}\pi_2^{e,0}\hat{\pi}_{2q+1}^{f,l}\kappa_{2,2q+1}^{i,j}(x_e,y_{2l-1+f},u_{e,2l-1+f}^j),$$

$$u_i(x_1,y_{2q+2})=g_{1,2q+2}^i+\sum_{j=1}^{m}\sum_{l=0}^{q}\int_a^{x_1}\int_{y_{2l}}^{y_{2l+2}}\frac{\kappa_{1,2q+2}^{i,j}(s,r,u_j(s,r))}{(x_1-s)^\alpha(y_{2q+2}-r)^\beta}drds$$

$$\approx g_{1,2q+2}^i+\sum_{j=1}^{m}\sum_{l=0}^{q}\sum_{e=0}^{2}\sum_{f=0}^{2}\varpi_1^{e,0}\hat{\pi}_{2q+2}^{f,l}\kappa_{1,2q+2}^{i,j}(x_e,y_{2l+f},u_{e,2l+f}^j), \tag{2.18}$$

$$u_i(x_2,y_{2q+2})\approx g_{2,2q+2}^i+\sum_{j=1}^{m}\sum_{l=0}^{q}\sum_{e=0}^{2}\sum_{f=0}^{2}\pi_2^{e,0}\hat{\pi}_{2q+2}^{f,l}\kappa_{2,2q+2}^{i,j}(x_e,y_{2l+f},u_{e,2l+f}^j), \tag{2.19}$$

where $\varpi_1^{e,0}$ and $\pi_2^{e,0}$ are defined in (2.5) and (2.9), respectively, and

$$\hat{\varpi}_{2q+1}^{f,0}=\int_c^{y_1}\frac{\gamma_{f,0}(r)}{(y_{2q+1}-r)^\beta}dr,\ \hat{\pi}_{2q+1}^{f,l}=\int_{y_{2l-1}}^{y_{2l+1}}\frac{\gamma_{f,2l-1}(r)}{(y_{2q+1}-r)^\beta}dr, f=0,1,2;l=1,\dots,q, \tag{2.20}$$

$$\hat{\pi}_{2q+2}^{f,l}=\int_{y_{2l}}^{y_{2l+2}}\frac{\gamma_{f,2l}(r)}{(y_{2q+2}-r)^\beta}dr, f=0,1,2;l=0,1,\dots,q. \tag{2.21}$$

Finally, when $v,n\geq 3$, we can divide it into four cases according to v,n are odd or even, namely $u_i(x_{2p+k_1},y_{2q+l_1}),k_1,l_1=1,2;i=1,\dots,m$. Let's approximate $u_i(x_{2p+k_1},y_{2q+l_1}),k_1,l_1=1,2$, assuming that the values of $u_{v,n}^i$, $u_{2p+k_1,n}^i$, $u_{v,2q+l_1}^i$, $k_1,l_1=1,2$; $v=0,\dots,2p$; $n=0,\dots,2q$; $p=1,\dots,N-1$; $q=1,\dots,L-1$ have been given. Therefore, when $k_1,l_1=1$, we have

$$u_i(x_{2p+1},y_{2q+1})=g_{2p+1,2q+1}^i+\sum_{j=1}^{m}\int_a^{x_1}\int_c^{y_1}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^\beta}\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))drds$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{p}\int_{x_{2k-1}}^{x_{2k+1}}\int_c^{y_1}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^\beta}\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))drds$$

$$+\sum_{j=1}^{m}\sum_{l=1}^{q}\int_a^{x_1}\int_{y_{2l-1}}^{y_{2l+1}}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^\beta}\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))drds$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}\int_{x_{2k-1}}^{x_{2k+1}}\int_{y_{2l-1}}^{y_{2l+1}}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^\beta}\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))drds$$

$$\approx g_{2p+1,2q+1}^i+\sum_{j=1}^{m}\int_a^{x_1}\int_c^{y_1}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^\beta}J_{[x_0,x_2]}^{[y_0,y_2]}\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))drds$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{p}\int_{x_{2k-1}}^{x_{2k+1}}\int_c^{y_1}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^\beta}J_{[x_{2k-1},x_{2k+1}]}^{[y_0,y_2]}\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))drds$$

$$+\sum_{j=1}^{m}\sum_{l=1}^{q}\int_{a}^{x_1}\int_{y_{2l-1}}^{y_{2l+1}}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^{\beta}}J_{[x_0,x_2]}^{[y_{2l-1},y_{2l+1}]}\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))drds$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}\int_{x_{2k-1}}^{x_{2k+1}}\int_{y_{2l-1}}^{y_{2l+1}}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^{\beta}}J_{[x_{2k-1},x_{2k+1}]}^{[y_{2l-1},y_{2l+1}]}\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))drds$$

$$=g_{2p+1,2q+1}^{i}+\sum_{j=1}^{m}\sum_{e=0}^{2}\sum_{f=0}^{2}\varpi_{2p+1}^{e,0}\hat{\varpi}_{2q+1}^{f,0}\kappa_{2p+1,2q+1}^{i,j}(x_e,y_f,u_{e,f}^{j})$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{e=0}^{2}\sum_{f=0}^{2}\pi_{2p+1}^{e,k}\hat{\varpi}_{2q+1}^{f,0}\kappa_{2p+1,2q+1}^{i,j}(x_{2k-1+e},y_f,u_{2k-1+e,f}^{j})$$

$$+\sum_{j=1}^{m}\sum_{l=1}^{q}\sum_{e=0}^{2}\sum_{f=0}^{2}\varpi_{2p+1}^{e,0}\hat{\pi}_{2q+1}^{f,l}\kappa_{2p+1,2q+1}^{i,j}(x_e,y_{2l-1+f},u_{e,2l-1+f}^{j})$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}\sum_{e=0}^{2}\sum_{f=0}^{2}\pi_{2p+1}^{e,k}\hat{\pi}_{2q+1}^{f,l}\kappa_{2p+1,2q+1}^{i,j}(x_{2k-1+e},y_{2l-1+f},u_{2k-1+e,2l-1+f}^{j}),\tag{2.22}$$

where $\varpi_{2p+1}^{e,0},\pi_{2p+1}^{e,k},\hat{\varpi}_{2q+1}^{f,0}$ and $\hat{\pi}_{2q+1}^{f,l}$ are defined in (2.11) and (2.20), respectively.

For $u_i(x_{2p+2},y_{2q+1})$ and $u_i(x_{2p+k_1},y_{2q+2}),k_1=1,2$, we use the same idea as in (2.22) to get their approximations, in other words, by using quadratic Lagrangian interpolation to approximate them and get the following results:

$$u_i(x_{2p+2},y_{2q+1})\approx g_{2p+2,2q+1}^{i}+\sum_{j=1}^{m}\sum_{k=0}^{p}\sum_{e=0}^{2}\sum_{f=0}^{2}\pi_{2p+2}^{e,k}\hat{\varpi}_{2q+1}^{f,0}\kappa_{2p+2,2q+1}^{i,j}(x_{2k+e},y_f,u_{2k+e,f}^{j})$$

$$+\sum_{j=1}^{m}\sum_{k=0}^{p}\sum_{l=1}^{q}\sum_{e=0}^{2}\sum_{f=0}^{2}\pi_{2p+2}^{e,k}\hat{\pi}_{2q+1}^{f,l}\kappa_{2p+2,2q+1}^{i,j}(x_{2k+e},y_{2l-1+f},u_{2k+e,2l-1+f}^{j}),\tag{2.23}$$

$$u_i(x_{2p+1},y_{2q+2})\approx g_{2p+1,2q+2}^{i}+\sum_{j=1}^{m}\sum_{l=0}^{q}\sum_{e=0}^{2}\sum_{f=0}^{2}\varpi_{2p+1}^{e,0}\hat{\pi}_{2q+2}^{f,k}\kappa_{2p+1,2q+2}^{i,j}(x_e,y_{2l+f},u_{e,2l+f}^{j})$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=0}^{q}\sum_{e=0}^{2}\sum_{f=0}^{2}\pi_{2p+1}^{e,k}\hat{\pi}_{2q+2}^{f,l}\kappa_{2p+1,2q+2}^{i,j}(x_{2k-1+e},y_{2l+f},u_{2k-1+e,2l+f}^{j}),\tag{2.24}$$

$$u_i(x_{2p+2},y_{2q+2})\approx g_{2p+2,2q+2}^{i}+\sum_{j=1}^{m}\sum_{k=0}^{p}\sum_{l=0}^{q}\sum_{e=0}^{2}\sum_{f=0}^{2}\pi_{2p+2}^{e,k}\hat{\pi}_{2q+2}^{f,l}\kappa_{2p+2,2q+2}^{i,j}(x_{2k+e},y_{2l+f},u_{2k+e,2l+f}^{j}),\tag{2.25}$$

where $\varpi_{2p+1}^{e,0}$, $\pi_{2p+1}^{e,k}$, $\pi_{2p+2}^{e,k}$, $\hat{\varpi}_{2q+1}^{f,0}$, $\hat{\pi}_{2q+1}^{f,l}$ and $\hat{\pi}_{2q+2}^{f,l}$ are shown in (2.11), (2.15), (2.20) and (2.21), respectively.

Summarizing all cases can get the complete scheme of Eq (2.1), that is, (2.4), (2.6)–(2.8), (2.10), (2.12)–(2.14), (2.16)–(2.19), (2.22) and (2.23)–(2.25) make up the numerical scheme of (2.1). Combined with the process of constructing the approximate value of $u_i(x_v,y_n)$, it can be known through calculation that for fixed k,l,e,v,n and f, the coefficients $\varpi_v^{e,0}$ and $\pi_v^{e,k}$ are proportional to $h_x^{1-\alpha}$, and $\hat{\varpi}_n^{f,0}$ and $\hat{\pi}_n^{f,l}$ are proportional to $h_y^{1-\beta}$. So, the scheme (2.4), (2.6)–(2.8), (2.10), (2.12)–(2.14), (2.16)–(2.19), (2.22) and (2.23)–(2.25) can be consistently written in the following form:

$$u^i_{v,n} = \begin{cases} g^i_{v,n} + h_x^{1-\alpha} h_y^{1-\beta} \sum\limits_{j=1}^{m} \sum\limits_{k=0}^{2} \sum\limits_{l=0}^{2} A^v_k \tilde{A}^n_l \kappa^{i,j}_{v,n}(x_k, y_l, u^j_{k,l}),\ v, n = 1, 2, \\ g^i_{v,n} + h_x^{1-\alpha} h_y^{1-\beta} \sum\limits_{j=1}^{m} \sum\limits_{k=0}^{v} \sum\limits_{l=0}^{2} \mathfrak{R}^v_k \tilde{A}^n_l \kappa^{i,j}_{v,n}(x_k, y_l, u^j_{k,l}),\ v = 2p+1, 2p+2; n = 1, 2, \\ g^i_{v,n} + h_x^{1-\alpha} h_y^{1-\beta} \sum\limits_{j=1}^{m} \sum\limits_{k=0}^{2} \sum\limits_{l=0}^{n} A^v_k \aleph^n_l \kappa^{i,j}_{v,n}(x_k, y_l, u^j_{k,l}),\ v = 1, 2; n = 2q+1, 2q+2, \\ g^i_{v,n} + h_x^{1-\alpha} h_y^{1-\beta} \sum\limits_{j=1}^{m} \sum\limits_{k=0}^{v} \sum\limits_{l=0}^{n} \mathfrak{R}^v_k \aleph^n_l \kappa^{i,j}_{v,n}(x_k, y_l, u^j_{k,l}), \\ \quad v = 2p+1, 2p+2; n = 2q+1, 2q+2, \end{cases} \tag{2.26}$$

where $p = 1, \ldots, N-1; q = 1, \ldots, L-1, A^v_k, \tilde{A}^n_l, \mathfrak{R}^v_k$ and $\aleph^n_l$ are defined by the following equations:

$$\begin{aligned}
&A^1_k = \hat{\Upsilon}_k, A^2_k = \tilde{\Upsilon}_k, \tilde{A}^1_l = \hat{\Theta}_l, \tilde{A}^2_l = \tilde{\Theta}_l, k, l = 0, 1, 2; \mathfrak{R}^{2p+1}_k = \bar{\Upsilon}^{(p)}_k, k = 0, \ldots, 2p+1, \\
&\mathfrak{R}^{2p+2}_k = \Upsilon^{(p)}_k, k = 0, \ldots, 2p+2; \aleph^{2q+1}_l = \bar{\Theta}^{(q)}_l, l = 0, \ldots, 2q+1; \aleph^{2q+2}_l = \Theta^{(q)}_l, l = 0, \ldots, 2q+2; \\
&\hat{\Upsilon}_k = \frac{\varpi^{k,0}_1}{h_x^{1-\alpha}}, \tilde{\Upsilon}_k = \frac{\pi^{k,0}_2}{h_x^{1-\alpha}}, k = 0, 1, 2; \Upsilon^{(p)}_0 = \frac{\pi^{0,0}_{2p+2}}{h_x^{1-\alpha}}, \Upsilon^{(p)}_{2k+1} = \frac{\pi^{1,k}_{2p+2}}{h_x^{1-\alpha}}, k = 0, \ldots, p, \\
&\Upsilon^{(p)}_{2k} = \frac{\pi^{2,k-1}_{2p+2} + \pi^{0,k}_{2p+2}}{h_x^{1-\alpha}}, k = 1, \ldots, p; \bar{\Upsilon}^{(p)}_0 = \frac{\varpi^{0,0}_{2p+1}}{h_x^{1-\alpha}}, \bar{\Upsilon}^{(p)}_1 = \frac{\varpi^{1,0}_{2p+1} + \pi^{0,1}_{2p+1}}{h_x^{1-\alpha}}, \\
&\bar{\Upsilon}^{(p)}_2 = \frac{\varpi^{2,0}_{2p+1} + \pi^{1,1}_{2p+1}}{h_x^{1-\alpha}}, \bar{\Upsilon}^{(p)}_{2k-1} = \Upsilon^{(p)}_{2k}, \bar{\Upsilon}^{(p)}_{2k} = \Upsilon^{(p)}_{2k+1}, k = 2, \ldots, p; \bar{\Upsilon}^{(p)}_{2p+1} = \Upsilon^{(p)}_{2p+2} = \frac{\pi^{2,p}_{2p+2}}{h_x^{1-\alpha}}; \\
&\hat{\Theta}_l = \frac{\hat{\varpi}^{l,0}_1}{h_y^{1-\beta}}, \tilde{\Theta}_l = \frac{\hat{\pi}^{l,0}_2}{h_y^{1-\beta}}, l = 0, 1, 2, \Theta^{(q)}_0 = \frac{\hat{\pi}^{0,0}_{2q+2}}{h_y^{1-\beta}}, \Theta^{(q)}_{2l+1} = \frac{\hat{\pi}^{1,l}_{2q+2}}{h_y^{1-\beta}}, l = 0, \ldots, q, \\
&\Theta^{(q)}_{2l} = \frac{\hat{\pi}^{2,l-1}_{2q+2} + \hat{\pi}^{0,l}_{2q+2}}{h_y^{1-\beta}}, l = 1, \ldots, q, \bar{\Theta}^{(q)}_0 = \frac{\hat{\varpi}^{0,0}_{2q+1}}{h_y^{1-\beta}}, \bar{\Theta}^{(q)}_1 = \frac{\hat{\varpi}^{1,0}_{2q+1} + \hat{\pi}^{0,1}_{2q+1}}{h_y^{1-\beta}}, \\
&\bar{\Theta}^{(q)}_2 = \frac{\hat{\varpi}^{2,0}_{2q+1} + \hat{\pi}^{1,1}_{2q+1}}{h_y^{1-\beta}}, \bar{\Theta}^{(q)}_{2l-1} = \Theta^{(q)}_{2l}, \bar{\Theta}^{(q)}_{2l} = \Theta^{(q)}_{2l+1}, l = 2, \ldots, q; \bar{\Theta}^{(q)}_{2q+1} = \Theta^{(q)}_{2q+2} = \frac{\hat{\pi}^{2,q}_{2q+2}}{h_y^{1-\beta}}.
\end{aligned} \tag{2.27}$$

3. Estimation of the truncation errors

To further derive the truncation error for scheme (2.26), we first define it as follows:

$$r^{(i)}_{v,n} := u_i(x_v, y_n) - \bar{u}^i_{v,n}, v, n \geq 1, i = 1, 2, \ldots, m. \tag{3.1}$$

The definition of $\bar{u}^i_{v,n}$ is to replace $u^i_{v,n}$ of (2.4), (2.6)–(2.8), (2.10), (2.12)–(2.14), (2.16)–(2.19), (2.22) and (2.23)–(2.25) with $u_i(x_v, y_n)$. For simplicity, we set $\frac{\partial^3 \kappa}{\partial s^3} = \partial^3_s \kappa$, $\frac{\partial^3 \kappa}{\partial r^3} = \partial^3_r \kappa$. Our estimates will use T_1 and T_2, which are defined as:

$$T_1 = \max_{\substack{x,s \in [a,b] \\ y,r \in [c,d]}} \{|\partial^3_s \kappa_{i,j}(x, y, s, r, u(s, r))|, |\partial^3_r \kappa_{i,j}(x, y, s, r, u(s, r))|, i, j = 1, 2, \ldots, m\}, \tag{3.2}$$

$$T_2 = \max_{\substack{x,s \in [a,b] \\ y,r \in [c,d]}} \{|\partial^4_s \kappa_{i,j}(x, y, s, r, u(s, r))|, |\partial^4_r \kappa_{i,j}(x, y, s, r, u(s, r))|, i, j = 1, 2, \ldots, m\}. \tag{3.3}$$

Lemma 1. *Suppose $r_{v,n}^{(i)}$ has the same meaning as we defined in (3.1), i.e., its truncation error defined in (3.1), if $\kappa(\cdot,\cdot,\cdot,\cdot,u(\cdot,\cdot))\in C^4(U)$, then*

$$|r_{v,n}^{(i)}|\le C(h_x^{4-\alpha}+h_y^{4-\beta}),v,n\ge 1,i=1,2\ldots,m,$$

where C only depends on T_1,T_2,m, α and β.

Proof. The error estimate will be based on the Lagrangian interpolated error term, which is

$$\begin{aligned}&\kappa_{v,n}^{i,j}(s,r,u_j(s,r))-J_{[x_k,x_{k+2}]}^{[y_l,y_{l+2}]}\kappa_{v,n}^{i,j}(s,r,u_j(s,r))\\&=\frac{1}{6}\frac{\partial^3\kappa_{v,n}^{i,j}(\xi_k(s),r,u_j(\xi_k(s),r))}{\partial s^3}\prod_{e=0}^{2}(s-x_{k+e})\\&+\sum_{e=0}^{2}\frac{\psi_{e,k}(s)}{6}\frac{\partial^3\kappa_{v,n}^{i,j}(x_{k+e},\varsigma_l(r),u_j(x_{k+e},\varsigma_l(r)))}{\partial r^3}\prod_{f=0}^{2}(r-y_{l+f}),\forall(s,r)\in[x_k,x_{k+2}]\times[y_l,y_{l+2}],\end{aligned}\tag{3.4}$$

here $\xi_k(s)$ is a function whose value range and definition domain are both in $[x_k,x_{k+2}]$, and $\varsigma_l(r)$ is a function whose value domain and definition domain are both in $[y_l,y_{l+2}]$.

Without loss of generality, we prove that the truncation error for the case of $v,n\ge 3$. According to the process of constructing the numerical scheme, we first estimate $r_{2p+1,2q+1}^{(i)},p,q\ge 1$ and get the following form:

$$\begin{aligned}|r_{2p+1,2q+1}^{(i)}|\le&|\sum_{j=1}^{m}\int_a^{x_1}\int_c^{y_1}(x_{2p+1}-s)^{-\alpha}(y_{2q+1}-r)^{-\beta}\\&\times[\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))-J_{[x_0,x_1]}^{[y_0,y_1]}\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))]drds|\\+&|\sum_{j=1}^{m}\sum_{k=1}^{p}\int_{x_{2k-1}}^{x_{2k+1}}\int_c^{y_1}(x_{2p+1}-s)^{-\alpha}(y_{2q+1}-r)^{-\beta}\\&\times[\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))-J_{[x_{2k-1},x_{2k+1}]}^{[y_0,y_1]}\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))]drds|\\+&|\sum_{j=1}^{m}\sum_{l=1}^{q}\int_a^{x_1}\int_{y_{2l-1}}^{y_{2l+1}}(x_{2p+1}-s)^{-\alpha}(y_{2q+1}-r)^{-\beta}\\&\times[\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))-J_{[x_0,x_1]}^{[y_{2l-1},y_{2l+1}]}\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))]drds|\\+&|\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}\int_{x_{2k-1}}^{x_{2k+1}}\int_{y_{2l-1}}^{y_{2l+1}}(x_{2p+1}-s)^{-\alpha}(y_{2q+1}-r)^{-\beta}\\&\times[\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))-J_{[x_{2k-1},x_{2k+1}]}^{[y_{2l-1},y_{2l+1}]}\kappa_{2p+1,2q+1}^{i,j}(s,r,u_j(s,r))]drds|\\\doteq&\, r_1+r_2+r_3+r_4.\end{aligned}\tag{3.5}$$

We first estimate r_1, and can directly obtain

$$r_1\le|\sum_{j=1}^{m}\int_a^{x_1}\int_c^{y_1}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^{\beta}}\frac{\partial_s^3\kappa_{2p+1,2q+1}^{i,j}(\xi_0(s),r,u_j(\xi_0(s),r))}{6}\prod_{e=0}^{2}(s-x_e)drds|$$

$$+\Big|\sum_{j=1}^{m}\int_{a}^{x_1}\int_{c}^{y_1}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^{\beta}}\sum_{e=0}^{2}\frac{\psi_{e,0}(s)}{6}\partial_t^3\kappa_{2p+1,2q+1}^{i,j}(x_e,\varsigma_0(r),u_j(x_e,\varsigma_0(r)))\prod_{f=0}^{2}(r-y_f)drds\Big|$$
$$\doteq r_1^{(1)}+r_1^{(2)}.$$

It can be calculated that $|\psi_{e,k}(s)|\le 1$, $\sum\limits_{e=0}^{2}|\psi_{e,k}(s)|\le 3$. For $r_1^{(1)}$ and $r_1^{(2)}$, by direct calculation, we can get

$$\begin{aligned}
r_1^{(1)} &\le T_1h_x^3\sum_{j=1}^{m}\Big|\int_{a}^{x_1}\int_{c}^{y_1}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^{\beta}}drds\Big|\\
&=\sum_{j=1}^{m}T_1h_x^{4-\alpha}h_y^{1-\beta}\frac{((2p+1)^{1-\alpha}-(2p)^{1-\alpha})((2q+1)^{1-\beta}-(2q)^{1-\beta})}{(1-\alpha)(1-\beta)}\\
&=\sum_{j=1}^{m}T_1h_x^{4-\alpha}h_y^{1-\beta}\frac{1}{\xi^{\alpha}\varsigma^{\beta}}\le\sum_{j=1}^{m}T_1h_x^{4-\alpha}h_y^{1-\beta},\quad \xi\in(2p,2p+1),\varsigma\in(2q,2q+1),\\
r_1^{(2)} &\le T_1h_y^3\sum_{j=1}^{m}\Big|\int_{a}^{x_1}\int_{c}^{y_1}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^{\beta}}drds\Big|\le\sum_{j=1}^{m}T_1h_x^{1-\alpha}h_y^{4-\beta},
\end{aligned}$$

where T_1 is defined with (3.2).

Therefore, we have

$$r_1\le T_1\sum_{j=1}^{m}(h_x^{4-\alpha}h_y^{1-\beta}+h_x^{1-\alpha}h_y^{4-\beta}).\tag{3.6}$$

Next, based on the error analysis formula of Lagrangian interpolation, we divide r_2 into two items and estimate them separately, and the specific derivation is as follows:

$$\begin{aligned}
r_2 &\le\sum_{j=1}^{m}\sum_{k=1}^{p}\Big|\int_{x_{2k-1}}^{x_{2k+1}}\int_{c}^{y_1}\frac{\partial_s^3\kappa_{2p+1,2q+1}^{i,j}(\xi_{2k-1}(s),r,u_j(\xi_{2k-1}(s),r))}{6(x_{2p+1}-s)^{\alpha}(y_{2q+1}-r)^{\beta}}\prod_{e=0}^{2}(s-x_{2k-1+e})drds\Big|\\
&+\sum_{j=1}^{m}\sum_{k=1}^{p}\Big|\int_{x_{2k-1}}^{x_{2k+1}}\int_{c}^{y_1}\frac{\sum\limits_{e=0}^{2}\psi_{e,2k-1}(s)\partial_t^3\kappa_{2p+1,2q+1}^{i,j}(x_{2k-1+e},\varsigma_0(r),u_j(x_{2k-1+e},\varsigma_0(r)))}{6(x_{2p+1}-s)^{\alpha}(y_{2q+1}-r)^{\beta}}\prod_{f=0}^{2}(r-y_f)drds\Big|\\
&\doteq r_2^{(1)}+r_2^{(2)}.
\end{aligned}$$

For $r_2^{(1)}$, we can obtain

$$\begin{aligned}
r_2^{(1)} &\le\sum_{j=1}^{m}\sum_{k=1}^{p}\Big|\int_{x_{2k-1}}^{x_{2k+1}}\int_{c}^{y_1}\frac{\partial_s^3\kappa_{2p+1,2q+1}^{i,j}(\xi_{2k-1}(s_k),r,u_j(\xi_{2k-1}(s_k),r))}{6(x_{2p+1}-s)^{\alpha}(y_{2q+1}-r)^{\beta}}\prod_{e=0}^{2}(s-x_{2k-1+e})drds\Big|\\
&+\sum_{j=1}^{m}\sum_{k=1}^{p}\Big|\int_{x_{2k-1}}^{x_{2k+1}}\int_{c}^{y_1}\frac{(x_{2p+1}-s)^{-\alpha}}{6(y_{2q+1}-r)^{\beta}}\prod_{e=0}^{2}(s-x_{2k-1+e})\\
&\times[\partial_s^3\kappa_{2p+1,2q+1}^{i,j}(\xi_{2k-1}(s),r,u_j(\xi_{2k-1}(s),r))-\partial_s^3\kappa_{2p+1,2q+1}^{i,j}(\xi_{2k-1}(s_k),r,u_j(\xi_{2k-1}(s_k),r))]drds\Big|,
\end{aligned}$$

where $s_k = x_{2k}$. We denote the first term on the right-hand side of the inequality as I_1 and the second term as I_2. First we compute I_1 and get

$$
\begin{aligned}
I_1 \leq\ & T_1 h_y^{1-\beta} \sum_{j=1}^{m} \sum_{k=1}^{p} \Big| \int_{x_{2k-1}}^{x_{2k}} \frac{\prod_{e=0}^{2}(s - x_{2k-1+e})}{(x_{2p+1} - s)^{\alpha}} ds + \int_{x_{2k}}^{x_{2k+1}} \frac{\prod_{e=0}^{2}(s - x_{2k-1+e})}{(x_{2p+1} - s)^{\alpha}} ds \Big| \\
\leq\ & T_1 h_y^{1-\beta} \sum_{j=1}^{m} \sum_{k=1}^{p-2} \Big| \int_{x_{2k-1}}^{x_{2k}} \frac{\prod_{e=0}^{2}(s - x_{2k-1+e})}{(x_{2p+1} - s)^{\alpha}} ds + \int_{x_{2k}}^{x_{2k+1}} \frac{\prod_{e=0}^{2}(s - x_{2k-1+e})}{(x_{2p+1} - s)^{\alpha}} ds \Big| \\
& + T_1 h_y^{1-\beta} \sum_{j=1}^{m} \Big(\Big| \int_{x_{2p-3}}^{x_{2p-1}} \frac{\prod_{e=0}^{2}(s - x_{2p-3+e})}{(x_{2p+1} - s)^{\alpha}} ds \Big| + \Big| \int_{x_{2p-1}}^{x_{2p+1}} \frac{\prod_{e=0}^{2}(s - x_{2p-1+e})}{(x_{2p+1} - s)^{\alpha}} ds \Big| \Big) \\
\doteq\ & T_1 h_y^{1-\beta} \sum_{j=1}^{m} I_{11} + T_1 h_y^{1-\beta} \sum_{j=1}^{m} I_{12},
\end{aligned}
$$

with

$$
\begin{aligned}
I_{11} &= \sum_{k=1}^{p-2} \Big| (x_{2p+1} - \hat{s}_k)^{-\alpha} \int_{x_{2k-1}}^{x_{2k}} \prod_{e=0}^{2} (s - x_{2k-1+e}) ds + (x_{2p+1} - \bar{s}_k)^{-\alpha} \int_{x_{2k}}^{x_{2k+1}} \prod_{e=0}^{2} (s - x_{2k-1+e}) ds \Big| \\
&= \frac{1}{4} h_x^4 \sum_{k=1}^{p-2} |(x_{2p+1} - \hat{s}_k)^{-\alpha} - (x_{2p+1} - \bar{s}_k)^{-\alpha}| = \frac{1}{4} h_x^4 \sum_{k=1}^{p-2} | -\alpha (x_{2p+1} - \tilde{s}_k)^{-\alpha-1} (\bar{s}_k - \hat{s}_k)| \\
&\leq \frac{\alpha}{4} h_x^4 \sum_{k=1}^{p-2} |2(x_{2p+1} - x_{2k+1})^{-\alpha-1} h_x| \leq \frac{\alpha}{4} h_x^4 \int_{x_3}^{x_{2p-1}} (x_{2p+1} - s)^{-\alpha-1} ds \\
&= \frac{1}{4} h_x^4 (2^{-\alpha} h_x^{-\alpha} - (2p-2)^{-\alpha} h_x^{-\alpha}) \leq \frac{1}{2} h_x^{4-\alpha}, \qquad (3.7) \\
I_{12} &\leq h_x^3 \Big(\int_{x_{2p-3}}^{x_{2p-1}} (x_{2p+1} - s)^{-\alpha} ds + \int_{x_{2p-1}}^{x_{2p+1}} (x_{2p+1} - s)^{-\alpha} ds \Big) \\
&= h_x^3 \int_{x_{2p-3}}^{x_{2p+1}} (x_{2p+1} - s)^{-\alpha} ds = \frac{4^{1-\alpha}}{1-\alpha} h_x^{4-\alpha}, \qquad (3.8)
\end{aligned}
$$

where $\hat{s}_k \leq \tilde{s}_k \leq \bar{s}_k$, $x_{2k-1} \leq \hat{s}_k \leq x_{2k} \leq \bar{s}_k \leq x_{2k+1}$. Substitute I_{11} and I_{22} into the original formula to get the direct result

$$
I_1 \leq T_1 \sum_{j=1}^{m} \Big(\frac{1}{2} + \frac{4^{1-\alpha}}{1-\alpha} \Big) h_x^{4-\alpha} h_y^{1-\beta}.
$$

We have completed the estimate of I_1, next we will estimate I_2, which is to say

$$
I_2 \leq T_2 h_x \sum_{j=1}^{m} \Big(\sum_{k=1}^{p} \int_{x_{2k-1}}^{x_{2k+1}} \frac{|\prod_{e=0}^{2}(s - x_{2k-1+e})|}{(x_{2p+1} - s)^{\alpha}} ds \times \int_{c}^{y_1} (y_{2q+1} - r)^{-\beta} dr \Big)
$$

$$\leq T_2 h_x^4 h_y^{1-\beta} \sum_{j=1}^{m} \sum_{k=1}^{p} \int_{x_{2k-1}}^{x_{2k+1}} (x_{2p+1}-s)^{-\alpha} ds \leq T_2 h_x^4 h_y^{1-\beta} \sum_{j=1}^{m} \int_{x_1}^{x_{2p+1}} (x_{2p+1}-s)^{-\alpha} ds$$
$$= T_2 h_x^4 h_y^{1-\beta} \sum_{j=1}^{m} \frac{(2ph_x)^{1-\alpha}}{1-\alpha} \leq \frac{T_2 b^{1-\alpha}}{1-\alpha} \sum_{j=1}^{m} h_x^4 h_y^{1-\beta},$$

where T_2 is shown in (3.3).

For $r_2^{(2)}$, we have

$$r_2^{(2)} \leq T_1 \sum_{j=1}^{m} (\sum_{k=1}^{p} | \int_{x_{2k-1}}^{x_{2k+1}} \frac{\sum_{e=0}^{2} \psi_{e,2k-1}(s)}{6(x_{2p+1}-s)^{\alpha}} ds| \cdot | \int_{c}^{y_1} \frac{\prod_{f=0}^{2}(r-y_f)}{(y_{2q+1}-r)^{\beta}} dr|)$$
$$\leq T_1 h_y^3 \sum_{j=1}^{m} \sum_{k=1}^{p} \int_{x_{2k-1}}^{x_{2k+1}} \int_{c}^{y_1} \frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^{\beta}} drds \leq \frac{b^{1-\alpha}T_1}{1-\alpha} \sum_{j=1}^{m} h_y^{4-\beta}.$$

Combining all the above calculation results, it can be concluded that

$$r_2 \leq \sum_{j=1}^{m} ([(\frac{1}{2} + \frac{4^{1-\alpha}}{1-\alpha}) T_1 h_x^{4-\alpha} + \frac{b^{1-\alpha}}{1-\alpha} T_2 h_x^4] h_y^{1-\beta} + \frac{b^{1-\alpha}}{1-\alpha} T_1 h_y^{4-\beta}). \tag{3.9}$$

For r_3, we estimate it using the same method as r_2, that is

$$r_3 \leq \sum_{j=1}^{m} \sum_{l=1}^{q} | \int_{a}^{x_1} \int_{y_{2l-1}}^{y_{2l+1}} \frac{\partial_s^3 \kappa_{2p+1,2q+1}^{i,j}(\xi_0(s), r, u_j(\xi_0(s), r))}{6(x_{2p+1}-s)^{\alpha}(y_{2q+1}-r)^{\beta}} \prod_{e=0}^{2}(s-x_e) drds|$$
$$+ \sum_{j=1}^{m} \sum_{l=1}^{q} | \int_{a}^{x_1} \int_{y_{2l-1}}^{y_{2l+1}} \frac{\sum_{e=0}^{2} \psi_{e,0}(s) \partial_t^3 \kappa_{2p+1,2q+1}^{i,j}(x_e, \varsigma_{2l-1}(r), u_j(x_e, \varsigma_{2l-1}(r)))}{6(x_{2p+1}-s)^{\alpha}(y_{2q+1}-r)^{\beta}} \prod_{f=0}^{2}(r-y_{2l-1+f}) drds|$$
$$\doteq r_3^{(1)} + r_3^{(2)}.$$

The calculation of $r_3^{(1)}$ is similar to that of $r_2^{(2)}$, so

$$r_3^{(1)} \leq T_1 h_x^3 \sum_{j=1}^{m} \sum_{l=1}^{q} | \int_{a}^{x_1} \int_{y_{2l-1}}^{y_{2l+1}} (x_{2p+1}-s)^{-\alpha} (y_{2q+1}-r)^{-\beta} drds|$$
$$\leq T_1 h_x^{4-\alpha} \sum_{j=1}^{m} \sum_{l=1}^{q} \int_{y_{2l-1}}^{y_{2l+1}} (y_{2q+1}-r)^{-\beta} dr \leq \frac{d^{1-\beta}}{1-\beta} T_1 \sum_{j=1}^{m} h_x^{4-\alpha}.$$

Similarly,

$$r_3^{(2)} \leq \sum_{j=1}^{m} \sum_{l=1}^{q} | \int_{a}^{x_1} \int_{y_{2l-1}}^{y_{2l+1}} \frac{\sum_{e=0}^{2} \psi_{e,0}(s) \partial_t^3 \kappa_{2p+1,2q+1}^{i,j}(x_e, \varsigma_{2l-1}(r_l), u_j(x_e, \varsigma_{2l-1}(r_l)))}{6(x_{2p+1}-s)^{\alpha}(y_{2q+1}-r)^{\beta}} \prod_{f=0}^{2}(r-y_{2l-1+f}) drds|$$

$$+\sum_{j=1}^{m}\sum_{l=1}^{q}\Big|\int_{a}^{x_1}\int_{y_{2l-1}}^{y_{2l+1}}\frac{(x_{2p+1}-s)^{-\alpha}}{6(y_{2q+1}-r)^{\beta}}\sum_{e=0}^{2}\psi_{e,0}(s)[\partial_t^3\kappa_{2p+1,2q+1}^{i,j}(x_e,\varsigma_{2l-1}(r),u_j(x_e,\varsigma_{2l-1}(r)))$$
$$-\partial_t^3\kappa_{2p+1,2q+1}^{i,j}(x_e,\varsigma_{2l-1}(r_l),u_j(x_e,\varsigma_{2l-1}(r_l)))]\prod_{f=0}^{2}(r-y_{2l-1+f})drds\Big|\doteq P_1+P_2,$$

where $r_l=y_{2l}$, and we can get

$$P_1\le T_1\sum_{j=1}^{m}\Big|\int_{a}^{x_1}\frac{\sum_{e=0}^{2}\psi_{e,0}(s)}{6(x_{2p+1}-s)^{\alpha}}ds\Big|\cdot\sum_{l=1}^{q}\Big|\int_{y_{2l-1}}^{y_{2l}}\frac{\prod_{f=0}^{2}(r-y_{2l-1+f})}{(y_{2q+1}-r)^{\beta}}dr+\int_{y_{2l}}^{y_{2l+1}}\frac{\prod_{f=0}^{2}(r-y_{2l-1+f})}{(y_{2q+1}-r)^{\beta}}dr\Big|$$
$$\le T_1h_x^{1-\alpha}\sum_{j=1}^{m}\sum_{l=1}^{q-2}\Big|\int_{y_{2l-1}}^{y_{2l}}\frac{\prod_{f=0}^{2}(r-y_{2l-1+f})}{(y_{2q+1}-r)^{\beta}}dr+\int_{y_{2l}}^{y_{2l+1}}\frac{\prod_{f=0}^{2}(r-y_{2l-1+f})}{(y_{2q+1}-r)^{\beta}}dr\Big|$$
$$+T_1h_x^{1-\alpha}\sum_{j=1}^{m}\Big(\Big|\int_{y_{2q-3}}^{y_{2q-1}}\frac{\prod_{f=0}^{2}(r-y_{2q-3+f})}{(y_{2q+1}-r)^{\beta}}dr\Big|+\Big|\int_{y_{2q-1}}^{y_{2q+1}}\frac{\prod_{f=0}^{2}(r-y_{2q-1+f})}{(y_{2q+1}-r)^{\beta}}dr\Big|\Big)$$
$$=T_1h_x^{1-\alpha}\sum_{j=1}^{m}P_{11}+T_1h_x^{1-\alpha}\sum_{j=1}^{m}P_{12}.$$

The same idea as (3.7) and (3.8) is used by us to estimate P_{11} and P_{12}, and the estimation process of P_2 and I_2 is similar, so it can be obtained

$$P_{11}\le\frac{1}{2}h_y^{4-\beta},P_{12}\le\frac{4^{1-\beta}}{1-\beta}h_y^{4-\beta},P_2\le\frac{T_2d^{1-\beta}}{1-\beta}\sum_{j=1}^{m}h_x^{1-\alpha}h_y^4,$$

combining P_1, P_2, $r_3^{(1)}$, and $r_3^{(2)}$, we can get the estimated result of r_3 as

$$r_3\le\sum_{j=1}^{m}\Big(\frac{T_1d^{1-\beta}}{1-\beta}h_x^{4-\alpha}+\Big[\Big(\frac{1}{2}+\frac{4^{1-\beta}}{1-\beta}\Big)T_1h_y^{4-\beta}+\frac{T_2d^{1-\beta}}{1-\beta}h_y^4\Big]h_x^{1-\alpha}\Big).\tag{3.10}$$

Finally we estimate r_4 and obtain

$$r_4\le\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}\Big|\int_{x_{2k-1}}^{x_{2k+1}}\int_{y_{2l-1}}^{y_{2l+1}}\frac{\partial_s^3\kappa_{2p+1,2q+1}^{i,j}(\xi_{2k-1}(s),r,u_j(\xi_{2k-1}(s),r))}{6(x_{2p+1}-s)^{\alpha}(y_{2q+1}-r)^{\beta}}\prod_{e=0}^{2}(s-x_{2k-1+e})drds\Big|$$
$$+\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}\Big|\int_{x_{2k-1}}^{x_{2k+1}}\int_{y_{2l-1}}^{y_{2l+1}}\frac{(x_{2p+1}-s)^{-\alpha}}{6(y_{2q+1}-r)^{\beta}}\sum_{e=0}^{2}\psi_{e,2k-1}(s)$$
$$\times\partial_t^3\kappa_{2p+1,2q+1}^{i,j}(x_{2k-1+e},\varsigma_{2l-1}(r),u_j(x_{2k-1+e},\varsigma_{2l-1}(r)))\prod_{f=0}^{2}(r-y_{2l-1+f})drds\Big|$$
$$\doteq r_4^{(1)}+r_4^{(2)},$$

where

$$r_4^{(1)} \leq \sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}|\int_{x_{2k-1}}^{x_{2k+1}}\int_{y_{2l-1}}^{y_{2l+1}} \frac{\partial_s^3\kappa_{2p+1,2q+1}^{i,j}(\xi_{2k-1}(s_k),r,u_j(\xi_{2k-1}(s_k),r))}{6(x_{2p+1}-s)^\alpha(y_{2q+1}-r)^\beta}\prod_{e=0}^{2}(s-x_{2k-1+e})drds|$$
$$+\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}|\int_{x_{2k-1}}^{x_{2k+1}}\int_{y_{2l-1}}^{y_{2l+1}} \frac{(x_{2p+1}-s)^{-\alpha}}{6(y_{2q+1}-r)^\beta}\prod_{e=0}^{2}(s-x_{2k-1+e})$$
$$\times[\partial_s^3\kappa_{2p+1,2q+1}^{i,j}(\xi_{2k-1}(s),r,u_j(\xi_{2k-1}(s),r))-\partial_s^3\kappa_{2p+1,2q+1}^{i,j}(\xi_{2k-1}(s_k),r,u_j(\xi_{2k-1}(s_k),r))]drds|,$$

where $s_k = x_{2k}$. We denote the two terms in the above inequality as Q_1 and Q_2. Estimates for Q_1 and Q_2 will be given below, where Q_1 is similar to I_1 estimation, so we have

$$Q_1 \leq T_1\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}|\int_{x_{2k-1}}^{x_{2k+1}}\int_{y_{2l-1}}^{y_{2l+1}}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^\beta}\prod_{e=0}^{2}(s-x_{2k-1+e})drds|$$
$$\leq \frac{T_1d^{1-\beta}}{1-\beta}\sum_{j=1}^{m}\sum_{k=1}^{p}|\int_{x_{2k-1}}^{x_{2k}}\frac{\prod_{e=0}^{2}(s-x_{2k-1+e})}{(x_{2p+1}-s)^\alpha}ds+\int_{x_{2k}}^{x_{2k+1}}\frac{\prod_{e=0}^{2}(s-x_{2k-1+e})}{(x_{2p+1}-s)^\alpha}ds|$$
$$\leq \frac{T_1d^{1-\beta}}{1-\beta}\sum_{j=1}^{m}(\frac{1}{2}+\frac{4^{1-\beta}}{1-\beta})h_x^{4-\alpha},$$
$$Q_2 \leq T_2h_x^4\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}\int_{x_{2k-1}}^{x_{2k+1}}\int_{y_{2l-1}}^{y_{2l+1}}\frac{(x_{2p+1}-s)^{-\alpha}}{(y_{2q+1}-r)^\beta}drds \leq \frac{T_2b^{1-\alpha}d^{1-\beta}}{(1-\alpha)(1-\beta)}\sum_{j=1}^{m}h_x^4.$$

Applying the same ideas and methods to $r_4^{(2)}$, it can be estimated, and the estimated results are as follows:

$$r_4^{(2)} \leq \sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}|\int_{x_{2k-1}}^{x_{2k+1}}\int_{y_{2l-1}}^{y_{2l+1}}\frac{(x_{2p+1}-s)^{-\alpha}}{6(y_{2q+1}-r)^\beta}\sum_{e=0}^{2}\psi_{e,2k-1}(s)$$
$$\times\partial_t^3\kappa_{2p+1,2q+1}^{i,j}(x_{2k-1+e},\varsigma_{2l-1}(r_l),u_j(x_{2k-1+e},\varsigma_{2l-1}(r_l)))\prod_{f=0}^{2}(r-y_{2l-1+f})drds|$$
$$+\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}|\int_{x_{2k-1}}^{x_{2k+1}}\int_{y_{2l-1}}^{y_{2l+1}}\sum_{e=0}^{2}\psi_{e,2k-1}(s)[\frac{\partial_t^3\kappa_{2p+1,2q+1}^{i,j}(x_{2k-1+e},\varsigma_{2l-1}(r),u_j(x_{2k-1+e},\varsigma_{2l-1}(r)))}{6(x_{2p+1}-s)^\alpha(y_{2q+1}-r)^\beta}$$
$$-\frac{\partial_t^3\kappa_{2p+1,2q+1}^{i,j}(x_{2k-1+e},\varsigma_{2l-1}(r_l),u_j(x_{2k-1+e},\varsigma_{2l-1}(r_l)))}{6(x_{2p+1}-s)^\alpha(y_{2q+1}-r)^\beta}]\prod_{f=0}^{2}(r-y_{2l-1+f})drds| \doteq W_1+W_2,$$

where $r_l = y_{2l}$. Handle W_1 with the same trick as P_1, that is to say

$$W_1 \leq T_1\sum_{j=1}^{m}\sum_{k=1}^{p}|\int_{x_{2k-1}}^{x_{2k+1}}\frac{\sum_{e=0}^{2}\psi_{e,2k-1}(s)}{6(x_{2p+1}-s)^\alpha}ds|\cdot\sum_{l=1}^{q}|\int_{y_{2l-1}}^{y_{2l+1}}\frac{\prod_{f=0}^{2}(r-y_{2l-1+f})}{(y_{2q+1}-r)^\beta}dr|$$

$$\leq \frac{T_1 b^{1-\alpha}}{1-\alpha}\sum_{j=1}^{m}\sum_{l=1}^{q}\left|\int_{y_{2l-1}}^{y_{2l}}\frac{\prod_{f=0}^{2}(r-y_{2l-1+f})}{(y_{2q+1}-r)^{\beta}}dr+\int_{y_{2l}}^{y_{2l+1}}\frac{\prod_{f=0}^{2}(r-y_{2l-1+f})}{(y_{2q+1}-r)^{\beta}}dr\right|$$
$$\leq \frac{T_1 b^{1-\alpha}}{1-\alpha}\sum_{j=1}^{m}(\frac{1}{2}+\frac{4^{1-\beta}}{1-\beta})h_y^{4-\beta},$$

$$W_2 \leq T_2 h_y^4\sum_{j=1}^{m}\sum_{k=1}^{p}\left|\int_{x_{2k-1}}^{x_{2k+1}}\frac{\sum_{e=0}^{2}\psi_{e,2k-1}(s)}{6(x_{2p+1}-s)^{\alpha}}ds\right|\cdot\sum_{l=1}^{q}\left|\int_{y_{2l-1}}^{y_{2l+1}}(y_{2q+1}-r)^{-\beta}dr\right| \leq \frac{T_2 b^{1-\alpha}d^{1-\beta}}{(1-\alpha)(1-\beta)}\sum_{j=1}^{m}h_y^4.$$

Therefore, we have

$$r_4 \leq \sum_{j=1}^{m}(T_1[(\frac{1}{2}+\frac{4^{1-\alpha}}{1-\alpha})\frac{d^{1-\beta}}{1-\beta}h_x^{4-\alpha}+(\frac{1}{2}+\frac{4^{1-\beta}}{1-\beta})\frac{b^{1-\alpha}}{1-\alpha}h_y^{4-\beta}]+\frac{T_2 b^{1-\alpha}d^{1-\beta}}{(1-\alpha)(1-\beta)}(h_x^4+h_y^4)). \tag{3.11}$$

Substitute (3.6) and (3.9)–(3.11) into (3.5), the final estimation result of $r^{(i)}_{2p+1,2q+1}$ is

$$|r^{(i)}_{2p+1,2q+1}| \leq C(h_x^{4-\alpha}+h_y^{4-\beta}),$$

where C only depends on T_1, T_2, m, α and β.

$|r^{(i)}_{2p+2,2q+1}|$ and $|r^{(i)}_{2p+k_1,2q+2}|, k_1 = 1,2$ can be proved similarly. Furthermore, $|r^{(i)}_{1,1}|$, $|r^{(i)}_{2,1}|$, $|r^{(i)}_{v,2}|$, $|r^{(i)}_{2p+k_1,n}|$ and $|r^{(i)}_{v,2q+l_1}|, v, n, k_1, l_1 = 1,2; p = 1,2,\ldots,N-1; q = 1,2,\ldots,L-1$, we can similarly give proofs for the cases and will not be repeated here. Finally, we can conclude that

$$|r^{(i)}_{v,n}| \leq C(h_x^{4-\alpha}+h_y^{4-\beta}), i = 1,\ldots,m; v = 1,\ldots,2N; n = 1,\ldots,2L. \tag{3.12}$$

Then the proof is complete. □

4. Convergence analysis

Next, Lemmas 1 and 2 will be used when analyzing the convergence of the scheme (2.26), where Lemma 1 is already given in Section 3, so we introduce Lemma 2 first.

Lemma 2. *The coefficients* $\bar{\Upsilon}^{(p)}_e, \bar{\Theta}^{(q)}_f, e, f = 0,1,2,$ $\Upsilon^{(p)}_k, k = 0,\ldots,2p+2; p = 1,\ldots,N-1$ *and* $\Theta^{(q)}_l, l = 0,\ldots,2q+2, q = 1,\ldots,L-1$ *defined in (2.27) satisfy:*

$$|\bar{\Upsilon}^{(p)}_e| \leq C(2p+2-k)^{-\alpha}, |\bar{\Theta}^{(q)}_f| \leq C(2q+2-f)^{-\beta}, |\Upsilon^{(p)}_k| \leq C(2p+3-k)^{-\alpha}, |\Theta^{(q)}_l| \leq C(2q+3-l)^{-\beta}.$$

Proof. We use the same techniques as in [8] to prove Lemma 2, and the detailed proof process is not shown here. □

Theorem 1. *Suppose u is the exact solution of (2.1) and $u^i_{v,n}$ is the approximate solution in (2.26). Let*

$$e^{(i)}_{v,n} = u_i(x_v, y_n) - u^i_{v,n}, i = 1,\ldots,m; v = 0,\ldots,2N; n = 0,\ldots,2L. \tag{4.1}$$

If $\kappa(\cdot,\cdot,\cdot,\cdot,u(\cdot,\cdot)) \in C^4(U \times R)$ meets the Lipschitz condition defined in (2.3), and the step size h_x and h_y satisfies:

$$\begin{aligned} &mLh_x^{1-\alpha}h_y^{1-\beta}\left|\Upsilon_{2p+2}^{(p)}\right|\left|\Theta_{2q+2}^{(q)}\right| < 1, 2mLh_x^{1-\alpha}h_y^{1-\beta}\left|\Upsilon_{2p+2}^{(p)}\right| < 1,\\ &2mLh_x^{1-\alpha}h_y^{1-\beta}\left|\Theta_{2q+2}^{(q)}\right| < 1, CmLh_y^{1-\beta}\frac{b^{1-\alpha}}{1-\alpha} < 1, \end{aligned} \tag{4.2}$$

then the following error estimates hold

$$|e_{v,n}^{(i)}| \le C(h_x^{4-\alpha} + h_y^{4-\beta}), \tag{4.3}$$

where C is only relevant to $m, L, b, d, \kappa, \alpha$ and β.

Proof. It is obvious from the definition in (4.1) that when $v = 0$ or $n = 0$, $e_{0,n}^{(i)} = e_{v,0}^{(i)} = 0$. From (2.26), when $v, n = 1, 2$, we can know that

$$\begin{aligned} e_{1,1}^{(i)} &= h_x^{1-\alpha}h_y^{1-\beta}\sum_{j=1}^{m}\sum_{k=0}^{2}\sum_{l=0}^{2}\hat{\Upsilon}_k\hat{\Theta}_l[\kappa_{1,1}^{i,j}(x_k,y_l,u_j(x_k,y_l)) - \kappa_{1,1}^{i,j}(x_k,y_l,u_{k,l}^{j})] + r_{1,1}^{(i)},\\ e_{2,1}^{(i)} &= h_x^{1-\alpha}h_y^{1-\beta}\sum_{j=1}^{m}\sum_{k=0}^{2}\sum_{l=0}^{2}\tilde{\Upsilon}_k\hat{\Theta}_l[\kappa_{2,1}^{i,j}(x_k,y_l,u_j(x_k,y_l)) - \kappa_{2,1}^{i,j}(x_k,y_l,u_{k,l}^{j})] + r_{2,1}^{(i)},\\ e_{1,2}^{(i)} &= h_x^{1-\alpha}h_y^{1-\beta}\sum_{j=1}^{m}\sum_{k=0}^{2}\sum_{l=0}^{2}\hat{\Upsilon}_k\tilde{\Theta}_l[\kappa_{1,2}^{i,j}(x_k,y_l,u_j(x_k,y_l)) - \kappa_{1,2}^{i,j}(x_k,y_l,u_{k,l}^{j})] + r_{1,2}^{(i)},\\ e_{2,2}^{(i)} &= h_x^{1-\alpha}h_y^{1-\beta}\sum_{j=1}^{m}\sum_{k=0}^{2}\sum_{l=0}^{2}\tilde{\Upsilon}_k\tilde{\Theta}_l[\kappa_{2,2}^{i,j}(x_k,y_l,u_j(x_k,y_l)) - \kappa_{2,2}^{i,j}(x_k,y_l,u_{k,l}^{j})] + r_{2,2}^{(i)}. \end{aligned}$$

It is known that $\kappa_{i,j}$ satisfies the condition (2.3). Combined with (2.27), after the calculation, it can be known that $\hat{\Upsilon}_k$, $\tilde{\Upsilon}_k$, $\hat{\Theta}_l$ and $\tilde{\Theta}_l$ are both bounded, so the above four formulas can be transformed into:

$$|e_{v,n}^{(i)}| \le CLh_x^{1-\alpha}h_y^{1-\beta}\sum_{j=1}^{m}\sum_{k=0}^{2}\sum_{l=0}^{2}|e_{k,l}^{(j)}| + |r_{v,n}^{(i)}|, \quad v,n = 1,2,$$

combining the above four inequalities and considering (4.2), we find that

$$|e_{v,n}^{(i)}| \le CL(|r_{1,1}^{(i)}| + |r_{2,1}^{(i)}| + |r_{1,2}^{(i)}| + |r_{2,2}^{(i)}|), \quad v,n = 1,2.$$

It is estimated that the process of e_{2p+k_1,l_1} and $e_{k_1,2q+l_1}, k_1, l_1 = 1, 2$ is the same, we only give the proof process of e_{2p+k_1,l_1}, and the other process is omitted. So, on the basis of (2.26) and (4.1), we directly get the following form:

$$\begin{aligned} u_{2p+1,1}^{i} &= h_x^{1-\alpha}h_y^{1-\beta}\sum_{j=1}^{m}\sum_{k=0}^{2p+1}\sum_{l=0}^{2}\bar{\Upsilon}_k^{(p)}\hat{\Theta}_l[\kappa_{2p+1,1}^{i,j}(x_k,y_l,u_j(x_k,y_l)) - \kappa_{2p+1,1}^{i,j}(x_k,y_l,u_{k,l}^{j})] + r_{2p+1,1}^{(i)},\\ u_{2p+2,1}^{i} &= h_x^{1-\alpha}h_y^{1-\beta}\sum_{j=1}^{m}\sum_{k=0}^{2p+2}\sum_{l=0}^{2}\Upsilon_k^{(p)}\hat{\Theta}_l[\kappa_{2p+2,1}^{i,j}(x_k,y_l,u_j(x_k,y_l)) - \kappa_{2p+2,1}^{i,j}(x_k,y_l,u_{k,l}^{j})] + r_{2p+2,1}^{(i)}, \end{aligned}$$

$$u^{i}_{2p+1,2} = h_x^{1-\alpha}h_y^{1-\beta}\sum_{j=1}^{m}\sum_{k=0}^{2p+1}\sum_{l=0}^{2}\bar{\Upsilon}^{(p)}_k\tilde{\Theta}_l[\kappa^{i,j}_{2p+1,2}(x_k,y_l,u_j(x_k,y_l)) - \kappa^{i,j}_{2p+1,2}(x_k,y_l,u^{j}_{k,l})] + r^{(i)}_{2p+1,2},$$

$$u^{i}_{2p+2,2} = h_x^{1-\alpha}h_y^{1-\beta}\sum_{j=1}^{m}\sum_{k=0}^{2p+2}\sum_{l=0}^{2}\Upsilon^{(p)}_k\tilde{\Theta}_l[\kappa^{i,j}_{2p+2,2}(x_k,y_l,u_j(x_k,y_l)) - \kappa^{i,j}_{2p+2,2}(x_k,y_l,u^{j}_{k,l})] + r^{(i)}_{2p+2,2},$$

where $\hat{\Theta}_l$ and $\tilde{\Theta}_l$ are the same as the previous definitions and they are both bounded. $\bar{\Upsilon}^{(p)}_k$ and $\Upsilon^{(p)}_k$ are in (2.27) and satisfy Lemma 2. In order to facilitate the calculation, we set $\|e_k\| = \max\{|e^{j}_{k,l}|, l = 1,2; j = 1,2,\ldots,m\}$, $\|r_k\| = \max\{|r^{(j)}_{k,l}|, l = 1,2; j = 1,2,\ldots,m\}$, after calculation, the above four formulas can be simplified into two, and they are

$$\left\|e_{2p+1}\right\| \le 2mCLh_x^{1-\alpha}h_y^{1-\beta}\sum_{k=0}^{2p}(2p+2-k)^{-\alpha}\|e_k\| + 2mLh_x^{1-\alpha}h_y^{1-\beta}|\Upsilon^{(p)}_{2p+2}|\left\|e_{2p+1}\right\| + \left\|r_{2p+1}\right\|,$$

$$\left\|e_{2p+2}\right\| \le 2mCLh_x^{1-\alpha}h_y^{1-\beta}\sum_{k=0}^{2p+1}(2p+3-k)^{-\alpha}\|e_k\| + 2mLh_x^{1-\alpha}h_y^{1-\beta}|\Upsilon^{(p)}_{2p+2}|\left\|e_{2p+2}\right\| + \left\|r_{2p+2}\right\|.$$

Therefore, we have

$$\left\|e_{2p+1}\right\| \le 2mCLh_x^{1-\alpha}h_y^{1-\beta}\sum_{k=0}^{2p}(2p+1-k)^{-\alpha}\|e_k\| + C\left\|r_{2p+1}\right\|, \tag{4.4}$$

$$\left\|e_{2p+2}\right\| \le 2mCLh_x^{1-\alpha}h_y^{1-\beta}\sum_{k=0}^{2p+1}(2p+2-k)^{-\alpha}\|e_k\| + C\left\|r_{2p+2}\right\|. \tag{4.5}$$

Applying the discrete Gronwall inequality [27] to (4.4) and (4.5) yields

$$\left\|e_{2p+1}\right\| \le C\left\|r_{2p+1}\right\| E_{1-\alpha}(2mCLh_y^{1-\beta}\Gamma(1-\alpha)b^{1-\alpha}),$$
$$\left\|e_{2p+2}\right\| \le C\left\|r_{2p+2}\right\| E_{1-\alpha}(2mCLh_y^{1-\beta}\Gamma(1-\alpha)b^{1-\alpha}).$$

Therefore, when $v \ge 1, n = 1,2$, the formula (4.3) holds. As $v = 1,2, n \ge 3$, the formula (4.3) also holds. Next, we will prove the case of $v, n \ge 3$. It can be directly derived based on (2.26), (2.27) and (4.1). Therefore, we have

$$e^{(i)}_{2p+1,2q+1} = h_x^{1-\alpha}h_y^{1-\beta}\sum_{j=1}^{m}\sum_{k=0}^{2p+1}\sum_{l=0}^{2q+1}\bar{\Upsilon}^{(p)}_k\bar{\Theta}^{(q)}_l[\kappa^{i,j}_{2p+1,2q+1}(x_k,y_l,u_j(x_k,y_l)) - \kappa^{i,j}_{2p+1,2q+1}(x_k,y_l,u^{j}_{k,l})]$$
$$+ r^{(i)}_{2p+1,2q+1},$$

$$e^{(i)}_{2p+2,2q+1} = h_x^{1-\alpha}h_y^{1-\beta}\sum_{j=1}^{m}\sum_{k=0}^{2p+2}\sum_{l=0}^{2q+1}\Upsilon^{(p)}_k\bar{\Theta}^{(q)}_l[\kappa^{i,j}_{2p+2,2q+1}(x_k,y_l,u_j(x_k,y_l)) - \kappa^{i,j}_{2p+2,2q+1}(x_k,y_l,u^{j}_{k,l})]$$
$$+ r^{(i)}_{2p+2,2q+1},$$

$$e^{(i)}_{2p+1,2q+2} = h_x^{1-\alpha}h_y^{1-\beta}\sum_{j=1}^{m}\sum_{k=0}^{2p+1}\sum_{l=0}^{2q+2}\bar{\Upsilon}^{(p)}_k\Theta^{(q)}_l[\kappa^{i,j}_{2p+1,2q+2}(x_k,y_l,u_j(x_k,y_l)) - \kappa^{i,j}_{2p+1,2q+2}(x_k,y_l,u^{j}_{k,l})]$$

$$+ r^{(i)}_{2p+1,2q+2},$$

$$e^{(i)}_{2p+2,2q+2} = h_x^{1-\alpha} h_y^{1-\beta} \sum_{j=1}^{m} \sum_{k=0}^{2p+2} \sum_{l=0}^{2q+2} \Upsilon^{(p)}_k \Theta^{(q)}_l [\kappa^{i,j}_{2p+2,2q+2}(x_k, y_l, u_j(x_k, y_l)) - \kappa^{i,j}_{2p+2,2q+2}(x_k, y_l, u^j_{k,l})]$$
$$+ r^{(i)}_{2p+2,2q+2},$$

where $\bar{\Upsilon}^{(p)}_k$, $\Upsilon^{(p)}_k$, $\bar{\Theta}^{(q)}_l$ and $\Theta^{(q)}_l$ satisfy Lemma 2. We first prove the situation of $v = 2p+1, n = 2q+1$, and the detailed proof process is as follows:

$$\left|e^{(i)}_{2p+1,2q+1}\right| \le CLh_x^{1-\alpha} h_y^{1-\beta} \sum_{j=1}^{m} \sum_{k=0}^{2p} \sum_{l=0}^{2q} (2p+2-k)^{-\alpha}(2q+2-l)^{-\beta} \left|e^{(j)}_{k,l}\right|$$
$$+ CLh_x^{1-\alpha} h_y^{1-\beta} \sum_{j=1}^{m} \sum_{k=0}^{2p} (2p+2-k)^{-\alpha} \left|e^{(j)}_{k,2q+1}\right| + CLh_x^{1-\alpha} h_y^{1-\beta} \sum_{j=1}^{m} \sum_{l=0}^{2q} (2q+2-l)^{-\beta} \left|e^{(j)}_{2p+1,l}\right|$$
$$+ Lh_x^{1-\alpha} h_y^{1-\beta} \sum_{j=1}^{m} \left|\Upsilon^{(p)}_{2p+2}\right| \left|\Theta^{(q)}_{2q+2}\right| \left|e^{(j)}_{2p+1,2q+1}\right| + \left|r^{(i)}_{2p+1,2q+1}\right|.$$

To use the relevant inequalities and to make the calculation simpler, we set $\left\|e_{2p+1,2q+1}\right\| = \max\left\{|e^{(j)}_{2p+1,2q+1}|, j = 1, 2, \ldots, m\right\}$, $\|\hat{r}_l\| = \max\left\{|r^{(i)}_{k,l}|, 0 \le k \le 2N; i = 1, 2, \ldots, m\right\}$, then the above formula becomes:

$$\left\|e_{2p+1,2q+1}\right\| \le CLh_x^{1-\alpha} h_y^{1-\beta} \sum_{j=1}^{m} \sum_{k=0}^{2p} \sum_{l=0}^{2q} (2p+2-k)^{-\alpha}(2q+2-l)^{-\beta} \left|e^{(j)}_{k,l}\right|$$
$$+ CLh_x^{1-\alpha} h_y^{1-\beta} \sum_{j=1}^{m} \sum_{k=0}^{2p} (2p+2-k)^{-\alpha} \left|e^{(j)}_{k,2q+1}\right| + CLh_x^{1-\alpha} h_y^{1-\beta} \sum_{j=1}^{m} \sum_{l=0}^{2q} (2q+2-l)^{-\beta} \left|e^{(j)}_{2p+1,l}\right|$$
$$+ mLh_x^{1-\alpha} h_y^{1-\beta} \left|\Upsilon^{(p)}_{2p+2}\right| \left|\Theta^{(q)}_{2q+2}\right| \left\|e_{2p+1,2q+1}\right\| + \left\|\hat{r}_{2q+1}\right\|.$$

By shifting the same terms on the right and left sides of the above inequality to the left and simplifying their coefficients, we get the following inequality

$$\left\|e_{2p+1,2q+1}\right\| \le CLh_x^{1-\alpha} h_y^{1-\beta} \sum_{j=1}^{m} \sum_{k=0}^{2p} \sum_{l=0}^{2q} (2p+2-k)^{-\alpha}(2q+2-l)^{-\beta} \left|e^{(j)}_{k,l}\right|$$
$$+ CLh_x^{1-\alpha} h_y^{1-\beta} \sum_{j=1}^{m} \sum_{k=0}^{2p} (2p+2-k)^{-\alpha} \left|e^{(j)}_{k,2q+1}\right|$$
$$+ CLh_x^{1-\alpha} h_y^{1-\beta} \sum_{j=1}^{m} \sum_{l=0}^{2q} (2q+2-l)^{-\beta} \left|e^{(j)}_{2p+1,l}\right| + C\left\|\hat{r}_{2q+1}\right\|.$$

For the sake of simplicity, we make $\|\hat{e}_l\| = \max\left\{|e^{(j)}_{k,l}|, 0 \le k \le 2N; j = 1, 2, \ldots, m\right\}$. Since the above formula holds for all $p = 1, 2, \ldots, N-1$, the above inequality can become

$$
\begin{aligned}
\left\|\hat{e}_{2q+1}\right\| &\leq CmLh_x^{1-\alpha}h_y^{1-\beta}\sum_{k=0}^{2p}\sum_{l=0}^{2q}(2p+2-k)^{-\alpha}(2q+2-l)^{-\beta}\left\|\hat{e}_l\right\| \\
&\quad + CmLh_x^{1-\alpha}h_y^{1-\beta}\sum_{k=0}^{2p}(2p+2-k)^{-\alpha}\left\|\hat{e}_{2q+1}\right\| \\
&\quad + CmLh_x^{1-\alpha}h_y^{1-\beta}\sum_{l=0}^{2q}(2q+2-l)^{-\beta}\left\|\hat{e}_l\right\| + C\left\|\hat{r}_{2q+1}\right\| \\
&\leq CmLh_y^{1-\beta}(\left\|\hat{e}_{2q+1}\right\| + \sum_{l=0}^{2q}(2q+2-l)^{-\beta}\left\|\hat{e}_l\right\|)\sum_{k=0}^{2p}([(2p+2-k)h_x]^{-\alpha}h_x) \\
&\quad + CmLh_x^{1-\alpha}h_y^{1-\beta}\sum_{l=0}^{2q}(2q+2-l)^{-\beta}\left\|\hat{e}_l\right\| + C\left\|\hat{r}_{2q+1}\right\| \\
&\leq CmLh_y^{1-\beta}(\left\|\hat{e}_{2q+1}\right\| + \sum_{l=0}^{2q}(2q+2-l)^{-\beta}\left\|\hat{e}_l\right\|)\int_{x_0}^{x_{2p+2}}(x_{2p+2}-s)^{-\alpha}ds \\
&\quad + CmLh_x^{1-\alpha}h_y^{1-\beta}\sum_{l=0}^{2q}(2q+2-l)^{-\beta}\left\|\hat{e}_l\right\| + C\left\|\hat{r}_{2q+1}\right\| \\
&= CmLh_y^{1-\beta}\frac{x_{2p+2}^{1-\alpha}}{1-\alpha}\sum_{l=0}^{2q}(2q+2-l)^{-\beta}\left\|\hat{e}_l\right\| + CmLh_y^{1-\beta}\frac{x_{2p+2}^{1-\alpha}}{1-\alpha}\left\|\hat{e}_{2q+1}\right\| \\
&\quad + CmLh_x^{1-\alpha}h_y^{1-\beta}\sum_{l=0}^{2q}(2q+2-l)^{-\beta}\left\|\hat{e}_l\right\| + C\left\|\hat{r}_{2q+1}\right\| \\
&\leq CmLh_y^{1-\beta}(h_x^{1-\alpha}+\frac{b^{1-\alpha}}{1-\alpha})\sum_{l=0}^{2q}(2q+2-l)^{-\beta}\left\|\hat{e}_l\right\| + CmLh_y^{1-\beta}\frac{b^{1-\alpha}}{1-\alpha}\left\|\hat{e}_{2q+1}\right\| + C\left\|\hat{r}_{2q+1}\right\|.
\end{aligned}
$$

Combining similar terms of the above inequalities, we get the following results:

$$
\begin{aligned}
\left\|\hat{e}_{2q+1}\right\| &\leq CmLh_y^{1-\beta}(h_x^{1-\alpha}+\frac{b^{1-\alpha}}{1-\alpha})\sum_{l=0}^{2q}(2q+2-l)^{-\beta}\left\|\hat{e}_l\right\| + C\left\|\hat{r}_{2q+1}\right\| \\
&\leq CmLh_y^{1-\beta}(h_x^{1-\alpha}+\frac{b^{1-\alpha}}{1-\alpha})\sum_{l=0}^{2q}(2q+1-l)^{-\beta}\left\|\hat{e}_l\right\| + C\left\|\hat{r}_{2q+1}\right\|.
\end{aligned} \tag{4.6}
$$

Applying the discrete Gronwall inequality [27] to (4.6) yields the following result

$$
\begin{aligned}
\left\|\hat{e}_{2q+1}\right\| &\leq C\left\|\hat{r}_{2q+1}\right\|E_{1-\beta}\left(CmL(\frac{b^{1-\alpha}}{1-\alpha}+h_x^{1-\alpha})\Gamma(1-\beta)((2q+1)h_y)^{1-\beta}\right) \\
&\leq C\left\|\hat{r}_{2q+1}\right\|E_{1-\beta}\left(CmL(\frac{b^{1-\alpha}}{1-\alpha}+h_x^{1-\alpha})\Gamma(1-\beta)d^{1-\beta}\right).
\end{aligned}
$$

The above formula combined with Lemma 1 has

$$|e_{2p+1,2q+1}| \leq C(h_x^{4-\alpha} + h_y^{4-\beta}).$$

For the cases of $v = 2p + 2, n = 2q + 1$ and $v = 2p + k_1, n = 2q + 2, k_1 = 1, 2$, we can give proofs similarly, and the detailed process is omitted. For all cases we have given the proof, so the proof of Theorem 1 is then completed. □

5. Numerical examples

In this part, we apply the numerical algorithm proposed in section 2 to solve the nonlinear fractional order Volterra integral equation system when $m = 2$, and through these numerical examples to verify whether the theoretical analysis of the error of the numerical scheme we have constructed is correct. A 3.40 GHz Intel(R)i5-7500 CPU and 8.00 GB of RAM ThinkCentre computer is used for the numerical implementation in Matlab of the following examples.

In all examples, we take $a = c = 0, b = d = 1$. The error is calculated by $e_h^u = \max\{|u_i(x_k, y_l) - u_{k,l}^i|$, $i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, 2N; l = 1, 2, \ldots, 2L\}$, and the convergence order is calculated by $\log_2(e_{2h}^u/e_h^u)$, where $2N = (b - a)/h_x$, $2L = (d - c)/h_y$.

Example 5.1. In Eq (2.1), we set

$$\kappa_{11}(x, y, s, r, u_1(s, r)) = u_1(s, r), \quad \kappa_{12}(x, y, s, r, u_2(s, r)) = -u_2(s, r),$$
$$\kappa_{21}(x, y, s, r, u_1(s, r)) = u_1(s, r), \quad \kappa_{22}(x, y, s, r, u_2(s, r)) = u_2(s, r),$$

and corresponding $g_1(x, y), g_2(x, y)$ as follows

$$g_1(x, y) = x^4y^4 + 144x^{5-\alpha}y^{4-\beta}(5 - \beta - 4y)\Pi_{i=1}^{5}(i - \alpha)^{-1}(i - \beta)^{-1},$$
$$g_2(x, y) = x^4y^3 - 144x^{5-\alpha}y^{4-\beta}(5 - \beta + 4y)\Pi_{i=1}^{5}(i - \alpha)^{-1}(i - \beta)^{-1}.$$

It is easy to check that its exact solutions is $u_1(x, y) = x^4y^4, u_2(x, y) = x^4y^3$.

In this example, we selected two groups of α and β for the experiment, the first group is $\alpha = 0.3, \beta = 0.6$, the second group is $\alpha = 0.1, \beta = 0.7$, and the results of the two groups are shown in Tables 1 and 2, respectively. We give the maximum error, CPU time and convergence order of $N = 2L, L = 8, 16, 32, 64, 128$. According to Theorem 1, the ideal order is $O(h_x^{4-\alpha} + h_y^{4-\beta})$. So if h_x is small enough compared to h_y, the order of the algorithm can reach $4 - \beta$. In Table 1, it can be seen that when $\alpha = 0.3, \beta = 0.6$, the result obtained after the experiment is almost 3.4. Similarly, in Table 2, when $\alpha = 0.1, \beta = 0.7$, the order is almost 3.3. In Tables 1 and 2, the CPU time shows an increasing trend, and as L gradually changes from 8 to 128, the CPU time also changes rapidly from 0.2 seconds to 2.2 hours. Next, we give the maximum error and convergence order for $L = [N^{(4-\alpha)/(4-\beta)}]$, $N = 8, 16, \cdots, 128$, where $[\cdot]$ points to round up. When taking appropriate step size h_x and h_y, the order can become $O(h_x^{4-\alpha} + h_y^{4-\beta}) = O(h_x^{4-\alpha})$. In Table 3, when $\alpha = 0.3, \beta = 0.6$, the order is almost 3.7, when $\alpha = 0.1, \beta = 0.7$, the order is almost 3.9, which is in accordance with our theoretical analysis.

Furthermore, we will discuss the time efficiency of the numerical scheme. From Table 1, it is easy to find that the CPU time of the scheme gradually increases from 0.27 second to about 2.12 hour when L take $8, 16, 32, 64, 128$. In the Table 4, we take the same parameters as table 1for the linear interpolation scheme for Eq (2.1), and find that the CPU time is bigger than Table 1 with the same L and α,β. Therefore, the scheme of this paper is an efficient numerical scheme.

Table 1. Maximum errors, convergence rate and CPU time when $N = 2L$ for Example 5.1.

L	$\alpha = 0.3, \beta = 0.6$	Rate	CPU time
8	8.71469811e-05	–	0.27051900 s
16	8.30528402e-06	3.39134927	2.04459220 s
32	7.91738036e-07	3.39093444	30.44934650 s
64	7.54232392e-08	3.39194214	8.06502670 min
128	7.17644876e-09	3.39366709	2.11983998 h

Table 2. Maximum errors, convergence rate and CPU time when $N = 2L$ for Example 5.1.

L	$\alpha = 0.1, \beta = 0.7$	Rate	CPU time
8	9.19713221e-05	–	0.25315950 s
16	9.65462986e-06	3.25189122	2.10016270 s
32	1.00080382e-06	3.27006174	30.79893399 s
64	1.02936973e-07	3.28132603	8.09805203 min
128	1.05426964e-08	3.28744545	2.191570 h

Table 3. Maximum errors and convergence rate when $L = [N^{(4-\alpha)/(4-\beta)}]$ for Example 5.1.

N	$\alpha = 0.3, \beta = 0.6$	Rate	$\alpha = 0.1, \beta = 0.7$	Rate
8	1.47566447e-04	–	5.99652448e-05	–
16	1.25967267e-05	3.55024392	4.87964376e-06	3.61927883
32	1.03971705e-06	3.59878603	3.64921135e-07	3.74111921
64	8.38328755e-08	3.63253104	2.64195326e-08	3.78790785
128	6.67860655e-09	3.64989708	1.87047577e-09	3.82012776

Table 4. Maximum errors and convergence rate and CPU time for using linear interpolation when $N = 2L$ for Example 5.1.

L	$\alpha = 0.3, \beta = 0.6$	Rate	CPU time
8	5.486647938442668e-03	-	0.287051800000000 s
16	1.408608301490477e-03	1.961654515170453	4.171250300000000 s
32	3.594643339853709e-04	1.970349951419168	1.08739484500000 min
64	9.125153844657952e-05	1.977927846533698	2.844682999166667 h
128	2.307246722643708e-05	1.983676596941022	4.476602595861111 h

Example 5.2. Taking

$$\kappa_{11}(x, y, s, r, u_1(s, r)) = xyu_1(s, r), \quad \kappa_{12}(x, y, s, r, u_2(s, r)) = -(s + r)u_2(s, r),$$
$$\kappa_{21}(x, y, s, r, u_1(s, r)) = xyu_1(s, r), \quad \kappa_{22}(x, y, s, r, u_2(s, r)) = (s + r)u_2(s, r),$$

in (2.1), and

$$g_1(x,y) = x^4y^4 + 144x^{5-\alpha}y^{4-\beta}(5(5-\beta)x(6-\alpha)^{-1} + 4y - 4xy^2)\Pi_{i=1}^{5}(i-\alpha)^{-1}(i-\beta)^{-1},$$
$$g_2(x,y) = x^4y^3 - 144x^{5-\alpha}y^{4-\beta}(5(5-\beta)x(6-\alpha)^{-1} + 4y + 4xy^2)\Pi_{i=1}^{5}(i-\alpha)^{-1}(i-\beta)^{-1}.$$

It is easy to check that its exact solution is $u_1(x,y) = x^4y^4, u_2(x,y) = x^4y^3$.

For this example, we test the equation system with the $\boldsymbol{\kappa}$ function, and the corresponding parameter selection is the same as that of Example 5.1. Based on the high-order numerical scheme (2.26) proposed in this paper, we repeat all the calculations in Example 5.1 to obtain the following experimental results, which are recorded in Tables 5 and 6. Tables 5 and 6 have the same meaning as Tables 1–3, respectively.

From the data summary in Table 5, it can be found that when $\alpha = 0.3, \beta = 0.6$, the order is almost 3.4, and when $\alpha = 0.1, \beta = 0.7$, the order almost becomes 3.3.

From the data in Table 6, it can be concluded that when $\alpha = 0.3, \beta = 0.6$ and $\alpha = 0.1, \beta = 0.7$, the experimentally obtained orders can almost reach 3.7 and 3.9, respectively.

Table 5. Maximum errors and convergence rate when $N = 2L$ for Example 5.2.

L	$\alpha = 0.3, \beta = 0.6$	Rate	$\alpha = 0.1, \beta = 0.7$	Rate
8	3.31798821e-04	–	2.16746075e-04	–
16	3.29804040e-05	3.33062778	2.27906591e-05	3.24949131
32	3.19745090e-06	3.36661495	2.35342130e-06	3.27561113
64	3.06031027e-07	3.38517237	2.40922619e-07	3.28811785
128	2.90800984e-08	3.39557387	2.45672984e-08	3.29375872

Table 6. Maximum errors and convergence rate when $L = [N^{(4-\alpha)/(4-\beta)}]$ for Example 5.2.

N	$\alpha = 0.3, \beta = 0.6$	Rate	$\alpha = 0.1, \beta = 0.7$	Rate
8	6.33914258e-04	–	2.63628865e-04	–
16	5.45874183e-05	3.53764734	2.01429527e-05	3.71016125
32	4.54514303e-06	3.58617086	1.47021666e-06	3.77617449
64	3.68622602e-07	3.62410925	1.05471302e-07	3.80110636
128	2.94802414e-08	3.64432428	7.43299233e-09	3.82676358

Example 5.3. We set

$$\kappa_{11}(x,y,s,r,u_1(s,r)) = u_1(s,r)\cos x, \quad \kappa_{12}(x,y,s,r,u_2(s,r)) = -xyu_2(s,r)\cos y,$$
$$\kappa_{21}(x,y,s,r,u_1(s,r)) = x^2u_1(s,r)\sin x, \quad \kappa_{22}(x,y,s,r,u_2(s,r)) = u_2(s,r)\cos y,$$

in (2.1), and corresponding $g_1(x,y), g_2(x,y)$ as follows

$$g_1(x,y) = x^4y^4 + 144x^{5-\alpha}y^{5-\beta}((5-\beta)x\cos y - 4\cos x)\Pi_{i=1}^{5}(i-\alpha)^{-1}(i-\beta)^{-1},$$
$$g_2(x,y) = x^4y^3 - 144x^{5-\alpha}y^{4-\beta}((5-\beta)\cos y + 4x^2y\sin x)\Pi_{i=1}^{5}(i-\alpha)^{-1}(i-\beta)^{-1},$$

it is easy to check that its exact solutions is $u_1(x,y) = x^4y^4, u_2(x,y) = x^4y^3$.

We repeat the operations above and present the corresponding results in the following two tables. According to the data in Table 7, the order obtained by the test is $4-\beta$, that is, when $\alpha = 0.3, \beta = 0.6$, it almost reaches 3.4, and when $\alpha = 0.1, \beta = 0.7$, it almost reaches 3.3. Likewise, the order in Table 8 is close to the convergence order $4-\alpha$, in other words, when $\alpha = 0.3, \beta = 0.6$, the order is almost 3.7, and when $\alpha = 0.1, \beta = 0.7$, the order is almost 3.9.

Table 7. Maximum errors and convergence rate when $N = 2L$ for Example 5.3.

L	$\alpha = 0.3, \beta = 0.6$	Rate	$\alpha = 0.1, \beta = 0.7$	Rate
8	7.85382137e-05	–	6.62809108e-05	–
16	7.81453794e-06	3.32916231	7.02665463e-06	3.23768353
32	7.60665918e-07	3.36082570	7.30739515e-07	3.26540886
64	7.31546448e-08	3.37824159	7.52112781e-08	3.28033631
128	6.98538205e-09	3.38853854	7.70071495e-09	3.28788471

Table 8. Maximum errors and convergence rate when $L = [N^{(4-\alpha)/(4-\beta)}]$ for Example 5.3.

N	$\alpha = 0.3, \beta = 0.6$	Rate	$\alpha = 0.1, \beta = 0.7$	Rate
8	9.11090974e-05	–	4.18616892e-05	–
16	7.79322194e-06	3.54730331	3.17701508e-06	3.71988669
32	6.47537360e-07	3.58918457	2.30521043e-07	3.78470158
64	5.22384402e-08	3.63177971	1.64884442e-08	3.80537127
128	4.15797352e-09	3.65115934	1.15324761e-09	3.83768107

Example 5.4. We set

$$\kappa_{11}(x, y, s, r, u_1(s, r)) = u_1(s, r), \quad \kappa_{12}(x, y, s, r, u_2(s, r)) = -u_2(s, r),$$
$$\kappa_{21}(x, y, s, r, u_1(s, r)) = u_1(s, r), \quad \kappa_{22}(x, y, s, r, u_2(s, r)) = -u_2^2(s, r),$$

in (2.1), and

$$g_1(x, y) = x^4y^4 + 36x^{4-\alpha}y^{4-\beta}(1 - 16xy(5-\alpha)^{-1}(5-\beta)^{-1})\Pi_{i=1}^{4}(i-\alpha)^{-1}(i-\beta)^{-1},$$
$$g_2(x, y) = x^3y^3 + 576x^{5-\alpha}y^{5-\beta}(900x^2y^2\Pi_{j=6}^{7}(j-\alpha)^{-1}(j-\beta)^{-1} - 1)\Pi_{i=1}^{5}(i-\alpha)^{-1}(i-\beta)^{-1}.$$

It is easy to check that its exact solution is $u_1(x, y) = x^4y^4, u_2(x, y) = x^3y^3$.

We similarly repeat all the computations of Example 5.1 and record the corresponding results below. In Table 9, when $\alpha = 0.3, \beta = 0.6$, the convergence order can reach 3.4, and when $\alpha = 0.1, \beta = 0.7$, the order can almost reach 3.3. In Table 10, when $\alpha = 0.3, \beta = 0.6$, the order differs very little from 3.7, and when $\alpha = 0.1, \beta = 0.7$, the order differs very little from 3.9.

From Table 1 to Table 10, all numerical results are as accurate as theoretical analysis results, which is the fact that the high-accuracy of the numerical approximation depends on the suitable regularity. In following Example 5.5, we will test the convergence of the numerical scheme for solving two-dimensional nonlinear fractional Volterra integral equations with low-regularity solution.

Table 9. Maximum errors and convergence rate when $N = 2L$ for Example 5.4.

L	$\alpha = 0.3, \beta = 0.6$	Rate	$\alpha = 0.1, \beta = 0.7$	Rate
8	8.11396294e-05	–	6.29251969e-05	–
16	8.43312827e-06	3.26626691	6.94409014e-06	3.17978024
32	8.44707515e-07	3.31954410	7.40619060e-07	3.22898209
64	8.27401234e-08	3.35179287	7.74194093e-08	3.25796447
128	7.99467692e-09	3.37147547	7.99505106e-09	3.27551614

Table 10. Maximum errors and convergence rate when $L = [N^{(4-\alpha)/(4-\beta)}]$ for Example 5.4.

N	$\alpha = 0.3, \beta = 0.6$	Rate	$\alpha = 0.1, \beta = 0.7$	Rate
8	9.59409088e-05	–	4.97564142e-05	–
16	8.29570414e-06	3.53170975	3.49561171e-06	3.83126566
32	6.99590747e-07	3.56778132	2.44323936e-07	3.83867782
64	5.71517175e-08	3.61364244	1.71666396e-08	3.83111564
128	4.58861881e-09	3.63866498	1.20192389e-09	3.83619021

Example 5.5. In Eq (2.1), we set

$$\kappa_{11}(x, y, s, r, u_1(s, r)) = u_1(s, r), \quad \kappa_{12}(x, y, s, r, u_2(s, r,)) = -u_2(s, r),$$
$$\kappa_{11}(x, y, s, r, u_1(s, r)) = u_1(s, r), \quad \kappa_{12}(x, y, s, r, u_2(s, r,)) = u_2(s, r),$$

corresponding $g_1(x, y), g_2(x, y)$ as follows

$$g_1(x, y) = x^{\alpha} y - \frac{xy}{\Gamma(2)}\Big(\frac{\Gamma(1-\alpha)\Gamma(1+\alpha)}{(1-\beta)(2-\beta)} y^{1-\beta} - \frac{\Gamma(1-\beta)\Gamma(1+\beta)}{(1-\alpha)(2-\alpha)} x^{1-\alpha}\Big),$$
$$g_2(x, y) = x y^{\beta} - \frac{xy}{\Gamma(2)}\Big(\frac{\Gamma(1-\alpha)\Gamma(1+\alpha)}{(1-\beta)(2-\beta)} y^{1-\beta} + \frac{\Gamma(1-\beta)\Gamma(1+\beta)}{(1-\alpha)(2-\alpha)} x^{1-\alpha}\Big).$$

It is easy to check that its exact solutions is $u_1(x, y) = x^{\alpha} y, u_2(x, y) = x y^{\beta}$.

From Table 11, when $\alpha = 0.3, \beta = 0.6$, the convergence order can reach 1.2. This is far from the convergence order what we want. This result seems reasonable since it can be verified that the regularity of solution is low.

Table 11. Maximum errors, convergence rate when $N = 2L$ for Example 5.5.

L	$\alpha = 0.3, \beta = 0.6$	Rate
8	8.207474789677494e-02	-
16	4.179280709050004e-02	0.973683750208453
32	1.894395669231108e-02	1.141516975226790
64	8.161427231980323e-03	1.214844316854944
128	3.425264414530949e-03	1.252606104237619

6. Conclusions

The research object of this paper is system of two-dimensional nonlinear fractional Volterra equations, which have singular kernels. To approximate its solution, we give an efficient block-by-block method based on the modified block-by-block method [8], and only two boundary layers are implicit in this method, other layers are explicit. The core of the method is that we divide its domain into a series of subdomains, that is, block it, and use piecewise quadratic Lagrangian interpolation on each subdomain to approximate $\kappa(x, y, s, r, u(s, r))$, and then calculate the corresponding integral to obtain the final result. Our proposed method has uniform accuracy and its convergence order is $O(h_x^{4-\alpha} + h_y^{4-\beta})$. We give a strict proof for the error analysis of the method, and give several numerical examples to verify the correctness of the theoretical analysis. From the examples in the paper, we can find that the convergence speed of the numerical scheme is relatively fast. When there are more and more discrete points, the computational complexity of the numerical scheme increases very quickly. Therefore, the scheme needs to improve the computational efficiency of the algorithm for calculating three-dimensional problems. In the future, we will establish a fast numerical scheme based on the special structure of the discrete matrix and fast Fourier transform.

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Conflict of interest

The authors declare that they have no competing interests.

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