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Research article

Differential equations of the neutral delay type: More efficient conditions for oscillation

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Abstract: In this article, we derive an optimized relationship between the solution and its corresponding function for second- and fourth-order neutral differential equations (NDE) in the canonical case. Using this relationship, we obtain new monotonic properties of the second-order equation. The significance of this paper stems from the fact that the asymptotic behavior and oscillation of solutions to NDEs are substantially affected by monotonic features. Based on the new relationships and properties, we obtain oscillation criteria for the studied equations. Finally, we present examples and review some previous theorems in the literature to compare our results with them.

Keywords: delay differential equation; neutral delay; monotonic properties; oscillation
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

In this article, we study the oscillatory nature of solutions of the neutral differential equations (NDE)

$$\frac{\mathrm{d}}{\mathrm{d}\iota}\left(\varrho(\iota)\left(\frac{\mathrm{d}}{\mathrm{d}\iota}z(\iota)\right)^{\alpha}\right)+q(\iota)\,x^{\alpha}(g(\iota))=0, \tag{1.1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\iota}\left(\varrho(\iota)\left(\frac{\mathrm{d}^3}{\mathrm{d}\iota^3}z(\iota)\right)^{\alpha}\right)+q(\iota)\,x^{\alpha}(g(\iota))=0, \tag{1.2}$$

where $\iota \geq \iota_0$, $z(\iota) = x(\iota) + p(\iota)\,x(h(\iota))$, and α is a ratio of two odd natural numbers. We use the following assumptions:

(A1) The functions ϱ, p and q are continuous on $[\iota_0, \infty)$ and satisfy the conditions: $\varrho(\iota) > 0$, $\varrho'(\iota) > 0$, $0 < p(\iota) < 1$, $q(\iota) > 0$, and q does not vanish identically on any half-line $[\iota_*, \infty)$, for $\iota_* \geq \iota_0$.

(A2) h and g are continuous delay functions on $[\iota_0, \infty)$ and fulfill the conditions: $h(\iota), g(\iota) \leq \iota\ g'(\iota) \geq 0$ and $\lim_{\iota\to\infty} h(\iota) = \lim_{\iota\to\infty} g(\iota) = \infty$.

Moreover, we consider the canonical case, that is,

$$\int_{\iota_0}^{\infty} \varrho^{-1/\alpha}(\mathrm{v})\,\mathrm{dv} = \infty. \tag{1.3}$$

By a solution of Eq (1.1) or (1.2), we mean a real function $x \in C^{m-1}([\iota_x, \infty))$ for some $\iota_x \geq \iota_0$, which has the property $\varrho \cdot \left(z^{m-1}\right)^{\alpha} \in C^1([\iota_x, \infty))$ and x satisfies Eq (1.1) on $[\iota_x, \infty)$, for $m = 2, 4$. Only solutions that satisfy the condition $\sup\{|x(\iota)| : \iota \geq \iota_*\} > 0$, for all $\iota_* \geq \iota_x$, will receive our attention. A solution of Eq (1.1) is called *non-oscillatory* if it is eventually positive or eventually negative; otherwise, it is called *oscillatory*.

Since the creation of the differentiation concept, ordinary differential equations have been utilized to model physical phenomena. As a result of the observation that most of the natural and physical phenomena contain a delay in time (different times), the so-called delay differential equations (DDE) have been established, which take into account the temporal memory of the phenomena. DDEs are functional differential equations in one independent variable, frequently time ι, and they contain late times as the highest derivative in them is on the solution without delay.

The property of oscillation is widespread in many physical, natural, and even social phenomena, so the study of oscillatory properties for solutions of differential equations is an interesting issue not only for its applied importance, but because it also contains many interesting analytical issues.

Sturm's paper [1] is one of the pioneering papers that contributed to the establishment of oscillation theory. He devised the comparative technique, which couples the oscillatory properties of solutions to one differential equation to another. Then, Kneser [2] completed the work in this field and deduced the type of solutions that have been known by his name so far. In 1921, Fite [3] presented the first results that included the oscillation of the solutions of differential equations with deviating arguments. Since then, many results, techniques, and approaches have been presented that have contributed to the development of oscillation theory, most of which have been compiled in monographs [4–8].

Neutral differential equations are a type of functional differential equation in which the highest derivative occurs on the solution with and without delay. This type of equation appears as a result of modeling many phenomena, such as electric networks containing lossless transmission lines (as in high speed computers), vibrating masses, and variational problems with time delays, see [9]. Interest in studying the qualitative behavior of DDE is increasing as a result of the creation of new models and the tremendous technical and scientific growth that the world is currently witnessing in engineering, biology, and physics, see [10–13].

Baculikova and Dzurina in [14] studied the oscillation of NDE

$$\frac{\mathrm{d}}{\mathrm{d}\iota}\left(\varrho(\iota)\left(\frac{\mathrm{d}}{\mathrm{d}\iota}z(\iota)\right)^{\alpha}\right) + q(\iota)\,x^{\beta}(g(\iota)) = 0,$$

when $\alpha \geq \beta$, g and h nondecreasing, $h(g(\iota)) = g(h(\iota))$. In [15–17], the oscillatory behavior of NDE

$$\left(\varrho(\iota)\,|z'(\iota)|^{\alpha-1}\,z'(\iota)\right)' + q(\iota)\,|x(g(\iota))|^{\beta-1}\,x(g(\iota)) = 0, \tag{1.4}$$

have been studied. Liu et al. [15], investigated the asymptotic behavior of (1.4), when $\alpha \geq \beta, \varrho'(\iota) > 0$ and $g'(\iota) > 0$. Wu et al. [16] and Zeng et al. [17] obtained oscillation criteria for (1.4), which develops the criteria in [15]. Grace et al. [18] developed criteria with more than one approach to test the oscillation of solutions of second-order NDEs. Recently, Pátíková and Fišnarová [19] used an improved Riccati substitution to obtain the oscillation criteria of (1.1). Jadlovská [20] provided sharp criteria to check the oscillation of the solutions of (1.1).

In 1998, Zafer [21] studied the oscillatory behavior of the NDE

$$\frac{\mathrm{d}^n}{\mathrm{d}\iota^n} z(\iota) + q(\iota) f(\iota, x(\iota), x(g(\iota))) = 0,$$

where there are $w, h \in \mathbf{C}^1([\iota_0, \infty), [0, \infty))$ such that $w(\iota) > 0$, $w'(\iota) > 0$, and

$$|f(\iota, u, v)| \geq h(\iota) w\left(\frac{|v|}{(1 - p(g(\iota)) g^{n-1}(\iota))}\right).$$

Later, Karpuz et al. [22] and Zhang et al. [23] used the principle of comparison to obtain an oscillation criterion for the NDE

$$\frac{\mathrm{d}^n}{\mathrm{d}\iota^n} z(\iota) + q(\iota) x(g(\iota)) = 0. \tag{1.5}$$

In [24], Zhang and Yan developed criteria of an iterative nature to test the oscillation of (1.5). Agarwal et al. [25] used the Riccati technique to study the oscillatory behavior of the NDE (1.5).

In 2012, Zhang et al. [26] examined the asymptotic behavior of

$$\frac{\mathrm{d}}{\mathrm{d}\iota}\left(\varrho(\iota)\left(\frac{\mathrm{d}^n}{\mathrm{d}\iota^n} x(\iota)\right)^{\alpha}\right) + q(\iota) x^{\beta}(g(\iota)) = 0, \tag{1.6}$$

in the noncanonical case. The results in [26] made sure that all nonoscillatory solutions of Eq (1.6) converge to zero. Zhang et al. [27] improved the results in [26]. By imposing the following conditions

$$f'(u) \geq 0 \text{ and } -f(-uv) \geq f(uv) \geq f(u) f(v), \text{ for } uv > 0.$$

Baculikova et al. [28] studied the oscillatory properties of

$$\frac{\mathrm{d}}{\mathrm{d}\iota}\left(\varrho(\iota)\left(\frac{\mathrm{d}^n}{\mathrm{d}\iota^n} x(\iota)\right)^{\alpha}\right) + q(\iota) f(x(g(\iota))) = 0,$$

in the canonical and noncanonical cases. Recently, there have been some studies concerned with the canonical case of Eq (1.6), see for examples [29–31].

In this paper, we derive new monotonic features of the second-order NDE (1.1). We then use these features to obtain optimized oscillation parameters. We use more than one approach to obtain oscillation parameters. Moreover, in the last section, we set new criteria that ensure the oscillation of solutions of the fourth order NDE (1.2). The new criteria are an extension and development of relevant previous studies.

2. Oscillation results for the second-order equation

In this section, we set out to investigate the monotonic properties and oscillatory behavior of solutions to Eq (1.1).

2.1. Preliminary results

Before looking at the oscillation of the DDE, it is known that determining the signs of derivatives of x or z is important and necessary. Establishing relationships between derivatives of various orders is also crucial, although doing so may impose further limitations on the study. The most influential factor in the relationships between derivatives is the monotonic properties of the solutions of these equations. Therefore, improving these properties or finding new properties of an iterative nature greatly affects the qualitative study of solutions to these equations.

The following notations will be required when presenting the results: $\mathcal{P}_s$: The set of all eventually positive solutions of (1.1), $h_0(\iota) := \iota$, $h_i = h \circ h_{i-1}$, for $i = 1, 2, ...,$

$$\mu_s(\iota) := \int_s^\iota \varrho^{-1/\alpha}(\mathrm{v})\,\mathrm{dv},$$

and

$$\widetilde{p}(\iota, m) := \left(\sum_{i=0}^{m}\left(\prod_{j=0}^{2i}\left(p \circ h_j\right)\right)\left[\frac{1}{(p \circ h_{2i})} - 1\right]\frac{(\mu_{\iota_1} \circ h_{2i})}{\mu_{\iota_1}}\right)^\alpha,\ \iota_1 \geq \iota_0.$$

Lemma 2.1. *The following properties are satisfied for each $x \in \mathcal{P}_s$:*

(P1) *z is non-decreasing,*
(P2) *$\frac{z}{\mu_{\iota_1}}$ is decreasing,*

for $\iota \geq \iota_1 \geq \iota_0$.

Proof. Assuming that $x \in \mathcal{P}_s$, we find, by taking into account (C2), that $x \circ h$, $x \circ g$ and z are also eventually positive. Hence, from Eq (1.1), the function $\varrho \cdot (z')^\alpha$ is decreasing, and so $\varrho \cdot (z')^\alpha$ is of fixed sign.

For the proof of (P1), we should consider two cases:
Case 1: $\varrho(\iota)(z'(\iota))^\alpha \geq 0$. Then, $z'(\iota) \geq 0$, and z is non-decreasing.
Case 2: $\varrho(\iota)(z'(\iota))^\alpha < 0$. Because z is positive and decreasing, there existis a constant L such that $\varrho(\iota)(z'(\iota))^\alpha \leq -L^2 < 0$ for $\iota \geq \iota_1$. Therefore, $z'(\iota) \leq -L^{2/\alpha}\varrho^{-1/\alpha}(\iota)$. By integrating this inequality from ι_1 to ∞ and using the canonical condition (1.3), we obtain $z(\iota_1) = \infty$, a contradiction.

Now, we have z is increasing for $\iota \geq \iota_1$. Thus,

$$z(\iota) \geq \int_{\iota_1}^{\iota} \varrho^{-1/\alpha}(\mathrm{v})\,\varrho^{1/\alpha}(\mathrm{v})\,z'(\mathrm{v})\,\mathrm{dv} \geq \mu_{\iota_1}(\iota)\,\varrho^{1/\alpha}(\iota)\,z'(\iota). \tag{2.1}$$

Then, $\left(\frac{1}{\mu_{\iota_1}}z\right)' = \frac{1}{\mu_{\iota_1}^2}\left(\mu_{\iota_1}z' - \varrho^{-1/\alpha}z\right) \leq 0$, (property (P2)).

Here, the proof ends. ∎

Lemma 2.2. *Assume that $x \in \mathcal{P}_s$. Then,*

$$\left(\varrho(\iota)(z'(\iota))^\alpha\right)' \leq -q(\iota)\,\widetilde{p}(g(\iota), m)\,z^\alpha(g(\iota)). \tag{2.2}$$

Proof. Assume that $x \in \mathcal{P}_s$. Based on the relationship between x and z, we obtain $x = z - p \cdot (x \circ h)$. Thus, $(x \circ h) = (z \circ h) - (p \circ h) \cdot (x \circ h_2)$. By substitution, we get

$$x = z - p \cdot (z \circ h) - p \cdot (p \circ h) \cdot (x \circ h_2).$$

By repeating this procedure, we arrive at

$$\begin{aligned} x &= z - p \cdot (z \circ h) + p \cdot (p \circ h) \cdot (z \circ h_2) - p \cdot (p \circ h) \cdot (p \circ h_2) \cdot (z \circ h_3) \\ &\quad + p \cdot (p \circ h) \cdot (p \circ h_2) \cdot (p \circ h_3) \cdot (x \circ h_4). \end{aligned}$$

Hence,

$$x > \sum_{i=0}^{m} \left(\prod_{j=0}^{2i} \left(p \circ h_j \right) \right) \left[\frac{(z \circ h_{2i})}{(p \circ h_{2i})} - (z \circ h_{2i+1}) \right]. \tag{2.3}$$

From Lemma 2.1, we obtain that (P1) and (P2) hold. Therefore, $(z \circ h_{2i}) \geq (z \circ h_{2i+1})$, and

$$(z \circ h_{2i}) \geq \frac{(\mu_{\iota_1} \circ h_{2i})}{\mu_{\iota_1}} z, \text{ for } i = 0, 1, ...,$$

for $\iota \geq \iota_1$. Then, (2.3) reduce to

$$x > z \sum_{i=0}^{m} \left(\prod_{j=0}^{2i} \left(p \circ h_j \right) \right) \left[\frac{1}{(p \circ h_{2i})} - 1 \right] \frac{(\mu_{\iota_1} \circ h_{2i})}{\mu_{\iota_1}}.$$

Now, Eq (1.1) becomes $(\varrho(\iota)(z'(\iota))^{\alpha})' \leq -q(\iota)\widetilde{p}(g(\iota), m) z^{\alpha}(g(\iota))$.

Here, the proof ends.

Lemma 2.3. *Assume that*

$$q(\iota)\widetilde{p}(g(\iota), m)\varrho^{1/\alpha}(\iota)\mu_{\iota_1}(\iota)\mu_{\iota_1}^{\alpha}(g(\iota)) \geq \alpha k \text{ for some positive constant } k, \tag{2.4}$$

and

$$\mu_{\iota_1}(\iota) \geq \delta \mu_{\iota_1}(g(\iota)) \text{ for some } 1 \leq \delta < \infty. \tag{2.5}$$

Then, the following properties are satisfied for each $x \in \mathcal{P}_s$:

(P3) $\lim_{\iota \to \infty} \frac{z(\iota)}{\mu_{\iota_1}(\iota)} = 0$,

(P4) $\frac{z}{\mu_{\iota_1}^{1-k}}$ *is decreasing,*

(P5) $\frac{z}{\mu_{\iota_1}^{\sqrt[\alpha]{k}\,\delta^k}}$ *is increasing.*

Proof. Assuming that $x \in \mathcal{P}_s$, we find, by taking into account (C2), that $x \circ h$, $x \circ g$ and z are also eventually positive.

From Lemma 2.1, we have that (P1) and (P2) hold.

Now, we have z/μ_{ι_1} is positive and decreasing. Then, $z/\mu_{\iota_1} \to c \geq 0$ as $\iota \to \infty$.

When, unlike property (P3), we assume that $c>0$, we find that there is a $\iota_1 \geq \iota_0$ such that $z/\mu_{\iota_1} \geq c$ for $\iota \geq \iota_1$. Thus, by integrating (2.2) from ι_1 to ι, we obtain

$$\varrho(\iota_1)(z'(\iota_1))^{\alpha} \geq c^{\alpha} \int_{\iota_1}^{\iota} q(\mathrm{v})\,\widetilde{p}(g(\mathrm{v}), m)\,\mu_{\iota_1}^{\alpha}(g(\mathrm{v}))\,d\mathrm{v}.$$

From (2.4), we arrive at

$$\begin{aligned} \varrho(\iota_1)(z'(\iota_1))^{\alpha} &\geq \alpha k c^{\alpha} \int_{\iota_1}^{\iota} \frac{1}{\varrho^{1/\alpha}(\mathrm{v})\,\mu_{\iota_1}(\mathrm{v})} d\mathrm{v} \\ &= k c^{\alpha} \ln \frac{\mu_{\iota_1}(\iota)}{\mu_{\iota_1}(\iota_1)} \to \infty \text{ as } \iota \to \infty, \end{aligned}$$

a contradiction. Then, $c=0$.

Next, using (2.1), (2.2), (2.4) and the fact that $\left(\varrho^{1/\alpha}(\iota)\,z'(\iota)\right)' \leq 0$, we find

$$\begin{aligned} \left(\varrho^{1/\alpha}(\iota)\,z'(\iota)\right)' &= \frac{1}{\alpha}\left(\varrho^{1/\alpha}(\iota)\,z'(\iota)\right)^{1-\alpha}\left(\varrho(\iota)(z'(\iota))^{\alpha}\right)' \\ &\leq -\frac{1}{\alpha}\left(\varrho^{1/\alpha}(\iota)\,z'(\iota)\right)^{1-\alpha} q(\iota)\,\widetilde{p}(g(\iota), m)\,z^{\alpha}(g(\iota)) \qquad (2.6) \\ &\leq -\frac{1}{\alpha}\left(\varrho^{1/\alpha}(\iota)\,z'(\iota)\right)^{1-\alpha} q(\iota)\,\widetilde{p}(g(\iota), m)\,\mu_{\iota_1}^{\alpha}(g(\iota))\,\varrho^{1/\alpha}(g(\iota))\,z'(g(\iota)) \\ &\leq -\frac{1}{\alpha} q(\iota)\,\widetilde{p}(g(\iota), m)\,\mu_{\iota_1}^{\alpha}(g(\iota))\,\varrho^{1/\alpha}(\iota)\,z'(\iota) \\ &\leq -\frac{k}{\varrho^{1/\alpha}(\iota)\,\mu_{\iota_1}(\iota)}\varrho^{1/\alpha}(\iota)\,z'(\iota) \\ &= -\frac{k}{\mu_{\iota_1}(\iota)} z'(\iota). \qquad (2.7) \end{aligned}$$

Here, we define the function $\phi := (1-k)\,z - \mu_{\iota_1} \cdot \varrho^{1/\alpha} \cdot z'$. By differentiating and using (2.7), we get

$$\begin{aligned} \phi' &= (1-k)\,z' - \mu_{\iota_1} \cdot \left(\varrho^{1/\alpha} \cdot z'\right)' - \varrho^{-1/\alpha}\left(\varrho^{1/\alpha} \cdot z'\right) \\ &= -kz' - \mu_{\iota_1} \cdot \left(\varrho^{1/\alpha} \cdot z'\right)' \\ &\geq -k\,z' + \mu_{\iota_1} \cdot \frac{k}{\mu_{\iota_1}} z' = 0. \end{aligned}$$

Now, we will prove that $\phi(\iota) > 0$. If we assume the contrary, then we find that $(1-k)\,z \leq \mu_{\iota_1} \cdot \left(\varrho^{1/\alpha} \cdot z'\right)$, and so $z/\mu_{\iota_1}^{1-k}$ is increasing. We note from (P3) that $\lim_{\iota\to\infty} \varrho^{1/\alpha}(\iota)\,z'(\iota) = 0$. Thus, by integrating (2.2) over $[\iota, \infty)$, we arrive at

$$\varrho(\iota)(z'(\iota))^{\alpha} \geq \int_{\iota}^{\infty} q(\mathrm{v})\,\widetilde{p}(g(\mathrm{v}), m)\,z^{\alpha}(g(\mathrm{v}))\,d\mathrm{v}. \qquad (2.8)$$

Hence, from (2.4) and (P2), we arrive at

$$\varrho(\iota)(z'(\iota))^{\alpha} \geq \alpha k \int_{\iota}^{\infty} \frac{1}{\varrho^{1/\alpha}(\mathrm{v})\,\mu_{\iota_1}(\mathrm{v})} \frac{z^{\alpha}(g(\mathrm{v}))}{\mu_{\iota_1}^{\alpha}(g(\mathrm{v}))} d\mathrm{v}$$

$$
\begin{aligned}
&\geq \alpha k \int_{\iota}^{\infty} \frac{1}{\varrho^{1/\alpha}(\mathrm{v})\,\mu_{\iota_1}^{\alpha+1}(\mathrm{v})} z^{\alpha}(\mathrm{v})\,\mathrm{dv} \\
&= \alpha k \int_{\iota}^{\infty} \frac{1}{\varrho^{1/\alpha}(\mathrm{v})\,\mu_{\iota_1}^{1+\alpha k}(\mathrm{v})} \left(\frac{z(\mathrm{v})}{\mu_{\iota_1}^{1-k}(\mathrm{v})}\right)^{\alpha} \mathrm{dv} \\
&\geq \alpha k \left(\frac{z(\iota)}{\mu_{\iota_1}^{1-k}(\iota)}\right)^{\alpha} \int_{\iota}^{\infty} \frac{1}{\varrho^{1/\alpha}(\mathrm{v})\,\mu_{\iota_1}^{1+\alpha k}(\mathrm{v})} \mathrm{dv} \\
&= \frac{z^{\alpha}(\iota)}{\mu_{\iota_1}^{\alpha}(\iota)},
\end{aligned}
$$

and hence $\mu_{\iota_1} \cdot \varrho^{1/\alpha} \cdot z' \geq z$, which contradicts (2.1). Thus, $\phi(\iota) > 0$, and then $z/\mu_{\iota_1}^{1-k}$ is decreasing.

Next, from (2.4) and (2.8), we have

$$
\begin{aligned}
\varrho(\iota)\left(z'(\iota)\right)^{\alpha} &\geq \alpha k \int_{\iota}^{\infty} \frac{1}{\varrho^{1/\alpha}(\mathrm{v})\,\mu_{\iota_1}(\mathrm{v})} \frac{z^{\alpha}(g(\mathrm{v}))}{\mu_{\iota_1}^{\alpha}(g(\mathrm{v}))} \mathrm{dv} \\
&= \alpha k \int_{\iota}^{\infty} \frac{1}{\varrho^{1/\alpha}(\mathrm{v})\,\mu_{\iota_1}(\mathrm{v})\,\mu_{\iota_1}^{\alpha k}(g(\mathrm{v}))} \left(\frac{z(g(\mathrm{v}))}{\mu_{\iota_1}^{1-k}(g(\mathrm{v}))}\right)^{\alpha} \mathrm{dv},
\end{aligned}
$$

which, with (P4) and (2.5), gives

$$
\begin{aligned}
\varrho(\iota)\left(z'(\iota)\right)^{\alpha} &\geq \alpha\, k\, z^{\alpha}(\iota) \int_{\iota}^{\infty} \frac{1}{\varrho^{1/\alpha}(\mathrm{v})\,\mu_{\iota_1}^{1+\alpha}(\mathrm{v})} \left(\frac{\mu_{\iota_1}(\mathrm{v})}{\mu_{\iota_1}(g(\mathrm{v}))}\right)^{\alpha k} \mathrm{dv} \\
&\geq k\, \delta^{\alpha k} \frac{z^{\alpha}(\iota)}{\mu_{\iota_1}^{\alpha}(\iota)}.
\end{aligned}
$$

Hence, $\mu_{\iota_1} \cdot \varrho^{1/\alpha} \cdot z' \geq k^{1/\alpha}\, \delta^{k}\, z$, and then $z/\mu_{\iota_1}^{\sqrt[\alpha]{k}\,\delta^{k}}$ is increasing.

Here, the proof ends.

Lemma 2.4. *Assume that $x \in \mathcal{P}_s$, (2.4) and (2.5) hold. Then,*

$$
\left(\varrho(\iota)\left(z'(\iota)\right)^{\alpha}\right)' \leq -q(\iota)\,\widehat{p}(g(\iota), m)\, z^{\alpha}(g(\iota)), \tag{2.9}
$$

where

$$
\widehat{p}(\iota, m) := \left(\sum_{i=0}^{m} \left(\prod_{j=0}^{2i} \left(p \circ h_j\right)\right) \left[\frac{1}{(p \circ h_{2i})} - \left(\frac{(\mu_{\iota_1} \circ h_{2i+1})}{(\mu_{\iota_1} \circ h_{2i})}\right)^{\sqrt[\alpha]{k}\,\delta^{k}}\right] \frac{\left(\mu_{\iota_1}^{1-k} \circ h_{2i}\right)}{\mu_{\iota_1}^{1-k}}\right)^{\alpha}.
$$

Proof. Proceeding as in the proof of Lemma 2.2, we arrive at (2.3). From Lemma 2.3, we have that (P4) and (P5) hold. Then, we get

$$
(z \circ h_{2i}) \geq \frac{\left(\mu_{\iota_1}^{1-k} \circ h_{2i}\right)}{\mu_{\iota_1}^{1-k}} z,
$$

and

$$
(z \circ h_{2i+1}) \leq \left(\frac{(\mu_{\iota_1} \circ h_{2i+1})}{(\mu_{\iota_1} \circ h_{2i})}\right)^{\sqrt[\alpha]{k}\,\delta^{k}} (z \circ h_{2i}),
$$

for $i = 0, 1, \ldots$. Thus, (2.3) transforms into

$$x > z \cdot \sum_{i=0}^{m} \left(\prod_{j=0}^{2i} \left(p \circ h_j \right) \right) \left[\frac{1}{(p \circ h_{2i})} - \left(\frac{(\mu_{\iota_1} \circ h_{2i+1})}{(\mu_{\iota_1} \circ h_{2i})} \right)^{\sqrt[\alpha]{k}\,\delta^k} \right] \frac{\left(\mu_{\iota_1}^{1-k} \circ h_{2i} \right)}{\mu_{\iota_1}^{1-k}},$$

which together with (1.1) gives (2.9).

Here, the proof ends.

In the following lemma, we formulate Eq (1.1) in linear form.

Lemma 2.5. *Assume that (2.4) and (2.5) hold. If $x \in \mathcal{P}_s$, then*

$$\left(\varrho^{1/\alpha}(\iota) z'(\iota) \right)' + Q(\iota, m) z(g(\iota)) \leq 0, \tag{2.10}$$

where

$$Q(\iota, m) = q(\iota) \widehat{p}(g(\iota), m) \times \begin{cases} \frac{1}{\alpha} \left((1-k) \delta^{-k} \right)^{1-\alpha} \mu_{\iota_1}^{\alpha-1}(g(\iota)) & \text{for } \alpha \geq 1, \\ \frac{1}{\alpha} \left(\sqrt[\alpha]{k} \delta^{k + \sqrt[\alpha]{k}\delta^k} \right)^{1-\alpha} \mu_{\iota_1}^{\alpha-1}(\iota) & \text{for } \alpha < 1. \end{cases}$$

Proof. Assuming that $x \in \mathcal{P}_s$, we find, by taking into account (C2), that $x \circ h$, $x \circ g$ and z are also eventually positive. From Lemmas 2.1 and 2.3, we have that (P1)–(P5) hold.

From Lemma 2.4, we have that (2.9) holds. Then,

$$\begin{aligned} \left(\varrho^{1/\alpha}(\iota) z'(\iota) \right)' &= \frac{1}{\alpha} \left(\varrho^{1/\alpha}(\iota) z'(\iota) \right)^{1-\alpha} \left(\varrho(\iota) (z'(\iota))^{\alpha} \right)' \\ &\leq -\frac{1}{\alpha} \left(\varrho^{1/\alpha}(\iota) z'(\iota) \right)^{1-\alpha} q(\iota) \widehat{p}(g(\iota), m) z^{\alpha}(g(\iota)). \end{aligned} \tag{2.11}$$

Assume first that $\alpha \geq 1$. Using (P4), we get that $(1-k) z \geq \mu_{\iota_1} \cdot \left(\varrho^{1/\alpha} \cdot z' \right)$. From the facts that $g(\iota) \leq \iota$, (P4) and (2.5), we obtain

$$\begin{aligned} \varrho^{1/\alpha}(\iota) z'(\iota) &\leq (1-k) \frac{1}{\mu_{\iota_1}(\iota)} z(\iota) \leq (1-k) \frac{\mu_{\iota_1}^k(g(\iota))}{\mu_{\iota_1}^k(\iota)} \frac{1}{\mu_{\iota_1}(g(\iota))} z(g(\iota)) \\ &\leq \frac{1-k}{\delta^k} \frac{1}{\mu_{\iota_1}(g(\iota))} z(g(\iota)), \end{aligned}$$

which with (2.11) gives

$$\left(\varrho^{1/\alpha}(\iota) z'(\iota) \right)' + \left(\frac{1-k}{\delta^k} \right)^{1-\alpha} \frac{q(\iota) \widehat{p}(g(\iota), m)}{\alpha \mu_{\iota_1}^{1-\alpha}(g(\iota))} z(g(\iota)) \leq 0. \tag{2.12}$$

Assume now that $\alpha < 1$. Using (P5) and (2.5), we arrive at

$$\begin{aligned} \varrho^{1/\alpha}(\iota) z'(\iota) &\geq \sqrt[\alpha]{k} \delta^k \frac{1}{\mu_{\iota_1}(\iota)} z(\iota) \geq \sqrt[\alpha]{k} \delta^k \frac{1}{\mu_{\iota_1}(\iota)} \left(\frac{\mu_{\iota_1}(\iota)}{\mu_{\iota_1}(g(\iota))} \right)^{\sqrt[\alpha]{k}\delta^k} z(g(\iota)) \\ &\geq \sqrt[\alpha]{k} \delta^{k + \sqrt[\alpha]{k}\delta^k} \frac{1}{\mu_{\iota_1}(\iota)} z(g(\iota)), \end{aligned}$$

which with (2.11) gives

$$\left(\varrho^{1/\alpha}(\iota)\, z'(\iota)\right)' + \frac{1}{\alpha}\left(\sqrt[\alpha]{k}\delta^{k+\sqrt[\alpha]{k}\delta^{k}}\right)^{1-\alpha} \frac{q(\iota)\,\widehat{p}(g(\iota), m)}{\mu_{\iota_1}^{1-\alpha}(\iota)} z(g(\iota)) \leq 0. \tag{2.13}$$

Combining (2.12) and (2.13), we get that (2.10) holds.

Here, the proof ends.

2.2. Oscillation criteria

Using the results in the previous section, we introduce new oscillation criteria for Eq (1.1) in the following theorems:

Theorem 2.1. *Assume that (2.4) and (2.5) hold. Then, Eq (1.1) is oscillatory if*

$$\limsup_{\iota\to\infty}\left[\mu_{\iota_1}^{k-1}(g(\iota))\int_{\iota_1}^{g(\iota)} \mu_{\iota_1}^{1-k}(g(\mathrm{v}))\,\mu_{\iota_1}(\mathrm{v})\, Q(\mathrm{v}, m)\,\mathrm{dv} + \mu_{\iota_1}^{k}(g(\iota))\int_{g(\iota)}^{\iota} \mu_{\iota_1}^{1-k}(g(\mathrm{v}))\, Q(\mathrm{v}, m)\,\mathrm{dv}\right.$$
$$\left. + \mu_{\iota_1}^{1-\sqrt[\alpha]{k}\delta^{k}}(g(\iota))\int_{\iota}^{\infty} \mu_{\iota_1}^{\sqrt[\alpha]{k}\delta^{k}}(g(\mathrm{v}))\, Q(\mathrm{v}, m)\,\mathrm{dv}\right] > 1\ . \tag{2.14}$$

Proof. On the basis of assuming the contrary, we assume that $x \in \mathcal{P}_s$. It follows from Lemmas 2.1 and 2.3 that (P1)–(P5) hold.

From Lemma 2.5, we have that (2.10) holds. Integrating (2.10) from ι to ∞ and using (P3), we obtain

$$\varrho^{1/\alpha}(\iota)\, z'(\iota) = \varrho^{1/\alpha}(\iota)\, z'(\iota) \geq \int_{\iota}^{\infty} Q(\mathrm{v}, m)\, z(g(\mathrm{v}))\,\mathrm{dv},$$

and so

$$z'(\iota) \geq \frac{1}{\varrho^{1/\alpha}(\iota)}\int_{\iota}^{\infty} Q(\mathrm{v}, m)\, z(g(\mathrm{v}))\,\mathrm{dv}.$$

Integrating once again from ι_1 to ι, we arrive at

$$\begin{aligned} z(\iota) &\geq \int_{\iota_1}^{\iota} \frac{1}{\varrho^{1/\alpha}(u)}\int_{u}^{\infty} Q(\mathrm{v}, m)\, z(g(\mathrm{v}))\,\mathrm{dv}\,\mathrm{d}u \\ &\geq \int_{\iota_1}^{\iota} \mu_{\iota_1}(\mathrm{v})\, Q(\mathrm{v}, m)\, z(g(\mathrm{v}))\,\mathrm{dv} + \mu_{\iota_1}(\iota)\int_{\iota}^{\infty} Q(\mathrm{v}, m)\, z(g(\mathrm{v}))\,\mathrm{dv}. \end{aligned}$$

Hence,

$$\begin{aligned} z(g(\iota)) &\geq \int_{\iota_1}^{g(\iota)} \mu_{\iota_1}(\mathrm{v})\, Q(\mathrm{v}, m)\, z(g(\mathrm{v}))\,\mathrm{dv} + \mu_{\iota_1}(g(\iota))\int_{g(\iota)}^{\infty} Q(\mathrm{v}, m)\, z(g(\mathrm{v}))\,\mathrm{dv} \\ &\geq \int_{\iota_1}^{g(\iota)} \mu_{\iota_1}(\mathrm{v})\, Q(\mathrm{v}, m)\, z(g(\mathrm{v}))\,\mathrm{dv} + \mu_{\iota_1}(g(\iota))\int_{g(\iota)}^{\iota} Q(\mathrm{v}, m)\, z(g(\mathrm{v}))\,\mathrm{dv} \\ &\quad + \mu_{\iota_1}(g(\iota))\int_{\iota}^{\infty} Q(\mathrm{v}, m)\, z(g(\mathrm{v}))\,\mathrm{dv}. \end{aligned}$$

Using (P4) and (P5), we conclude that

$$1 \geq \mu_{\iota_1}^{k-1}(g(\iota))\int_{\iota_1}^{g(\iota)} \mu_{\iota_1}^{1-k}(g(\mathrm{v}))\,\mu_{\iota_1}(\mathrm{v})\, Q(\mathrm{v}, m)\,\mathrm{dv} + \mu_{\iota_1}^{k}(g(\iota))\int_{g(\iota)}^{\iota} \mu_{\iota_1}^{1-k}(g(\mathrm{v}))\, Q(\mathrm{v}, m)\,\mathrm{dv}$$

$$+\mu_{\iota_1}^{1-\sqrt[\alpha]{k}\delta^k}\left(g\left(\iota\right)\right)\int_{\iota}^{\infty}\mu_{\iota_1}^{\sqrt[\alpha]{k}\delta^k}\left(g\left(\mathrm{v}\right)\right)Q\left(\mathrm{v},m\right)\mathrm{dv}.$$

Taking $\limsup_{\iota\to\infty}$ of the previous inequality, we arrive at a contradiction with (2.14).

Here, the proof ends.

Theorem 2.2. *Assume that (2.4) and (2.5) hold. Then, Eq (1.1) is oscillatory if*

$$\liminf_{\iota\to\infty}\int_{g(\iota)}^{\iota}Q\left(\mathrm{v},m\right)\mu_{\iota_1}\left(g\left(\mathrm{v}\right)\right)\mathrm{dv}>\frac{1-k}{\mathrm{e}}. \tag{2.15}$$

Proof. On the basis of assuming the contrary, we assume that $x\in\mathcal{P}_s$. It follows from Lemmas 2.1 and 2.3 that (P1)–(P5) hold. From Lemma 2.5, we have that (2.10) holds.

Using (P4), we have $(1-k)\,z\geq\mu_{\iota_1}\cdot\left(\varrho^{1/\alpha}\cdot z'\right)$, which (2.10) gives

$$\left(\varrho^{1/\alpha}\left(\iota\right)z'\left(\iota\right)\right)'+\frac{1}{1-k}Q\left(\iota,m\right)\mu_{\iota_1}\left(g\left(\iota\right)\right)\varrho^{1/\alpha}\left(g\left(\iota\right)\right)z'\left(g\left(\iota\right)\right)\leq 0. \tag{2.16}$$

Then, $\varrho^{1/\alpha}\cdot z'$ is a positive solution of the delay differential inequality of first-order (2.16). It follows from Theorem 1 in [32] that the delay differential equation

$$\left(\varrho^{1/\alpha}\left(\iota\right)z'\left(\iota\right)\right)'+\frac{1}{1-k}Q\left(\iota,m\right)\mu_{\iota_1}\left(g\left(\iota\right)\right)\varrho^{1/\alpha}\left(g\left(\iota\right)\right)z'\left(g\left(\iota\right)\right)=0, \tag{2.17}$$

has also a positive solution. From Theorem 2 in [33], Eq (2.17) is oscillatory under condition (2.15), a contradiction.

Here, the proof ends.

Theorem 2.3. *Assume that (2.4) and (2.5) hold. Then, Eq (1.1) is oscillatory if there is a* $\rho\in\left(\left[\iota_1,\infty\right),\mathbb{R}^+\right)$ *such that*

$$\limsup_{\iota\to\infty}\int_{\iota_1}^{\iota}\left(\rho\left(\mathrm{v}\right)Q\left(\mathrm{v},m\right)\frac{\mu_{\iota_1}^{1-k}\left(g\left(\mathrm{v}\right)\right)}{\mu_{\iota_1}^{1-k}\left(\mathrm{v}\right)}-\frac{\left(\rho_+'\left(\mathrm{v}\right)\right)^2}{4\rho\left(\mathrm{v}\right)}\varrho^{1/\alpha}\left(\mathrm{v}\right)\right)\mathrm{dv}=\infty. \tag{2.18}$$

Proof. On the basis of assuming the contrary, we assume that $x\in\mathcal{P}_s$. It follows from Lemma 2.3 that (P4) holds. From Lemma 2.5, we have that (2.10) holds.

Now, we define the function $w:=\rho\cdot\left(\left(\varrho^{1/\alpha}\cdot z'\right)/z\right)$. Then $w\geq 0$, for $\iota\geq\iota_1$. It follows from (2.10) that

$$\begin{aligned}
w' &= \frac{\rho'}{\rho}w+\rho\cdot\frac{\left(\varrho^{1/\alpha}\cdot z'\right)'}{z}-\rho\cdot\frac{\left(\varrho^{1/\alpha}\cdot z'\right)}{z^2}\cdot\frac{\left(\varrho^{1/\alpha}\cdot z'\right)}{\varrho^{1/\alpha}}\\
&\leq -\rho\cdot Q\cdot\frac{z\circ g}{z}+\frac{\rho'}{\rho}\cdot w-\frac{1}{\varrho^{1/\alpha}\cdot\rho}\cdot w^2\\
&\leq -\rho\cdot Q\cdot\frac{z\circ g}{z}+\frac{\left(\rho_+'\right)^2}{4\rho}\cdot\varrho^{1/\alpha},
\end{aligned}$$

by using the fact that $\varrho w-Bw^2\leq\frac{1}{4}\varrho^2B^{-1}$. Thus, from (P4), we get

$$w'\leq-\rho\cdot Q\cdot\frac{\mu_{\iota_1}^{1-k}\circ g}{\mu_{\iota_1}^{1-k}}+\frac{\left(\rho_+'\right)^2}{4\rho}\cdot\varrho^{1/\alpha}. \tag{2.19}$$

Integrating (2.19) from ι_1 to ι, we arrive at

$$w(\iota_1) - w(\iota) \geq \int_{\iota_1}^{\iota} \left(\rho(\mathrm{v}) \, Q(\mathrm{v}, m) \frac{\mu_{\iota_1}^{1-k}(g(\mathrm{v}))}{\mu_{\iota_1}^{1-k}(\mathrm{v})} - \frac{(\rho'_{+}(\mathrm{v}))^2}{4\rho(\mathrm{v})} \varrho^{1/\alpha}(\mathrm{v}) \right) \mathrm{dv}.$$

Taking $\limsup_{\iota \to \infty}$ of the previous inequality, we arrive at a contradiction with (2.18).

Here, the proof ends.

Remark 2.1. *Using formula (2.2) instead of (2.9), we can obtain the same oscillation criteria by replacing $\widehat{q}$ with $\widetilde{q}$. It is also easy to verify that $\widetilde{p}(\iota, 0) = (1 - p(\iota))^{\alpha}$.*

Corollary 2.1. *Assume that (2.5) hold and*

$$q(\iota) \varrho^{1/\alpha}(\iota) \mu_{\iota_1}(\iota) \mu_{\iota_1}^{\alpha}(g(\iota)) \geq \alpha k \ \textit{ for some positive constant } k,$$

Then, equation

$$\frac{\mathrm{d}}{\mathrm{d}\iota} \left(\varrho(\iota) \left(\frac{\mathrm{d}}{\mathrm{d}\iota} x(\iota) \right)^{\alpha} \right) + q(\iota) \, x^{\alpha}(g(\iota)) = 0,$$

is oscillatory if there is a $\rho \in ([\iota_1, \infty), \mathbb{R}^{+})$ such that

$$\limsup_{\iota \to \infty} \int_{\iota_1}^{\iota} \left(\rho(\mathrm{v}) \, \phi(\mathrm{v}) \, q(\mathrm{v}) \frac{\mu_{\iota_1}^{1-k}(g(\mathrm{v}))}{\mu_{\iota_1}^{1-k}(\mathrm{v})} - \frac{(\rho'_{+}(\mathrm{v}))^2}{4\rho(\mathrm{v})} \varrho^{1/\alpha}(\iota) \right) \mathrm{dv} = \infty,$$

where

$$\phi(\iota) = \begin{cases} \frac{1}{\alpha} \left((1-k) \, \delta^{-k} \right)^{1-\alpha} \mu_{\iota_1}^{\alpha-1}(g(\iota)) & \textit{for } \alpha \geq 1, \\ \\ \frac{1}{\alpha} \left(\sqrt[\alpha]{k} \delta^{k + \sqrt[\alpha]{k} \delta^{k}} \right)^{1-\alpha} \mu_{\iota_1}^{\alpha-1}(\iota) & \textit{for } \alpha < 1. \end{cases}$$

Theorem 2.4. *Assume that (2.4) and (2.5) hold. Then, Eq (1.1) is oscillatory if*

$$\limsup_{\iota \to \infty} \frac{1}{\mu_{\iota_1}^{\alpha}(\iota)} \int_{\iota_1}^{\iota} \left[q(\mathrm{v}) \, \widehat{p}(\mathrm{v}, m) \, \mu_{\iota_1}^{\alpha}(g(\mathrm{v})) - \frac{\beta}{\varrho^{1/\alpha}(\mathrm{v}) \, \mu_{\iota_1}(\mathrm{v})} \right] \mathrm{dv} > 0, \tag{2.20}$$

where $\beta = (\alpha / (\alpha + 1))^{\alpha+1}$.

Proof. On the basis of assuming the contrary, we assume that $x \in \mathcal{P}_s$. It follows from Lemmas 2.1 and 2.3 that (P1)–(P5) hold.

We define $w := \left(\varrho^{1/\alpha} \cdot z' \right)^{\alpha} / z^{\alpha}$, then

$$\begin{aligned} w' &= \frac{\left(\left(\varrho^{1/\alpha} \cdot z' \right)^{\alpha} \right)'}{z^{\alpha}} - \frac{\left(\varrho^{1/\alpha} \cdot z' \right)^{\alpha}}{z^{\alpha+1}} \cdot \alpha z' \\ &\leq -q \cdot \widehat{p} \cdot \frac{z^{\alpha} \circ g}{z^{\alpha}} - \frac{\alpha}{\varrho^{1/\alpha}} \cdot \frac{\left(\varrho^{1/\alpha} \cdot z' \right)^{\alpha+1}}{z^{\alpha+1}}, \end{aligned}$$

which with (P2) gives

$$\mu_{\iota_1}^{\alpha}(\iota) \, w'(\iota) \leq -q(\iota) \, \widehat{p}(g(\iota), m) \, \mu_{\iota_1}^{\alpha}(g(\iota)) - \frac{\alpha \mu_{\iota_1}^{\alpha}(\iota)}{\varrho^{1/\alpha}(\iota)} w^{1+1/\alpha}(\iota). \tag{2.21}$$

Integrating (2.21) from ι_1 to ι, we have

$$\frac{\mu_{\iota_1}^{\alpha}(\iota_1)\,w(\iota_1)}{\mu_{\iota_1}^{\alpha}(\iota)} \geq \frac{1}{\mu_{\iota_1}^{\alpha}(\iota)}\left[\int_{\iota_1}^{\iota} q(\mathrm{v})\,\widehat{p}(\mathrm{v},m)\,\mu^{\alpha}(g(\mathrm{v}))\,\mathrm{dv} - \alpha\int_{\iota_1}^{\iota}\frac{\mu_{\iota_1}^{\alpha-1}(\mathrm{v})}{\varrho^{1/\alpha}(\mathrm{v})}\left[w(\mathrm{v}) - \mu_{\iota_1}(\mathrm{v})\,w^{1+1/\alpha}(\mathrm{v})\right]\mathrm{dv}\right].$$

Using the inequality

$$Au - Bu^{1+1/\alpha} \leq (\alpha/(\alpha+1))^{\alpha+1}A^{\alpha+1}B^{-\alpha}, \tag{2.22}$$

we obtain

$$\frac{\mu_{\iota_1}^{\alpha}(\iota_1)\,w(\iota_1)}{\mu_{\iota_1}^{\alpha}(\iota)} \geq \frac{1}{\mu_{\iota_1}^{\alpha}(\iota)}\int_{\iota_1}^{\iota}\left[q(\mathrm{v})\,\widehat{p}(\mathrm{v},m)\,\mu^{\alpha}(g(\mathrm{v})) - \frac{\beta}{\varrho^{1/\alpha}(\mathrm{v})\,\mu_{\iota_1}(\mathrm{v})}\right]\mathrm{dv}.$$

Taking $\limsup_{\iota\to\infty}$ of the previous inequality, we arrive at a contradiction with (2.20).

Here, the proof ends.

Corollary 2.2. *Assume that (2.5) hold. Then, Eq (1.1) is oscillatory if*

$$\liminf_{\iota\to\infty} q(\iota)\,\widehat{p}(g(\iota),m)\,\varrho^{1/\alpha}(\iota)\,\mu_{\iota_1}(\iota)\,\mu_{\iota_1}^{\alpha}(g(\iota)) > \beta. \tag{2.23}$$

Proof. It is easy to note that condition (2.23) guarantees both conditions (2.4) and (2.20).

Example 2.1. *Consider the NDE*

$$\frac{\mathrm{d}}{\mathrm{d}\iota}\left(\left(\frac{\mathrm{d}}{\mathrm{d}\iota}\left[x(\iota) + p_0 x(a\iota)\right]\right)^{\alpha}\right) + \frac{q_0}{\iota^{\alpha+1}}x^{\alpha}(b\iota) = 0, \tag{2.24}$$

where $\iota > 0$, $p_0 \in [0,1)$, $q_0 > 0$, $a \in (0,1)$, *and* $0 < b < \min\left\{1, \sqrt[\alpha]{\frac{1}{\alpha}q_0(1-p_0)}\right\}$. *It is easy to check that* $h_i = a^i\iota$, *and*

$$\widetilde{p}(\iota,m) = \left(\left[\frac{1}{p_0} - 1\right]\sum_{i=0}^{m} p_0^{2i+1}a^{2i}\right)^{\alpha} := A_{0,m}.$$

By choosing $\delta = 1/b$, *and* $k = \frac{1}{\alpha}A_{0,m}q_0 b^{\alpha}$, *we have that (2.4) and (2.5) hold. Then*

$$\widehat{p}(\iota,m) = \left(\left[\frac{1}{p_0} - a^{\sqrt[\alpha]{k}\,\delta^k}\right]\sum_{i=0}^{m} p_0^{2i+1}a^{2(1-k)i}\right)^{\alpha} := A_{1,m},$$

and

$$\begin{aligned} Q(\iota,m) &= A_{1,m}\frac{q_0}{\iota^2}\times\begin{cases} \frac{1}{\alpha}b^{\alpha-1}\left((1-k)\,\delta^{-k}\right)^{1-\alpha} & \text{for } \alpha \geq 1 \\ \frac{1}{\alpha}\left(\sqrt[\alpha]{k}\delta^{k+\sqrt[\alpha]{k}\delta^k}\right)^{1-\alpha} & \text{for } \alpha < 1 \end{cases} \\ &= A_{1,m}\frac{q_0}{\iota^2}B. \end{aligned}$$

Hence, condition (2.14) becomes

$$A_{1,m}Bq_0\left[\frac{b^{1-k}}{1-k} + \frac{b}{k}\left(b^{-k}-1\right) + \frac{b}{1-\sqrt[\alpha]{k}b^{-k}}\right] > 1. \tag{2.25}$$

Moreover, condition (2.15) becomes

$$A_{1,m}Bbq_0 \ln \frac{1}{b} > \frac{1-k}{\mathrm{e}}. \tag{2.26}$$

On the other hand, by choosing $\rho(\iota) = \iota$, condition (2.18) reduces to

$$A_{1,m}Bb^{1-k}q_0 > \frac{1}{4}. \tag{2.27}$$

Remark 2.2. *Corollary 1 in [19] confirms that Eq (2.24) is oscillatory if*

$$q_0 > \frac{\beta}{b^{\alpha}(1-p_0)^{\alpha}}. \tag{H1}$$

Using Theorem 6 in [18], we get that (2.24) is oscillatory if

$$q_0 > \frac{\beta}{b^{\alpha\kappa}(1-p_0)^{\alpha}}, \tag{H2}$$

where $\kappa = (\alpha/(\alpha + (1-p_0)^{\alpha} q_0 b^{\alpha}))^{\alpha}$. Consider the special case of (2.24) when $p_0 = 1/2$, $a = 0.9$, and $\alpha = 1$, namely,

$$\frac{\mathrm{d}^2}{\mathrm{d}\iota^2}\left(x(\iota) + \frac{1}{2}x\left(\frac{9}{10}\iota\right)\right) + \frac{q_0}{\iota^2}x(b\iota) = 0. \tag{2.28}$$

It is easy to check that $A_{0,m} = 0.62696$, $k = 0.62696q_0b$, and

$$A_{1,m} = \left[2 - (0.9)^{k\,b^{-k}}\right]\sum_{i=0}^{m}(0.5)^{2i+1}(0.9)^{2(1-k)i}.$$

Applying conditions (2.25)–(2.27), we obtain that (2.28) is oscillatory if one of the following conditions is satisfied:

$$A_{1,m}q_0\left[\frac{b^{1-k}}{1-k} + \frac{b}{k}\left(b^{-k}-1\right) + \frac{b}{1-kb^{-k}}\right] > 1, \tag{H3}$$

$$A_{1,m}bq_0 \ln \frac{1}{b} > \frac{1-k}{\mathrm{e}}, \tag{H4}$$

or

$$A_{1,m}b^{1-k}q_0 > \frac{1}{4}. \tag{H5}$$

Figure 1 shows a comparison of regions where conditions (H1)–(H5) are satisfied for Eq (2.28). It is easy to see that our criterion (H5) provides the best results for the oscillation of (2.28). For example, we find that criterion (H5) ensures that the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\iota^2}\left(x(\iota) + \frac{1}{2}x\left(\frac{9}{10}\iota\right)\right) + \frac{1}{2\iota^2}x\left(\frac{8\iota}{10}\right) = 0,$$

is oscillatory, while the rest of the criteria fail to do so. Figure 2 shows the lower bounds of the regions in which condition (H5) is satisfied for $m = 0, 1, 5$.

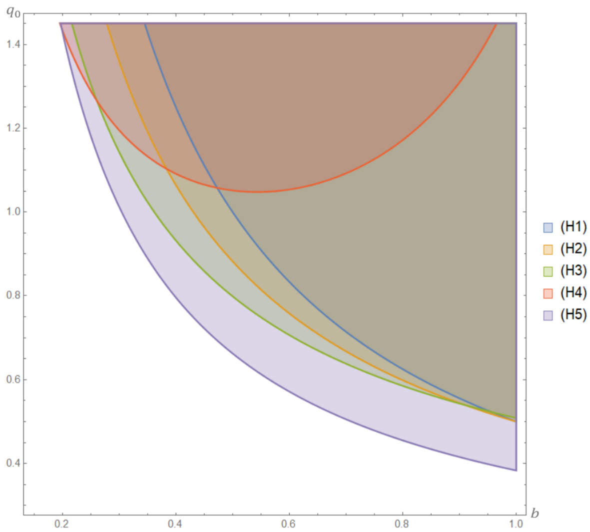

Figure 1. Comparison between the criteria (H1)–(H5) for $i = 0, 1, 2, 3$.

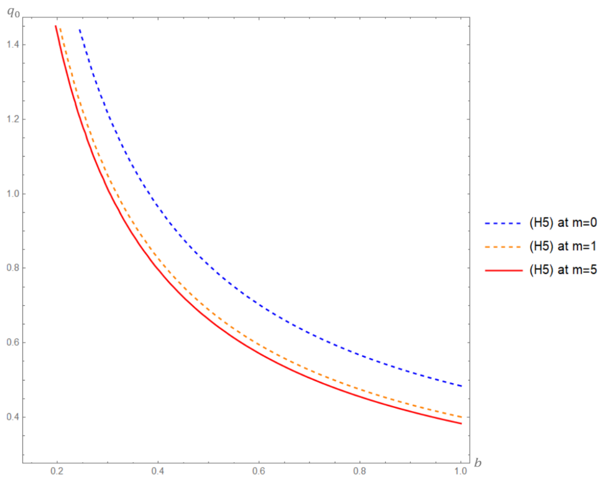

Figure 2. Lower bounds of the regions in which condition (H5) is satisfied for $m = 0, 1, 5$.

3. Oscillation results for the fourth-order equation

The following notations will be required when presenting the results: $\mathcal{F}_s$: The set of all eventually positive solutions of (1.2),

$$\overline{\mu}_s(\iota) := \int_s^{\iota} \mu_{\iota_1}(\mathrm{v})\,\mathrm{dv},$$

and

$$\varphi_c(\iota, m) := \left(\sum_{i=0}^{m}\left(\prod_{j=0}^{2i}\left(p\circ h_j\right)\right)\left[\frac{1}{(p\circ h_{2i})}-1\right]\left(\frac{h_{2i}}{\iota}\right)^{c}\right)^{\alpha}.$$

Lemma 3.1. *[38] If $H \in C^r([\iota_0,\infty),(0,\infty))$, $H^{(j)}(\iota) > 0$ for $j = 1, 2, ..., r$, and $H^{(r+1)}(\iota) \le 0$, then $H(\iota) \ge \frac{1}{r}\iota H'(\iota)$, eventually.*

The following lemma determines the sign of the derivatives of the positive solutions of (1.2), which comes directly from applying Lemma 2.2.1 in [35] to Eq (1.2).

Lemma 3.2. *The following properties are satisfied for each* $x \in \mathcal{F}_s$*:*

$$\begin{aligned} &\text{(i)} \quad z(\iota) > 0,\ z'(\iota) > 0,\ z'''(\iota) > 0 \textit{ and } \left(\varrho(\iota)(z'''(\iota))^{\alpha}\right)' \leq 0, \\ &\text{(ii)} \quad z'' \textit{ is of fixed sign.} \end{aligned}$$

Lemma 3.3. *Assume that* $x \in \mathcal{F}_s$*. If* $z''(\iota) > 0$*, eventually, then,*

$$\begin{aligned} &\text{(F1)} \quad z(\iota) \geq \frac{1}{3}\iota z'(\iota), \\ &\text{(F2)} \quad \left(\varrho(\iota)(z'''(\iota))^{\alpha}\right)' \leq -q(\iota)\varphi_3(g(\iota), m) z^{\alpha}(g(\iota)). \end{aligned}$$

On the other hand, if $z''(\iota) < 0$*, eventually, then*

$$\begin{aligned} &\text{(F3)} \quad z(\iota) \geq \epsilon\iota z'(\iota), \\ &\text{(F4)} \quad \left(\varrho(\iota)(z'''(\iota))^{\alpha}\right)' \leq -q(\iota)\varphi_{1/\epsilon}(g(\iota), m) z^{\alpha}(g(\iota)), \end{aligned}$$

for all $\epsilon \in (0,1)$.

Proof. Assume that $x \in \mathcal{F}_s$ and $z''(\iota) > 0$ for $\iota \geq \iota_1$. By using Lemma 3.1 with $H = z$ and $r = 3$, we get that $z \geq \frac{1}{3}\iota z'$. Next, proceeding as in the proof Lemma 2.2, we arrive at (2.3). Using the facts that $z'(\iota) > 0$ and (F1), we obtain $(z \circ h_{2i}) \geq (z \circ h_{2i+1})$ and

$$(z \circ h_{2i}) \geq \frac{h_{2i}^3}{\iota^3} z.$$

Then, (2.3) becomes

$$x > z \sum_{i=0}^{m} \left(\prod_{j=0}^{2i} \left(p \circ h_j\right) \right) \left[\frac{1}{(p \circ h_{2i})} - 1 \right] \left(\frac{h_{2i}}{\iota} \right)^3,$$

which together with (1.2) gives (F2).

Next, assume that $z''(\iota) < 0$ for $\iota \geq \iota_1$. Then, there is $\iota_2 > \iota_1$ such that

$$z(\iota) \geq \int_{\iota_1}^{\iota} z'(\mathrm{v})\, \mathrm{dv} \geq (\iota - \iota_1) z'(\iota) \geq \epsilon\iota z'(\iota),$$

for all $\iota \geq \iota_2$ and $\epsilon \in (0,1)$. Using the previous fact and $z'(\iota) > 0$, (2.3) reduces to

$$x > z \sum_{i=0}^{m} \left(\prod_{j=0}^{2i} \left(p \circ h_j\right) \right) \left[\frac{1}{(p \circ h_{2i})} - 1 \right] \left(\frac{h_{2i}}{\iota} \right)^{1/\epsilon},$$

which together with (1.2) gives (F4).

Here, the proof ends.

3.1. Oscillation criteria

Lemma 3.4. *Assume that $x \in \mathcal{F}_s$, and there is a $\rho \in C([\iota_0, \infty), (0, \infty))$ such that*

$$\limsup_{\iota\to\infty} \int_{\iota_1}^{\iota} \left(\rho(\mathrm{v}) q(\mathrm{v}) \varphi_3(g(\mathrm{v}), m) \left(\frac{g(\mathrm{v})}{\mathrm{v}} \right)^{3\alpha} - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\rho'(\mathrm{v}))^{\alpha+1}}{\rho^{\alpha}(\mathrm{v}) \overline{\mu}_{\iota_1}^{\alpha}(\mathrm{v})} \right) \mathrm{dv} = \infty. \tag{3.1}$$

Then $z''(\iota) < 0$, eventually.

Proof. Assume that $x \in \mathcal{F}_s$. From Lemma 3.2, we have that (i) and (ii) hold.

Suppose the contrary that $z''(\iota) > 0$ for $\iota \geq \iota_1$. Then, we find

$$z''(\iota) \geq \int_{\iota_1}^{\iota} \frac{\varrho^{1/\alpha}(\mathrm{v}) z'''(\mathrm{v})}{\varrho^{1/\alpha}(\mathrm{v})} \mathrm{dv} \geq \varrho^{1/\alpha}(\iota) z'''(\iota) \mu_{\iota_1}(\iota).$$

Hence, z''/μ_{ι_1} is decreasing, and so

$$z'(\iota) \geq \int_{\iota_1}^{\iota} \frac{z''(\mathrm{v})}{\mu_{\iota_1}(\mathrm{v})} \mu_{\iota_1}(\mathrm{v}) \mathrm{dv} \geq \frac{\overline{\mu}_{\iota_1}(\iota)}{\mu_{\iota_1}(\iota)} z''(\iota) \geq \overline{\mu}_{\iota_1}(\iota) \varrho^{1/\alpha}(\iota) z'''(\iota). \tag{3.2}$$

Moreover, from (F1), we get

$$\frac{z \circ g}{z} \geq \left(\frac{g}{\iota} \right)^3. \tag{3.3}$$

Now, we define the function

$$w := \rho \cdot \varrho \cdot \left(\frac{z'''}{z} \right)^{\alpha} > 0.$$

Then, from (F2), (3.2) and (3.3), we find

$$\begin{aligned} w' &= \frac{\rho'}{\rho} \cdot w + \rho \cdot \frac{(\varrho \cdot (z''')^{\alpha})'}{z^{\alpha}} - \alpha \rho \cdot \varrho \cdot \frac{(z''')^{\alpha}}{z^{\alpha+1}} z' \\ &\leq \frac{\rho'}{\rho} \cdot w - \rho \cdot q \cdot \varphi_3(g, m) \cdot \frac{z^{\alpha} \circ g}{z^{\alpha}} - \alpha \rho \cdot \varrho^{1+1/\alpha} \cdot \overline{\mu}_{\iota_1} \cdot \frac{(z''')^{\alpha+1}}{z^{\alpha+1}} \\ &\leq -\rho \cdot q \cdot \varphi_3(g, m) \cdot \left(\frac{g}{\iota} \right)^{3\alpha} + \frac{\rho'}{\rho} \cdot w - \alpha \frac{\overline{\mu}_{\iota_1}}{\rho^{1/\alpha}} \cdot w^{1+1/\alpha}. \end{aligned}$$

By using inequality (2.22), we obtain

$$w' \leq -\rho \cdot q \cdot \varphi_3(g, m) \cdot \left(\frac{g}{\iota} \right)^{3\alpha} + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\rho')^{\alpha+1}}{\rho^{\alpha} \cdot \overline{\mu}_{\iota_1}^{\alpha}}.$$

By integrating this inequality from ι_1 to ι, we conclude that

$$w(\iota_1) \geq \int_{\iota_1}^{\iota} \left(\rho(\mathrm{v}) q(\mathrm{v}) \varphi_3(g(\mathrm{v}), m) \left(\frac{g(\mathrm{v})}{\mathrm{v}} \right)^{3\alpha} - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\rho'(\mathrm{v}))^{\alpha+1}}{\rho^{\alpha}(\mathrm{v}) \overline{\mu}_{\iota_1}^{\alpha}(\mathrm{v})} \right) \mathrm{dv},$$

which contradicts (3.1).

Here, the proof ends.

Theorem 3.1. *Assume that* $g'(\iota) > 0$ *and there is a* $\rho \in C([\iota_0, \infty), (0, \infty))$ *such that (3.1) holds. Then, Eq (1.2) is oscillatory if the equation*

$$\left(\frac{1}{g'(\iota)} y'(\iota)\right)' + y(\iota) \int_{\iota}^{\infty} \frac{1}{\varrho^{1/\alpha}(v)} \left(\int_{v}^{\infty} q(\mathrm{v}) \varphi_{1/\epsilon}(g(\mathrm{v}), m) \, \mathrm{dv}\right)^{1/\alpha} \mathrm{d}v = 0 \tag{3.4}$$

is oscillatory.

Proof. Assume the contrary that $x \in \mathcal{F}_s$. From Lemma 3.2, we have that (H1) and (H2) hold. It follows from Lemma 3.4, $z''(\iota) < 0$, eventually. By integrating (F4) twice from ι to ∞, we conclude that

$$\frac{z''(\iota)}{z(g(\iota))} \leq - \int_{\iota}^{\infty} \frac{1}{\varrho^{1/\alpha}(v)} \left(\int_{v}^{\infty} q(\mathrm{v}) \varphi_{1/\epsilon}(g(\mathrm{v}), m) \, \mathrm{dv}\right)^{1/\alpha} \mathrm{d}v. \tag{3.5}$$

We define the function $\omega = z' / (z \circ g)$. Then, from (3.5), we find

$$\begin{aligned} \omega' &= \frac{z''}{z \circ g} - \frac{z'}{(z \circ g)^2} \cdot (z' \circ g) \cdot g' \\ &\leq -\int_{\iota}^{\infty} \frac{1}{\varrho^{1/\alpha}(v)} \left(\int_{v}^{\infty} q(\mathrm{v}) \varphi_{1/\epsilon}(g(\mathrm{v}), m) \, \mathrm{dv}\right)^{1/\alpha} \mathrm{d}v - \frac{(z')^2}{(z \circ g)^2} \cdot g' \\ &\leq -\int_{\iota}^{\infty} \frac{1}{\varrho^{1/\alpha}(v)} \left(\int_{v}^{\infty} q(\mathrm{v}) \varphi_{1/\epsilon}(g(\mathrm{v}), m) \, \mathrm{dv}\right)^{1/\alpha} \mathrm{d}v - g' \cdot \omega^2, \end{aligned}$$

and so,

$$\omega' + \int_{\iota}^{\infty} \frac{1}{\varrho^{1/\alpha}(v)} \left(\int_{v}^{\infty} q(\mathrm{v}) \varphi_{1/\epsilon}(g(\mathrm{v}), m) \, \mathrm{dv}\right)^{1/\alpha} \mathrm{d}v + g' \cdot \omega^2 \leq 0. \tag{3.6}$$

In view of [36, 37], Eq (3.4) has a non-oscillatory solution if and only if there exists a function ω satisfying (3.6), a contradiction.

Here, the proof ends.

3.2. Comparison with previous results

In the following, we review some theorems from previous studies that dealt with the oscillation of the NDE

$$\frac{\mathrm{d}^4}{\mathrm{d}\iota^4} z(\iota) + q(\iota) x(g(\iota)) = 0, \tag{3.7}$$

by using different techniques, so that we can compare our results with them.

Theorem 3.2. *[21, Theorem 2] Suppose that*

$$\liminf_{\iota \to \infty} \int_{g(\iota)}^{\iota} g^3(\mathrm{v}) (1 - p(g(\mathrm{v}))) q(\mathrm{v}) \, \mathrm{dv} > \frac{384}{\mathrm{e}}.$$

Then (3.7) is oscillatory.

Theorem 3.3. *[22, 23, Corollary 1] Suppose that*

$$\liminf_{\iota \to \infty} \int_{g(\iota)}^{\iota} g^3(\mathrm{v}) (1 - p(g(\mathrm{v}))) q(\mathrm{v}) \, \mathrm{dv} > \frac{6}{\mathrm{e}}.$$

Then (3.7) is oscillatory.

Theorem 3.4. *[24, Theorem 2] Suppose that there exists a $m \in \mathbb{Z}^+$ such that*

$$\liminf_{\iota\to\infty} G_m(\iota) > \frac{1}{\mathrm{e}^m},$$

where $\eta(\iota) = \max\{g(s),\ s \in [\iota_0, \iota]\}$, $\eta_{-1}(\iota) = \sup\{s \geq \iota_0 : \eta(s) = \iota\}$, $\eta_{-(i+1)} = \eta_{-1}(\eta_{-i}(\iota))$, $G(\iota) = \frac{1}{6}q(\iota)\,g^3(\iota)(1 - p(g(\iota)))$

$$G_1(\iota) = \int_{\eta(\iota)}^{\iota} G(\iota)\,\mathrm{dv},\ \iota \geq \eta_{-1}(\iota_0),$$

and

$$G_{i+1}(\iota) = \int_{\eta(\iota)}^{\iota} G(\mathrm{v})\,G_k(\mathrm{v})\,\mathrm{dv},\ \ \iota \geq \eta_{-(i+1)}(\iota_0),\ \textit{for } i = 1, 2, \ldots.$$

Then (3.7) is oscillatory.

Theorem 3.5. *[25, Theorem 2.1] Suppose that there exist* $\theta_1, \theta_2 \in \mathbf{C}^1([\iota_0, \infty), (0, \infty))$ *such that*

$$\int_{\iota_0}^{\infty}\left(\theta_1(\mathrm{v})\,q(\mathrm{v})(1 - p(g(\mathrm{v})))\frac{g^3(\mathrm{v})}{\mathrm{v}^3} - \frac{1}{2\epsilon}\frac{\left(\theta_1'(\mathrm{v})\right)^2}{\iota^2\theta_1(\mathrm{v})}\right)\mathrm{dv} = \infty,$$

and

$$\int_{\iota_0}^{\infty}\left(\theta_2(\mathrm{v})\left(\int_{\mathrm{v}}^{\infty}(s - \mathrm{v})\,q(s)(1 - p(g(s)))\frac{g(s)}{s}\mathrm{d}s\right) - \frac{\left(\theta_2'(\mathrm{v})\right)^2}{4\theta_2(\mathrm{v})}\right)\mathrm{dv} = \infty.$$

Then (3.7) is oscillatory.

Example 3.1. *Consider the NDE*

$$\frac{\mathrm{d}^4}{\mathrm{d}\iota^4}\left[x(\iota) + p_0 x(a\iota)\right] + \frac{q_0}{\iota^4}x(b\iota) = 0, \tag{3.8}$$

where $\iota > 0$, $p_0, a, b \in (0, 1)$, *and* $q_0 > 0$. *It is easy to check that*

$$\varphi_c(\iota, m) := [1 - p_0]\sum_{i=0}^{m} p_0^{2i} a^{2ci} := \mathrm{v}_c. \tag{3.9}$$

By choosing $\rho(\iota) = \iota^3$, *condition (3.1) reduce to*

$$\limsup_{\iota\to\infty}\int_{\iota_1}^{\iota}\left(q_0\mathrm{v}_3 b^3 - \frac{9}{2}\right)\frac{1}{\mathrm{v}}\mathrm{dv} = \infty,$$

which is satisfied if $q_0 L_3 b^3 > \frac{9}{2}$. *From Theorem 3.1, Eq (3.8) is oscillatory if the equation*

$$y''(\iota) + \frac{q_0 L_{1/\epsilon} b}{6\iota^2}y(\iota) = 0 \tag{3.10}$$

is oscillatory. Using Corollary 2.1, Eq (3.10) is oscillatory if $q_0 > 3/(2L_{1/\epsilon}b)$. *Therefore, Eq (3.8) is oscillatory if*

$$q_0 > \max\left\{\frac{9}{2L_3 b^3}, \frac{3}{2L_{1/\epsilon}b}\right\}. \tag{C1}$$

Remark 3.1. *Consider the NDE (3.8). By applying Theorems 3.2–3.5, we get respectively the following results:*

– *Eq (3.8) is oscillatory if*

$$q_0 > \frac{384}{eb^3(1-p_0)\ln(1/b)}; \tag{C2}$$

– *Eq (3.8) is oscillatory if*

$$q_0 > \frac{6}{eb^3(1-p_0)\ln(1/b)}; \tag{C3}$$

– *We have* $\eta(\iota) = b\iota$, $G(\iota) = q_0 b^3(1-p_0)\frac{1}{6\iota}$ *and*

$$G_i(\iota) = \left(\frac{1}{6}q_0 b^3(1-p_0)\ln\frac{1}{b}\right)^i, \text{ for } i = 1,2,...;$$

Then Eq (3.8) is oscillatory if

$$q_0 > \frac{6}{eb^3(1-p_0)\ln(1/b)};$$

– *By choosing* $\theta_1(\iota) = \iota^3$ *and* $\theta_2(\iota) = \iota$, *Eq (3.8) is oscillatory if*

$$q_0 > \max\left\{\frac{9}{2b^3(1-p_0)}, \frac{3}{2b(1-p_0)}\right\}. \tag{C4}$$

From the aforementioned, we observe that

(1) *Since* $L_c \geq (1-p_0)$, *criterion (C1) is an improvement of (C4).*
(2) *Criterion (C3) is an improvement of (C2).*
(3) *The results of Theorem 3.4 are the same as those of Theorem 3.3, although Theorem 3.3 is easier to apply.*
(4) *Setting* $a = 0.9$ *and* $p = 0.8$, *Figure 3 shows the lower bounds of* q_0 *values at which criteria (C1), (C3) and (C4) are satisfied. We note that (C3) provides the best results for the oscillation of (3.8) when* $b \in (0, 0.476)$, *and (C1) provides the best results for the oscillation of (3.8) when* $b \in (0.476, 1)$.

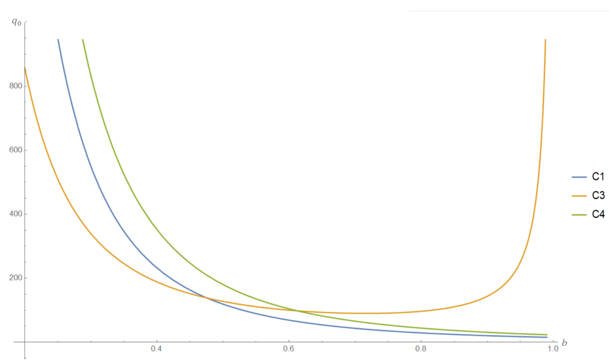

Figure 3. Lower bounds of q_0 values at which criteria (C1), (C3) and (C4) are satisfied.

4. Conclusions

In this work, the oscillatory behavior of second- and fourth-order half-linear neutral differential equations is studied in the canonical case. For the second-order equation, we obtained improved monotonic properties based on establishing a new relationship between the solution and its corresponding function. We then used the new relationships and properties to infer a set of oscillation criteria by using different methods. At the end of this part of the paper, we presented examples and remarks that illustrate the importance of the results and compare them with relevant results in the literature. For the fourth-order equation, after obtaining new relationships between x and z in each case of positive solutions, we introduced a new criterion to test the oscillation of the studied equation. Then, we reviewed some previous theorems in the literature and compared our results with them using an example.

We notice through the results that improving the relationship between the solution and the corresponding function of the neutral differential equations contributes to obtaining new monotonic properties for the positive solutions of these equations, which in turn leads to the development of oscillation criteria. It would be interesting to extend this improvement to higher-order differential equations in the non-canonical case.

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Conflict of interest

The authors declare no conflict of interest.

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