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Research article

Discrete quasiprobability distributions involving Bernoulli polynomials

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Abstract: The aim of this short paper is to present a new family of discrete densities with two parameters based on Bernoulli numbers and polynomials. We use the properties of such numbers in order to compute the first moments and the density of a finite sum of such independent variables.

Keywords: quasiprobability; Bernoulli numbers; Stirling numbers; Planck's law; poisson distribution
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1. Introduction

The axioms of the probability theory in the sense of A. Kolmogorov required three conditions in order to define a rigorous notion of the probability measure on a measurable space. To this end we introduce a set Ω called the universe, and the sets of all possible events are encoded by a Borel σ-algebra $\mathcal{B}(\Omega)$. Among these three classical conditions, is the requirement that a probability measure takes nonnegative values. If we drop-off this positivity condition, this gives rise to the notion of quasiprobability.

Definition 1.1 (Quasiprobability)**.** *A quasiprobability measure on* $(\Omega, \mathcal{B}(\Omega))$ *is a real valued measure defined on the Borel* σ*-algebra of* Ω*, in other words a map*

$$\widetilde{\boldsymbol{P}} : \mathcal{B}(\Omega) \to \mathbb{R}$$

satisfying the conditions.

(1) $\widetilde{\boldsymbol{P}} : (\Omega) = 1.$

(2) For any countable sequence $A_1, \dots, A_n, \dots$ *of disjoint sets,*

$$\widetilde{\boldsymbol{P}}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n \geq 1} \widetilde{\boldsymbol{P}}(A_n).$$

In the sequel, we will only consider discrete quasiprobabilities i.e. $\Omega = \mathbb{N}$. Given any discrete random variable $X : \Omega \to \mathbb{N}$ which is distributed following a density f_X i.e. $f_X(n) = \widetilde{\mathbf{P}}(X = n)$ then the condition (1) of the definition above just reads,

$$\sum_{n\geq 1} f_X(n) = 1.$$

The main issue is that $f_X(n)$ is now allowed to take negative values. This idea to relax the axioms of probabilities leading to the notion of negative probabilities has been already raised by P.A.M. Dirac and was also formulated in a more precise way by R. Feynmann. We have chosen to focus on is based on Bernoulli numbers and Bernoulli polynomials. These objects unexpectedly appear within the field of quantum statistical physics (see e.g. [14] §2.3.1). Indeed, a central result in this theory is given by Planck's law of energy radiation of a black-body. It states that the density of energy radiation in function of the wave frequency ν at constant temperature T is given by the formula

$$f(\nu) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}} - 1},$$

where h the Planck constant, k the Boltzmann constant, and c the speed of light.

Using Eq (4) we obtain a series expansion for $f(\nu)$ for some constant C independent of ν,

$$f(\nu) = C\sum_{n\geq 0}\frac{B_n}{n!}\left(\frac{h\nu}{kT}\right)^{n+2},$$

where $(B_n)_{n\geq 0}$ is the sequence of Bernoulli numbers defined by the relation

$$\frac{te^t}{e^t - 1} = \sum_{n\geq 0} B_n\frac{t^n}{n!}. \tag{1}$$

The sum obtained is among all the energy microstates in the quantum formalism.

Let us define for each $n \geq 0$ and a fixed ν, the function

$$f(n;\nu) = \frac{1}{f(\nu)}C\frac{B_n}{n!}\left(\frac{h\nu}{kT}\right)^{n+2}.$$

By definition $\sum_{n\geq 0} f(n;\nu) = 1$ so that can interpret $f_n(\nu)$ as a *local density of energy radiation* for a microstate at fixed frequency ν. This observation suggests that $f(n,\nu)$ defines a discrete density of probability. Unfortunately, a major obstacle is that Bernoulli numbers assume both positive and negative values, and therefore bringing us outside the field of the probability theory. Therefore this density can be seen as the distribution of a quasiprobability discrete variable in the sense of Definition 1.1 given above.

2. The Poly-Bernoulli quasiprobability distribution

The so-called Bernoulli polynomials and their related numbers arose in many parts in mathematics. Their definition is simply characterised by the relation:

$$\frac{te^{tx}}{e^t - 1} = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}. \tag{2}$$

The series in the right hand side is entire in the open disc $|t| < 2\pi$. These polynomials enjoy nice properties, an important one is the following differential equation

$$B_n'(x) = nB_{n-1}(x). \tag{3}$$

Thus $B_n(x)$ defines a polynomial of degree n which can be computed by induction, the first Bernoulli polynomials are given by $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$ and $B_2(x) = x^2 - x + \frac{1}{6}$. The Bernoulli polynomials have received considerable attention giving rise to a plethora of remarkable relations (see e.g. [12] Chap. 24). The Bernoulli numbers are just the constant terms of the Bernoulli polynomial i.e. $B_n = B_n(0)$, accordingly the first terms are given by $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_2 = \frac{1}{6}$. A remarkable fact is that $B_{2n+1} = 0$ for any integer $n \geq 1$. In some sense Bernoulli numbers are considered as much important as the polynomials.

We introduce the main object which is a discrete quasiprobability distribution with two parameters.

Definition 2.1. *Given two real parameters $r \geq 0$ and $\theta \in (0; 2\pi)$, we define a quasiprobability distribution X supported on $\mathbb{N}$ given by the density function,*

$$f_X(n) := \frac{e^\theta - 1}{\theta e^{\theta r}} B_n(r)\frac{\theta^n}{n!}.$$

We denote by $PB(r, \theta)$ such distribution and we call it the poly-Bernoulli distribution with parameters r and θ.

The prefix poly-simply means polynomial and it has nothing to do with the classical Bernoulli probability distributions. The fact that $\sum_{n\in\mathbb{N}} f_X(n) = 1$ comes immediately from (4) and our definition involves two parameters θ and r. If we consider the case when $r = 0$, then the distribution $P(0, \theta)$ has the property that only even integers contribute in our computation, in other words, $f_X(2k + 1) = 0$ for any positive integer k. Moreover, using relation (8) we remark that the density f_X assumes both negative and positive value and thus it defines a quasiprobability distribution in the sense of Definition 1.1.

The use of special numbers and special functions in order to define new distribution is not new. Regarding random distribution involving special numbers one has the work of Kim et al. (see for example [5–7, 10, 11]). For another kind of random variables involving polynomials one has [8] for Dowling polynomials, [9] for Lah-Bell polynomials and [3] for derangements polynomials.

We give the first moments of this distribution, the mean and the variance in the sense of quasiprobrobilities. We also focus on the case when $r = 0$ which is seems already giving interesting relations regarding the distribution of the sum of such distributions.

3. General properties of Bernoulli polynomials and numbers

Bernoulli polynomials are defined by the generating series

$$\frac{te^{tx}}{e^t - 1} = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}. \tag{4}$$

The corresponding Bernoulli numbers which are the constant coefficients of $B_n(x)$, namely $B_n = B_n(0)$ has generating series obtained by specialiazing the Eq (4) for $x = 0$, thus

$$\frac{t}{e^t - 1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}. \tag{5}$$

Let us consider

$$u(t) = \frac{t}{e^t - 1} - \frac{t}{e^{-t} - 1} = \frac{t}{e^t - 1} - \frac{te^t}{1 - e^t} = \frac{t(e^t + 1)}{e^t - 1} = \sum_{n\geq 0} B_n \frac{(1 + (-1)^n)t^n}{n!}.$$

Thus,

$$\frac{t(e^{t/2} + e^{-t/2})}{e^{t/2} - e^{-t/2}} = 2\sum_{n\geq 0} B_{2n} \frac{t^{2n}}{(2n)!}. \tag{6}$$

From (6), we can easily deduces the series expansion

$$\frac{t}{2}\coth(\frac{t}{2}) = \sum_{k\geq 0} B_{2k} \frac{t^{2k}}{(2k)!}. \tag{7}$$

The even Bernoulli numbers coming into play in (11) are related to the Riemann-zeta function, indeed using basic Fourier analysis (see [2]), one can obtain that for every $n \geq 1$

$$\frac{B_{2n}}{(2n)!} = (-1)^{n+1} \frac{2\zeta(2n)}{(2\pi)^{2n}}. \tag{8}$$

One has the following nice relation (e.g. 24.14.2 of [12])

$$\sum_{k=0}^{n} \binom{n}{k} B_k\, B_{n-k} = (1 - n)B_n - nB_{n-1}. \tag{9}$$

3.1. Density of PB-distributions

The following proposition provides the properties of PB-distributions.

Proposition 3.1. *Given any nonnegative integer n, and X a $PB(0, \theta)$ distribution. Then, one has the following properties,*

(1) $f_X(2n + 1) = 0$.

(2) $f_X(2n) = (-1)^{n+1} \frac{e^\theta - 1}{\theta} 2\zeta(2n) \left(\frac{\theta}{2\pi}\right)^{2n}$.

(3) sgn $(f_X(2n)) = (-1)^{n+1}$.

Proof. (1) This is an immediate consequence of the fact that $B_{2n+1} = 0$ for any positive integer $n > 0$.
(2) By definition,

$$f_X(2n) := \frac{e^\theta - 1}{\theta} B_{2n} \frac{\theta^{2n}}{(2n)!}.$$

Using (8), we have the required identity.
(3) Since $\theta > 0$ and $\zeta(2n) > 0$, the previous identity in (2) proves that the sign of $f_X(2n)$ is given by $(-1)^{n+1}$.

3.2. Comparison with the Poisson distribution

Let Y be a Poisson random variable of parameter $\theta > 0$ with density function f_Y (see e.g. [1] 20.7), and X with density $f(n;r,\theta)$, then we can write

$$f(n;r,\theta) = m_n(r,\theta) f_Y(n) \text{ where } m_n(r,\theta) = \frac{e^\theta - 1}{\theta e^{\theta(r-1)}} B_n(r).$$

In other words, the densites of X and Y only differ up to a multiplicative factor depending on n, r, θ. For an adequacy with Poisson's distribution, one has to find the values of the parameters (r,θ) for which $m_n(r,\theta) = 1$, $n \geq 1$. As we have done before we consider the case $r = 0$, thus we get the factor $m_n(\theta) = B_n e^\theta (e^\theta - 1)/\theta$. For small values of θ we have that $m_n(\theta) \approx B_n$ so that $f_X(n)$ differs from a Poisson density $\mathcal{P}(\theta)$, only by a factor given by B_n. To sum up, our distribution in the case where $r = 0$ satisfies the following asymptotic property with respect to the Poisson distribution,

$$\lim_{\theta \to 0^+} \frac{f_X(n)}{f_Y(n)} = B_n.$$

3.3. Asymptotic properties of the density of X when the parameter r = 0.

Let us simply denote $f_\theta(n)$ instead of $f(n;0,\theta)$. As we have seen above the density f_θ is supported only at even nonnegative integers,

$$f_\theta(2k) = (e^\theta - 1)\frac{B_{2k}}{(2k)!}\theta^{2k-1}.$$

Then by using Eq (8) we infer that

$$f_\theta(2k) = (-1)^{k+1}\frac{2\zeta(2k)}{(2\pi)^{2k}}\theta^{2k-1}(e^\theta - 1).$$

In particular the cumulative distribution function $F_X(x) = \sum_{n \leq x} f_\theta(n)$ is an alternating series. Since $\lim_k \zeta(2k) = 1 + \sum_{n\geq 2} \lim_k n^{-2k} = 1$, we get

$$|f_\theta(2k)| \sim \frac{2(2k)!}{(2\pi)^{2k}}\theta^{2k-1}(e^\theta - 1) \text{ as } k \to \infty.$$

Using Striling's formula we obtain the following asymptotic estimate for $|f_\theta(2k)|$.

Proposition 3.2. *For any $|\theta| < 2\pi$, we have*

$$|f_X(2n)| \sim \left(\frac{n}{\pi e}\right)^{2n} \sqrt{16n}\,\theta^{2n-1}(e^\theta - 1) \text{ as } n \to \infty.$$

3.4. The expectation value of X

The evaluation of the expectation value of X requires to use specific properties of Bernoulli polynomials which are given in section §2. We first prove the following useful lemma.

Lemma 3.3. *We have the following relation*

$$\frac{\partial}{\partial r} f(n;r,\theta) = \theta\,(f(n-1;r,\theta) - f(n;r,\theta)).$$

Proof. Using Definition 2.1 and Eq (3) we have,

$$\frac{\partial}{\partial r}f(n;r,\theta)=\left(\frac{B_n'(r)e^{\theta r}-\theta B_n(r)e^{r\theta}}{e^{2\theta r}}\right)\frac{(e^\theta-1)\theta^{n-1}}{n!}$$

$$=\left(\frac{nB_{n-1}(r)-\theta B_n(r)}{e^{\theta r}}\right)\frac{(e^\theta-1)\theta^{n-1}}{n!}=\frac{e^\theta-1}{e^{r\theta}}\left(nB_{n-1}(r)\frac{\theta^{n-1}}{n!}-\theta B_n(r)\frac{\theta^{n-1}}{n!}\right)$$

$$=\frac{e^\theta-1}{e^{r\theta}}\left(B_{n-1}(r)\frac{\theta^{n-1}}{(n-1)!}-\theta B_n(r)\frac{\theta^{n-1}}{n!}\right)=\frac{e^\theta-1}{e^{r\theta}}\theta\left(B_{n-1}(r)\frac{\theta^{n-2}}{(n-1)!}-B_n(r)\frac{\theta^{n-1}}{n!}\right).$$

Hence

$$\frac{\partial}{\partial r}f(n;r,\theta)=\theta\left(f(n-1;r,\theta)-f(n;r,\theta)\right).$$

This proves the lemma.

Theorem 3.4. *Let X be a random variable with density distribution $f(n;r,\theta)$, then the mean of X is given by*

$$\mathbf{E}(X)=\theta r+1-\frac{\theta}{1-e^{-\theta}}.$$

Proof. Let us compute the derivative of the mean of X with respect to the parameter r using the previous lemma, for convenience we write $f(n)$ instead of $f(n;r,\theta)$,

$$\frac{\partial}{\partial r}\mathbf{E}(X)=\frac{\partial}{\partial r}\sum_{n\geq 1}n\mathbf{P}\left(X=n\right)=\sum_{n\geq 1}n\frac{\partial}{\partial r}f(n)=\theta\sum_{n\geq 1}n\left(f(n-1)-f(n)\right)$$

$$=\theta f(0)+\theta\sum_{n\geq 1}(n+1)f(n)-nf(n)=\theta f(0)+\theta\sum_{n\geq 1}f(n)=\theta\sum_{n\geq 0}f(n).$$

Thus we get $\frac{\partial}{\partial r}\mathbf{E}(X)=\theta$. Now we perform an integration wrt r and we get that

$$\mathbf{E}(X)=\theta r+\mathbf{E}(X)|_{r=0}.$$

Let us explicit the term $\mathbf{E}(X)|_{r=0}$ which denotes the evaluation of $\mathbf{E}(X)$ when $r=0$.

$$\mathbf{E}(X)|_{r=0}=\sum_{n\geq 1}nf(n;0,\theta)=(e^\theta-1)\sum_{n\geq 1}B_n\frac{\theta^{n-1}}{(n-1)!}$$

and since $B_n=0$ for any odd integer $n>1$ and $B_1=-1/2$, we obtain that

$$\mathbf{E}(X)|_{r=0}=-\frac{1}{2}(e^\theta-1)+(e^\theta-1)\sum_{k\geq 1}B_{2k}\frac{\theta^{2k-1}}{(2k-1)!}.\tag{10}$$

From (4), we have the series expansion

$$\frac{\theta}{2}\coth(\frac{\theta}{2})=\sum_{k\geq 0}B_{2k}\frac{\theta^{2k}}{(2k)!}.\tag{11}$$

The differentiation of Eq (11) with respect to θ yields,

$$\sum_{k\geq 1} B_{2k}\frac{\theta^{2k-1}}{(2k-1)!} = \frac{\partial}{\partial\theta}\left(\frac{\theta}{2}\coth(\frac{\theta}{2})\right) = \frac{\partial}{\partial\theta}\left(\frac{\theta(e^\theta+1)}{2(e^\theta-1)}\right) = \frac{1}{2} + \frac{1}{e^\theta-1} - \frac{\theta e^\theta}{(e^\theta-1)^2}.$$

By replacing in (10) we obtain

$$\mathbf{E}(X)|_{r=0} = -\frac{1}{2}(e^\theta-1) + (e^\theta-1)\left(\frac{1}{2} + \frac{1}{e^\theta-1} - \frac{\theta e^\theta}{(e^\theta-1)^2}\right).$$

Hence we finally obtain $\mathbf{E}(X)|_{r=0} = 1 - \dfrac{\theta}{1-e^{-\theta}}$ and the proof follows.

3.5. The moment generating function and the variance of X

Proposition 3.5. *Let be given $s \in \mathbb{R}$ such that $s \neq 0$. Then the probability generating function*

$$g_X(s) = se^{(s-1)\theta r}\frac{e^\theta-1}{e^{s\theta}-1}.$$

Proof.

$$g_X(s) = \sum_{n\geq 0} s^n\mathbf{P}(X=n) = \frac{e^\theta-1}{\theta e^{\theta r}}\sum_{n\geq 0} s^n B_n(r)\frac{\theta^n}{n!} = \frac{e^\theta-1}{\theta e^{\theta r}}\sum_{n\geq 0} B_n(r)\frac{(s\theta)^n}{n!}.$$

By using (4) we obtain the required result

$$g_X(s) = \frac{e^\theta-1}{\theta e^{\theta r}}\frac{s\theta e^{s\theta r}}{e^{s\theta}-1} = se^{r\theta(s-1)}\frac{e^\theta-1}{e^{s\theta}-1}.$$

Actually we can use this formula to recover the expectation value of X, although the next proof below is in our opinion less instructive than the first one. Indeed let us consider the log-derivative of $g_X(s)$,

$$\frac{g'_X(s)}{g_X(s)} = \frac{1}{s} + \theta r - \frac{\theta e^{\theta s}}{e^{\theta s}-1}. \tag{12}$$

Taking the limit when $s \to 1^-$, we recover immediately the expectation value of X,

$$\mathbf{E}(X) = 1 + \theta r - \frac{\theta e^\theta}{e^\theta-1}.$$

If we proceed to the differentiation of (12) we get,

$$\frac{g''_X(s)g_X(s) - g'_X(s)^2}{g_X(s)^2} = -\frac{1}{s^2} - \frac{d}{ds}\left(\frac{\theta}{1-e^{-\theta s}}\right) = -\frac{1}{s^2} + \frac{\theta^2 e^{-\theta s}}{(1-e^{-\theta s})^2}. \tag{13}$$

In particular if we let $s \to 1^-$ in Eq (13) we get

$$g''_X(1) - g'_X(1)^2 = -1 + \frac{\theta^2 e^{-\theta}}{(1-e^{-\theta})^2}.$$

Reminding that $\mathrm{Var}(X) = g''_X(1) - g'_X(1)^2 + \mathbf{E}(X)$, one obtains the following result

Corollary 3.6. *The variance of X is given by,*

$$\mathrm{Var}(X) = \theta\left(r + \frac{(\theta-1)e^{-\theta}-1}{(1-e^{-\theta})^2}\right).$$

4. The distribution of sums of i.i.d. PB(θ)-distributions

The density of the sum of two independent and identically distributed (i.i.d.) variables in the general case, leads to a quite complicated expression. In this section we assume that $r = 0$. In that case we are able to find a closed form for the distribution of the sum of a finte number of $PB(\theta)$-distribution. Similary to probability theory, one defines the notion of independence of two quasiprobabilities $PB(\theta)$-distributions X and Y, we say that X and Y are i.i.d. if $X, Y \sim PB(\theta)$ and $f_{X+Y} = f_X * f_Y$ as usual. This definition applies analogously to finite sums with more than two variables.

Proposition 4.1. *Let X and Y two i.i.d. random variables following the distribution $PB(\theta)$. Then the law of the sum is given by*

$$f_{X+Y}(n) = \left(\frac{1-e^{\theta}}{\theta}\right)((n-1)f(n) + \theta\, f(n-1)) .$$

Proof. By independence,

$$\begin{aligned} f_{X+Y}(n) = f_X * f_Y(n) &= \sum_{k=0}^{n} \widetilde{\mathbf{P}}(X=k)\, \widetilde{\mathbf{P}}(Y=n-k) \\ &= \left(\frac{e^{\theta}-1}{\theta}\right)^2 \sum_{k=0}^{n} B_k \frac{\theta^k}{k!} B_{n-k} \frac{\theta^{n-k}}{(n-k)!} \\ &= \left(\frac{e^{\theta}-1}{\theta}\right)^2 \left(\sum_{k=0}^{n} \binom{n}{k} B_k\, B_{n-k} \right) \frac{\theta^n}{n!}. \end{aligned}$$

Now we use the well-known formula 9. Therefore we get

$$\begin{aligned} f_{X+Y}(n) &= \left(\frac{e^{\theta}-1}{\theta}\right)^2 ((1-n)B_n - nB_{n-1}) \frac{\theta^n}{n!} \\ &= \left(\frac{e^{\theta}-1}{\theta}\right)^2 (1-n)B_n \frac{\theta^n}{n!} - \frac{(e^{\theta}-1)^2}{\theta} B_{n-1} \frac{\theta^{n-1}}{(n-1)!} \\ &= \left(\frac{e^{\theta}-1}{\theta}\right)((1-n)f(n) - \theta\, f(n-1)) . \end{aligned}$$

4.1. Generalization to the sum of n i.i.d. PB(θ)-distributions

In order to generalize the previous result we need the following Lemma

Lemma 4.2 (Vandiver)**.**

$$\sum_{k_1+\ldots+k_n=k} \frac{k!}{k_1!\ldots k_n!} B_{k_1} \ldots B_{k_n} = (-1)^{n-1} \binom{k}{n} \sum_{i=0}^{n-1} \left[\begin{matrix} n \\ n\text{-}i \end{matrix} \right] B_{k-i}$$

where the numbers $\left[\begin{matrix} n \\ p \end{matrix} \right]$ *are unsigned Stirling numbers of the first kind defined by the generating function* $x(x+1)\ldots(x+n-1) = \sum_{p=0}^{n} \left[\begin{matrix} n \\ p \end{matrix} \right] x^p$.

Proof of the Lemma. See [13] Eq (140) or e.g. [4] Eq (1.5).

The distribution of the the sum of n i.i.d. with density PB(θ) is given by the following formula

Theorem 4.3. *For any integer $k \geq n$, we have*

$$f_{S_n}(k) = \left(\frac{1-e^{\theta}}{\theta}\right)^n \begin{pmatrix} k \\ n \end{pmatrix} \frac{\theta^k}{k!} \sum_{i \equiv k[2]} \begin{bmatrix} n \\ n-i \end{bmatrix} (-1)^{(k-i)/2+1} (k-i)! \frac{\zeta(k-i)}{(2\pi)^{k-i}}.$$

We have

Proof.

$$f_{S_n}(k) = f_{X_1} * \ldots * f_{X_n}(k) = \sum_{k_1+\ldots+k_n=k} f_{X_1}(k_1)\ldots f_{X_n}(k_n)$$

$$= \left(\frac{e^{\theta}-1}{\theta}\right)^n \sum_{k_1+\ldots+k_n=k} B_{k_1}\frac{\theta^{k_1}}{k_1!}\ldots B_{k_n}\frac{\theta^{k_n}}{k_n!}$$

$$= \left(\frac{e^{\theta}-1}{\theta}\right)^n \left(\sum_{k_1+\ldots+k_n=k} \frac{k!}{k_1!\ldots k_n!} B_{k_1}\ldots B_{k_n}\right)\frac{\theta^k}{k!}.$$

By Lemma 4.2 we get,

$$f_{S_n}(k) = \left(\frac{e^{\theta}-1}{\theta}\right)^n \left((-1)^{n-1} \begin{pmatrix} k \\ n \end{pmatrix} \sum_{i=0}^{n-1} \begin{bmatrix} n \\ n-i \end{bmatrix} B_{k-i}\right)\frac{\theta^k}{k!}$$

$$= \left(\frac{e^{\theta}-1}{\theta}\right)^n (-1)^{n-1} \begin{pmatrix} k \\ n \end{pmatrix} \sum_{i=0}^{n-1} \begin{bmatrix} n \\ n-i \end{bmatrix} B_{k-i}\frac{\theta^{k-i}}{(k-i)!}\frac{\theta^i}{k(k-1)\ldots(k-i+1)}.$$

Hence,

$$f_{S_n}(k) = \left(\frac{1-e^{\theta}}{\theta}\right)^{n-1} \begin{pmatrix} k \\ n \end{pmatrix} \sum_{i=0}^{n-1} \begin{bmatrix} n \\ n-i \end{bmatrix} \frac{\theta^i}{k(k-1)\ldots(k-i+1)} f(k-i).$$

As we know the densities $f(n)$ are supported by nonnegative even integers thus the expression of the density of the sum takes the following accurate form

$$f_{S_n}(k) = \left(\frac{1-e^{\theta}}{\theta}\right)^{n-1} \begin{pmatrix} k \\ n \end{pmatrix} \sum_{i \equiv k[2]} \begin{bmatrix} n \\ n-i \end{bmatrix} \frac{\theta^i}{k(k-1)\ldots(k-i+1)} f(k-i).$$

Using the relation $f(2k) = (-1)^{k+1}\dfrac{2\zeta(2k)}{(2\pi)^{2k}}\theta^{2k-1}(e^{\theta}-1)$ we get

$$f_{S_n}(k) = \left(\frac{1-e^{\theta}}{\theta}\right)^{n-1} \begin{pmatrix} k \\ n \end{pmatrix} \sum_{i \equiv k[2]} \begin{bmatrix} n \\ n-i \end{bmatrix} \frac{(-1)^{(k-i)/2+1}\theta^i\theta^{k-i-1}}{k(k-1)\ldots(k-i+1)} \frac{\zeta(k-i)}{(2\pi)^{k-i}}(e^{\theta}-1)$$

$$= \left(\frac{1-e^{\theta}}{\theta}\right)^{n} \begin{pmatrix} k \\ n \end{pmatrix} \theta^k \sum_{i \equiv k[2]} \begin{bmatrix} n \\ n-i \end{bmatrix} (-1)^{(k-i)/2+1} \frac{1}{k(k-1)\ldots(k-i+1)} \frac{\zeta(k-i)}{(2\pi)^{k-i}}.$$

Hence we obtain

$$f_{S_n}(k) = \left(\frac{1-e^{\theta}}{\theta}\right)^n \begin{pmatrix} k \\ n \end{pmatrix} \frac{\theta^k}{k!} \sum_{i \equiv k[2]} \begin{bmatrix} n \\ n-i \end{bmatrix} (-1)^{(k-i)/2+1} (k-i)! \frac{\zeta(k-i)}{(2\pi)^{k-i}}.$$

5. Conclusions

We have introduced a new class of random distributions which are have total mass one but are not necessarily nonnegative. These quasiprobability distributions are based on Bernoulli polynomials. One feature of this distribution is that it does not charge odd positive integers. Due to the numerous relations involving Bernoulli polynomials, one is able to compute the expected value and the density of the sum of independent identical PB-distributions. Also, we have obtained an asymptotic estimate of the density as n tends to infinity. That such distribution can serve as a model for discrete statistical distribution which charges only even integers.

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Conflict of interest

The author declares no conflict of interest.

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