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Research article

A study of nonlocal fractional delay differential equations with hemivariational inequality

Ebrahem A. Algehyne[1,*], **Abdur Raheem**[2,*], **Mohd Adnan**[2], **Asma Afreen**[2] **and Ahmed Alamer**[1]

[1] Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk-71491, Saudi Arabia
[2] Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

* **Correspondence:** Email: e.algehyne@ut.edu.sa, araheem.iitk3239@gmail.com.

Abstract: In this paper, we study an abstract system of fractional delay differential equations of order $1 < q < 2$ with a hemivariational inequality in Banach spaces. To establish the existence of a solution to the abstract inequality, we employ the Rothe technique in conjunction with the surjectivity of multivalued pseudomonotone operators and features of the Clarke generalized gradient. Further, to show the existence of the fractional differential equation, we use the fractional cosine family and fixed point theorem. Finally, we include an example to elaborate the effectiveness of the findings.

Keywords: Caputo fractional derivative; hemivariational inequality; Rothe method; Clarke subdifferential; pseudomonotone
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1. Introduction

In recent decades, the theory of fractional-order differential systems has attracted considerable attention mainly to its impressive applicability in various scientific and engineering domains, such as medical model simulations and electrical engineering. The nonlocal properties of fractional differential and integral operators have made them valuable tools. It has been found that arbitrary-order fractional differential equations provide a more accurate description of the dynamic response of real-world objects. The importance of a thorough understanding of fractional calculus has increasingly been recognized. Please see [10, 16, 18] for further reading.

Essential applications in physics, engineering, economics, game theory, etc., have attracted much attention to the theory of variational and hemivariational inequalities. Variational and hemivariational inequalities have been the topic of substantial research in the literature, with numerous theoretical conclusions, numerical analyses, and applications to contact mechanics. Constructing and studying the existence and uniqueness of solutions to variational and hemivariational inequalities is one of the most promising areas of study in this field (see, for example, [6, 12, 15, 19, 22, 23] and the references therein). Properties of the Clarke generalized gradient, established for locally Lipschitz functions, form the basis of the theory of hemivariational inequalities, which is another active research topic. Starting with Panagiotopoulos's works, this theory has seen significant growth over the past 30 years. See [4, 5, 8, 11, 19, 20] and the references therein for a sampling of the mechanical, physical, and engineering problems solved with the help of hemivariational inequalities.

Initial work on systems of variational inequalities and differential equations was done by Aubin and Cellina [2] in 1984. From a side perspective, Pang and Stewart [17] in 2008 were the first to study them within the context of finite-dimensional spaces. Differential variational inequality (DVI) is the term used to describe this organized system. They also gave examples of how and where DVI could be useful, including in mechanical impact problems, economic dynamics, etc. Since then, most researchers have worked on DVI. Only a limited number of DVI's real-world applications in infinite-dimensional spaces were studied to support these theoretical results. In addition, hemivariational inequalities for fractional differentials have not been studied in both finite and infinite dimensions until now. Therefore, we aim to address this knowledge gap by creating novel mathematical tools and approaches for solving fractional differential hemivariational inequality.

People recognize several problems that arise in the real world as time delay problems because of their connections to the present and the past. It has been noticed that the delays are either time-dependent or constant. There has been a rapid increase in the study of delay differential equations and their applications. Tumwiine et al. [21], Dehghan and Salehi [7] investigated many real-world applications that show the effects of delay. HIV infection of $CD4^+$ T-cells was modeled by Yan and Kou [24] using a fractional-order time-delayed approach.

Rothe came up with the method of semi-discretization in 1930 to solve a second-order scalar parabolic initial value problem. This method is also called the "Rothe method". Researchers have utilized and enhanced this technique; for more information, see [13, 22, 23, 26]. The method of Rothe is useful for examining the existence of solutions to differential equations. This method is also used to study diffusion problems. There are numerous works that discuss this topic; see [3, 10, 25].

Many authors examined the existence of a solution of a system with nonlocal conditions as its application results are more favorable than those of the classical initial conditions [12].

In light of the above discussion, this study will focus on a specific class of nonlocal fractional delay differential equations of order $1 < q < 2$ with a variational inequality in Banach spaces. Let $\mathfrak{X}$, $\mathcal{Y}$, $\mathfrak{Z}$, and $\mathcal{E}$ be reflexive, real separable Banach spaces. Let $\mathfrak{Z}^*$ be the dual space of $\mathfrak{Z}$ and $\mathfrak{U}$ be a real separable Hilbert space with $\mathfrak{Z} \subset \mathfrak{U} \subset \mathfrak{Z}^*$. Consider the following system of a fractional delay differential evolutionary hemivariational inequality (FDEHVI):

$$\begin{cases} {}^C D_0^q \vartheta(t) = A\vartheta(t) + \varrho(t, \vartheta(t), \vartheta(t-\mu))z(t) \ \text{ for a.e. } t \in [0, l], \\ \langle \mathcal{B}_1 z''(t) + \mathcal{B}_2 z'(t), v\rangle + J^0(\mathcal{M}z(t), \mathcal{M}v) \geq \langle \wp(t), v\rangle, \ \forall\, v \in \mathfrak{Z} \text{ and for a.e. } t \in [0, l], \\ \vartheta(t) + \varpi(\vartheta)(t) = \varphi(t), \ t \in [-\mu, 0], \ \ \vartheta'(0) = \psi_0, \\ z(0) = 0, \ \ z'(0) = \zeta, \end{cases} \tag{1.1}$$

where ${}^C D_0^q$ denotes the fractional derivative of Caputo type of order $1 < q < 2$. A densely defined closed linear operator $A : D(A) \subset \mathfrak{X} \to \mathfrak{X}$ generates a strongly continuous q-order cosine family $\{C_q(t)\}_{t\geq 0}$ in $\mathfrak{X}$. For a prefixed $l > 0$, $\varrho : [0, l] \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$, $\varpi : \mathfrak{X} \to \mathfrak{X}$, $\mathcal{B}_1, \mathcal{B}_2 : \mathfrak{Z} \to \mathfrak{Z}^*$, $\mathcal{M} : \mathfrak{Z} \to \mathcal{E}$, $J : \mathcal{E} \to \mathbb{R}$, $\wp : [0, l] \to \mathfrak{Z}^*$ be given maps.

2. Preliminaries

We recall some basic notations:
$C([0,l],\mathfrak{X})$: The space $\{\vartheta : [0,l] \to \mathfrak{X} \mid \vartheta \text{ is continuous}\}$;
$C^1([0,l],\mathfrak{X})$: The space $\{\vartheta : [0,l] \to \mathfrak{X} \mid \vartheta, \vartheta' \in C([0,l],\mathfrak{X})\}$;
$L^p([0,l],\mathfrak{X})$: The space $\left\{\vartheta : [0,l] \to \mathfrak{X} \mid \vartheta \text{ is Bochner integrable and } \int_0^l \|\vartheta(t)\|^p dt < \infty\right\}$;
$AC([0,l],\mathfrak{X})$: The space of all absolutly continuous function from $[0,l]$ to $\mathfrak{X}$;
$\mathcal{P}(\mathfrak{X})$: The nonempty subsets of $\mathfrak{X}$;
$\mathcal{B}(\mathfrak{X})$: The space of all bounded linear operators on $\mathfrak{X}$;
$\mathcal{L}(\mathfrak{X},\mathfrak{Y})$: The space of bounded linear operators from a Banach space $\mathfrak{X}$ to $\mathfrak{Y}$.
For a set $\Omega \subset \mathfrak{X}$, we define $\|\vartheta\|_\Omega = \sup\{\|\vartheta\|_{\mathfrak{X}} \mid \vartheta \in \Omega\}$. The norm in spaces $C([0,l],\mathfrak{X})$ and $L^p([0,l],\mathfrak{X})$, $p \geq 1$ are defined, respectively, by

$$\|\vartheta\|_{C([0,l],\mathfrak{X})} = \max_{t\in[0,l]} \|\vartheta(t)\|_{\mathfrak{X}} \text{ and } \|\vartheta\|_{L^p([0,l],\mathfrak{X})} = \left(\int_{[0,l]} \|\vartheta(t)\|_{\mathfrak{X}}^p dt\right)^{\frac{1}{p}}.$$

Definition 2.1. *A multimap $\mathcal{F} : \mathfrak{Z}_1 \subseteq \mathfrak{Z} \to \mathcal{P}(\mathfrak{Z})$ is said to be condensing with respect to an MNC Λ (Λ-condensing) if for any bounded set $\Omega \subset \mathfrak{Z}$, the relation*

$$\Lambda(\Omega) \leq \Lambda(\mathcal{F}(\Omega))$$

implies the relative compactness of Ω.

Definition 2.2. *[16] For a function $y \in C^N([0,T],\mathfrak{Z})$, Caputo fractional derivative of order $\alpha \in (N-1,N)$ is defined by*

$${}_0^C D_t^\alpha y(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t (t-s)^{N-1-\alpha} y^{(N)}(s) ds.$$

A map $J : \mathcal{E} \to \mathbb{R}$ is called locally Lipschitz, i.e, for every $e \in \mathcal{E}$, there exists a neighborhood O_e and a constant M_e, such that

$$|J(x) - J(y)| \leq M_e \|x - y\|_{\mathcal{E}}, \quad \forall\, x, y \in O_e.$$

Definition 2.3. *For a locally Lipschitz $J : \mathcal{E} \to \mathbb{R}$, the following limit*

$$J^0(e, e_0) = \limsup_{v\to e,\ \mu\to 0^+} \frac{J(v + \mu e_0) - J(v)}{\mu}$$

is said to be the generalized directional derivative of J at $e \in \mathcal{E}$ in the direction $e_0 \in \mathcal{E}$ and set

$$\partial J(e) = \{\xi \in \mathcal{E}^* | J^0(e; v) \geq \langle \xi, v\rangle, \forall v \in \mathcal{E}\}$$

is said to be the generalized subdifferential of J.

3. Differential hemivariational inequality

Let E $= L^2([0,l],\mathcal{E})$, £ $= L^2([0,l],\mathfrak{Z})$, $\mathcal{V} = L^2([0,l],\mathfrak{U})$, £$^* = L^2([0,l],\mathfrak{Z}^*)$, $H^1([0,l],\mathfrak{Z}) = \{z \in$ £ $\mid z' \in$ £$\}$. We denote $\varsigma : \mathfrak{Z} \to \mathfrak{U}$ the embedding operator between $\mathfrak{Z}$ and $\mathfrak{U}$, and by $\langle \cdot, \cdot \rangle_{£\times£^*}$ the duality between £ and £*.

Consider the following assumptions:

(HB1) $\mathcal{B}_1 \in L(\mathfrak{Z},\mathfrak{Z}^*)$, such that

- (i) there exists $m_{\mathcal{B}_1} > 0$, such that $\langle \mathcal{B}_1 z, z\rangle \geq m_{\mathcal{B}_1}\|z\|^2_{\mathfrak{Z}}$ for all $z \in \mathfrak{Z}$;
- (ii) $\langle \mathcal{B}_1 z_1, z_2\rangle = \langle z_1, \mathcal{B}_1 z_2\rangle$ for all $z_1, z_2 \in \mathfrak{Z}$.

(HB2) $\mathcal{B}_2 \in L(\mathfrak{Z},\mathfrak{Z}^*)$, such that

- (i) there exists $m_{\mathcal{B}_2} > 0$, such that $\langle \mathcal{B}_2 z, z\rangle \geq m_{\mathcal{B}_2}\|z\|^2_{\mathfrak{Z}}$ for all $z \in \mathfrak{Z}$;
- (ii) $\langle \mathcal{B}_2 z_1, z_2\rangle = \langle z_1, \mathcal{B}_2 z_2\rangle$ for all $z_1, z_2 \in \mathfrak{Z}$.

(HM) $\mathcal{M} \in L(\mathfrak{Z},\mathcal{E})$ is a compact operator.

(HJ) $J : \mathcal{E} \to \mathbb{R}$ is locally Lipschitz and there exists $m_J > 0$, such that $\|\partial J(e)\|_{\mathcal{E}^*} \leq m_J(1 + \|e\|_{\mathcal{E}})$ for all $e \in \mathcal{E}$.

(H℘) $\wp \in L^2([0,l],\mathfrak{Z}^*)$.

(HO) $m_{\mathcal{B}_1} + \tau m_{\mathcal{B}_2} > \tau^2 m_J \|\mathcal{M}\|^2$ for all $\tau \in (0,\tau_0)$.

Problem 3.1. *Find $z \in \mathfrak{Z}$, such that $z(0) = 0$, $z'(0) = \zeta$ and for a.e. $t \in (0,l)$, we have*

$$\langle \mathcal{B}_1 z''(t) + \mathcal{B}_2 z'(t), v\rangle + J^0(\mathcal{M}z(t), \mathcal{M}v) \geq \langle \wp(t), v\rangle$$

for all $v \in \mathfrak{Z}$.

Problem 3.1 has another equivalent form as follows:

Problem 3.2. *Find $z \in \mathfrak{Z}$, such that*

$$\mathcal{B}_1 z''(t) + \mathcal{B}_2 z'(t) + \mathcal{M}^*\partial J(\mathcal{M}z(t)) \ni \wp(t), \quad \textit{for a.e. } t \in (0,l).$$

Definition 3.1. *An element $z \in \mathfrak{Z}$ is a solution to Problem 3.1 or Problem 3.2, if and only if, there exists $\xi \in \mathcal{E}^*$, such that*

$$\mathcal{B}_1 z''(t) + \mathcal{B}_2 z'(t) + \mathcal{M}^*\xi(t) = \wp(t) \ \textit{ for a.e. } t \in (0,l)$$

with $\xi(t) \in \partial J(\mathcal{M}z(t))$.

We define the sequence of time step $\tau_n \to 0$, such that the value $\frac{l}{\tau_n}$ is an integer. For the sake of convenience, the subscript n is omitted in the sequel.

The approximation of $\wp$ is given by

$$\mathfrak{Z}^* \ni \bar{\wp}_\tau(t) := \wp^k_\tau = \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} \wp(t)dt, \ \text{ for } t \in ((k-1)\tau, k\tau], \ k \in 1,2,\ldots,N.$$

We have $\bar{\wp}_\tau \to \wp$ in $\mathfrak{Z}^*$ when $\tau \to 0$. We consider the following approximation of natural order derivatives $z''(t)$ and $z'(t)$ as

$$z''(t) = \frac{z^k_\tau - 2z^{k-1}_\tau + z^{k-2}_\tau}{\tau^2},$$

$$z'(t) = \frac{z_\tau^k - z_\tau^{k-1}}{\tau},$$

and $z(t) = z_\tau^k$.

We apply Rothe's method on Problem 3.2 to define following Rothe problem:

Problem 3.3. *Find* $\{z_\tau^k\}_{k=1}^N \subset \mathfrak{Z}$*, such that*

$$\mathcal{B}_1 \frac{z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2}}{\tau^2} + \mathcal{B}_2 \frac{z_\tau^k - z_\tau^{k-1}}{\tau} + \mathcal{M}^* \partial J(\mathcal{M} z_\tau^k) \ni \wp_\tau^k, \quad \textit{for } k = 1, 2, \ldots, N. \tag{3.1}$$

Lemma 3.1. *Assume that (HB1), (HB2), (HM), (HJ) and (H℘) hold. Then, there exists* τ_0*, such that for all* $\tau \in (0, \tau_0)$*, Problem 3.3 has at least one solution.*

Proof. Given $z_\tau^0, z_\tau^1, \ldots, z_\tau^{k-1}$, we will find $z_\tau^k \in \mathfrak{Z}$ which satisfies (3.1), or equivalently

$$\mathcal{B}_1 z_\tau^k + \tau \mathcal{B}_2 z_\tau^k + \tau^2 \mathcal{M}^* \partial J(\mathcal{M} z_\tau^k) \ni \tau^2 \wp_\tau^k + (2\mathcal{B}_1 + \tau \mathcal{B}_2) z_\tau^{k-1} - \mathcal{B}_1 z_\tau^{k-2}. \tag{3.2}$$

For this aim, we shall introduce the multivalued operator $S : \mathfrak{Z} \to 2^{\mathfrak{Z}^*}$ by

$$Sy = \mathcal{B}_1 y + \tau \mathcal{B}_2 y + \tau^2 \mathcal{M}^* \partial J(\mathcal{M} y) \tag{3.3}$$

for $y \in \mathfrak{Z}$. We shall show that operator S is surjective. By [19, Theorem 73], it is enough to prove that S is pseudomonotone and coercive. First we show that there exists a constant $\tau_0' > 0$, such that S is pseudomonotone for all $\tau \in (0, \tau_0')$. Now, by (HB1) and (HB2), the operator

$$y \to \mathcal{B}_1 y + \tau \mathcal{B}_2 y \tag{3.4}$$

is continuous, monotone and bounded for $\tau \in (0, \bar{\tau}_0)$, where $\bar{\tau}_0 = m_{\mathcal{B}_2} + \sqrt{(m_{\mathcal{B}2})^2 + 4 m_{\mathcal{B}1} m_J \|\mathcal{M}\|^2}$. Thus, from [19, Lemma 3], the operator given in (3.4) is pseudomonotone. Moreover, by the hypotheses (HM), (HJ) and [9, Proposition 5.6], the operator

$$y \to \mathcal{M}^* \partial J(\mathcal{M} y) \tag{3.5}$$

is also pseudomonotone. Therefore, by [15, Proposition 3.59(ii)], we can say that the operator S is pseudomonotone. Subsequently, we establish that S is coercive.

$$\langle Sy, y \rangle = \langle \mathcal{B}_1 y, y \rangle + \langle \tau \mathcal{B}_2 y, y \rangle + \left\langle \tau^2 \mathcal{M}^* \partial J(\mathcal{M} y), y \right\rangle$$

for all $y \in \mathfrak{Z}$. From hypothesis (HJ), we find out

$$\left\langle \tau^2 \mathcal{M}^* \partial J(\mathcal{M} y), y \right\rangle \geq -\tau^2 m_J \|\mathcal{M}\|^2 \|y\|^2 - \tau^2 m_J \|\mathcal{M}\| \, \|y\|$$

for all $y \in \mathfrak{Z}$. By the hypotheses (HB1), (HB2) and (HM), we have

$$\langle Sy, y \rangle \geq \left(m_{\mathcal{B}1} + \tau m_{\mathcal{B}2} - \tau^2 m_J \|\mathcal{M}\|^2 \right) \|y\|^2 - \tau^2 m_J \|\mathcal{M}\| \, \|y\|.$$

From the condition (HO), we can choose $\tau_0 = \frac{m_{\mathcal{B}2} + \sqrt{(m_{\mathcal{B}2})^2 + 4 m_{\mathcal{B}1} m_J \|\mathcal{M}\|^2}}{2 m_J \|\mathcal{M}\|^2} > 0$. Clearly $\tau_0 < \bar{\tau}_0$. Thus, for any given $\tau \in (0, \tau_0)$, S is coercive.

Due to [19, Theorem 73], we can conclude that S is surjective, i.e., there exists z_τ^k, such that $\xi_\tau^k \in \partial J(\mathcal{M} z_\tau^k)$ and the problem 3.3 holds for all $\tau \in (0, \tau_0)$.

Lemma 3.2. *Under the hypotheses (HB1), (HB2), (HM), (H℘), (HJ) and (HO), there exist $\tau_0 > 0$ and $\Lambda > 0$ independent of τ, such that for all $\tau \in (0, \tau_0)$, the solution to Problem 3.3 satisfies*

$$\max_{k=1,2,\dots,N} \left\| z_\tau^k \right\|_3 \leq \Lambda, \tag{3.6}$$

$$\sum_{k=1}^{N} \left\| z_\tau^k - z_\tau^{k-1} \right\|_3^2 \leq \Lambda, \tag{3.7}$$

$$\tau \sum_{k=1}^{N} \left\| \frac{z_\tau^k - z_\tau^{k-1}}{\tau} \right\|_3^2 \leq \Lambda, \tag{3.8}$$

$$\left\| z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2} \right\|_3 \leq \Lambda, \tag{3.9}$$

$$\max_{k=1,2,\dots,N} \left\| \xi_\tau^k \right\|_{\mathcal{E}^*} \leq \Lambda, \tag{3.10}$$

where $\xi_\tau^k \in \mathcal{E}^*$ is, such that $\xi_\tau^k \in \partial J(\mathcal{M} z_\tau^k)$ and

$$\mathcal{B}_1 \frac{z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2}}{\tau^2} + \mathcal{B}_2 \frac{z_\tau^k - z_\tau^{k-1}}{\tau} + \mathcal{M}^* \xi_\tau^k = \wp_\tau^k, \quad \textit{for } k = 1, 2, \dots, N.$$

Proof. For all $1 \leq k \leq N$, we have

$$\mathcal{B}_1 \frac{z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2}}{\tau^2} + \mathcal{B}_2 \frac{z_\tau^k - z_\tau^{k-1}}{\tau} + \mathcal{M}^* \xi_\tau^k = \wp_\tau^k, \quad \text{for } k = 1, 2, \dots, N, \tag{3.11}$$

where $\xi_\tau^k \in \partial J\left(\mathcal{M} z_\tau^k\right)$. Multiplying the above equation by z_τ^k, then we get the equality

$$\left\langle \wp_\tau^k, z_\tau^k \right\rangle = \left\langle \mathcal{B}_1 \frac{z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2}}{\tau^2} + \mathcal{B}_2 \frac{z_\tau^k - z_\tau^{k-1}}{\tau} + \mathcal{M}^* \xi_\tau^k, z_\tau^k \right\rangle,$$

and apply the hypotheses (HB1), (HB2), (HJ) and (HM), and the equality

$$2 \left\langle \mathcal{B}_2 \left(z_\tau^k - z_\tau^{k-1} \right), z_\tau^k \right\rangle = \left\langle \mathcal{B}_2 z_\tau^k, z_\tau^k \right\rangle - \left\langle \mathcal{B}_2 z_\tau^{k-1}, z_\tau^{k-1} \right\rangle + \left\langle \mathcal{B}_2 \left(z_\tau^k - z_\tau^{k-1} \right), z_\tau^k - z_\tau^{k-1} \right\rangle,$$

and

$$\begin{aligned} 2 \left\langle \mathcal{B}_1 \left(z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2} \right), z_\tau^k \right\rangle = {} & \left\langle \mathcal{B}_1 z_\tau^k, z_\tau^k \right\rangle - \left\langle \mathcal{B}_1 \left(2z_\tau^{k-1} - z_\tau^{k-2} \right), 2z_\tau^{k-1} - z_\tau^{k-2} \right\rangle \\ & + \left\langle \mathcal{B}_1 \left(z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2} \right), z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2} \right\rangle, \end{aligned}$$

we obtain

$$\begin{aligned} & \frac{1}{2} \left\langle \mathcal{B}_1 z_\tau^k, z_\tau^k \right\rangle - \frac{1}{2} \left\langle \mathcal{B}_1 \left(2z_\tau^{k-1} - z_\tau^{k-2} \right), 2z_\tau^{k-1} - z_\tau^{k-2} \right\rangle \\ & + \frac{1}{2} \left\langle \mathcal{B}_1 \left(z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2} \right), z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2} \right\rangle \\ & + \tau \Big[\frac{1}{2} \left\langle \mathcal{B}_2 z_\tau^k, z_\tau^k \right\rangle + \frac{1}{2} \left\langle \mathcal{B}_2 \left(z_\tau^k - z_\tau^{k-1} \right), z_\tau^k - z_\tau^{k-1} \right\rangle - \frac{1}{2} \left\langle \mathcal{B}_2 z_\tau^{k-1}, z_\tau^{k-1} \right\rangle \Big] \\ & - \tau^2 m_J \|\mathcal{M}\| \, \|z_\tau^k\| - \tau^2 m_J \|\mathcal{M}\|^2 \|z_\tau^k\|^2 \leq \tau^2 \|\wp_\tau^k\| \, \|z_\tau^k\|. \end{aligned} \tag{3.12}$$

Applying Cauchy's inequality with $\varepsilon > 0$, we have

$$m_J\|\mathcal{M}\|\,\|z_\tau^k\| \leq \frac{m_J^2\|\mathcal{M}\|^2}{4\varepsilon} + \varepsilon\|z_\tau^k\|^2, \tag{3.13}$$

$$\|\wp_\tau^k\|\,\|z_\tau^k\| \leq \frac{\|\wp_\tau^k\|^2}{\varepsilon} + \varepsilon\|z_\tau^k\|^2. \tag{3.14}$$

Combining (3.12) to (3.14), we get

$$\begin{aligned}
&\frac{1}{2}\left\langle \mathcal{B}_1 z_\tau^k, z_\tau^k\right\rangle - \frac{1}{2}\left\langle \mathcal{B}_1\left(2z_\tau^{k-1} - z_\tau^{k-2}\right), 2z_\tau^{k-1} - z_\tau^{k-2}\right\rangle \\
&+\frac{1}{2}\left\langle \mathcal{B}_1\left(z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2}\right), z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2}\right\rangle \\
&+\tau\frac{1}{2}m_{\mathcal{B}2}\left(\|z_\tau^k\|^2 + \|z_\tau^k - z_\tau^{k-1}\|^2 - \|z_\tau^{k-1}\|^2\right) - \tau^2 m_J\|\mathcal{M}\|^2\|z_\tau^k\|^2 \\
&\leq \tau^2\left(\frac{\|\wp_\tau^k\|^2}{4\varepsilon} + \varepsilon\|z_\tau^k\|^2\right) + \tau^2\left(\frac{m_J^2\|\mathcal{M}\|^2}{4\varepsilon} + \varepsilon\|z_\tau^k\|^2\right).
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\left\langle \mathcal{B}_1 z_\tau^k, z_\tau^k\right\rangle - \frac{1}{2}\left\langle \mathcal{B}_1\left(2z_\tau^{k-1} - z_\tau^{k-2}\right), 2z_\tau^{k-1} - z_\tau^{k-2}\right\rangle \\
&+\frac{1}{2}\left\langle \mathcal{B}_1\left(z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2}\right), z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2}\right\rangle \leq \tau^2\left(\frac{\|\wp_\tau^k\|^2}{4\varepsilon} + \frac{m_J^2\|\mathcal{M}\|^2}{4\varepsilon}\right) \\
&+\tau^2\|z_\tau^k\|^2\Big(m_J\|\mathcal{M}\|^2 + 2\varepsilon\Big) - \tau\frac{1}{2}m_{\mathcal{B}2}\left(\|z_\tau^k\|^2 + \|z_\tau^k - z_\tau^{k-1}\|^2 - \|z_\tau^{k-1}\|^2\right).
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\left\langle \mathcal{B}_1 z_\tau^k, z_\tau^k\right\rangle - \tau^2\|z_\tau^k\|^2\Big(m_J\|\mathcal{M}\|^2 + 2\varepsilon\Big) + \frac{1}{2}\left\langle \mathcal{B}_1\left(z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2}\right), z_\tau^k - 2z_\tau^{k-1} + z_\tau^{k-2}\right\rangle \\
&\quad\leq \tau^2\left(\frac{\|\wp_\tau^k\|^2}{4\varepsilon} + \frac{m_J^2\|\mathcal{M}\|^2}{4\varepsilon}\right) + \tau\frac{1}{2}m_{\mathcal{B}2}\|z_\tau^{k-1}\|^2 + \frac{1}{2}\left\langle \mathcal{B}_1\left(2z_\tau^{k-1} - z_\tau^{k-2}\right), 2z_\tau^{k-1} - z_\tau^{k-2}\right\rangle.
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\left(\frac{1}{2}m_{\mathcal{B}_1} - \tau^2\Big(m_J\|\mathcal{M}\|^2 + 2\varepsilon\Big)\right)\|z_\tau^k\|^2 \quad\leq\quad & \tau^2\left(\frac{\|\wp_\tau^k\|^2}{4\varepsilon} + \frac{m_J^2\|\mathcal{M}\|^2}{4\varepsilon}\right) + \tau\frac{1}{2}m_{\mathcal{B}2}\|z_\tau^{k-1}\|^2 \\
&+\frac{1}{2}m_{\mathcal{B}_1}\|2z_\tau^{k-1} - z_\tau^{k-2}\|^2.
\end{aligned}$$

We choose τ_0, such that

$$\left(\frac{1}{2}m_{\mathcal{B}_1} - \tau_0^2\Big(m_J\|\mathcal{M}\|^2 + 2\varepsilon\Big)\right)\|z_\tau^k\|^2 \quad\leq\quad \overline{\Lambda}\Big(1 + \|\wp_\tau\|_{\pounds}^2\Big). \tag{3.16}$$

Moreover, $\wp_\tau \to \wp \in \mathfrak{Z}^*$ as $\tau \to 0$ and, thus, $\wp_\tau$ is bounded in $\mathfrak{Z}^*$.
Therefore, from (3.16), we have

$$\max_{k=1,2,\ldots,N}\left\|z_\tau^k\right\|^2 \leq \Lambda.$$

Using (3.11) and (3.15), one can easily prove the remaining estimations.

Next, we define the piecewise function z_τ and piecewise constant interpolation functions $\bar{z}_\tau, \xi_\tau, f_\tau$ as follows:

$$z_\tau(t) = z_\tau^k + \frac{t-t_k}{\tau}(z_\tau^k - z_\tau^{k-1}), \quad \xi_\tau(t) = \xi_\tau^k, \quad \forall\, t \in (t_{k-1}, t_k],$$

$$\bar{z}_\tau(t) = \begin{cases} z_\tau^k, & t \in (t_{k-1}, t_k], \\ 0, & t = 0. \end{cases}$$

$$f_\tau(t) = \begin{cases} f_\tau^k, & t \in (t_{k-1}, t_k], \\ f(0), & t = 0. \end{cases}$$

$$\partial z_\tau(t) = \partial z_\tau^k + \frac{t-t_k}{\tau}(\partial z_\tau^k - \partial z_\tau^{k-1}), \quad \xi_\tau(t) = \xi_\tau^k, \quad \forall\, t \in (t_{k-1}, t_k],$$

$$\partial\bar{z}_\tau(t) = \begin{cases} \partial z_\tau^k, & t \in (t_{k-1}, t_k], \\ \zeta, & t = 0, \end{cases}$$

for $k = 1, 2, \ldots, N$.

Lemma 3.3. *Assume that (HB1), (HB2), (H℘), (HJ), (HM) and (HO) hold. Then, there exist* $\tau_0 > 0$ *and* $\Lambda > 0$*, such that for any* $\tau \in (0, \tau_0)$ *the functions* z_τ*,* $\bar{z}_\tau$ *and* ξ_τ *satisfy*

$$\begin{aligned} \|z_\tau\|_{C([0,l],\mathfrak{Z})} &\le \Lambda, \\ \|z_\tau\|_{£} &\le \Lambda, \\ \left\|z_\tau'\right\|_{£} &\le \Lambda, \\ \left\|z_\tau''\right\|_{£} &\le \Lambda, \\ \|\bar{z}_\tau\|_{£} &\le \Lambda, \\ \|\xi_\tau\|_{\mathrm{E}^*} &\le \Lambda. \end{aligned}$$

Theorem 3.1. *Assume that (HB1), (HB2), (H℘), (HM), (HJ) and (HO) hold. Let* $\{\tau_N\}$ *be a sequence with* $\tau_N \to 0$ *as* $N \to \infty$*. Then, there is a subsequence, still denoted by* $\{\tau_N\}$*, such that*

$$\begin{aligned} &\bar{z}_{\tau_k} \to z \text{ weakly in } £, \\ &z_{\tau_k} \to z \text{ weakly in } £, \\ &z'_{\tau_k} \to z' \text{ weakly in } £, \\ &z''_{\tau_k} \to z'' \text{ weakly in } £, \\ &\xi_{\tau_k} \to \xi \text{ weakly in } \mathrm{E}^*, \\ &\vartheta_{\tau_k} \to \vartheta \text{ weakly in } C([0,l], \mathfrak{X}), \end{aligned}$$

where $(\vartheta, z) \in C([0,l], \mathfrak{X}) \times H^1([0,l], \mathfrak{Z})$ *is the solution of (1.1).*

Note 3.1. *For the proof of Lemma 3.3 and Theorem 3.1, please refer to [22].*

4. Existence of mild solutions

Denote $C_l = C([0,l],\mathfrak{X})$, $C_\mu = C([-\mu,0],\mathfrak{X})$, $\text{Ç} = C([-\mu,l],\mathfrak{X})$.

Let χ_l and χ_μ be the Hausdorff MNCs [1,12] on Ç and C_μ, respectively. Consider

(H1) $\varrho : [0,l]\times\mathfrak{X}\times C_\mu$ is continuous map, such that there exists $\eta_\varrho \in L^p(0,l)$, $p > \frac{1}{q}$ and increasing continuous function Ψ_ϱ, such that $\|\varrho(t,\vartheta,w)\| \le \eta_\varrho(t)\Psi_\varrho(\|\vartheta\| + \|w\|_{C_\mu})$ for all $\vartheta\in\mathfrak{X}$, $w\in C_\mu$.

(H2) $\varpi : \text{Ç} \to C_\mu$ is continuous map, such that there is an increasing continuous function Ψ_ϖ such that,

(i) $\|\varpi(\vartheta)\|_{C_\mu} \le \Psi_\varpi(\|\vartheta\|_C)$, for all $\vartheta\in\text{Ç}$;

(ii) there exists $\eta_\varpi \ge 0$, such that $\chi_\mu(\varpi(\Omega)) \le \eta_\varpi\chi_l(\Omega)$, for all bounded sets $\Omega\subset\text{Ç}$.

Definition 4.1. *Mild solution of problem (1.1) on* $[-\mu,l]$ *is a function* $\vartheta\in\text{Ç}$*, for which there exists an integrable function* $z:[0,l]\to K$*, where* K *is closed convex subset of* $\mathfrak{Z}$*, such that*

$$\begin{aligned}
&\vartheta(t) = C_q(t)[\varphi(0)-\varpi(\vartheta)(0)] + S_q(t)\psi_0 + \int_0^t P_q(t-s)\varrho(s,\vartheta(s),\vartheta(s-\mu))z(s)ds, \quad t\in[0,l],\\
&\langle\mathcal{B}_1 z''(t) + \mathcal{B}_2 z'(t), v\rangle + J^0(\mathcal{M}z(t),\mathcal{M}v) \ge \langle\wp(t),v\rangle \text{ for all } v\in\mathfrak{Z} \text{ and for a.e. } t\in[0,l],\\
&\vartheta(t) + \varpi(\vartheta)(t) = \varphi(t), \quad t\in[-\mu,0], \quad \vartheta'(0)=\psi_0,\\
&z(0)=0,\ z'(0)=\zeta.
\end{aligned}$$

Define the solution set

$$\Theta(t) = \left\{z\in K, \langle\mathcal{B}_1 z''(t) + \mathcal{B}_2 z'(t), v\rangle + J^0(\mathcal{M}z(t),\mathcal{M}v) \ge \langle\wp(t),v\rangle, \forall\, v\in K\right\}.$$

Clearly, $\Theta:\mathfrak{Z}\to 2^{\mathfrak{Z}^*}$ is upper semi continuous.
Define $\Upsilon : [0,l]\times\mathfrak{X}\times C_\mu \to 2^{\mathfrak{X}}$ as

$$\Upsilon(t,v,w) = \{\varrho(t,v,w)y; y\in\Theta(t)\}. \tag{4.1}$$

Since Θ has closed convex values, so does Υ. Moreover, from the continuity of ϱ, the composition multimap Υ is upper semi continuous. For $\vartheta\in C$, we define

$$\mathcal{G}_\Upsilon(\vartheta) = \{\wp\in L^p([0,l],\mathfrak{X}) : \wp(t)\in\Upsilon(t,\vartheta(t),\vartheta(t-\mu)) \text{ for a.e. } t\in[0,l]\}.$$

Thus, the solution of problem (1.1) becomes

$$\vartheta(t) = C_q(t)[\varphi(0)-\varpi(\vartheta)(0)] + S_q(t)\psi_0 + \int_0^t P_q(t-s)\wp(s)ds,\ \wp\in\mathcal{G}_\Upsilon(\vartheta), \quad t\in[0,l]. \tag{4.2}$$

$$\vartheta(t) + \varpi(\vartheta)(t) = \varphi(t), \quad t\in[-\mu,0]. \tag{4.3}$$

Let us define $\mathfrak{I} : L^p([0,l],\mathfrak{X}) \to C_l$ by

$$\mathfrak{I}(\wp)(t) = \int_0^t P_q(t-s)\wp(s)ds. \tag{4.4}$$

Define the multioperator $\Gamma : \text{Ç} \to 2^{\text{Ç}}$ as below.
For given $\varphi \in C_\mu$

$$\Gamma(\vartheta)(t) = \begin{cases} \varphi(t) - \varpi(\vartheta)(t), & t \in [-\mu, 0], \\ \{C_q(t)[\varphi(0) - \varpi(\vartheta)(0)] + \Im(\wp)(t) : \wp \in \mathcal{G}_\Upsilon(\vartheta)\}, & t \in [0, l]. \end{cases} \tag{4.5}$$

Then, $\vartheta \in \text{Ç}$ is a solution of (4.2) and (4.3), if ϑ is a fixed point of Γ. We will use [11, Corollary 3.3.1], to show that $Fix(\Gamma) \neq \emptyset$.

Lemma 4.1. *[12] Under the assumptions (H1) and (HJ), $\mathcal{G}_\Upsilon$ is well-defined and weakly upper semi continuous.*

Lemma 4.2. *[12] The operator $\Im$ defined by (4.4) is compact.*

Lemma 4.3. *Let (H1) and (HJ) hold. Then, solution multioperator Γ is quasicompact and closed.*

Proof. Since ϖ is continuous and $\Im$ is compact, therefore $\Gamma(\mho)$ is relatively compact for any compact set $\mho \subset \text{Ç}$. Therefore, it is a quasicompact multimap.

Let $\{\vartheta_k\} \subset \text{Ç}$, $\vartheta_k \to \vartheta^*$, $\nu_k \in \Gamma(\vartheta_k)$ and $\nu_k \to \nu^*$. We will prove that $\nu^* \in \Gamma(\vartheta^*)$. Take $\wp_k \in \mathcal{G}_\Upsilon(\vartheta_k)$, such that

$$\nu_k(t) = \varphi(t) - \varpi(\vartheta_k)(t), \quad t \in [-\mu, 0], \tag{4.6}$$

$$\nu_k(t) = C_q(t)[\varphi(0) - \varpi(\vartheta_k)] + S_q(t)\psi_0 + \Im(\wp_k)(t), \quad t \in [0, l]. \tag{4.7}$$

Since $\mathcal{G}_\Upsilon$ is weakly upper semi continuous and $\{\vartheta_k\}$ is compact, $\{\wp_k\}$ is weakly compact and suppose that $\wp_k \rightharpoonup \wp^*$ in $L^p([0, l], \mathfrak{X})$. Furthermore, $\wp^* \in \mathcal{G}_\Upsilon(\vartheta^*)$. By the compactness of $\Im$, we obtain that $\Im(\wp_k) \to \Im(\wp^*)$ in C_l. Taking limits of (4.6)-(4.7) as $k \to \infty$, we get

$$\nu^*(t) = \varphi(t) - \varpi(\vartheta^*)(t), \quad t \in [-\mu, 0],$$

$$\nu^*(t) = C_q(t)[\varphi(0) - \varpi(\vartheta^*)(0)] + \Im(\wp^*)(t), \quad t \in [0, l],\ \wp^* \in \mathcal{G}_\Upsilon(\vartheta^*).$$

Thus, $\nu^* \in \Gamma(\vartheta^*)$.

Lemma 4.4. *Assume that (H1), (H2) and (HJ) hold. If $\eta_\varpi C_q^l < 1$, then Γ is χ_l-condensing, here $C_q^l = \sup_{t \in [0,l]} \|C_q^l(t)\|$.*

Proof. Let $\Xi \subset \text{Ç}$ be a bounded set, then we have

$$\Gamma(\Xi) = \Gamma_1(\Xi) + \Gamma_2(\Xi),$$

where

$$\Gamma_1(\vartheta)(t) = \begin{cases} C_q(t)[\varphi(0) - \varpi(\vartheta)(0)] + S_q(t)\psi_0, & t \in [0, l], \\ \varphi(t) - \varpi(\vartheta)(t), & t \in [-\mu, 0], \end{cases}$$

$$\Gamma_2(\vartheta)(t) = \begin{cases} \{\Im(\wp)(t) : \wp \in \mathcal{G}_\Upsilon(\vartheta)\}, & t \in [0, l], \\ 0, & t \in [-\mu, 0]. \end{cases}$$

From the property of χ_l, we have

$$\chi_l(\Gamma(\Xi)) \le \chi_l(\Gamma_1(\Xi)) + \chi_l(\Gamma_2(\Xi)). \tag{4.8}$$

For $z_1, z_2 \in \Gamma_1(\Xi)$, there exist $\vartheta_1, \vartheta_2 \in \Xi$, such that

$$z_1(t) = \begin{cases} C_q(t)[\varphi(0) - \varpi(\vartheta_1)(0)] + S_q(t)\psi_0, & t \in [0, l], \\ \varphi(t) - \varpi(\vartheta_1)(t), & t \in [-\mu, 0], \end{cases}$$

$$z_2(t) = \begin{cases} C_q(t)[\varphi(0) - \varpi(\vartheta_2)(0)] + S_q(t)\psi_0, & t \in [0, l], \\ \varphi(t) - \varpi(\vartheta_2)(t), & t \in [-\mu, 0]. \end{cases}$$

Then

$$\|z_1(t) - z_2(t)\| \le \begin{cases} \|C_q(t)\| \, \|\varpi(\vartheta_1) - \varpi(\vartheta_2)\|_{C_\mu}, & t \in [0, l], \\ \|\varpi(\vartheta_1) - \varpi(\vartheta_2)\|_{C_\mu}, & t \in [-\mu, 0]. \end{cases}$$

Therefore,

$$\|z_1 - z_2\|_{\c{C}} \le C_q^l \|\varpi(\vartheta_1) - \varpi(\vartheta_2)\|_{C_\mu},$$

as $C_q^l \ge 1$. This implies $\chi_l(\Gamma_1(\Xi)) \le C_q^l \chi_\mu(\varpi(\Xi))$. Using (H2)(ii), we get

$$\chi_l(\Gamma_1(\Xi)) \le \eta_\varpi C_q^l \chi_l(\Xi). \tag{4.9}$$

Concerning Γ_2, we know that $\mathcal{G}_\Upsilon(\Xi)$ is bounded. Then, using the compactness of $\Im$, we see that $\Gamma_2(\Xi)$ is relatively compact set. So, $\chi(\Gamma_2(\Xi)) = 0$. From (4.8) and (4.9), we have

$$\chi_l(\Gamma(\Xi)) \le \eta_\varpi C_q^l \chi_l(\Xi).$$

If $\chi_l(\Xi) \le \chi_l(\Gamma(\Xi))$, then $\chi_l(\Xi) \le \eta_\varpi C_q^l \chi_l(\Xi)$. This implies $\chi_l(\Xi) = 0$, since $\eta_\varpi C_q^l < 1$. From the regularity of χ_l, Ξ is relatively compact. This completes the proof.

Theorem 4.1. *Suppose that (H1), (H2) and (HJ) hold. Then, problem (4.2)-(4.4) has at least one mild solution on* $[-\mu, l]$ *provided that* $\eta_\varpi C_q^l < 1$, *and*

$$\liminf_{k\to\infty} \frac{\|\nu_k\|_{\c{C}}}{k} \le \liminf_{k\to\infty} \left[C_q^l \frac{\Psi_\varpi(k)}{k} + \eta_J(1 + l)\frac{\Psi(2k)}{k} \sup_{t\in[0,l]} \int_0^t \left\| P_q(t-s) \right\| \eta_\varrho(s) ds \right] < 1. \tag{4.10}$$

Proof. The assumption $\eta_\varpi C_q^l < 1$ guarantees that Γ is χ_l-condensing. Further, by Lemma 4.3 and [11, Theorem 1.1.12], Γ is upper semi continuous. To apply [11, Corollary 3.3.1], it is enough to show that there exists $r > 0$, such that $\Gamma(\mathcal{B}_r) \subset \mathcal{B}_r$, where $\mathcal{B}_r$ is the ball of radius r in Ç centered at origin. Contrarily, suppose that there exists $\{\vartheta_k\} \subset$ Ç, such that $\|\vartheta_k\|_{\c{C}} \le k$ and $\nu_k \in \Gamma(\vartheta_k)$ with $\|\nu_k\|_{\c{C}} > k$. From the definition of Γ, we choose $\wp_k \in \mathcal{G}_\Upsilon(\vartheta_k)$, such that

$$\nu_k(t) = \varphi(t) - \varpi(\vartheta_k)(t), \quad t \in [-\mu, 0],$$
$$\nu_k(t) = C_q(t)[\varphi(0) - \varpi(\vartheta_k)(0)] + S_q(t)\psi_0 + \Im(\wp_k)(t), \quad t \in [0, l].$$

Then, for $t \in [-\mu, 0]$, we get

$$\begin{aligned} \|\nu_k(t)\| &\leq \|\varphi\|_{C_\tau} + \|\varpi(\vartheta_k)\|_{C_\mu} \\ &\leq \|\varphi\|_{C_\mu} + \Psi_\varpi(\|\vartheta_k\|_{Ç}) \\ &\leq \|\varphi\|_{C_\mu} + \Psi_\varpi(k), \end{aligned}$$

thanks to (H2)(i). For $t \in [0, l]$, we have

$$\begin{aligned} \|\nu_k(t)\| &\leq C_q^l[\|\varphi\|_{C_\mu} + \|\varpi(\vartheta_k)\|_{C_\mu}] + S_q^l\|\psi_0\| + \sup_{t\in[0,l]} \|\mathfrak{I}(\wp_k)(t)\| \\ &\leq C_q^l[\|\varphi\|_{C_\mu} + \Psi_\varpi(\|\vartheta_k\|)_{C_\mu}] + S_q^l\|\psi_0\| + \sup_{t\in[0,l]} \int_0^t \|P_q(t-s)\| \, \|\wp_k(s)\| ds \\ &\leq C_q^l[\|\varphi\|_{C_\mu} + \Psi_\varpi(k)] + S_q^l\|\psi_0\| + \eta_J(1+l)\Psi(2k) \sup_{t\in[0,l]} \int_0^t \|P_q(t-s)\|\eta_\varrho(s) ds. \end{aligned}$$

Then,

$$\liminf_{k\to\infty} \frac{\|\nu_k\|_{Ç}}{k} \leq \liminf_{k\to\infty} \left[C_q^l \frac{\Psi_\varpi(k)}{k} + \eta_J(1+l)\frac{\Psi(2k)}{k} \sup_{t\in[0,l]} \int_0^t \left\| P_q(t-s) \right\| \eta_\varrho(s) ds \right].$$

Thus, $\liminf_{k\to\infty} \dfrac{\|\nu_k\|_{Ç}}{k} < 1$ due to (4.10), and we get a contradiction. The proof is complete.

5. Application

Example. Consider the following system

$$\begin{cases} {}^C D_t^{\frac{5}{4}} \vartheta(t,y) = \dfrac{\partial^2 \vartheta(t,y)}{\partial y^2} + \dfrac{t}{2e}[\vartheta(t,y) + \vartheta(t-6,y)]z(t,y), \quad \forall\, y \in [0,1], \ \forall\, t \in [0,2], \\ \langle 3z_{tt}(t,y) + z_t(t,y), v\rangle + J^0(Mz(t,y), Mv) \geq \langle \wp(t,y), v\rangle, \quad \forall\, v \in [3,5], \ \forall\, t \in [0,2], \\ \vartheta(t,0) = \vartheta(t,1) = 0, \quad \forall\, t \in [0,2], \\ \vartheta(t,y) + \varpi(\vartheta(t,y)) = \varphi(t,y), \ \ \vartheta'(0,y) = 0, \quad \forall\, y \in [0,1], \ \forall\, t \in [-6,0], \\ z(0,y) = 0, \ \ z'(0,y) = 1. \end{cases} \tag{5.1}$$

Let $\mathfrak{X} = \mathcal{L}^2[0,1]$, $\mathcal{E} = \mathbb{R}$, $\mathfrak{Z} = [3,5]$.
Define the operator $A : D(A) \subset \mathfrak{X} \to \mathfrak{X}$ as $A\vartheta(t,y) = \dfrac{\partial^2 \vartheta(t,y)}{\partial y^2}$ with

$$D(A) = \left\{ \vartheta \in \mathfrak{X} : \vartheta, \vartheta_y, \vartheta_{yy} \text{ are absolutely continuous, } \vartheta_{yy} \in \mathfrak{X}, \text{and } \vartheta(t,0) = \vartheta(t,1) = 0 \right\}.$$

The operator A has discrete spectrum with normalized eigenvectors $e_n(\vartheta) = \sqrt{\frac{2}{\pi}} \sin(n\vartheta)$ corresponding to the eigenvalues $\lambda_n = -n^2$, where $n \in \mathbb{N}$. Moreover, $\{e_n : n \in \mathbb{N}\}$ forms an orthogonal basis for $\mathfrak{X}$. Thus, we have

$$A\vartheta = \sum_{n\in\mathbb{N}} -n^2 \langle \vartheta, e_n \rangle e_n, \ \vartheta \in D(A).$$

Clearly, A generates a strongly continuous cosine family given by

$$C(t)\vartheta = \sum_{n\in\mathbb{N}} \cos(nt)\langle\vartheta, e_n\rangle e_n.$$

Moreover, A generates a strongly continuous exponentially bounded fractional cosine family $C_q(t)$, such that $C_q(0) = I$ and

$$C_q(t) = \int_0^\infty \psi_{t,\frac{q}{2}}(s)C(s)ds, \quad t > 0,$$

where $\psi_{t,\frac{q}{2}}(s) = t^{\frac{q}{2}}\chi_{\frac{q}{2}}(st^{-\frac{q}{2}})$ and

$$\chi_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)}, \quad 0 < \alpha < 1.$$

Clearly, $\|C_q\| \leq 1$.

Define $\vartheta(t)(y) = \vartheta(t, y)$, $z(t)(y) = z(t, y)$, $\varrho(t, \vartheta(t), \vartheta(t-\tau))(y) = \dfrac{t}{2e}[\vartheta(t, y) + \vartheta(t-6, y)]$, $\wp(t)(y) = \wp(t, y)$, $\mathcal{B}_1 z''(t)(y) = 3z_{tt}(t, y)$, $\mathcal{B}_2 z'(t)(y) = z_t(t, y)$, $J(v) = \min\{g_1(v), g_2(v)\}$, where $g_i : \mathbb{R} \to \mathbb{R}$ ($i = 1, 2$) are convex quadratic functions. Choose continuous function ϖ, such that (H2) hold. Moreover, $\mathcal{M} \in L(\mathfrak{Z}, \mathcal{E})$ is a compact operator and φ is a continuous function. In view of [14, Example 12], we can easily show that J satisfies (HJ).

If all the conditions of Theorem 4.1 hold, then the above system has at least one mild solution on $[-6, 2]$.

6. Conclusions

The main objective of the present paper is to study the existence of a mild solution for a class of nonlocal fractional delay differential equations of order $1 < q < 2$ with a hemivariational inequality in Banach spaces. First, we used Rothe's method of semidiscretization to show that there is a solution to variational inequality. For this, we have used some properties of Clarke generalized gradient and pseudomonotone operator. Next, we have obtained sufficient conditions for the existence of a mild solution to the considered system. We plan to look into coupled systems of fractional hemivariational inequalities in the future.

Conflict of interest

The authors declare no conflict of interest.

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