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Research article

On a fuzzy bipolar metric setting with a triangular property and an application on integral equations

Gunaseelan Mani[1], Arul Joseph Gnanaprakasam[2], Khalil Javed[3], Eskandar Ameer[4], Saber Mansour[5], Hassen Aydi[6,7,8,*] and Wajdi Kallel[5]

[1] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602 105, Tamil Nadu, India

[2] Department of Mathematics, College of Engineering and Technology, Faculty of Engineering and Tevhnology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur, Kanchipuram, Chennai, Tamil Nadu 603203, India

[3] Department of Mathematics and Statistics, International Islamic University Islamabad, Pakistan

[4] Department of Mathematics, Taiz University, Taiz, Yeman

[5] Department of Mathematics, Umm Al-Qura University, Faculty of Applied sciences, P.O. Box 14035, Holly Makkah 21955, Saudi Arabia; samansour@uqu.edu.sa; wfkallel@uqu.edu.sa

[6] Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia

[7] China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

[8] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa

* **Correspondence:** Email: hassen.aydi@isima.rnu.tn.

Abstract: In this manuscript, fixed point results without continuity via triangular notion on fuzzy bipolar metric spaces are established. The paper includes tangible examples which display the motivation for such investigations as those presented here. We solve an integral equation in this setting. The present work is a generalization of some published works.

Keywords: fixed point; fuzzy metric spaces; triangular notion; fuzzy bipolar metric space
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

In 1960, Schweizer et al. [1] first initiated the concept of a continuous t-norm. In 1965, Zadeh [2] introduced a fuzzy set and its properties. After that, Kramosil et al. [3] established the fuzzy sets into fuzzy metric spaces in 1975 using the concept of continuous t-norms. In 1994, George et al. [6] introduced the modified version of fuzzy metric spaces. Grabeic [4] established and improved the well-known Banach's fixed point theorem (FPT) to fuzzy metric spaces in the visual of Karamosil et al. [3]. Following that, Gregori et al. [5] gave an extension of the Banach contraction theorem using a fuzzy metric space in the same way as George et al. [6], Mutlu et al. [7] generalized a metric space, also known as a bipolar metric space. Bartwal et al. [8] proposed and proved FPTs using the fuzzy bipolar metric space (FBMS). Very recently, Shamas et al. [9] demonstrated fixed-point results without continuity in the setting of fuzzy metric spaces using the triangular property. In this paper, we will use the triangular property on fuzzy bipolar metric spaces to prove fixed point theorems without continuity. For further fixed point results using fuzzy setting, see the works [10–12].

Now, let's include some basic definitions and useful lemmas.

Definition 1.1. [13] A triangular norm (shortly, ł-norm) is a binary operation on the unit interval $[0, 1]$, i.e., a function $* : [0, 1] \times [0, 1] \to [0, 1]$ such that for all $a, b, c \in [0, 1]$, the following four axioms are satisfied:
$(Tn1)$ $a * (b * c) = (a * b) * c$ (associativity);
$(Tn2)$ $a * b = b * a$ (commutativity);
$(Tn3)$ $a * 1 = a$ (boundary condition);
$(Tn4)$ $a * b \le a * c$, whenever $b \le c$ (monotonicity).

Definition 1.2. [6] The 3-tuple $(X, \mu, *)$ is a fuzzy metric space if X is an arbitrary set, $*$ is a t-norm and $\mu : X^2 \times (0, \infty) \to [0, 1]$ is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, \kappa \in X$ and $r, t > 0$:
$(f1)$ $\mu(x, y, t) > 0$;
$(f2)$ $\mu(x, y, t) = 1 \Leftrightarrow x = y$;
$(f3)$ $\mu(x, y, t) = \mu(y, x, t)$;
$(f4)$ $\mu(x, y, t) * \mu(y, \kappa, r) \le \mu(x, \kappa, t + r)$;
$(f5)$ $\mu(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Definition 1.3. [8] Let $\Phi \ne \emptyset$ and $\Psi \ne \emptyset$. A quadruple $(\Phi, \Psi, \Gamma_\beta, *)$ is called a FBMS, where $*$ is a continuous ł-norm and Γ_β is a fuzzy set on $\Phi \times \Psi \times (0, \infty)$, if for all $ł, \rho, \delta > 0$:

(1) $\Gamma_\beta(\sigma, \mu, ł) > 0$ for all $(\sigma, \mu) \in \Phi \times \Psi$;

(2) $\Gamma_\beta(\sigma, \mu, ł) = 1$ iff $\sigma = \mu$ for $\sigma \in \Phi$ and $\mu \in \Psi$;

(3) $\Gamma_\beta(\sigma, \mu, ł) = \Gamma_\beta(\mu, \sigma, ł)$ for all $\sigma, \mu \in \Phi \cap \Psi$;

(4) $\Gamma_\beta(\sigma_1, \mu_2, ł + \rho + \delta) \ge \Gamma_\beta(\sigma_1, \mu_1, ł) * \Gamma_\beta(\sigma_2, \mu_1, \rho) * \Gamma_\beta(\sigma_2, \mu_2, \delta)$ for all $\sigma_1, \sigma_2 \in \Phi$ and $\mu_1, \mu_2 \in \Psi$;

(5) $\Gamma_\beta(\sigma, \mu, .) : [0, \infty) \longrightarrow [0, 1]$ is left continuous;

(6) $\Gamma_\beta(\sigma, \mu, .)$ is non-decreasing for all $\sigma \in \Phi$ and $\mu \in \Psi$.

Definition 1.4. [8] Let $(\Phi, \Psi, \Gamma_\beta, *)$ be a FBMS.

(1) An element $\sigma \in \Phi \cup \Psi$ is referred to as a left point if $\sigma \in \Phi$, a right point if $\sigma \in \Psi$, and a central point if both are satisfied. Similarly, a sequence $\{\sigma_\gamma\}$ on Φ and a sequence $\{\mu_\gamma\}$ on Ψ are referred to as left and right sequences, respectively;

(2) A sequence $\{\sigma_\gamma\}$ is convergent to a point σ iff ($\{\sigma_\gamma\}$ is a left sequence, σ is a right point and $\lim\limits_{\gamma\to\infty} \Gamma_\beta(\sigma_\gamma, \sigma, \mathfrak{l}) = 1$ for all $\mathfrak{l} > 0$) or ($\{\sigma_\gamma\}$ is a right sequence, σ is a left point and $\lim\limits_{\gamma\to\infty} \Gamma_\beta(\sigma, \sigma_\gamma, \mathfrak{l}) = 1$ for all $\mathfrak{l} > 0$);

(3) A bisequence $(\{\sigma_\gamma\}, \{\mu_\gamma\})$ is a sequence on the set $\Phi \times \Psi$. If the sequences $\{\sigma_\gamma\}$ and $\{\mu_\gamma\}$ are convergent, then the bisequence $(\{\sigma_\gamma\}, \{\mu_\gamma\})$ is said to be convergent. $(\{\sigma_\gamma\}, \{\mu_\gamma\})$ is a Cauchy bisequence, if $\lim\limits_{\gamma,\mathfrak{m}\to\infty} \Gamma_\beta(\sigma_\gamma, \mu_\mathfrak{m}, \mathfrak{l}) = 1$ for all $\mathfrak{l} > 0$;

(4) $(\Phi, \Psi, \Gamma_\beta, *)$ is complete, if every Cauchy bisequence is convergent in $\Phi \times \Psi$.

Lemma 1.5. *[8] Let $(\Phi, \Psi, \Gamma_\beta, *)$ be a FBMS and $\mathfrak{r} \in \Phi \cap \Psi$ be a limit of a sequence, then it is its unique limit.*

Definition 1.6. Let $(\Phi, \Psi, \Gamma_\beta, *)$ be a FBMS. The FBM Γ_β is triangular if

$$\frac{1}{\Gamma_\beta(\sigma_1, \mu_2, \mathfrak{l})} - 1 \le \left(\frac{1}{\Gamma_\beta(\sigma_1, \mu_1, \mathfrak{l})} - 1\right) + \left(\frac{1}{\Gamma_\beta(\sigma_2, \mu_1, \mathfrak{l})} - 1\right) + \left(\frac{1}{\Gamma_\beta(\sigma_2, \mu_2, \mathfrak{l})} - 1\right).$$

Example 1.7. Let $(\Phi, \Psi, |.|)$ be a bipolar metric space (where Φ and Ψ are subsets of the real number). Let $\Gamma_\beta : \Phi \times \Psi \times (0, \infty) \to [0, 1]$ be a FBM defined by

$$\Gamma_\beta(\sigma, \mu, \mathfrak{l}) = \frac{\mathfrak{l}}{\mathfrak{l} + |\sigma - \mu|}$$

for all $\mathfrak{l} > 0$, $\sigma \in \Phi$ and $\mu \in \Psi$. The FBM Γ_β is triangular.

Proof. We have for all $\mathfrak{l} > 0$, $\sigma \in \Phi$ and $\mu \in \Psi$,

$$\begin{aligned}
\frac{1}{\Gamma_\beta(\sigma_1, \mu_2, \mathfrak{l})} - 1 &= \frac{|\sigma_1 - \mu_2|}{\mathfrak{l}} = \frac{|\sigma_1 - \mu_1 + \mu_1 - \sigma_2 + \sigma_2 - \mu_2|}{\mathfrak{l}} \\
&\le \frac{|\sigma_1 - \mu_1|}{\mathfrak{l}} + \frac{|\sigma_2 - \mu_1|}{\mathfrak{l}} + \frac{|\sigma_2 - \mu_2|}{\mathfrak{l}} \\
&= \left(\frac{1}{\Gamma_\beta(\sigma_1, \mu_1, \mathfrak{l})} - 1\right) + \left(\frac{1}{\Gamma_\beta(\sigma_2, \mu_1, \mathfrak{l})} - 1\right) \\
&+ \left(\frac{1}{\Gamma_\beta(\sigma_2, \mu_2, \mathfrak{l})} - 1\right),
\end{aligned}$$

which implies that

$$\frac{1}{\Gamma_\beta(\sigma_1,\mu_2,\mathfrak{l})}-1 \le \Big(\frac{1}{\Gamma_\beta(\sigma_1,\mu_1,\mathfrak{l})}-1\Big)+\Big(\frac{1}{\Gamma_\beta(\sigma_2,\mu_1,\mathfrak{l})}-1\Big) + \Big(\frac{1}{\Gamma_\beta(\sigma_2,\mu_2,\mathfrak{l})}-1\Big), \text{ for all } \mathfrak{l}>0.$$

Hence, the FBM Γ_β is triangular. □

Motivated by Shamas et al. [9], we demonstrate FPTs omitting continuity and using triangular property on FBMSs with an application.

2. Main results

In this section, we use the triangular property to prove FPTs on FBMSs without continuity. The following result investigates for fixed points of maps Υ satisfying $\Upsilon(\Phi)\subseteq\Phi$ and $\Upsilon(\Psi)\subseteq\Psi$. These maps are referred in the literature as noncyclic maps, introduced by Fernandez-Leon and Gabeleh [14].

Theorem 2.1. *Let $(\Phi,\Psi,\Gamma_\beta,*)$ be a complete FBMS and the mapping $\Upsilon:\Phi\cup\Psi\to\Phi\cup\Psi$ be such that*

(1) $\Upsilon(\Phi)\subseteq\Phi$ and $\Upsilon(\Psi)\subseteq\Psi$ (i.e., the map Υ is noncyclic);

(2) $\frac{1}{\Gamma_\beta(\Upsilon(\sigma),\Upsilon(\mu),\mathfrak{l})}-1\le \Bbbk\Big(\frac{1}{\Gamma_\beta(\sigma,\mu,\mathfrak{l})}-1\Big)$, for all $\mathfrak{l}>0$, where $\Bbbk\in(0,1)$;

(3) Γ_β is triangular.

Then Υ has a UFP (unique fixed point).

Proof. Fix $\sigma_0\in\Phi$ and $\mu_0\in\Psi$. Assume that $\Upsilon(\sigma_\gamma)=\sigma_{\gamma+g}$ and $\Upsilon(\mu_\gamma)=\mu_{\gamma+g}$ for all $\gamma\in\mathbb{N}\cup\{0\}$. Then

$$\begin{aligned}\frac{1}{\Gamma_\beta(\sigma_{\gamma+1},\mu_{\gamma+1},\mathfrak{l})}-1 &= \frac{1}{\Gamma_\beta(\Upsilon(\sigma_\gamma),\Upsilon(\mu_\gamma),\mathfrak{l})}-1\\ &\le \Bbbk\Big(\frac{1}{\Gamma_\beta(\sigma_\gamma,\mu_\gamma,\mathfrak{l})}-1\Big)\\ &= \Bbbk\Big(\frac{1}{\Gamma_\beta(\Upsilon\sigma_{\gamma-1},\Upsilon\mu_{\gamma-1},\mathfrak{l})}-1\Big)\\ &\vdots\\ &\le \Bbbk^\gamma\Big(\frac{1}{\Gamma_\beta(\sigma_0,\mu_0,\mathfrak{l})}-1\Big).\end{aligned}$$

As $\gamma\to\infty$, we derive that

$$\lim_{\gamma\to\infty}\Gamma_\beta(\sigma_\gamma,\mu_\gamma,\mathfrak{l})=1, \text{ for } \mathfrak{l}>0.$$

We have

$$
\begin{aligned}
\frac{1}{\Gamma_\beta(\sigma_{\gamma+1},\mu_\gamma,\mathfrak{l})}-1 &= \frac{1}{\Gamma_\beta(\Upsilon(\sigma_\gamma),\Upsilon(\mu_{\gamma-1}),\mathfrak{l})}-1\\
&\le \Bbbk\Big(\frac{1}{\Gamma_\beta(\sigma_\gamma,\mu_{\gamma-1},\mathfrak{l})}-1\Big)\\
&= \Bbbk\Big(\frac{1}{\Gamma_\beta(\Upsilon\sigma_{\gamma-1},\Upsilon\mu_{\gamma-2},\mathfrak{l})}-1\Big)\\
&\vdots\\
&\le \Bbbk^\gamma\Big(\frac{1}{\Gamma_\beta(\sigma_1,\mu_0,\mathfrak{l})}-1\Big).
\end{aligned}
$$

As $\gamma \to \infty$, we derive that

$$
\lim_{\gamma\to\infty}\Gamma_\beta(\sigma_{\gamma+1},\mu_\gamma,\mathfrak{l}) = 1, \text{ for all } \mathfrak{l} > 0.
$$

Let $\gamma, \mathfrak{m} \in \mathbb{N}$ with $\gamma < \mathfrak{m}$. Then by Γ_β is triangular, we get

$$
\begin{aligned}
\frac{1}{\Gamma_\beta(\sigma_\gamma,\mu_\mathfrak{m},\mathfrak{l})}-1 \le& \Big(\frac{1}{\Gamma_\beta(\sigma_\gamma,\mu_\gamma,\mathfrak{l})}-1\Big)+\Big(\frac{1}{\Gamma_\beta(\sigma_{\gamma+1},\mu_\gamma,\mathfrak{l})}-1\Big)\\
&+\Big(\frac{1}{\Gamma_\beta(\sigma_{\gamma+1},\mu_\mathfrak{m},\mathfrak{l})}-1\Big)\\
&\vdots\\
&\le \Big(\frac{1}{\Gamma_\beta(\sigma_\gamma,\mu_\gamma,\mathfrak{l})}-1\Big)+\Big(\frac{1}{\Gamma_\beta(\sigma_{\gamma+1},\mu_\gamma,\mathfrak{l})}-1\Big)\\
&+\cdots+\Big(\frac{1}{\Gamma_\beta(\sigma_{\mathfrak{m}-1},\mu_{\mathfrak{m}-1},\mathfrak{l})}-1\Big)\\
&+\Big(\frac{1}{\Gamma_\beta(\sigma_\mathfrak{m},\mu_{\mathfrak{m}-1},\mathfrak{l})}-1\Big)+\Big(\frac{1}{\Gamma_\beta(\sigma_\mathfrak{m},\mu_\mathfrak{m},\mathfrak{l})}-1\Big)\\
&\le \Bbbk^\gamma\Big(\frac{1}{\Gamma_\beta(\sigma_0,\mu_0,\mathfrak{l})}-1\Big)+\Bbbk^\gamma\Big(\frac{1}{\Gamma_\beta(\sigma_1,\mu_0,\mathfrak{l})}-1\Big)\\
&+\cdots+\Bbbk^{m-1}\Big(\frac{1}{\Gamma_\beta(\sigma_0,\mu_0,\mathfrak{l})}-1\Big)+\Bbbk^\mathfrak{m}\Big(\frac{1}{\Gamma_\beta(\sigma_1,\mu_0,\mathfrak{l})}-1\Big)\\
&+\Bbbk^\mathfrak{m}\Big(\frac{1}{\Gamma_\beta(\sigma_0,\mu_0,\mathfrak{l})}-1\Big)\\
&\le \Bbbk^\gamma(1+\Bbbk+\Bbbk^2+\cdots+\Bbbk^{\mathfrak{m}-\gamma})\Big(\frac{1}{\Gamma_\beta(\sigma_0,\mu_0,\mathfrak{l})}-1\Big)\\
&+\Bbbk^\gamma(1+\Bbbk+\Bbbk^2+\cdots+\Bbbk^{\mathfrak{m}-\gamma})\Big(\frac{1}{\Gamma_\beta(\sigma_1,\mu_0,\mathfrak{l})}-1\Big)\\
&\le \frac{\Bbbk^\gamma}{1-\Bbbk}\Big(\frac{1}{\Gamma_\beta(\sigma_0,\mu_0,\mathfrak{l})}-1\Big)+\frac{\Bbbk^\gamma}{1-\Bbbk}\Big(\frac{1}{\Gamma_\beta(\sigma_1,\mu_0,\mathfrak{l})}-1\Big).
\end{aligned}
$$

As $\gamma, \mathfrak{m} \to \infty$, we get

$$\lim_{\gamma, \mathfrak{m} \to \infty} \Gamma_\beta(\sigma_\gamma, \mu_{\mathfrak{m}}, \mathfrak{l}) = 1, \text{ for } \mathfrak{l} > 0.$$

Thus, $(\{\sigma_\gamma\}, \{\mu_\gamma\})$ is a Cauchy bisequence. Since $(\Phi, \Psi, \Gamma_\beta, *)$ is complete, $(\{\sigma_\gamma\}, \{\mu_\gamma\})$ is a convergent bisequence. We know that the bisequence $(\{\sigma_\gamma\}, \{\mu_\gamma\})$ is biconvergent, so $\{\sigma_\gamma\} \to \mathfrak{r}$ and $\{\mu_\gamma\} \to \mathfrak{r}$ for all $\mathfrak{r} \in \Phi \cap \Psi$. By Lemma 1.5, both sequences $\{\sigma_\gamma\}$ and $\{\mu_\gamma\}$ have a unique limit.

Next, we prove that $\mathfrak{r} \in \Phi \cap \Psi$ is a fixed point of Υ. Since Γ_β is triangular, we derive that

$$\begin{aligned}\frac{1}{\Gamma_\beta(\Upsilon(\mathfrak{r}), \mathfrak{r}, \mathfrak{l})} - 1 &\leq \left(\frac{1}{\Gamma_\beta(\Upsilon(\mathfrak{r}), \Upsilon(\mu_\gamma), \mathfrak{l})} - 1\right) + \left(\frac{1}{\Gamma_\beta(\Upsilon(\sigma_\gamma), \Upsilon(\mu_\gamma), \mathfrak{l})} - 1\right) \\ &+ \left(\frac{1}{\Gamma_\beta(\Upsilon(\sigma_\gamma), \mathfrak{r}, \mathfrak{l})} - 1\right) \\ &\leq \Bbbk\left(\frac{1}{\Gamma_\beta(\mathfrak{r}, \mu_\gamma, \mathfrak{l})} - 1\right) + \Bbbk\left(\frac{1}{\Gamma_\beta(\sigma_\gamma, \mu_\gamma, \mathfrak{l})} - 1\right) \\ &+ \left(\frac{1}{\Gamma_\beta(\sigma_{\gamma+1}, \mathfrak{r}, \mathfrak{l})} - 1\right).\end{aligned}$$

As $\gamma \to \infty$, the three right-hand terms go to zero. We deduce that

$$\Gamma_\beta(\Upsilon(\mathfrak{r}), \mathfrak{r}, \mathfrak{l}) = 1.$$

Therefore, $\Upsilon(\mathfrak{r}) = \mathfrak{r}$. Let $\mathfrak{v} \in \Phi \cap \Psi$ be another fixed point of Υ. Then

$$\frac{1}{\Gamma_\beta(\mathfrak{r}, \mathfrak{v}, \mathfrak{l})} - 1 = \frac{1}{\Gamma_\beta(\Upsilon(\mathfrak{r}), \Upsilon(\mathfrak{v}), \mathfrak{l})} - 1 \leq \Bbbk\left(\frac{1}{\Gamma_\beta(\mathfrak{r}, \mathfrak{v}, \mathfrak{l})} - 1\right).$$

As $\Bbbk \in (0, 1)$, therefore

$$\Gamma_\beta(\mathfrak{r}, \mathfrak{v}, \mathfrak{l}) = 1.$$

Hence, $\mathfrak{r} = \mathfrak{v}$. □

Example 2.2. Let $\Phi = [0, 1]$ and $\Psi = \{0\} \cup \mathbb{N} - \{1\}$ be equipped with a continuous $\mathfrak{l}$-norm. Define $\Gamma_\beta(\mathfrak{r}, \mathfrak{v}, \mathfrak{l}) = \frac{\mathfrak{l}}{\mathfrak{l}+|\sigma-\mu|}$, for all $\mathfrak{l} > 0$, $\sigma \in \Phi$ and $\mu \in \Psi$. Clearly, $(\Phi, \Psi, \Gamma_\beta, *)$ is a complete FBMS. Note that Γ_β is triangular. Define $\Upsilon : \Phi \cup \Psi \to \Phi \cup \Psi$ by

$$\Upsilon(\mathfrak{r}) = \begin{cases} \frac{\mathfrak{r}}{4}, & \text{if } \mathfrak{r} \in [0, 1], \\ 0, & \text{if } \mathfrak{r} \in \mathbb{N} - \{1\}. \end{cases}$$

Then,

$$\begin{aligned}\frac{1}{\Gamma_\beta(\Upsilon\sigma, \Upsilon\mu, \mathfrak{l})} - 1 &= \frac{|\Upsilon\sigma - \Upsilon\mu|}{\mathfrak{l}} \\ &= \frac{|\sigma - \mu|}{4\mathfrak{l}} \\ &\leq \frac{|\sigma - \mu|}{2\mathfrak{l}}\end{aligned}$$

$$= \frac{1}{2}\Big(\frac{1}{\Gamma_\beta(\sigma,\mu,\mathfrak{l})} - 1\Big).$$

Therefore, all the conditions of Theorem 2.1 are fulfilled with $\mathbb{k} = \frac{1}{2} \in (0,1)$. Hence, Υ has a UFP, i.e.,$\sigma = 0$.

The notion of cyclic maps was first introduced in by Kirk el al. [15]. Later, the notion for cyclic maps and fixed points for cyclic maps have been further developed by Eldred and Veeramani [16] by introducing the notion of best proximity points.

Theorem 2.3. *Let $(\Phi, \Psi, \Gamma_\beta, *)$ be a complete FBMS and the mapping $\Upsilon : \Phi \cup \Psi \to \Phi \cup \Psi$ be such that*

(1) $\Upsilon(\Phi) \subseteq \Psi$ and $\Upsilon(\Psi) \subseteq \Phi$ (i.e. the map Υ is cyclic);

(2) $\frac{1}{\Gamma_\beta(\Upsilon(\sigma),\Upsilon(\mu),\mathfrak{l})} - 1 \leq \mathbb{k}\Big(\frac{1}{\Gamma_\beta(\sigma,\mu,\mathfrak{l})} - 1\Big)$, for all $\mathfrak{l} > 0$, where $\mathbb{k} \in (0,1)$;

(3) Γ_β is triangular.

Then Υ has a UFP.

Proof. Fix $\sigma_0 \in \Phi$. Assume that $\Upsilon(\sigma_\gamma) = \mu_\gamma$ and $\Upsilon(\mu_\gamma) = \sigma_{\gamma+1}$ for all $\gamma \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
\frac{1}{\Gamma_\beta(\sigma_\gamma,\mu_\gamma,\mathfrak{l})} - 1 &= \frac{1}{\Gamma_\beta(\Upsilon(\mu_{\gamma-1}),\Upsilon(\sigma_\gamma),\mathfrak{l})} - 1 \\
&\leq \mathbb{k}\Big(\frac{1}{\Gamma_\beta(\sigma_\gamma,\mu_{\gamma-1},\mathfrak{l})} - 1\Big) \\
&= \mathbb{k}\Big(\frac{1}{\Gamma_\beta(\Upsilon\mu_{\gamma-1},\Upsilon\sigma_{\gamma-1},\mathfrak{l})} - 1\Big) \\
&\vdots \\
&\leq \mathbb{k}^{2\gamma}\Big(\frac{1}{\Gamma_\beta(\sigma_0,\mu_0,\mathfrak{l})} - 1\Big).
\end{aligned}$$

As $\gamma \to \infty$, we derive that

$$\lim_{\gamma\to\infty} \Gamma_\beta(\sigma_\gamma,\mu_\gamma,\mathfrak{l}) = 1, \text{ for } \mathfrak{l} > 0.$$

$$\begin{aligned}
\frac{1}{\Gamma_\beta(\sigma_{\gamma+1},\mu_\gamma,\mathfrak{l})} - 1 &= \frac{1}{\Gamma_\beta(\Upsilon(\mu_\gamma),\Upsilon(\sigma_\gamma),\mathfrak{l})} - 1 \\
&\leq \mathbb{k}\Big(\frac{1}{\Gamma_\beta(\sigma_\gamma,\mu_{\gamma-1},\mathfrak{l})} - 1\Big) \\
&= \mathbb{k}\Big(\frac{1}{\Gamma_\beta(\Upsilon\mu_{\gamma-1},\Upsilon\sigma_{\gamma-1},\mathfrak{l})} - 1\Big) \\
&\vdots \\
&\leq \mathbb{k}^{2\gamma+1}\Big(\frac{1}{\Gamma_\beta(\sigma_0,\mu_0,\mathfrak{l})} - 1\Big).
\end{aligned}$$

Again,

$$\lim_{\gamma\to\infty} \Gamma_\beta(\sigma_{\gamma+1}, \mu_\gamma, \mathfrak{l}) = 1, \text{ for } \mathfrak{l} > 0.$$

Let $\gamma, \mathfrak{m} \in \mathbb{N}$ with $\gamma < \mathfrak{m}$. Since Γ_β is triangular, we get

$$\begin{aligned}
\frac{1}{\Gamma_\beta(\sigma_\gamma, \mu_\mathfrak{m}, \mathfrak{l})} - 1 \leq & \Big(\frac{1}{\Gamma_\beta(\sigma_\gamma, \mu_\gamma, \mathfrak{l})} - 1\Big) + \Big(\frac{1}{\Gamma_\beta(\sigma_{\gamma+1}, \mu_\gamma, \mathfrak{l})} - 1\Big) \\
& + \Big(\frac{1}{\Gamma_\beta(\sigma_{\gamma+1}, \mu_\mathfrak{m}, \mathfrak{l})} - 1\Big) \\
& \vdots \\
& \leq \Big(\frac{1}{\Gamma_\beta(\sigma_\gamma, \mu_\gamma, \mathfrak{l})} - 1\Big) + \Big(\frac{1}{\Gamma_\beta(\sigma_{\gamma+1}, \mu_\gamma, \mathfrak{l})} - 1\Big) \\
& + \cdots + \Big(\frac{1}{\Gamma_\beta(\sigma_{\mathfrak{m}-1}, \mu_{\mathfrak{m}-1}, \mathfrak{l})} - 1\Big) \\
& + \Big(\frac{1}{\Gamma_\beta(\sigma_\mathfrak{m}, \mu_{\mathfrak{m}-1}, \mathfrak{l})} - 1\Big) + \Big(\frac{1}{\Gamma_\beta(\sigma_\mathfrak{m}, \mu_\mathfrak{m}, \mathfrak{l})} - 1\Big) \\
& \leq \Bbbk^{2\gamma}\Big(\frac{1}{\Gamma_\beta(\sigma_0, \mu_0, \mathfrak{l})} - 1\Big) + \Bbbk^{2\gamma+1}\Big(\frac{1}{\Gamma_\beta(\sigma_0, \mu_0, \mathfrak{l})} - 1\Big) \\
& + \cdots + \Bbbk^{2m-2}\Big(\frac{1}{\Gamma_\beta(\sigma_0, \mu_0, \mathfrak{l})} - 1\Big) + \Bbbk^{2\mathfrak{m}-1}\Big(\frac{1}{\Gamma_\beta(\sigma_0, \mu_0, \mathfrak{l})} - 1\Big) \\
& + \Bbbk^{2\mathfrak{m}}\Big(\frac{1}{\Gamma_\beta(\sigma_0, \mu_0, \mathfrak{l})} - 1\Big) \\
& \leq \Bbbk^{2\gamma}(1 + \Bbbk + \Bbbk^2 + \cdots + \Bbbk^{2\mathfrak{m}-2\gamma})\Big(\frac{1}{\Gamma_\beta(\sigma_0, \mu_0, \mathfrak{l})} - 1\Big) \\
& \leq \frac{\Bbbk^{2\gamma}}{1-\Bbbk}\Big(\frac{1}{\Gamma_\beta(\sigma_0, \mu_0, \mathfrak{l})} - 1\Big).
\end{aligned}$$

As $\gamma, \mathfrak{m} \to \infty$, we get

$$\lim_{\gamma,\mathfrak{m}\to\infty} \Gamma_\beta(\sigma_\gamma, \mu_\mathfrak{m}, \mathfrak{l}) = 1, \text{ for } \mathfrak{l} > 0.$$

Thus, $(\{\sigma_\gamma\}, \{\mu_\gamma\})$ is a Cauchy bisequence. Since $(\Phi, \Psi, \Gamma_\beta, *)$ is complete, $(\{\sigma_\gamma\}, \{\mu_\gamma\})$ is a convergent bisequence. Since the bisequence $(\{\sigma_\gamma\}, \{\mu_\gamma\})$ is biconvergent sequence, $\{\sigma_\gamma\} \to \mathfrak{r}$ and $\{\mu_\gamma\} \to \mathfrak{r}$ for all $\mathfrak{r} \in \Phi \cap \Psi$. By Lemma 1.5, both sequences $\{\sigma_\gamma\}$ and $\{\mu_\gamma\}$ have a unique limit.

Next, we prove that $\mathfrak{r} \in \Phi \cap \Psi$ is a fixed point of Υ. Since Γ_β is triangular, we derive that

$$\begin{aligned}
\frac{1}{\Gamma_\beta(\Upsilon(\mathfrak{r}), \mathfrak{r}, \mathfrak{l})} - 1 \leq & \Big(\frac{1}{\Gamma_\beta(\Upsilon(\mathfrak{r}), \Upsilon(\mu_\gamma), \mathfrak{l})} - 1\Big) + \Big(\frac{1}{\Gamma_\beta(\Upsilon(\sigma_\gamma), \Upsilon(\mu_\gamma), \mathfrak{l})} - 1\Big) \\
& + \Big(\frac{1}{\Gamma_\beta(\Upsilon(\sigma_\gamma), \mathfrak{r}, \mathfrak{l})} - 1\Big) \\
& \leq \Bbbk\Big(\frac{1}{\Gamma_\beta(\mathfrak{r}, \mu_\gamma, \mathfrak{l})} - 1\Big) + \Bbbk\Big(\frac{1}{\Gamma_\beta(\sigma_\gamma, \mu_\gamma, \mathfrak{l})} - 1\Big)
\end{aligned}$$

$$+\Big(\frac{1}{\Gamma_\beta(\sigma_{\gamma+1},\mathfrak{r},\mathfrak{l})}-1\Big).$$

Again, all the right-terms go to zero when $\gamma\to\infty$. Consequently, as $\gamma\to\infty$, we get

$$\Gamma_\beta(\Upsilon(\mathfrak{r}),\mathfrak{r},\mathfrak{l})=1.$$

Therefore, $\Upsilon(\mathfrak{r})=\mathfrak{r}$. Let $\mathfrak{v}\in\Phi\cap\Psi$ is another fixed point of Υ. Then

$$\frac{1}{\Gamma_\beta(\mathfrak{r},\mathfrak{v},\mathfrak{l})}-1=\frac{1}{\Gamma_\beta(\Upsilon(\mathfrak{r}),\Upsilon(\mathfrak{v}),\mathfrak{l})}-1\leq\Bbbk\Big(\frac{1}{\Gamma_\beta(\mathfrak{r},\mathfrak{v},\mathfrak{l})}-1\Big).$$

As $\Bbbk\in(0,1)$, therefore

$$\Gamma_\beta(\mathfrak{r},\mathfrak{v},\mathfrak{l})=1.$$

Hence, $\mathfrak{r}=\mathfrak{v}$. □

Example 2.4. Let $\Phi=\{0,1,2,7\}$ and $\Psi=\{0,\frac{1}{4},\frac{1}{2},3\}$ be equipped with a continuous $\mathfrak{l}$-norm. Define $\Gamma_\beta(\mathfrak{r},\mathfrak{v},\mathfrak{l})=\frac{\mathfrak{l}}{\mathfrak{l}+|\sigma-\mu|}$ for all $\mathfrak{l}>0$, $\sigma\in\Phi$ and $\mu\in\Psi$. Clearly, $(\Phi,\Psi,\Gamma_\beta,*)$ is a complete FBMS. Note that Γ_β is triangular. Define $\Upsilon:\Phi\cup\Psi\to\Phi\cup\Psi$ by

$$\Upsilon(\mathfrak{r})=\begin{cases}\frac{\mathfrak{r}}{5}, & \text{if } \mathfrak{r}\in\{0,7,2\},\\ 0, & \text{if } \mathfrak{r}\in\{\frac{1}{4},\frac{1}{2},1,3\}.\end{cases}$$

Then,

$$\begin{aligned}\frac{1}{\Gamma_\beta(\Upsilon\sigma,\Upsilon\mu,\mathfrak{l})}-1&=\frac{|\Upsilon\sigma-\Upsilon\mu|}{\mathfrak{l}}\\&=\frac{|\sigma-\mu|}{5\mathfrak{l}}\\&\leq\frac{|\sigma-\mu|}{2\mathfrak{l}}\\&=\frac{1}{2}\Big(\frac{1}{\Gamma_\beta(\sigma,\mu,\mathfrak{l})}-1\Big).\end{aligned}$$

Therefore, all the conditions of Theorem 2.3 are fulfilled with $\Bbbk=\frac{1}{2}\in(0,1)$. Hence, Υ has a UFP, i.e., $\sigma=0$.

3. Application

The applications of cyclic maps are presented in the investigation of market equilibrium in duopoly markets, see [17,18]. These applications allow us to find the exact solutions of systems of transcendent equations with the help of cyclic maps, for which the computer algebra systems can present only an approximation solutions. For more details, see [19].

In this section, we are going to present an application of the result for fixed points for noncyclic maps in solving integral equations. Let us consider the integral equation

$$\sigma(\mathfrak{b})=\beta(\mathfrak{b})+\int_{\mathcal{H}_1\cup\mathcal{H}_2}\Omega(\mathfrak{b},\rho,\sigma(\rho))d\rho,\ \ \mathfrak{b}\in\mathcal{H}_1\cup\mathcal{H}_2, \tag{3.1}$$

where $\mathcal{H}_1\cup\mathcal{H}_2$ is a Lebesgue measurable set.

Theorem 3.1. *Suppose that:*

(1) $\Omega : (\mathcal{H}_1^2 \cup \mathcal{H}_2^2) \times [0, \infty) \to [0, \infty)$ *and* $b \in L^\infty(\mathcal{H}_1) \cup L^\infty(\mathcal{H}_2)$*;*

(2) $\exists$ *a continuous function* $\theta : \mathcal{H}_1^2 \cup \mathcal{H}_2^2 \to [0, \infty)$ *and* $\Bbbk \in (0, 1)$ *such that*

$$|\Omega(\mathfrak{b}, \rho, \sigma(\rho)) - \Omega(\mathfrak{b}, \rho, \mu(\rho))| \leq \Bbbk\theta(\mathfrak{b}, \rho)(|\sigma(\mathfrak{b}) - \mu(\mathfrak{b})|),$$

for $\mathfrak{b}, \rho \in \mathcal{H}_1^2 \cup \mathcal{H}_2^2$*;*

(3) $\int_{\mathcal{H}_1 \cup \mathcal{H}_2} \theta(\mathfrak{b}, \rho) d\rho \leq 1.$

Then the integral equation (3.1) *has a unique solution in* $L^\infty(\mathcal{H}_1) \cup L^\infty(\mathcal{H}_2)$.

Proof. Let $\Phi = L^\infty(\mathcal{H}_1)$ and $\Psi = L^\infty(\mathcal{H}_2)$ be two normed linear spaces, where $\mathcal{H}_1, \mathcal{H}_2$ are Lebesgue measurable sets and $m(\mathcal{H}_1 \cup \mathcal{H}_2) < \infty$. Consider $\Gamma_\beta : \Phi \times \Psi \times (0, \infty) \to [0, 1]$ by

$$\Gamma_\beta(\sigma, \mu, \mathfrak{l}) = \frac{\mathfrak{l}}{\mathfrak{l} + |\sigma - \mu|}$$

for all $\sigma \in \Phi, \mu \in \Psi$. Then $(\Phi, \Psi, \Gamma_\beta, \star)$ is a complete FBMS. Define $\Upsilon : L^\infty(\mathcal{H}_1) \cup L^\infty(\mathcal{H}_2) \to L^\infty(\mathcal{H}_1) \cup L^\infty(\mathcal{H}_2)$ by

$$\Upsilon(\sigma(\mathfrak{b})) = \beta(\mathfrak{b}) + \int_{\mathcal{H}_1 \cup \mathcal{H}_2} \Omega(\mathfrak{b}, \rho, \sigma(\rho)) d\rho, \ \ \mathfrak{b} \in \mathcal{H}_1 \cup \mathcal{H}_2.$$

Now,

$$\begin{aligned}
\frac{1}{\Gamma_\beta(\Upsilon\sigma(\mathfrak{b}), \Upsilon\mu(\mathfrak{b}), \mathfrak{l})} - 1 &= \frac{|\Upsilon\sigma(\mathfrak{b}) - \Upsilon\mu(\mathfrak{b})|}{\mathfrak{l}} \\
&= \frac{\left|\beta(\mathfrak{b}) + \int_{\mathcal{H}_1 \cup \mathcal{H}_2} \Omega(\mathfrak{b}, \rho, \sigma(\rho)) d\rho - \left(\beta(\mathfrak{b}) + \int_{\mathcal{H}_1 \cup \mathcal{H}_2} \Omega(\mathfrak{b}, \rho, \mu(\rho)) d\rho\right)\right|}{\mathfrak{l}} \\
&= \frac{\left|\int_{\mathcal{H}_1 \cup \mathcal{H}_2} (\Omega(\mathfrak{b}, \rho, \sigma(\rho)) - \Omega(\mathfrak{b}, \rho, \mu(\rho))) d\rho\right|}{\mathfrak{l}} \\
&\leq \frac{\left|\int_{\mathcal{H}_1 \cup \mathcal{H}_2} \Bbbk\theta(\mathfrak{b}, \rho)(|\sigma(\mathfrak{b}) - \mu(\mathfrak{b})|) d\rho\right|}{\mathfrak{l}} \\
&\leq \frac{\Bbbk|\sigma(\mathfrak{b}) - \mu(\mathfrak{b})|}{\mathfrak{l}} \\
&= \Bbbk\left(\frac{1}{\Gamma_\beta(\sigma(\mathfrak{b}), \mu(\mathfrak{b}), \mathfrak{l})} - 1\right).
\end{aligned}$$

Hence, all the conditions of Theorem 2.1 hold. Here, the integral equation has a unique solution. □

4. Conclusions

In this paper, we defined FBMSs and investigated some of their properties. The characteristic features of FBMSs have developed and proved fixed point theorems without continuity. Hereafter, we ensured the existence of a solution of an integral equation via the FBMS setting.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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