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*Research article*

## On periodic Ambrosetti-Prodi-type problems

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**Abstract:** This work presents a discussion of Ambrosetti-Prodi-type second-order periodic problems. In short, the existence, non-existence, and multiplicity of solutions will be discussed on the parameter  $\lambda$ . The arguments rely on a Nagumo condition, to guarantee an a priori bound on the first derivative, lower and upper-solutions method, and the Leray-Schauder's topological degree theory. There are two types of new results based on the parameter's variation: An existence and non-existence theorem and a multiplicity theorem, proving the existence of a bifurcation point. An application to a damped and forced pendulum is studied, suggesting a method to estimate the critical values of the parameter.

**Keywords:** Nagumo-type condition; lower and upper solutions; Leray-Schauder degree; Ambrosetti-Prodi problems

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### 1. Introduction

In this paper, we consider the second-order periodic Ambrosetti-Prodi-type problem composed by the differential equation

$$v''(x) + f(x, v(x), v'(x)) = \lambda g(x), \quad x \in [a, b], \tag{1.1}$$

where  $f : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $g : [a, b] \rightarrow \mathbb{R}^+$  are continuous functions,  $\lambda \in \mathbb{R}$ , and the periodic boundary conditions

$$\begin{aligned} v(a) &= v(b), \\ v'(a) &= v'(b). \end{aligned} \tag{1.2}$$

Periodic problems are an important field of research, not only from a theoretical point of view but also by the huge sort of applications. They have been studied by many authors with different goals and applying a large panoply of methods. We mention only a few examples: results of nonexistence and multiplicity of solutions for strongly nonlinear differential equations [3]; sufficient conditions for the existence of periodic orbits as solutions of the  $\phi$ -Laplacian generalized Liénard equations [9], or as limit cycles [6]; solvability of higher-order fully differential equations [10], nonlinear oscillations of even-order differential equations [11]; cones theory applied to singular third order problems [12]; problems with anti-periodic boundary conditions [21].

Equations involving parameters as in (1.1) allow a so-called Ambrosetti-Prodi alternative, as it was introduced in [1]: There are values  $\lambda_0$  and  $\lambda_1$ , of the real parameter  $\lambda$ , such that the problem has no solution for  $\lambda < \lambda_0$ , at least one if  $\lambda = \lambda_0$ , or two solutions, for  $\lambda_0 < \lambda < \lambda_1$ . Since then, these problems have been studied in several cases and contexts, such as separated two-point or three-point boundary value problems [13, 20]; Neuman's type conditions [19]; the periodic case [8, 16, 22]; parametric problems with the nonlinear Robin  $(p, q)$ -Laplace operator [18]; with adequate asymptotic properties [5]; applied to the fractional Laplacian [2], with asymptotic sign-changed nonlinearities [17], among others.

Recently, in [14, 15], the authors suggested a method to apply lower and upper solutions to third-order periodic Ambrosetti-Prodi problems, based on a speed-growth condition on the nonlinearity, requiring that it grows with different velocities in the several variables. In this paper, we prove that such speed condition is no need for second-order periodic problems and the monotone properties on the nonlinearities are more general and not necessarily periodic.

This work is organized as follows: Section 2 contains the definitions of lower and upper functions, the Nagumo condition used, the corresponding *a priori* bound for the first derivative, and a classic localization theorem existent in the literature. In Section 3, a first discussion on the parameter is done, about the existence and non-existence of solution, and, in Section 4, this discussion is extended to the multiplicity of solutions. The last section contains a model to study the oscillation of a damped and forced pendulum, using a technique to estimate the critical values  $\lambda_0$  and  $\lambda_1$  of the parameter.

## 2. Definitions and preliminary results

The functional frame work, followed in the paper, is the usual space of continuous functions  $C^2[a, b]$  with the correspondent norm

$$\max \{ \|u^{(i)}\|, i = 0, 1, 2 \},$$

where

$$\|w\| := \max_{t \in [a, b]} |w(t)|.$$

Lower and upper functions are considered as in the next definition:

**Definition 2.1.** *The function  $\alpha \in C^2[a, b]$  is a lower solution of problems (1.1) and (1.2) if:*

- (i)  $\alpha''(x) + f(x, \alpha(x), \alpha'(x)) \geq \lambda g(x)$ ,
- (ii)  $\alpha(a) \leq \alpha(b)$ ,  $\alpha'(a) \geq \alpha'(b)$ .

*The function  $\beta \in C^2[a, b]$  is an upper solution of problems (1.1) and (1.2) if:*

- (iii)  $\beta''(x) + f(x, \beta(x), \beta'(x)) \leq \lambda g(x)$ ,

(iv)  $\beta(a) \geq \beta(b), \beta'(a) \leq \beta'(b)$ .

The only growth condition to require to the nonlinearity in (1.1) is given by a Nagumo-type condition:

**Definition 2.2.** A continuous function  $h : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  verifies a Nagumo-type condition relatively to some continuous functions  $\gamma, \Gamma$ , such that

$$\gamma(x) \leq \Gamma(x), \text{ for every } x \in [a, b],$$

in the set

$$S = \{(x, y_0, y_1) \in [a, b] \times \mathbb{R}^2 : \gamma(x) \leq y_0 \leq \Gamma(x)\},$$

if there is a continuous function  $\psi_S : [0, +\infty[ \rightarrow ]0, +\infty[$  such that

$$|h(x, y_0, y_1)| \leq \psi_S(|y_1|), \forall (x, y_0, y_1) \in S, \quad (2.1)$$

with

$$\int_0^{+\infty} \frac{\tau}{\psi_S(\tau)} d\tau = +\infty. \quad (2.2)$$

The *a priori* estimation provided by the Nagumo condition is given by next lemma, which proof is a particular case ( $n = 2$ ) of [7], Lemma 1.

**Lemma 2.1.** Let  $h : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuous function verifying the Nagumo-type conditions (2.1) and (2.2) in  $S$ . Then there is  $r > 0$  such that every solution  $y(x)$  of (1.1) verifying

$$\gamma(x) \leq y(x) \leq \Gamma(x), \forall x \in [a, b] \quad (2.3)$$

satisfies

$$\|y'\| < r.$$

**Remark 2.1.** This “*a priori*” bound is independently of  $\lambda$ , since  $\lambda$  belongs to a bounded set.

The existence and localization result will be based on Theorem 5.3 of [4] adapted to the Nagumo conditions (2.1) and (2.2), for the values of the parameter  $\lambda$  such that there are upper and lower solutions of (1.1) and (1.2) according to Definition 2.1.

**Theorem 2.1.** [4] Let  $f : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $g : [a, b] \rightarrow \mathbb{R}^+$  be continuous functions. Assume that there are lower and upper solutions of problems (1.1) and (1.2),  $\alpha(x)$  and  $\beta(x)$ , respectively, accordingly Definition 2.1, such that

$$\alpha(x) \leq \beta(x), \text{ for } x \in [a, b],$$

and  $f$  verifies Nagumo-type conditions (2.1) and (2.2) in

$$S = \{(x, y_0, y_1) \in [a, b] \times \mathbb{R}^2 : \alpha(x) \leq y_0 \leq \beta(x)\}.$$

Then (1.1) and (1.2) have at least a solution  $v(x) \in C^2([0, 1])$  such that

$$\alpha(x) \leq v(x) \leq \beta(x), \forall x \in [a, b].$$

The following example stresses how the localization provided by the lower and upper solutions technique could be helpful for the estimation of the parameter variation.

**Example 2.1.** Consider the problem composed by the differential equation

$$v''(x) + \sqrt[3]{v(x)} - 5(v'(x))^2 = \lambda, \quad x \in [0, \pi], \quad (2.4)$$

together with the periodic conditions

$$v(0) = v(\pi), \quad v'(0) = v'(\pi). \quad (2.5)$$

The functions  $\alpha, \beta : [0, \pi] \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} \alpha(x) &= -\left(x - \frac{\pi}{2}\right)^2 + \frac{\pi}{4}, \\ \beta(x) &= (x - 5)^2 - 2.7, \end{aligned}$$

are, respectively, lower and upper solutions of problems (2.4) and (2.5) for

$$-66.164 \leq \lambda \leq -52.537.$$

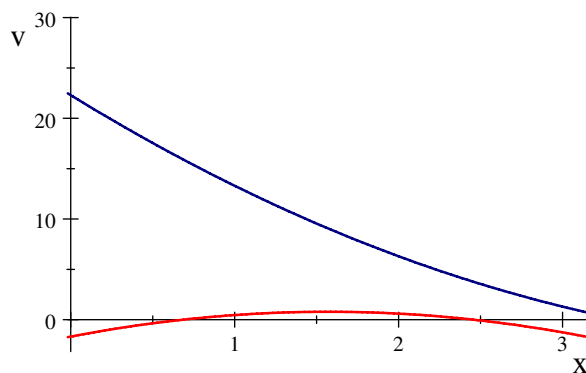
The above problem is a particular case of the initial one (1.1) and (1.2) with

$$\begin{aligned} f(x, v, v') &= \sqrt[3]{v} - 5(v')^2, \\ g(x) &\equiv 1, \quad a = 0, \quad \text{and } b = \pi. \end{aligned}$$

As the assumptions of Theorem 2.1 are verified, then there is a solution  $v(x)$  of (2.4) and (2.5), for  $-66.164 \leq \lambda \leq -52.537$ , lying in the strip

$$-\left(x - \frac{\pi}{2}\right)^2 + \frac{\pi}{4} \leq v(x) \leq (x - 5)^2 - 2.7, \quad \forall x \in [0, \pi].$$

As it is illustrated graphically in Figure 1, this solution is not a trivial one, as no constant function is inside the strip.



**Figure 1.** Localization of  $v_1$ .

### 3. Existence and non-existence result

A first discussion on  $\lambda$  about the existence and non-existence of a solution will be done in this section.

**Theorem 3.1.** Consider the assumptions of Theorem 2.1,  $f : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $g : [a, b] \rightarrow \mathbb{R}^+$  continuous functions, satisfying a Nagumo-type condition and assume that there are  $\lambda_1 \in \mathbb{R}$  and  $m > 0$  such that

$$\frac{f(x, 0, 0)}{g(x)} < \lambda_1 < \frac{f(x, y_0, 0)}{g(x)}, \quad (3.1)$$

for every  $x \in [a, b]$  and  $y_0 \leq -m$ .

Then there is  $\lambda_0 < \lambda_1$  (eventually  $\lambda_0 = -\infty$ ) such that:

- for  $\lambda < \lambda_0$ , the problems (1.1) and (1.2) has no solution;
- for  $\lambda_0 < \lambda \leq \lambda_1$ , the problems (1.1) and (1.2) has at least one solution.

*Proof.* Defining

$$\lambda^* := \max \left\{ \frac{f(x, 0, 0)}{g(x)}, x \in [a, b] \right\},$$

by (3.1), there exists  $x^* \in [a, b]$  such that

$$\frac{f(x, 0, 0)}{g(x)} \leq \lambda^* = \frac{f(x^*, 0, 0)}{g(x^*)} < \lambda_1, \forall x \in [a, b].$$

Thus  $\beta(x) \equiv 0$  is an upper solution of (1.1) and (1.2). Moreover, the function  $\alpha(x) \equiv -m$  is a lower solution of (1.1) and (1.2), as by (3.1),

$$\begin{aligned} \alpha''(x) &= 0 > \lambda_1 g(x) - f(x, -m, 0) \\ &> \lambda^* g(x) - f(x, -m, 0). \end{aligned} \quad (3.2)$$

Therefore, by Theorem 2.1, there is at least a solution of (1.1) and (1.2) with  $\lambda = \lambda^*$ .

Next we show that the set of the parameters for which there is a solution, is a continuous set, that is assuming that the problem (1.1) and (1.2) have a solution for  $\lambda = \xi < \lambda_1$ , then it has at least one solution for  $\lambda \in [\xi, \lambda_1]$ .

Suppose that (1.1) and (1.2) have a solution  $v_\xi(x)$ , for  $\xi \leq \lambda_1$ , that is

$$\begin{cases} v_\xi''(x) + f(x, v_\xi(x), v_\xi'(x)) = \xi g(x), \\ v_\xi(a) = v_\xi(b), v_\xi'(a) = v_\xi'(b). \end{cases}$$

For  $m > 0$  given by (3.1), take  $M > 0$ , large enough such that

$$M \geq m, v_\xi(a) = v_\xi(b) \geq -M. \quad (3.3)$$

As in (3.2), the function  $\alpha(x) = -M$  is a lower solution of (1.1) and (1.2) for  $\lambda \leq \lambda_1$ , and  $\beta(x) = v_\xi(x)$  is an upper solution of (1.1) and (1.2), for every  $\lambda \in [\xi, \lambda_1]$ , because, for  $\xi \leq \lambda$ , we have

$$v_\xi''(x) + f(x, v_\xi(x), v_\xi'(x)) = \xi g(x) \leq \lambda g(x).$$

To apply Theorem 2.1, it remains to prove that

$$-M \leq v_\xi(x), \forall x \in [a, b].$$

Suppose the inequality is not true. So, there is  $x \in [a, b]$  such that

$$-M > v_\xi(x), \quad (3.4)$$

and define

$$v_\xi(x_0) := \min_{x \in [a, b]} v_\xi(x) < -M.$$

By (3.3),  $x_0 \in ]a, b[$ ,  $v'_\xi(x_0) = 0$  and  $v''_\xi(x_0) \geq 0$ . So, by (3.1), we obtain the following contradiction:

$$\begin{aligned} 0 &\leq v''_\xi(x_0) = \xi g(x_0) - f(x_0, v_\xi(x_0), v'_\xi(x_0)) \\ &\leq \lambda g(x_0) - f(x_0, v_\xi(x_0), 0) \\ &\leq \lambda_1 g(x_0) - f(x_0, v_\xi(x_0), 0) < 0. \end{aligned}$$

Therefore, by Theorem 2.1, the problems (1.1) and (1.2) have at least a solution  $v(x)$  for every  $\lambda$  such that  $\xi \leq \lambda \leq \lambda_1$ .

Consider now the set

$$S = \{\lambda \in \mathbb{R} : (1.1), (1.2) \text{ has at least a solution}\}$$

As  $\lambda^* \in S$ , then  $S \neq \emptyset$  and define

$$\lambda_0 := \inf S.$$

Clearly, (1.1) and (1.2) have no solution for  $\lambda < \lambda_0$  and, by the continuity of the parameters set, it has at least a solution for  $\lambda \in [\lambda_0, \lambda_1]$ .

Remark that, if  $\lambda_0 = -\infty$  then (1.1) and (1.2) have a solution for every  $\lambda < \lambda_1$ .

#### 4. Multiplicity theorem

The multiplicity of solutions is proven by topological degree theory applied to a homotopic, modified and perturbed problem.

**Theorem 4.1.** *Let  $f : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuous function verifying the assumptions of Theorem 3.1, together with the monotone condition*

$$y_1 \geq y_2 \implies f(x, y_1, z) \geq f(x, y_2, z), \forall (x, z) \in [a, b] \times \mathbb{R}.$$

*If there are  $B > 0$  such that every solution  $v$  of (1.1) and (1.2) with  $\lambda \leq \lambda_1$ , verifies*

$$|v(x)| \leq \frac{B}{2}, \forall x \in [a, b], \quad (4.1)$$

*and  $\theta \in \mathbb{R}$  such that*

$$f(x, y_0, y_1) \geq \theta g(x), \quad (4.2)$$

*for every  $(x, y_0, y_1) \in [a, b] \times [-m, B] \times \mathbb{R}$ , with  $m$  given by (3.1), then  $\lambda_0$ , in Theorem 3.1, is finite and:*

- (1) If  $\lambda < \lambda_0$ , (1.1) and (1.2) have no solution;*
- (2) If  $\lambda = \lambda_0$ , (1.1) and (1.2) have at least one solution;*
- (3) If  $\lambda \in ]\lambda_0, \lambda_1]$ , (1.1) and (1.2) have at least two solutions.*

*Proof.* For clarity, we divide the proof into several steps.

**Step 1.** Every solution  $v$  of problems (1.1) and (1.2) for  $\lambda \in [\lambda_0, \lambda_1]$ , satisfies

$$-m < v(x) < \frac{B}{2}, \forall x \in [a, b],$$

with  $m$  given by (3.1) and  $B$  by (4.1).

Assume, by contradiction, that there is a solution  $v$  of (1.1) and (1.2) for  $\lambda \in [\lambda_0, \lambda_1]$ , and  $\tau \in [a, b]$  such that

$$v(\tau) := \min_{x \in [a, b]} v(x) \leq -m.$$

If  $\tau \in ]a, b[$ , then  $v'(\tau) = 0$ ,  $v''(\tau) \geq 0$ , and, by (3.1) this contradiction holds

$$\begin{aligned} 0 &\leq v''(\tau) = \lambda g(\tau) - f(\tau, v(\tau), v'(\tau)) \\ &\leq \lambda_1 g(\tau) - f(\tau, v(\tau), v'(\tau)) < 0. \end{aligned}$$

If  $\tau = a$  or  $\tau = b$ ,

$$\min_{x \in [a, b]} v(x) = v(a) = v(b),$$

then

$$0 \leq v'(a) = v'(b) \leq 0,$$

and therefore,

$$v'(a) = v'(b) = 0, \quad v''(a) \geq 0 \text{ and } v''(b) \geq 0.$$

Applying an analogous technique as in the previous theorem, it can be proved similar contradictions. So, every solution  $v$  of (1.1) and (1.2) with  $\lambda \in [\lambda_0, \lambda_1]$ , verifies

$$v(x) > -m, \forall x \in [a, b],$$

and, therefore, by (4.1),

$$-m < v(x) < \frac{B}{2}, \forall x \in [a, b].$$

**Step 2.**  $\lambda_0$  is finite.

Assume that  $\lambda_0 = -\infty$ . So, by Theorem 3.1, for every  $\lambda \leq \lambda_1$ , the problems (1.1) and (1.2) has at least a solution.

Define

$$g_1 := \min_{x \in [a, b]} g(x) > 0,$$

and take  $\lambda$  sufficiently small such that

$$\theta - \lambda > 0 \text{ and } \left( \frac{b-a}{4} \right)^2 (\theta - \lambda) g_1 > B. \quad (4.3)$$

For every solution  $v$  of (1.1) and (1.2), by (4.2),

$$v''(x) = \lambda g(x) - f(x, v(x), v'(x)) \leq (\lambda - \theta) g(x),$$

and, by (1.2), there exists  $\xi \in ]a, b[$  such that  $v'(\xi) = 0$ .

For  $x < \xi$

$$v'(x) = - \int_x^\xi v''(\tau) d\tau \geq \int_x^\xi (\theta - \lambda) g(\tau) d\tau \geq (\theta - \lambda)(\xi - x) g_1.$$

For  $x \geq \xi$

$$v'(x) = \int_\xi^x v''(\tau) d\tau \leq (\lambda - \theta)(x - \xi) g_1.$$

Choose  $J = [a, \frac{3a+b}{4}]$ , or  $J = [\frac{a+3b}{4}, b]$ , such that  $|\xi - x| \geq \frac{b-a}{4}$ , for every  $x \in J$ . If  $J = [a, \frac{3a+b}{4}]$ , then

$$v'(x) \geq \frac{b-a}{4} (\theta - \lambda) g_1, \forall x \in J,$$

for  $J = [\frac{a+3b}{4}, b]$ ,

$$v'(x) \leq \frac{b-a}{4} (\lambda - \theta) g_1, \forall x \in J.$$

In the first case, by (4.1) and (4.3),

$$\begin{aligned} 0 &= \int_a^b v'(\tau) d\tau = \int_a^{\frac{3a+b}{4}} v'(\tau) d\tau + \int_{\frac{3a+b}{4}}^b v'(\tau) d\tau \\ &\geq \int_a^{\frac{3a+b}{4}} \left( \frac{b-a}{4} (\theta - \lambda) g_1 \right) dt + v(b) - v\left(\frac{3a+b}{4}\right) \\ &\geq \frac{b-a}{4} (\theta - \lambda) \left( \frac{b-a}{4} \right) g_1 - \frac{B}{2} - v\left(\frac{3a+b}{4}\right) \\ &= \left( \frac{b-a}{4} \right)^2 (\theta - \lambda) g_1 - \frac{B}{2} - v\left(\frac{3a+b}{4}\right) \\ &> B - \frac{B}{2} - v\left(\frac{3a+b}{4}\right) = \\ &= \frac{B}{2} - v\left(\frac{3a+b}{4}\right), \end{aligned}$$

which contradicts (4.1).

For  $J = [\frac{a+3b}{4}, b]$  a similar contradiction is achieved, and, therefore,  $\lambda_0$  is finite.

**Step 3.** For  $\lambda \in ]\lambda_0, \lambda_1]$ , (1.1) and (1.2) has at least two solutions.

As  $\lambda_0$  is finite, by Theorem 3.1, for  $\lambda_{-1} < \lambda_0$ , the problems (1.1) and (1.2), have no solution for  $\lambda = \lambda_{-1}$ .

By Lemma 2.1 and Step 1, we can take  $m_1 > 0$  large enough such that  $\|v'\| < m_1$ , for every solution  $v$  of (1.1) and (1.2), with  $\lambda \in ]\lambda_{-1}, \lambda_1]$ .

Consider the operators

$$\mathcal{L} : C^2([a, b]) \subset C^1([a, b]) \mapsto C([a, b]) \times \mathbb{R}^2,$$

and, for  $\mu \in [0, 1]$ ,

$$\Theta_\mu : C^1([a, b]) \mapsto C([a, b]) \times \mathbb{R}^2,$$



where

$$\mathcal{L}v = (v'', v(a), v'(a)),$$

and

$$\Theta_\mu v = \begin{pmatrix} \mu\lambda g(x) - \mu f(x, \delta(x, v(x)), v'(x)) + v'(x) - \mu\delta(x, v(x)), \\ v(b), v'(b) \end{pmatrix}.$$

As  $\mathcal{L}$  has a compact inverse it can be considered the completely continuous operator

$$\Psi_\mu : (C^1([a, b]), \mathbb{R}) \mapsto (C^1([a, b]), \mathbb{R})$$

defined by

$$\Psi_\mu v = (\mathcal{L}^{-1}\Theta_\mu)(v).$$

Let  $B_1 := \max\{m, |B|\}$  and define the set

$$\Omega = \{v \in \text{dom}\mathcal{L} : \|v\| < B_1, \|v'\| < m_1\}.$$

By Step 1, if  $v$  is a solution of (1.1) and (1.2) with  $\lambda \in ]\lambda_{-1}, \lambda_1]$ , then  $v \notin \partial\Omega$ , the topological degree is well defined, and

$$d(\Psi_{\lambda_{-1}}, \Omega) = 0. \quad (4.4)$$

Considering the convex combination of  $\lambda_{-1}$  and  $\lambda_1$ ,

$$\Pi(\eta) = (1 - \eta)\lambda_{-1} + \eta\lambda_1, \text{ with } \eta \in [0, 1],$$

and the corresponding homotopic problems (1.1) and (1.2) for  $\lambda = \Pi(\eta)$ , the topological degree  $d(\Psi_{\Pi(\eta)}, \Omega_2)$  is well defined for  $\eta \in [0, 1]$  and  $\lambda \in ]\lambda_{-1}, \lambda_1]$ .

By (4.4) and the invariance of the degree under homotopy

$$0 = d(\Psi_{\lambda_{-1}}, \Omega) = d(\Psi_\lambda, \Omega), \quad (4.5)$$

with  $\lambda \in ]\lambda_{-1}, \lambda_1]$ .

Take  $s \in ]\lambda_0, \lambda_1] \subset ]\lambda_{-1}, \lambda_1]$  and, by Theorem 3.1, let  $v_s$  be the correspondent solution of (1.1) and (1.2) for  $\lambda = s$ .

Consider  $\delta > 0$ , small enough, such that

$$|v_s(x) + \delta| < B_1, \forall x \in [a, b]. \quad (4.6)$$

Then  $v_*(x) := v_s(x) + \delta, \forall x \in [a, b]$ , is an upper solution of (1.1) and (1.2), with  $s < \lambda < \lambda_1$ , as

$$\begin{aligned} v_*''(x) &= v_s''(x) = sg(x) - f(x, v_s(x), v_s'(x)) \\ &< \lambda g(x) - f(x, v_s(x), v_s'(x)) \\ &\leq \lambda g(x) - f(x, v_s(x) + \delta, v_s'(x)) \\ &= \lambda g(x) - f(x, v_*(x), v_*'(x)), \end{aligned}$$

and, for the boundary conditions,

$$v_*(a) = v_s(a) + \delta = v_s(b) + \delta = v_*(b),$$

$$v'_*(a) = v'_s(a) = v'_s(b) = v'_*(b).$$

The constant function equal to  $-m$ , is a lower solution of (1.1), (1.2), with  $\lambda \in [s, \lambda_1]$ , as, by (3.1),

$$0 > \lambda_1 g(x) - f(x, -m, 0) > sg(x) - f(x, -m, 0),$$

and the boundary conditions trivially hold.

So, by Theorem 2.1,

$$-m \leq v(x) < v(x) + \delta := v_*(x), \forall x \in [a, b],$$

and as  $\lambda$  belongs to a bounded set, by Lemma 2.1 and Remark 2.1, there are  $m_\delta > 0$  and a set

$$\Omega_\delta = \{y \in \text{dom } \mathcal{L} : -m < y < v_s(x) + \delta, \|y\| < m_\delta\},$$

such that

$$d(\Psi_\lambda, \Omega_\delta) \neq 0, \text{ for } \lambda \in [\lambda_0, \lambda_1].$$

Considering  $m_\delta$  sufficiently large such that  $\Omega_\delta \subset \Omega_2$ , then, by (4.5) and the additivity of the topological degree,

$$d(\Psi_\lambda, \Omega - \Omega_\delta) \neq 0, \text{ for } \lambda \in [\lambda_0, \lambda_1].$$

So, the problems (1.1) and (1.2) has at least two solutions  $u$  and  $v$  such that  $u \in \Omega_\delta$  and  $v \in \Omega - \overline{\Omega_\delta}$  for  $\lambda \in [s, \lambda_1]$ , for  $s$  is arbitrary in  $] \lambda_0, \lambda_1 ]$ .

**Step 4.** For  $\lambda = \lambda_0$ , (1.1) and (1.2) has at least one solution.

Take a sequence  $(\lambda_n)$  with  $\lambda_n \in ] \lambda_0, \lambda_1 ]$  and  $\lim \lambda_n = \lambda_0$ . By Theorem 3.1, for each  $\lambda = \lambda_n$ , the problems (1.1) and (1.2) has a solution  $v_n$ . From the bounds given in Step 1,  $\|v_n\| < B_1$ ,  $\|v'_n\| < B_1$  independently of  $n$ , and, by Remark 1, there is  $k > 0$  sufficiently large such that  $\|v''_n\| < k$ , independently of  $n$ .

Then sequences  $(v_n)$  and  $(v'_n)$ ,  $n \in \mathbb{N}$ , are bounded in  $C([a, b])$ . By the Arzèla-Ascoli theorem, we can take a subsequence of  $(v_n)$  that converges in  $C^2([a, b])$  to a solution  $v_0(t)$  of (1.1) and (1.2) for  $\lambda = \lambda_0$ .

Therefore, there exists at least one solution for  $\lambda = \lambda_0$ .

## 5. Oscillation of a damped and forced pendulum

Consider the oscillation of a damped and forced pendulum given by the differential equation

$$u''(x) + \frac{k}{m}u'(x) - \frac{g}{r}\sqrt[3]{u(x)} = \lambda p(x), \quad x \in [0, 1],$$

with  $k < 0$ , where  $u(x)$  represents the angle between the string and the vertical,  $m$ , the mass of the pendulum,  $g$ , the gravity acceleration,  $r$  the string length,  $\lambda$  a real parameter, and  $p(x)$  a weight positive and continuous function, subject to the periodic conditions

$$u(0) = u(1) \text{ and } u'(0) = u'(1). \quad (5.1)$$

Remark that this is a particular case of the problems (1.1) and (1.2) with

$$f(x, y_0, y_1) = \frac{k}{m}y_1 - \frac{g}{r}\sqrt[3]{y_0},$$

$a = 0$  and  $b = 1$ .

As a numerical example consider  $k = -1, m = 1, r = 5, g = 9.8$  and  $p(x) = x^2 + 3$ , that is, the equation

$$u''(x) - u'(x) - \frac{9.8}{5} \sqrt[3]{u(x)} = \lambda(x^2 + 3), \quad x \in [0, 1]. \quad (5.2)$$

Therefore the functions

$$\begin{aligned} \alpha(x) &= \frac{x}{5}, \\ \beta(x) &= 2\left(x - \frac{2}{3}\right)^2 + 0.19 \end{aligned}$$

are, respectively, lower and upper solutions of (5.1) and (5.2), for  $-0.33656 \leq \lambda \leq 0.30199$ .

The function  $f$  verifies a Nagumo-type growth condition in the set

$$\bar{S} = \left\{ (x, y_0, y_1) \in [0, 1] \times \mathbb{R}^2 : \frac{x}{5} \leq y_0 \leq 2\left(x - \frac{2}{3}\right)^2 + 0.19 \right\},$$

as

$$\begin{aligned} |f(x, y_0, y_1)| &\leq \left| -y_1 - \frac{9.8}{5} \sqrt[3]{y_0} \right| \\ &\leq |y_1| + \frac{9.8}{5} \sqrt[3]{2\left(x - \frac{2}{3}\right)^2 + 0.19} \\ &\leq |y_1| + 2.02 := \psi_{\bar{S}}(|y_1|), \end{aligned}$$

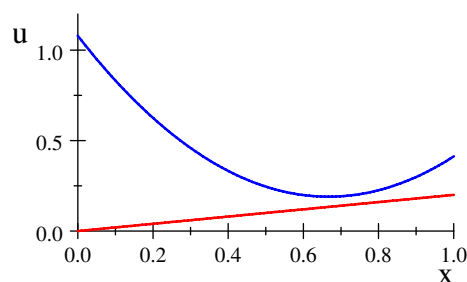
and

$$\int_0^{+\infty} \frac{\tau}{|\tau| + 2.02} d\tau = +\infty.$$

By Theorem 2.1, there is a solution  $u(x)$  such that

$$\frac{x}{5} \leq u(x) \leq 2\left(x - \frac{2}{3}\right)^2 + 0.19, \quad \forall x \in [0, 1].$$

Remark that, from this localization property, it is clear that this solution  $u(x)$  is not a constant function, that is, a trivial periodic solution, as it can be seen in Figure 2.



**Figure 2.** Oscillation estimation

Moreover the assumptions of Theorem 3.1 are satisfied for

$$0 < \lambda_1 < 0.49 \sqrt[3]{m},$$

and  $\lambda_0 \leq -0.33656$ .

Restricting the set of solutions  $u$  of (5.1) and (5.2) to  $\bar{S}$ , then  $|u(x)| \leq 1.0789$ . Condition (4.2) holds with  $r > 0$  given by Lemma 2.1, and  $\theta \leq -\frac{r+2}{3} < 0$ , we have

$$\begin{aligned} f(x, y_0, y_1) &= -y_1 - \frac{9.8}{5} \sqrt[3]{2 \left(x - \frac{2}{3}\right)^2 + 0.19} \\ &\geq \theta(x^2 + 3) = \theta p(x), \end{aligned}$$

for  $(x, y_0, y_1) \in [0, 1] \times [0, 1.0789] \times [-r, r]$ .

So, Theorem 4.1 holds,  $\lambda_0$  is finite and for  $\lambda \in ]\lambda_0, \lambda_1]$ , the problems (5.1) and (5.2) have at least two solutions.

## 6. Discussion

The lower and upper solutions method is an adequate tool to study periodic problems, or other types of boundary value problems, mainly because it provides, not only the existence but also, the solutions localization. This technique plays a key role to prove the Ambrosetti-Prodi alternative, not only for the existence and non-existence cases but also for the multiplicity.

Moreover, from the localization part, it can be easily seen if the periodic solution is, eventually, trivial or if it is nontrivial. This happens, in the periodic case, since there are no horizontal lines inside the admissible region.

## 7. Conclusions

In recent papers on third-order periodic problems [14, 15], it was shown that the nonlinearity must have different growth speeds on some variables, to have more than one solution.

This work proves that, for second-order periodic problems, such property is not needed to discuss the solution multiplicity, based on the parameter variation.

The example and the application suggest a method to estimate the critical values of the parameter, exploiting the localization region, given by the lower and upper solutions method.

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## Conflict of interest

There is no conflict of interest.

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