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Research article

Minimum of heavy-tailed random variables is not heavy tailed

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Abstract: By constructing an appropriate example, we show that the class of heavy-tailed distributions is not closed under minimum. We provide two independent heavy-tailed random variables, such that their minimum is not heavy tailed. In addition, we establish a few properties of the distributions considered in the example.

Keywords: heavy-tailed distribution; closure properties; minimum of random variables; closure under minimum; generalized long-tailed distribution
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1. Introduction

We say that distribution F is *heavy-tailed* and write $F \in \mathscr{H}$ if

$$\int_{-\infty}^{\infty} \mathrm{e}^{\lambda x} \mathrm{d}F(x) = \infty \quad \text{for any} \quad \lambda > 0.$$

If $F(x) = \mathbb{P}(X \leqslant x)$, then random variable X is called heavy-tailed. It is well known (see, for instance, Theorem 2.6 in [10]) that $F \in \mathscr{H}$ if and only if

$$\limsup_{x\to\infty} \mathrm{e}^{\delta x}\overline{F}(x) = \infty \quad \text{for any} \quad \delta > 0.$$

Here $\overline{F}(x) = 1 - F(x)$ denotes the right tail of $F(x)$. We say that distribution F is *strongly heavy-tailed* and write $F \in \mathscr{H}^*$ if

$$\lim_{x\to\infty} \mathrm{e}^{\delta x}\overline{F}(x) = \infty \quad \text{for any} \quad \delta > 0.$$

Obviously, $\mathscr{H}^* \subset \mathscr{H}$ and one can check that $\mathscr{H} \backslash \mathscr{H}^* \neq \emptyset$. For discussion on classes $\mathscr{H}$, $\mathscr{H}^*$ and examples $F \in \mathscr{H} \setminus \mathscr{H}^*$ see [2, 15, 16] among others.

Concerning other properties of heavy-tailed distribution class, it is easy to see that $\mathscr{H}$ is closed under convolution, mixing, maximum and product-convolution.

Let us denote the convolution of distributions F_1 and F_2 by

$$F_1 * F_2(x) = \int_{-\infty}^{\infty} F_1(x-y)\mathrm{d}F_2(y).$$

We say that some class of distributions $\mathscr{B}$ is closed under convolution if for any two distributions F_1 and F_2 it holds that

$$F_1 \in \mathscr{B}, F_2 \in \mathscr{B} \;\Rightarrow\; F_1 * F_2 \in \mathscr{B}. \tag{1.1}$$

The relation (1.1) for class of distributions $\mathscr{B} = \mathscr{H}$ follows immediately from definition of $\mathscr{H}$. Namely, by supposing that F_1, F_2 are distributions of independent random variable X_1 and X_2, we get

$$\begin{aligned} F_1 * F_2 \in \mathscr{H} \;&\Leftrightarrow\; \mathbb{E}\mathrm{e}^{\lambda(X_1+X_2)} = \mathbb{E}\mathrm{e}^{\lambda X_1}\mathbb{E}\mathrm{e}^{\lambda X_2} = \infty \text{ for any } \lambda > 0 \\ &\Leftrightarrow\; F_1 \in \mathscr{H} \text{ or } F_2 \in \mathscr{H}. \end{aligned}$$

Similarly, we say that a class of distributions $\mathscr{B}$ is closed under mixing if for $p \in (0,1)$

$$F_1 \in \mathscr{B}, F_2 \in \mathscr{B} \;\Rightarrow\; pF_1 + (1-p)F_2 \in \mathscr{B}.$$

Since for any $\lambda > 0$

$$\int_{-\infty}^{\infty} \mathrm{e}^{\lambda x}\mathrm{d}(pF_1 + (1-p)F_2)(x) = p\int_{-\infty}^{\infty} \mathrm{e}^{\lambda x}\mathrm{d}F_1(x) + (1-p)\int_{-\infty}^{\infty} \mathrm{e}^{\lambda x}\mathrm{d}F_2(x),$$

we get a stronger assertion

$$F_1 \in \mathscr{H} \text{ or } F_2 \in \mathscr{H} \;\Leftrightarrow\; pF_1 + (1-p)F_2 \in \mathscr{H} \text{ for } p \in (0,1).$$

It is said that class of distributions $\mathscr{B}$ is closed under maximum if $F_1, F_2 \in \mathscr{B}$ implies

$$F_{X_1 \vee X_2} = F_1F_2 \in \mathscr{B}.$$

Like in the case of convolution, a stronger assertion on closure under maximum follows

$$F_1 \in \mathscr{H} \text{ or } F_2 \in \mathscr{H} \;\Leftrightarrow\; F_1F_2 \in \mathscr{H}$$

because

$$\begin{aligned} \overline{F_1F_2}(x) &= \overline{F}_1(x) + \overline{F}_2(x) - \overline{F}_1(x)\overline{F}_2(x) \\ &\underset{x\to\infty}{\sim} \overline{F}_1(x) + \overline{F}_2(x). \end{aligned}$$

Considering the closure under the product-convolution, we present the following result:

$$F_1 \in \mathscr{H}, F_2(-0) = 0, F_2(0) < 1 \;\Rightarrow\; F_1 \otimes F_2 \in \mathscr{H}, \tag{1.2}$$

where symbol $\otimes$ denotes the product-convolution, i.e., $F_1 \otimes F_2(x) = \mathbb{P}(X_1X_2 \leqslant x)$ for independent random variables X_1 and X_2 with distributions F_1 and F_2. For the proof of (1.2) it suffices to observe that

$$\mathbb{E}\,\mathrm{e}^{\lambda X_1 X_2} \geqslant \mathbb{E}\,\mathrm{e}^{\lambda X_1^+ X_2} \geqslant \mathbb{E}\,\mathrm{e}^{\lambda X_1^+ X_2}\mathbb{1}_{\{X_2>a\}} \geqslant \mathbb{E}\,\mathrm{e}^{\lambda a X_1^+}\mathbb{P}(X_2 > a),$$

where $\lambda > 0$ is an arbitrary constant, and $a > 0$ is such that $\mathbb{P}(X_2 > a) > 0$.

Studies of other interesting properties of heavy-tailed distributions can be found in [2–4, 7–10] among others.

The problem whether class $\mathscr{H}$ is closed with respect to minimum is much more difficult and, to our knowledge, was not solved. In this paper, we prove that class $\mathscr{H}$ is not closed under minimum. We construct two independent random variables X and Y with the corresponding distributions $F \in \mathscr{H}$ and $G \in \mathscr{H}$, such that their minimum $X \wedge Y = \min\{X, Y\}$ is not heavy tailed, i.e., $F_{X\wedge Y} = 1 - \overline{F}\,\overline{G} = F + G - FG \notin \mathscr{H}$.

2. Main results

Consider the distribution tail $\overline{F}(x)$ of the following form:

$$\begin{aligned}\overline{F}(x) = \mathbb{1}_{(-\infty,0)}(x) + \mathrm{e}^{-x}\mathbb{1}_{[0,1)}(x) + \sum_{n=1}^{\infty}\mathrm{e}^{-x}\prod_{j=1}^{n}\mathrm{e}^{(2j)!-(2j-1)!}\mathbb{1}_{[(2n)!,(2n+1)!)}(x)\\ + \sum_{n=1}^{\infty}\mathrm{e}^{-(2n-1)!}\prod_{j=1}^{n-1}\mathrm{e}^{(2j)!-(2j-1)!}\mathbb{1}_{[(2n-1)!,(2n)!)}(x).\end{aligned} \tag{2.1}$$

This distribution and distribution in (2.5) below will be used for the main result on the minimum of heavy-tailed r.v.s. Our first result yields several properties of the distribution F.

Theorem 2.1. *Assume that F is defined in* (2.1). *Then $F \in \mathscr{H}$, $F \notin \mathscr{H}^*$ and*

$$\limsup_{x\to\infty}\frac{\overline{F}(x-1)}{\overline{F}(x)} < \infty. \tag{2.2}$$

The property in (2.2) defines the class of *generalized long-tailed distributions*, $\mathscr{OL}$, introduced in [13]. Recall that a distribution F on $\mathbb{R}$ belongs to the class $\mathscr{OL}$, if for any (or some) $y > 0$

$$\limsup_{x\to\infty}\frac{\overline{F}(x-y)}{\overline{F}(x)} < \infty. \tag{2.3}$$

Thus, Theorem 2.1 says that

$$(\mathscr{H} \cap \mathscr{OL}) \setminus \mathscr{H}^* \neq \varnothing. \tag{2.4}$$

By Proposition 2.2(ii) in [13], $F \in \mathscr{OL}$ implies that $\lim_{x\to\infty}\mathrm{e}^{\delta x}\overline{F}(x) = \infty$ for some $\delta > 0$, and $\mathscr{OL}$ also admits some light-tailed distributions. Various results related to class $\mathscr{OL}$ can be found in [1, 5, 6, 19, 20]. In particular, authors of [20] showed that $\mathscr{H}^*\setminus\mathscr{OL} \neq \varnothing$, cf. (2.4). Note that class $\mathscr{OL}$ was also introduced in [14], where it was called a Semi-$\mathscr{L}$ class of distributions.

Consider now another distribution with the tail $\overline{G}(x)$ of the following form:

$$\overline{G}(x) = \mathbb{1}_{(-\infty,1)}(x) + \sum_{n=1}^{\infty} \mathrm{e}^{-x+1} \prod_{j=2}^{n} \mathrm{e}^{(2j-1)!-(2j-2)!} \mathbb{1}_{[(2n-1)!,(2n)!)}(x)$$
$$+ \sum_{n=1}^{\infty} \mathrm{e}^{-(2n)!+1} \prod_{j=2}^{n} \mathrm{e}^{(2j-1)!-(2j-2)!} \mathbb{1}_{[(2n)!,(2n+1)!)}(x). \tag{2.5}$$

Analogously to the result in Theorem 2.1, it holds that $G \in \mathscr{H}$, $G \notin \mathscr{H}^*$ and $G \in \mathscr{OL}$.

The main result of the paper says that the distribution $F_{X\wedge Y}(x) = 1-\overline{F}(x)\overline{G}(x)$ is light-tailed. Indeed, by construction of $\overline{F}$ and $\overline{G}$, we have

$$\overline{F}(x)\overline{G}(x) = \mathbb{1}_{(-\infty,0)}(x) + \mathrm{e}^{-x}\mathbb{1}_{[0,\infty,0)}(x)$$

and we obtain the following assertion.

Theorem 2.2. *Assume that X and Y are independent r.v.s with distribution tails $\overline{F}$ in* (2.1) *and $\overline{G}$ in* (2.5)*, respectively. Then*

$$F_{X\wedge Y} \notin \mathscr{H}.$$

Remark 2.1. We mention two related results, which follow easily from definitions. First result says that, although class $\mathscr{H}$ is not closed under minimum, it is closed in the class $\mathscr{H}^*$, i.e.,

$$F_1 \in \mathscr{H}, F_2 \in \mathscr{H}^* \ \Rightarrow\ F_{X_1\wedge X_2} \in \mathscr{H},$$

where X_1 and X_2 are random variables with corresponding distributions F_1 and F_2. Second result says that class $\mathscr{OL}$ is closed under minimum:

$$F_1 \in \mathscr{OL}, F_2 \in \mathscr{OL} \Rightarrow F_{X_1\wedge X_2} \in \mathscr{OL}.$$

The study of the minimum of random variables is important for problems related to various stochastic models. For example it concerns the order statistics $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n}$ of random variables $X_1, X_2, \ldots, X_n$. It is obvious that

$$F_{k:n}(x) = \mathbb{P}(X_{k:n} \leqslant x) = \sum_{j=0}^{k-1} \binom{n}{j} (F_X(x))^j (\overline{F}_X(x))^{n-j}$$

in the case of independent and identically distributed random variables with common distribution F_X. We can see from this expression that properties of order statistics are related to the closure property of random variables under minimum. The order statistics properties for various subclasses of $\mathscr{H}$ were considered in [11, 12, 17, 18], for instance. The definition of the class $\mathscr{H}$ implies immediately the following assertion.

Theorem 2.3. *Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common distribution F_X. Then $F_{X_{k,n}} \in \mathscr{H}$ for $k \in \{1, 2, \ldots, n\}$ if and only if $F_X \in \mathscr{H}$.*

While, it follows from Theorem 2.2 that the analogous statement to Theorem 2.3 fails even in the case $n = 2$ if the random variables $X_1, X_2, \ldots, X_n$ are independent but possibly differently distributed.

3. Proof of Theorem 2.1

Take the sequence $x_n = (2n)!$, $n \geq 1$. For any $\lambda > 0$ we have

$$\begin{aligned}
\mathrm{e}^{\lambda x_n}\overline{F}(x_n) &= \mathrm{e}^{\lambda(2n)!}\exp\{-(2n)! + (2n)! - (2n-1)! + \cdots + 2! - 1!\} \\
&= \exp\{\lambda(2n)! - (2n-1)! + (2n-2)! - (2n-3)! + \cdots + 2! - 1!\} \\
&\geqslant \exp\{(2n-1)!(2n\lambda - 1)\} \ \to \ \infty
\end{aligned}$$

as $n \to \infty$. Hence,

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\overline{F}(x) \geqslant \lim_{n\to\infty} \mathrm{e}^{\lambda x_n}\overline{F}(x_n) \ = \ \infty,$$

implying $F \in \mathscr{H}$.

To show that $F \notin \mathscr{H}^*$, define the sequence $y_n = ((2n)! + (2n+1)!)/2$, $n \geqslant 1$. Then

$$\begin{aligned}
\mathrm{e}^{\lambda y_n}\overline{F}(y_n) = \exp\Big\{&\lambda\frac{(2n)! + (2n+1)!}{2} - \frac{(2n)! + (2n+1)!}{2} \\
&+ \ (2n)! - (2n-1)! + \cdots + 2! - 1!\Big\} \\
= \exp\{&(2n)!(n(\lambda-1) + \lambda) - ((2n-1)! - (2n-2)!) - \cdots - (3! - 2!) - 1\} \\
\leqslant \exp\{&(2n)!(n(\lambda-1) + \lambda)\} \ \to \ 0
\end{aligned}$$

as $n \to \infty$ for $0 < \lambda < 1$. Hence, for such λ,

$$\liminf_{x\to\infty} \mathrm{e}^{\lambda x}\overline{F}(x) \leqslant \lim_{n\to\infty} \mathrm{e}^{\lambda y_n}\overline{F}(y_n) \ = \ 0.$$

It remains to prove that $F \in \mathscr{OL}$. Take $x \in [(2n)!, (2n+2)!)$ and consider the following four cases:

$$\text{(a)} \ \begin{cases} x & \in [(2n+1)!, (2n+2)!), \\ x-1 & \in [(2n+1)!, (2n+2)!), \end{cases} \quad \text{(b)} \ \begin{cases} x & \in [(2n+1)!, (2n+2)!), \\ x-1 & \in [(2n)!, (2n+1)!), \end{cases}$$

$$\text{(c)} \ \begin{cases} x & \in [(2n)!, (2n+1)!), \\ x-1 & \in [(2n)!, (2n+1)!), \end{cases} \quad \text{(d)} \ \begin{cases} x & \in [(2n)!, (2n+1)!), \\ x-1 & \in [(2n-1)!, (2n)!). \end{cases}$$

In case (a) we have

$$\frac{\overline{F}(x-1)}{\overline{F}(x)} = 1.$$

In case (b),

$$\overline{F}(x-1) = \mathrm{e}^{-(x-1)}\prod_{j=1}^{n}\mathrm{e}^{(2j)!-(2j-1)!}, \ \ \overline{F}(x) = \mathrm{e}^{-(2n+1)!}\prod_{j=1}^{n}\mathrm{e}^{(2j)!-(2j-1)!},$$

and, therefore,

$$\frac{\overline{F}(x-1)}{\overline{F}(x)} = \mathrm{e}^{-(x-(2n+1)!)+1} \leqslant \mathrm{e}.$$

In case (c),

$$\frac{\overline{F}(x-1)}{\overline{F}(x)} = \mathrm{e}.$$

In case (d),

$$\overline{F}(x-1) = \mathrm{e}^{-(2n-1)!}\prod_{j=1}^{n-1}\mathrm{e}^{(2j)!-(2j-1)!},\quad \overline{F}(x) = \mathrm{e}^{-x}\prod_{j=1}^{n}\mathrm{e}^{(2j)!-(2j-1)!}$$

and, because $x < (2n)! + 1$, it holds

$$\frac{\overline{F}(x-1)}{\overline{F}(x)} = \mathrm{e}^{x-(2n)!} \;<\; \mathrm{e}.$$

These four estimates yield

$$\limsup_{x\to\infty}\frac{\overline{F}(x-1)}{\overline{F}(x)} = \mathrm{e}.$$

Thus, $F \in \mathscr{OL}$. □

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Conflict of interest

The authors declare that they have no conflicts of interest.

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