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Research article

Characterizations of modules definable in o-minimal structures

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Abstract: Let $\mathfrak{M}$ be an o-minimal expansion of a densely linearly ordered set and $(S, +, \cdot, 0_S, 1_S)$ be a ring definable in $\mathfrak{M}$. In this article, we develop two techniques for the study of characterizations of S-modules definable in $\mathfrak{M}$. The first technique is an algebraic technique. More precisely, we show that every S-module definable in $\mathfrak{M}$ is finitely generated. For the other technique, we prove that every S-module definable in $\mathfrak{M}$ admits a unique definable S-module manifold topology. As consequences, we obtain the following: (1) if S is finite, then a module A is isomorphic to an S-module definable in $\mathfrak{M}$ if and only if A is finite; (2) if S is an infinite ring without zero divisors, then a module A is isomorphic to an S-module definable in $\mathfrak{M}$ if and only if A is a finite dimensional free module over S; and (3) if $\mathfrak{M}$ is an expansion of an ordered divisible abelian group and S is an infinite ring without zero divisors, then every S-module definable in $\mathfrak{M}$ is definably connected with respect to the unique definable S-module manifold topology.

Keywords: o-minimality; definable group; definable ring; definable module; manifold topology
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1. Introduction

Throughout this paper, let $\mathfrak{M}$ be a fixed (but arbitrary) *o-minimal* expansion of a densely linearly ordered set $(M, <)$ (that is, every unary definable set is a finite union of open intervals and points). We assume the reader's familiarity with basic model theory and o-minimality. We refer to [1, 7] for more on model theory and [2, 6, 13, 14] for more on o-minimality. Here, the word "definable" means "definable in $\mathfrak{M}$ possibly with parameters" and the word "0-definable" means "definable in $\mathfrak{M}$ without parameters". Recall that we may equip M with the order topology induced by $<$; therefore, every subset of M^n can be equipped with the subspace topology induced by the product topology on M^n. Unless indicated otherwise, topological properties on a subset of M^n are considered with respect to this topology. For natural numbers $m \leq n$, let $\Pi(n, m)$ denote the set of all coordinate projections from M^n to M^m. For any set $X \subseteq M^n$, let $\dim X$ denote the largest natural number m such that $\pi(X)$ has nonempty interior for some $\pi \in \Pi(n, m)$. We say that a subset Y of X is a *large* subset of X if $\dim(X \setminus Y) < \dim X$.

Let $(G, *, e)$ be a group with the group operation $*$ and the identity e. We say that the group $(G, *, e)$ is a *definable group* if the set G and the group operation $*$ are definable. We will simply write G if the group operation and the identity are clear from the context. Note that every finite group is isomorphic to a definable group. In [5], E. Hrushovski showed that an algebraic group can be recovered from birational data. Inspired by this result, A. Pillay introduced manifold topologies on definable groups and used them to study characterizations of infinite definable groups (see [12]).

Let X be a definable set and τ be a topology on X. For a definable set I, we say that a collection $\{Y_i\}_{i\in I}$ of subsets of X is a *definable collection* if $\bigcup\{\{i\} \times Y_i : i \in I\}$ is definable. We say that τ is a *definable topology* if there is a definable collection of subsets of X that generates τ. We call every element of τ a τ-open set. A map from X^n to X^m is *τ-continuous* if the map is continuous with respect to the product topologies on X^n and X^m generated by τ. Next, let G be a definable group. Obviously, we may equip G with the subspace topology induced by the order topology on $(M, <)$ or the discrete topology. These topologies are definable topologies on G. In addition, for each $k \in \mathbb{N}$, we say that a definable topology τ on G is a *definable group k-manifold topology* if both the group operation and the inversion map are τ-continuous, and there exist definable τ-open subsets $D_1, \ldots, D_n$ of G and definable maps $\phi_1, \ldots, \phi_n$ such that $\bigcup\{D_i : i = 1, \ldots, n\} = G$ and each $\phi_i \colon D_i \to M^k$ is a homeomorphism from D_i onto its image. Interestingly, it has been shown in [12] that every definable group admits a unique definable group ($\dim G$)-manifold topology, τ_G. In [15], V. Razenj proved that if $\dim G = 1$ and G is definably τ_G-connected, then G is isomorphic to either $\bigoplus_{i\in I} \mathbb{Q}$ or $\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_{p^\infty} \oplus \bigoplus_{i\in I} \mathbb{Q}$, for some index set I, where $\mathbb{P}$ is the set of all primes. Characterizations of 2-dimensional and 3-dimensional definable groups are studied in [8]. We know that if $\dim G = 2$ and G is a definably τ_G-connected, non-abelian definable group, then there is a real closed field T such that G is isomorphic to a semidirect product of the additive group of T and the multiplicative group of the positive elements of T; and if $\dim G = 3$ and G is a non-solvable, centerless, definably τ_G-connected definable group, then there is a real closed field T such that G is isomorphic to either $PSL_2(T)$ or $SO_3(T)$. In [3], M. Edmundo introduced a notion of definable G-modules and used them to study definable solvable groups.

Analogously, definable rings are also studied in [9]. Let $(S, +, \cdot, 0_S, 1_S)$ (or simply write S if it is clear from the context) be a ring. We say that S is a *definable ring* if the set S, the addition $+$ and the multiplication $\cdot$ are definable. For each $k \in \mathbb{N}$, a topology τ on S is a *definable ring k-manifold topology* if the addition, the additive inversion and the multiplication are τ-continuous and there exist definable τ-open subsets $D_1, \ldots, D_n$ of S and definable maps $\phi_1, \ldots, \phi_n$ such that $\bigcup\{D_i : i = 1, \ldots, n\} = S$ and each $\phi_i \colon D_i \to M^k$ is a homeomorphism from D_i onto its image. It has been shown in [9] that S admits a unique definable ring ($\dim S$)-manifold topology, τ_S. In [10], Y. Peterzil and C. Steinhorn proved that if S is an infinite definable ring without zero divisors, then there is a real closed field T such that S is definably isomorphic to either T, $T(\sqrt{-1})$, or $\mathbb{H}(T)$ where $\mathbb{H}(T)$ denote the ring of quaternions over T; therefore, S is a division ring.

Inspired by these results, we are interested in an intermediate step. To be more precise, the main question of this article is to find characterizations of definable modules. Let $(S, +, \cdot, 0_S, 1_S)$ be a definable ring and $(A, \oplus, 0_A, \lambda_S)$ be a left (right) S-module where $\lambda_S \colon S \times A \to A$ is the left (right) scalar multiplication. We say that A is a *definable left (right) S-module* if $(A, \oplus, 0_A)$ is a definable group and λ_S is definable. For the sake of readability, we will write λ instead of λ_S if the ring S is clear from the context. To study characterizations of definable left (right) S-modules, we develop two techniques. The following techniques work for both left and right S-modules. For simplicity, we will consider only left S-modules and, from now on, the word "S-module" means "left S-module". For the first approach, we consider the generators of A as S-module. The key step is to show that every definable S-module is finitely generated (see Section 2). As a result, we obtain:

Theorem A. *1. If S is a finite ring and A is an S-module, then A is isomorphic to a definable S-module if and only if A is finite.*

2. Suppose S is an infinite definable ring without zero divisors and A is an S-module. Then A is isomorphic to a definable S-module if and only if A is a finite dimensional free module over S.

In addition, by the fundamental theorem of finite abelian groups, the characterization of infinite definable rings without zero divisors, and Theorem A, we have:

Corollary A. *1. Suppose S is a finite ring and A is a definable S-module. Then A is isomorphic as a group to a direct product of cyclic groups of prime-power order.*

2. Suppose S is an infinite definable ring without zero divisors and A is a definable S-module. Then there exist a definable real closed field T and a natural number k such that T is a subring of S and A is definably isomorphic (as S-modules) to either T^k, $T(\sqrt{-1})^k$ or $\mathbb{H}(T)^k$.

Next, since manifold topologies on algebraic structures are important tools to study characterizations, we also develop a result on the existence of definable module manifold topologies, which will be introduced in Section 3, and use it to give an alternative proof of (2) in Theorem A (when $\mathfrak{M}$ is an expansion of an ordered divisible abelian group). Interestingly, this proof implies that if $\mathfrak{M}$ is an expansion of an ordered divisible abelian group, then every definable module over infinite definable ring without zero divisors is connected with respect to the unique definable group manifold topology.

Conventions and notations

In this paper, k, m, n and p will range over the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers. Throughout, we fix a definable ring $(S, +, \cdot, 0_S, 1_S)$ and an S-module $(A, \oplus, 0_A, \lambda)$.

2. Generators of modules

Let $a_1, \dots, a_n \in A$. The *span* of $\{a_1, \dots, a_n\}$ is the set

$$\mathrm{Span}_S\{a_1, \dots, a_n\} = \{\lambda(s_1, a_1) \oplus \cdots \oplus \lambda(s_n, a_n) : s_1, \dots, s_n \in S\}.$$

We will say that A is *finitely generated* if there exist $a_1, \dots, a_n \in A$ such that $\mathrm{Span}_S\{a_1, \dots, a_n\} = A$. It is easy to see that if A is a definable S-module, then $\mathrm{Span}_S\{a_1, \dots, a_n\}$ is a definable subgroup of A.

In [11], Y. Peterzil and S. Starchenko proved:

Lemma 2.1. *[11, Lemma 2.16] Suppose $\mathfrak{M}$ is $\aleph_0$-saturated and G is a definable group. Then there exist $g_1, \dots, g_k \in G$ such that the only definable subgroup of G containing $g_1, \dots, g_k$ is G.*

Note that such $g_1, \ldots, g_k$ in the above lemma are not generators of the group G in the sense of classical group theory since every finitely generated group must be countable. However, when we consider in the context of definable S-modules, the above result gives us more descriptive information.

Theorem B. *Suppose A is a definable S-module. Then A is finitely generated.*

Proof. Let $b \in M^p$ and $\varphi(x, z), \psi(y, z)$ be formulas such that $\varphi(x, b)$ defines the set S and $\psi(y, b)$ defines the set A. Let $\mathfrak{N}$ be an elementary extension of $\mathfrak{M}$ that is $\aleph_0$-saturated. Then $\varphi(x, b)$ defines the underlying set S' of a ring in $\mathfrak{N}$ and $\psi(y, b)$ defines the underlying set A' of an S'-module in $\mathfrak{N}$. By Lemma 2.1, there exist $d_1, \ldots, d_k \in A'$ such that the only definable subgroup of A' containing $d_1, \ldots, d_k$ is A'. Since $\mathrm{Span}_{S'}\{d_1, \ldots, d_k\}$ is a definable subgroup of A', we have

$$\mathrm{Span}_{S'}\{d_1, \ldots, d_k\} = A'.$$

Then $y \in A'$ if and only if there exist $s_1, \ldots, s_k \in S'$ such that $y = \lambda(s_1, d_1) \oplus \cdots \oplus \lambda(s_k, d_k)$. Let $\chi(y, y_1, \ldots, y_k)$ be the formula representing

$$\psi(y, b) \leftrightarrow \exists x_1 \ldots \exists x_k \bigwedge_{i=1}^{k} \varphi(x_i, b) \wedge y = \lambda(x_1, y_1) \oplus \cdots \oplus \lambda(x_k, y_k)$$

Therefore, $\mathfrak{N} \models \forall y \chi(y, d_1, \ldots, d_k)$. Hence,

$$\mathfrak{N} \models \exists y_1 \ldots \exists y_k \forall y \chi(y, y_1, \ldots, y_k).$$

Since $\mathfrak{M}$ is an elementary substructure of $\mathfrak{N}$ and b is in M,

$$\mathfrak{M} \models \exists y_1 \ldots \exists y_k \forall y \chi(y, y_1, \ldots, y_k).$$

Therefore, A is finitely generated. □

We now give the first proof of Theorem A.

Proof of Theorem A. Obviously, every finite S-module is isomorphic to a definable S-module. If S is finite and A is a definable S-module, by Theorem B, we have that A is also finite. Therefore, we obtain (1) in Theorem A.

To prove (2), suppose S is an infinite definable ring without zero divisors. Obviously, each S^k is a definable S-module and every finite dimensional free module over S is isomorphic to S^k (for some k) as S-modules. Suppose A is isomorphic to a definable S-module. Without loss of generality, we assume that A is a definable S-module. Recall that every infinite definable ring without zero divisors is a division ring and every module over a division ring is free. By Theorem B, we have A is a finitely generated module over S; hence, A is a finite dimension free module over S. □

In addition, Theorem B also provides information about definable ideals of S. Observe that every definable ideal of S is a definable S-module with respect to the induced operators from S. The following is an immediate consequence of Theorem B and this observation.

Corollary B. *Every definable ideal of S is a finitely generated ideal.*

3. Definable S-module manifold topologies

From now on, we assume A is a definable S-module and S is infinite. By [9, Lemma 4.1], let τ_S be the unique definable ring $(\dim S)$-manifold topology on S. For each topology τ on A, we say a map $f\colon S \times A \to A$ is *τ-continuous* if f is continuous with respect to the product topology $\tau_S \times \tau$ on $S \times A$ and the topology τ on A. Let $k \in \mathbb{N}$. A definable topology τ on A is a *definable S-module k-manifold topology* if the addition, the additive inversion, and the scalar multiplication are τ-continuous and there exist definable τ-open subsets $D_1, \ldots, D_n$ of A and definable maps $\phi_1, \ldots \phi_n$ such that $\bigcup\{D_i : i = 1, \ldots, n\} = A$ and each $\phi_i\colon D_i \to M^k$ is a homeomorphism from D_i onto its image.

For a definable topology τ, we say that a set is *definably τ-connected* if it is not a disjoint union of two definable τ-open sets. Observe that for definable topologies τ_1 and τ_2, the product of a definably τ_1-connected set and a definably τ_2-connected set is definably $(\tau_1 \times \tau_2)$-connected. We know that, by [12, Corollary 2.10] and Cell Decomposition Theorem, if τ is a definable group $(\dim A)$-manifold topology on A, then the definably τ-connected component containing the identity 0_A, denoted by A^0, exists.

Lemma 3.1. *If A admits a definable S-module $(\dim A)$-manifold topology, then A^0 is a definable S-submodule of A.*

Proof. Let τ be a definable S-module $(\dim A)$-manifold topology on A. By [12, Proposition 2.12], we have A^0 is the smallest definable subgroup of finite index in A. Therefore, $\dim A^0 = \dim A$. Recall that S has only finitely many definably τ_S-connected components. Let $S_1, \ldots, S_k$ enumerate all definably τ_S-connected components of S. Therefore, each $S_i \times A^0$ is definably $(\tau_S \times \tau)$-connected. Since λ is τ-continuous and $0_A \in A^0$, each image $\lambda(S_i \times A^0)$ is a definably τ-connected set containing 0_A. Therefore, $\lambda(S \times A^0) = \bigcup\{\lambda(S_i \times A^0) : i = 1, \ldots, k\} \subseteq A^0$. It follows immediately that A^0 is an S-submodule of A. □

Recall that definable groups admit the descending chain condition on definable subgroups, i.e., every descending family $(G_i)_{i\in\mathbb{N}}$ of definable groups is eventually constant (see e.g., [12, Remark 2.13]). As a consequence of this result, we obtain:

Lemma 3.2. *Let H be a definable group and G be a definable subgroup of H. Assume that there is $b \in H$ such that $kb \notin G$ for every positive integer k. Then there exists the smallest definable subgroup G' of H containing $G \cup \{b\}$. In addition, we have $\dim G < \dim G' \leq \dim H$.*

Proof. Suppose to the contrary that there is no smallest definable subgroup of H containing $G \cup \{b\}$. We recursively define a sequence $(H_i)_{i\in\mathbb{N}}$ of definable subgroups of H as follows:

Set $H_0 = H$. Suppose $H_0, \ldots, H_i$ have been constructed. Then there exists a definable subgroup H_i' of H containing $G \cup \{b\}$ such that H_i is not a subgroup of H_i'. Set $H_{i+1} = H_i \cap H_i'$. Then H_{i+1} is a proper definable subgroup of H_i containing $G \cup \{b\}$.

Therefore $(H_i)_{i\in\mathbb{N}}$ is an infinite proper descending chain of definable subgroups of H. This contradicts the descending chain condition of definable groups.

Let G' be the smallest definable subgroup of H containing $G \cup \{b\}$. Since there is no positive integer k such that $kb \in G$, we have G is of infinite index in G'. By [12, Lemma 2.11], we have $\dim G < \dim G' \leq \dim H$. □

By the above lemmas, we can prove a key step towards an alternative proof of (2) in Theorem A.

Lemma 3.3. *Suppose $\mathfrak{M}$ is an expansion of an ordered divisible abelian group. If A admits a definable S-module* $(\dim A)$*-manifold topology, then A is a finitely generated module over S. Moreover, if A is a free module over S, then A is a finite dimensional free module over S.*

Proof. Without loss of generality, we assume that $\mathfrak{M}$ is $\aleph_1$-saturated. Note that A^0 is infinite and abelian.

Claim. Let G be a definable subgroup of A^0. Suppose for any $a \in A^0$, there is a positive integer k such that $ka \in G$. Then $G = A^0$.

Proof of Claim. By saturation and compactness theorem, there is a positive integer k such that $ka \in G$ for all $a \in A^0$. Since $k(a \oplus G) = ka \oplus G = G$ for all $a \in G$, the quotient group A^0/G is of bounded exponent. By definable choice (see e.g., [2]), we have that A^0/G is isomorphic to a definable abelian group. By [16, Lemma 5.7], we have A^0/G is finite. Since A^0 is a subgroup of A of finite index, G also has finite index in A. Since $G \subseteq A^0$ and A^0 is the smallest definable subgroup of A of finite index, we have $G = A^0$. □

We recursively construct a sequence $(a_i)_{i\in\mathbb{N}}$ as follows:

Set $a_0 = 0_A$. Suppose $a_0, \dots, a_i$ have been constructed. If the smallest definable subgroup of A^0 containing $a_0, \dots, a_i$ is A^0, then let $a_{i+1} = 0_A$. Otherwise, by the above claim, let $a_{i+1} \in A^0$ be such that ka_{i+1} is not contained in the smallest definable subgroup containing $a_0, \dots, a_i$ for any positive integer k.

For each $i \in \mathbb{N}$, let A_i be the smallest definable subgroup of A^0 containing $a_0, \dots, a_i$. Observe that for every $i \in \mathbb{N}$, $A_i \subseteq \mathrm{Span}_S\{a_0, \dots, a_i\}$ and, by Lemma 3.2, if $a_{i+1} \notin A_i$, then $\dim A_i < \dim A_{i+1} \le \dim A^0$. Let $n' = \dim A^0$. For every $j \ge n'$, we have $\dim A_j = n'$ and so $A_j = A^0$. Since $A_{n'} \subseteq \mathrm{Span}_S\{a_0, \dots, a_{n'}\}$ and $a_0, \dots, a_{n'} \in A^0$, by Lemma 3.1, we get $A^0 = \mathrm{Span}_S\{a_0, \dots, a_{n'}\}$. Since A^0 is of finite index in A, there exist $b_0, \dots, b_p \in A$ such that $A = \bigcup\{b_j \oplus A^0 : j = 0, \dots, p\}$. Hence $A = \mathrm{Span}_S\{a_0, \dots, a_{n'}, b_0, \dots, b_p\}$. This completes the proof. □

Remark. *Since every finite dimensional free module over S is isomorphic to S^k for some $k \in \mathbb{N}$, if S is definably τ_S-connected, then A is definably τ_A-connected.*

To complete this alternative proof of (2) of Theorem A (when $\mathfrak{M}$ is an expansion of an ordered divisible abelian group), it suffices to prove the following:

Theorem C. *The definable module A admits a unique definable S-module* $(\dim A)$*-manifold topology.*

Proof. First, we may assume that $\mathfrak{M}$ is $\aleph_1$-saturated, $A \subseteq M^n$, $\dim A = n$, $S \subseteq M^m$, and $\dim S = m$. Let τ_A be the unique definable group $(\dim A)$-manifold topology on A. By the proof of [12, Proposition 2.5], there exist a definable large open subset X of S and a definable large open subset V of A such that

- for every definable subset U of X and $s \in S$, $s + U$ is τ_S-open if and only if U is open in X; and
- for every definable subset U' of V and $a \in A$, $a \oplus U'$ is τ_A-open if and only if U' is open in V.

To complete this proof, it is enough to prove that the scalar multiplication λ is τ_A-continuous.

Let $(x_0, v_0) \in S \times A$. It suffices to find a $(\tau_S \times \tau_A)$-open neighbourhood of (x_0, v_0) where λ is τ_A-continuous. Note that $\dim(\lambda(Y \times W)) = \dim A$ for every definable τ_S-open Y and definable τ_A-open

W. By cell decomposition theorem, there exists a large open definable subset P of $X \times V$ such that λ is continuous (with respect to the induced topology on the ambient space) on P and the image $\lambda(P)$ is a subset of V. Let U_S be an open definable subset of X and U_A be an open definable subset of V such that $U_S \times U_A \subseteq P$. Then there exist $s \in S$ and $a \in A$ such that $(x_0 + s, v_0 \oplus a) \in U_S \times U_A \subseteq P$. Observe that the map $(x, v) \mapsto \lambda(x, v)\colon U_S \times U_A \to V$ is τ_A-continuous and

$$\lambda(x, v) = \lambda(x + s, v \oplus a) \oplus \lambda(x, \ominus a) \oplus \lambda(s, \ominus a) \oplus \lambda(-s, v)$$

for every $(x, y) \in S \times A$. Without loss of generality, we may assume that $a \neq 0_A$ and $s \neq 0_S$. Since the map $x \mapsto \lambda(x, \ominus a)$ from the additive group of S to A and the map $v \mapsto \lambda(-s, v)\colon A \to A$ are group homomorphisms (by [4, Theorem 2.6]), they are continuous with respect to their group manifold topologies. Hence, λ is τ_A-continuous on $((-s) + U_S) \times ((\ominus a) \oplus U_A)$. This completes the proof. □

We end this section by an immediate consequence of Theorem C and the remark after Lemma 3.3.

Corollary 3.4. *If $\mathfrak{M}$ is an expansion of an ordered divisible abelian group and S is an infinite definable ring without zero divisors, then A is definably τ_A-connected.*

4. Open questions

4.1. *Suppose S is an infinite ring. Here, we obtain a complete characterization of definable S-modules when S has no zero divisors. However, the question is still open when S (possibly) has zero divisors.*

4.2. *Suppose A is a definable abelian group. Obviously, if $|A| = n$ for some positive integer n, then A is an $\mathbb{Z}/n\mathbb{Z}$-module. This gives rise to the question:*

If A is infinite, how to determine whether A is a definable S-module for some definable ring S?

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Conflict of interest

The authors declare no conflict of interest.

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