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Research article

A solution of a fractional differential equation via novel fixed-point approaches in Banach spaces

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Abstract: This manuscript is devoted to presenting some convergence results of a three-step iterative scheme under the Chatterjea–Suzuki–C ((CSC), for short) condition in the setting of a Banach space. Also, an example of mappings satisfying the (CSC) condition with a unique fixed point is provided. This example proves that the proposed scheme converges to a fixed point of a weak contraction faster than some known and leading schemes. Finally, our main results will be applied to find a solution to functional and fractional differential equations (FDEs) as an application.

Keywords: three-step iteration; differential equation; (CSC) condition; weak and strong convergence; Banach space
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1. Introduction and preliminaries

Let $\mathcal{Y}$ be a linear space with a norm $\|.\|$ and $\mathcal{G}$ be a non-empty subset of $\mathcal{Y}$. A mapping $\mathcal{J}:\mathcal{G}\to\mathcal{G}$ is called a contraction mapping on $\mathcal{G}$ if for all $x, y \in \mathcal{G}$,

$$\|\mathcal{J}x-\mathcal{J}y\| \leq \alpha\|x-y\|, \text{ for some fixed } \alpha\in[0,1). \tag{1.1}$$

A point $s_0 \in \mathcal{G}$, which satisfies the equation $s_0 = \mathcal{J} s_0$ is called a fixed point for the mapping $\mathcal{J}$. The fixed point set of $\mathcal{J}$ is normally denoted by $F_{\mathcal{J}}$.

Construction of fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and their applications in a number of applied areas. In such cases, one tries to find the approximate values of the such solutions by means of iterative schemes. For this purpose, we first express the sought solution of the given problem as a fixed point of a certain self mapping. The fixed point set in this case is now the same as the solution set of the given problem. The Banach contraction principle (BCP) [1] is one of the celebrated tools to provide the existence of a unique fixed point s_0 if the associated operator $\mathcal{J}$ is a contraction and $\mathcal{G}$ is some closed subset of a Banach space. Moreover, the BCP suggests a Picard [2] iterative scheme, that is, $a_{r+1} = \mathcal{J} a_r$, to find the approximate value of the point s_0.

On the uniformly convex Banach (UCB) space $\mathcal{Y}$, if $\mathcal{J}$ is a nonexpansive mapping, then $F_{\mathcal{J}} \neq \emptyset$ provided that $\mathcal{G}$ is a closed, convex and bounded subset of $\mathcal{Y}$. See for example Browder [3], Göhde [4] and others. Kirk [5], generalized the results of Browder and Göhde in a reflexive Banach (RB) space. It is known that there is a clear deficiency of Picard iteration in convergence in general on the set $F_{\mathcal{J}}$ for a nonexpansive operator $\mathcal{J}$, as shown in the following example:

Example 1.1. *Suppose that* $\mathcal{G} = [0, 1]$, *and* $\mathcal{J} x = -(x - 1)$. *Then,* $\mathcal{J}$ *is nonexpansive but not a contraction. According to the Browder and Göhde result,* $\mathcal{J}$ *admits a fixed point. In this case, we see that* $s_0 = 0.5$ *is the unique fixed point of* $\mathcal{J}$. *Notice that, for each* $a_1 = a \in \mathcal{G} - \{0.5\}$, *Picard [2] iteration generates the following sequence:*

$$a, 1 - a, a, 1 - a, ...$$

This sequence does not converge to the fixed point $s_0 = 0.5$ *of* $\mathcal{J}$.

On the other hand, the nonexpansive mappings have many important applications in the various applied sciences. For this reason, this science has spread and expanded on a large scale. In 2008, Suzuki [6] generalized nonexpansive mappings on Banach spaces. He proved that any mapping in this class admits a fixed point under the same assumptions of Browder and Göhde [4]. Moreover, he proved that all nonexpansive mappings are properly contained in this new class of mappings. Unlike nonexpansive mappings, Suzuki mappings do not have be continuous (see, e.g., [6–8] and others). Notice that, a mapping $\mathcal{J} : \mathcal{G} \to \mathcal{G}$ is said to be endowed with a condition (C) (or said to be a Suzuki mapping) if the following condition holds:

$$\frac{1}{2}\|x - \mathcal{J} x\| \leq \|x - y\| \Rightarrow \|\mathcal{J} x - \mathcal{J} y\| \leq \|x - y\|. \tag{C}$$

Researchers have extensively studied mappings with the Suzuki (C) condition, and many different results are now available in the literature. Karapinar [9] suggested a new condition for mappings on Banach spaces. These mappings are also discontinuous in general, as shown by a numerical example in this manuscript. The mapping $\mathcal{J} : \mathcal{G} \to \mathcal{G}$ is said to be satisfy a Chatterjea–Suzuki–C (CSC) condition if the inequality below is true.

$$\frac{1}{2}\|x - \mathcal{J} x\| \leq \|x - y\| \Rightarrow \|\mathcal{J} x - \mathcal{J} y\| \leq \frac{1}{2}(\|x - \mathcal{J} y\| + \|y - \mathcal{J} x\|). \tag{CSC}$$

In recent years, iterative schemes have been used for finding approximate solutions of nonlinear problems. For example, in [10], a novel iterative approach for finding approximate solutions of a special type of FDE was introduced. As we have seen in Example 1.1, the Picard iterative approach diverges in the fixed point set of nonexpansive mappings in general. This example suggests that we use other iterative methods to find fixed points of nonexpansive (and generalized nonexpansive) mappings, like the iterative methods due to Mann [11] , Ishikawa [12], Noor [13], Agarwal [14], Abbas [15], Thakur [7], Wairojjana et al. [16], Khatoon and Uddin [17] and Hasanen et al. [18–20].

Recently, Ullah and Arshad [8] suggested an M-iterative scheme as follows:

$$\left.\begin{array}{l} a_1 = a \in \mathcal{G}, \\ c_r = (1-\alpha_r)a_r + \alpha_r \mathcal{J} a_r, \\ b_r = \mathcal{J} c_r, \\ a_{r+1} = \mathcal{J} b_r. \end{array}\right\} \tag{1.2}$$

The same authors discussed this scheme via the (C)−condition and found that this scheme is faster than the schemes due to Agarwal [14] and Thakur [7] by a numerical example. In this paper, we use (CSC)−condition, and prove some convergence theorems with illustrative examples. In addition, we apply the theoretical results to find the existence of a solution for a FDE.

Definition 1.2. *Let $\mathcal{Y}$ be a Banach space and $\{a_r\} \subseteq \mathcal{Y}$ be a bounded set. If $\emptyset \neq \mathcal{G} \subseteq \mathcal{Y}$ is convex and closed, then, the asymptotic radius of $\{a_r\}$ corresponding to $\mathcal{G}$ is essentially denoted and defined by the formula $\mathcal{R}(\mathcal{G}, \{a_r\}) = \inf\{\limsup_{r\to\infty} \|a_r - s\| : s \in \mathcal{G}\}$. Similarly, the asymptotic center of the sequence $\{a_r\}$ corresponding to $\mathcal{G}$ is denoted and defined by the formula $\mathcal{A}(\mathcal{G}, a_r\}) = \{s \in \mathcal{G} : \limsup_{r\to\infty} \|a_r - s\| = \mathcal{R}(\mathcal{G}, a_r)\}$.*

Remark 1.3. *If $\mathcal{Y}$ is a UCB space [21], then it is well known that $\mathcal{A}(\mathcal{G}, \{a_r\})$ contains one element. Also, the set $\mathcal{A}(\mathcal{G}, \{a_r\})$ is convex when $\mathcal{G}$ is weakly compact and convex; for more details, see [22, 23].*

Definition 1.4. *[24] A Banach space $\mathcal{Y}$ is said to be satisfy Opial's condition if and only if the sequence $\{a_r\} \subseteq \mathcal{Y}$ converges in the weak sense to $s_0 \in \mathcal{G}$, and the following inequality holds:*

$$\limsup_{r\to\infty} \|a_r - s_0\| < \limsup_{r\to\infty} \|a_r - e_0\| \ \forall e_0 \in \mathcal{Y} - \{s_0\}.$$

Clearly, every Hilbert space meets Opial's condition.

Definition 1.5. *[25] A mapping $\mathcal{J}$ defined on a subset $\mathcal{G}$ of a Banach space $\mathcal{Y}$ is said to be satisfy the condition (I) if and only if one has a function $q : [0, \infty) \to [0, \infty)$ such that $q(0) = 0$, $q(u) > 0$ for every $u \in [0, \infty) - \{0\}$, and $\|x - \mathcal{J}x\| \geq q(d(x, F_{\mathcal{J}}))$, where $x \in \mathcal{G}$, and $d(x, F_{\mathcal{J}})$ represents the distance between x and $F_{\mathcal{J}}$.*

Some facts are combined in the following propositions, which can be found in [9].

Proposition 1.6. *Let $\mathcal{Y}$ be a Banach space and $\mathcal{G}$ be a non-empty closed subset of $\mathcal{Y}$. For the self-mapping $\mathcal{J} : \mathcal{G} \to \mathcal{G}$, the following properties hold:*

(i) If $\mathcal{J}$ is enriched with the (CSC) condition, and $F_{\mathcal{J}} \neq \emptyset$, then $\|\mathcal{J}x - s\| \leq \|x - s\|$ for each $x \in \mathcal{G}$ and $s \in F_{\mathcal{J}}$.

(ii) *If $\mathcal{J}$ is enriched with the (CSC) condition, then $F_{\mathcal{J}}$ is closed. Furthermore, $F_{\mathcal{G}}$ is convex if $\mathcal{G}$ is convex and $\mathcal{Y}$ is strictly convex.*

(iii) *If $\mathcal{J}$ is enriched with the (CSC) condition, then for arbitrary $x, y \in \mathcal{G}$,*

$$\|x - \mathcal{J}y\| \leq 5\|x - \mathcal{J}x\| + \|x - y\|.$$

(iv) *If $\mathcal{J}$ is enriched with the (CSC) condition, $\{a_r\}$ is weakly convergent to s, and $\lim_{r\to\infty}\|\mathcal{J}a_r - a_r\| = 0$, then $s \in F_{\mathcal{J}}$ provided that $\mathcal{Y}$ satisfies Opial's condition.*

The following lemma is very important in the sequel, and it was introduced by [26]

Lemma 1.7. *Let $0 < q \leq k_r \leq p < 1$ and $\mathcal{Y}$ be a UCB space. If there exists the real number $e \geq 0$ such that $\{a_r\}$ and $\{b_r\}$ in $\mathcal{Y}$ satisfy $\limsup_{r\to\infty}\|a_r\| \leq e$, $\limsup_{r\to\infty}\|b_r\| \leq e$ and $\lim_{r\to\infty}\|(1-k_r)a_r + k_r b_r\| = e$, then one has, $\lim_{r\to\infty}\|a_r - b_r\| = 0$.*

2. Main results

Now, we are in a position to connect an $M-$iterative scheme (1.2) with the class of mappings enriched with the (CSC) condition. The first result of this section is the following key lemma:

Lemma 2.1. *Let $\mathcal{Y}$ be a UCB space and $\emptyset \neq \mathcal{G} \subseteq \mathcal{Y}$ be a closed and convex set. Suppose that $\mathcal{J} : \mathcal{G} \to \mathcal{G}$ is enriched with the (CSC) condition satisfying $F_{\mathcal{J}} \neq \emptyset$. If $\{a_r\}$ is a sequence generated by iteration (1.2), then $\lim_{r\to\infty}\|a_r - s_0\|$ exists, for each $s_0 \in F_{\mathcal{J}}$.*

Proof. Let $s_0 \in F_{\mathcal{J}}$ be an arbitrary element, and then by Proposition 1.6(i), we get

$$\begin{aligned}
\|c_r - s_0| &\leq \|(1-\alpha_r)a_r + \alpha_r\mathcal{J}a_r - s_0\| \\
&\leq (1-\alpha_r)\|a_r - s_0\| + \alpha_r\|\mathcal{J}a_r - s_0\| \\
&\leq (1-\alpha_r)\|a_r - s_0\| + \alpha_r\|a_r - s_0\| \\
&\leq \|a_r - s_0\|.
\end{aligned} \tag{2.1}$$

Moreover,

$$\|b_r - s_0\| = \|\mathcal{J}c_r - s_0\| \leq \|c_r - s_0\|. \tag{2.2}$$

It follows from (2.2) that

$$\|a_{r+1} - s_0\| = \|\mathcal{J}b_r - s_0\| \leq \|b_r - s_0\| \leq \|c_r - s_0\|. \tag{2.3}$$

By combining (2.1), (2.2) and (2.3), it is seen that $\|a_{r+1} - s_0\| \leq \|a_r - s_0\|$, that is, $\{\|a_r - s_0\|\}$ is essentially bounded and also non-increasing. This means that $\lim_{r\to\infty}\|a_r - s_0\|$ exists for each element s_0 belonging to $F_{\mathcal{J}}$. □

The next theorem gives the necessary and sufficient conditions for the existence of a fixed point.

Theorem 2.2. *Let $\mathcal{Y}$ be a UCB space and $\emptyset \neq \mathcal{G} \subseteq \mathcal{Y}$ be a closed and convex set. Assume that $\mathcal{J} : \mathcal{G} \to \mathcal{G}$ is enriched with the (CSC) condition satisfying $F_{\mathcal{J}} \neq \emptyset$. If $\{a_r\}$ is a sequence made by $M-$iterations (1.2), then, $F_{\mathcal{J}} \neq \emptyset$ if and only if the sequence $\{a_r\}$ is bounded, and $\lim_{r\to\infty}\|a_r - \mathcal{J}a_r\| = 0$.*

Proof. First, assume that $F_{\mathcal{J}} \neq \emptyset$. Hence, for any $s_0 \in F_{\mathcal{J}}$, Lemma 2.1 suggests that $\{a_r\}$ is bounded, and $\lim_{r\to\infty} \|a_r - s_0\|$ exists. Consider

$$\lim_{r\to\infty} \|a_r - s_0\| = e. \tag{2.4}$$

We want to prove $\lim_{r\to\infty} \|a_r - \mathcal{J}a_r\| = 0$. From (2.1), one can write

$$\|c_r - s_0\| \leq \|a_r - s_0\|$$

$$\Rightarrow \limsup_{r\to\infty} \|c_r - s_0\| \leq \limsup_{r\to\infty} \|a_r - s_0\| = e. \tag{2.5}$$

Since $s_0 \in F_{\mathcal{J}}$, by Proposition 1.6(i), one has

$$\|\mathcal{J}a_r - s_0\| \leq \|a_r - s_0\|,$$

which implies that

$$\limsup_{r\to\infty} \|\mathcal{J}a_r - s_0\| \leq \limsup_{r\to\infty} \|a_r - s_0\| = e. \tag{2.6}$$

From (2.3), we have

$$\|a_{r+1} - s_0\| \leq \|c_r - s_0\|.$$

Using this together with (2.4), we obtain that

$$e \leq \liminf_{r\to\infty} \|c_r - s_0\|. \tag{2.7}$$

From (2.5) and (2.7), one can write

$$\lim_{r\to\infty} \|c_r - s_0\| = e. \tag{2.8}$$

Since $\|c_r - s_0\| = \|(1-\alpha_r)(a_r - s_0) + \alpha_r(\mathcal{J}a_r - s_0)\|$, so by (2.8), one has

$$e = \lim_{r\to\infty} \|(1-\alpha_r)(a_r - s_0) + \alpha_r(\mathcal{J}a_r - s_0)\|. \tag{2.9}$$

Considering (2.4), (2.6) and (2.9) along with Lemma 1.7, one gets

$$\lim_{r\to\infty} \|a_r - \mathcal{J}a_r\| = 0.$$

Conversely, assume that $\{a_r\}$ is bounded with $\lim_{r\to\infty} \|a_r - \mathcal{J}a_r\| = 0$. We shall prove that $F_{\mathcal{J}} \neq \emptyset$. For this, let $s_0 \in A(\mathcal{G}, \{a_r\})$. Using Proposition 1.6(iii), one can obtain

$$\begin{aligned}
\mathcal{A}(\mathcal{J}s_0, \{a_r\}) &= \limsup_{r\to\infty} \|a_r - \mathcal{J}s_0\| \\
&\leq 5\limsup_{r\to\infty} \|a_r - \mathcal{J}a_r\| + \limsup_{r\to\infty} \|a_r - s_0\| \\
&= \limsup_{r\to\infty} \|a_r - s_0\| \\
&= \mathcal{A}(s_0, \{s_r\}).
\end{aligned}$$

Hence, $\mathcal{J}s_0 \in \mathcal{A}(\mathcal{G}, \{a_r\})$. Since the set $\mathcal{A}(\mathcal{G}, \{a_r\}$ contains a singleton point, then $\mathcal{J}s_0 = s_0$. It follows that $s_0 \in F_{\mathcal{J}}$, i.e., $F_{\mathcal{J}} \neq \emptyset$. This completes the proof. □

Now, we will study the convergence. We start with the weak convergence as follows:

Theorem 2.3. *Let $\mathcal{Y}$ be a UCB space and $\emptyset \neq \mathcal{G} \subseteq \mathcal{Y}$ be a weakly compact and convex set. If $\mathcal{J} : \mathcal{G} \to \mathcal{G}$ is enriched with the (CSC) condition satisfying $F_{\mathcal{J}} \neq \emptyset$, and $\{a_r\}$ is a sequence of M iterates (1.2), then $\{a_r\}$ converges weakly to a point in $F_{\mathcal{J}}$ provided that $\mathcal{Y}$ satisfies Opial's condition.*

Proof. Since $\mathcal{G}$ is weakly compact, there exists a subsequence $\{a_{r_t}\}$ of $\{a_r\}$ and a point, namely, $a_0 \in \mathcal{G}$, such that $\{a_{r_t}\}$ converges weakly to a_0. In the view of Theorem 2.2, one can note that $\lim_{t\to\infty} \|a_{r_t} - \mathcal{J}a_{r_t}\| = 0$. All the requirements of Proposition 1.6(ii) are now available, so $a_0 \in F_{\mathcal{J}}$. The aim is to show that the point a_0 is only a weak limit of $\{a_r\}$. On the contrary, we may suppose that a_0 cannot become a weak limit for $\{a_r\}$, that is, there exists another subsequence, namely, $\{a_{r_s}\}$ of $\{a_r\}$, with a weak limit, namely, $a_0' \neq a_0$. Again in the view of Theorem 2.2, one can note that $\lim_{s\to\infty} \|a_{r_s} - \mathcal{J}a_{r_s}\| = 0$. All the requirements of Proposition 1.6(ii) are now available, so $a_0' \in F_{\mathcal{J}}$. Using Opial's condition of $\mathcal{Y}$ along with Lemma 2.1, we get

$$\begin{aligned}
\lim_{r\to\infty} \|a_r - a_0\| &= \lim_{t\to\infty} \|a_{r_t} - a_0\| < \lim_{t\to\infty} \|a_{r_t} - a_0'\| \\
&= \lim_{r\to\infty} \|a_r - a_0'\| = \lim_{s\to\infty} \|a_{r_s} - a_0'\| \\
&< \lim_{s\to\infty} \|a_{r_s} - a_0\| = \lim_{r\to\infty} \|a_r - a_0\|.
\end{aligned}$$

Thus, we get $\lim_{r\to\infty} \|a_r - a_0\| < \lim_{r\to\infty} \|a_r - a_0\|$, which is a contradiction. This completes the proof. □

If we replaced the weak compactness of the domain with compactness, we have the following strong convergence result.

Theorem 2.4. *Let $\mathcal{Y}$ be a UCB space and $\emptyset \neq \mathcal{G} \subseteq \mathcal{Y}$ be a compact and convex set. If $\mathcal{J} : \mathcal{G} \to \mathcal{G}$ is enriched with the (CSC) condition satisfying $F_{\mathcal{J}} \neq \emptyset$, and $\{a_r\}$ is a sequence generated by $M-$ iteration (1.2), then $\{a_r\}$ converges strongly to a point in $F_{\mathcal{J}}$.*

Proof. Since $\{a_r\} \in \mathcal{G}$, and $\mathcal{G}$ is a compact set, then $\{a_r\}$ has a strongly convergent subsequence $\{a_{r_t}\}$ such that $\lim_{t\to\infty} \|a_{r_t} - z_0\| = 0$ for some element $z_0 \in \mathcal{G}$. Hence, by Theorem 2.2, we conclude that $\lim_{t\to\infty} \|a_{r_t} - \mathcal{J}a_{r_t}\| = 0$. Applying Proposition 1.6(iii), we get

$$\|a_{r_t} - \mathcal{J}z_0\| \leq 5\|a_{r_t} - \mathcal{J}a_{r_t}\| + \|a_{t_r} - z_0\|,$$

which implies that $a_{r_t} \to \mathcal{J}z_0$ as $t \to \infty$. Also, we get $\mathcal{J}z_0 = z_0$, that is, $z_0 \in F_{\mathcal{J}}$. Based on Lemma 2.1, $\lim_{t\to\infty} \|a_t - a_0\|$ exists. Hence, the element z_0 is a strong limit point for $\{a_r\}$. □

Now, we remove the compactness of $\mathcal{G}$ in the above result and establish the following other strong convergence theorem as follows:

Theorem 2.5. *Let $\mathcal{Y}$ be a Banach space and $\emptyset \neq \mathcal{G} \subseteq \mathcal{Y}$ be a closed and convex set. If $\mathcal{J} : \mathcal{G} \to \mathcal{G}$ is enriched with the (CSC) condition satisfying $F_{\mathcal{J}} \neq \emptyset$, and $\{a_r\}$ is a sequence of iterates by (1.2), then $\{a_r\}$ converges strongly to a point in $F_{\mathcal{J}}$ whenever* $\liminf_{r\to\infty} d(a_r, F_{\mathcal{J}}) = 0$.

Proof. For all $s_0 \in F_{\mathcal{J}}$, Lemma 2.1 suggests the existence of $\lim_{r\to\infty} \|a_r - s_0\|$. By assumption, it follows that

$$\lim_{r\to\infty} dist(a_r, F_{\mathcal{J}}) = 0.$$

By Proposition 1.6(ii), the set $F_{\mathcal{J}}$ is closed in $\mathcal{G}$. Accordingly, the remaining proof now closely follows the proof of [Theorem 2, [25]] and hence is omitted. □

Now, we suggest another strong convergence theorem without assuming the compactness of the domain.

Theorem 2.6. *Let $\mathcal{Y}$ be a UCB space and $\emptyset \neq \mathcal{G} \subseteq \mathcal{Y}$ be a closed and convex. If $\mathcal{J} : \mathcal{G} \to \mathcal{G}$ is enriched with the (CSC) condition satisfying $F_{\mathcal{J}} \neq \emptyset$, and $\{a_r\}$ is a sequence of iterates by (1.2), then $\{a_r\}$ converges strongly to an elements in $F_{\mathcal{J}}$ whenever $\mathcal{J}$ satisfies the condition (I).*

Proof. In view of Theorem 2.2, we conclude that $\liminf_{r\to\infty} \|a_r - \mathcal{J}a_r\| = 0$. Applying the condition (I), we obtain $\liminf_{r\to\infty} d(a_r, F_{\mathcal{J}}) = 0$. Therefore, all assumptions of Theorem 2.5 are now successfully proved, and hence the sequence $\{a_r\}$ essentially converges strongly in $F_{\mathcal{J}}$. □

3. Application to a fractional differential equation

FDEs gained the attention of researchers because these equations have many interesting applications in areas of science and engineering like electromagnetic theory, fluid flow, electrical networks, and probability theory. As we discussed at the outset of this paper, many problems are difficult, if not impossible, to solve using analytical techniques. Hence, it is necessary to find approximate values for these solutions by alternative methods. FDEs have recently been solved by some authors using the techniques of fixed points for nonexpansive operators; see, for examples, [27–32].

Now, under the (CSC) condition, we apply an M-iterative scheme (1.2) to find the solution for the following FDE:

$$\left.\begin{array}{l} D^{\xi} h(u) + \Upsilon(u, h(u)) = 0, \\ h0) = h(1) = 0, \end{array}\right\} \tag{3.1}$$

where $(0 \leq u \leq 1)$, $(1 < \xi < 2)$, D^{ξ} stands for the Caputo fractional derivative endowed with the order ξ, and $\Upsilon : [0, 1] \times \mathbb{R} \to \mathbb{R}$.

Let $\mathcal{Y} = C[0, 1]$ be the set of all real continuous functions on $[0, 1]$ to $\mathbb{R}$ equipped with the usual maximum norm. The corresponding Green's function associated with (3.1) is defined by

$$G(u, v) = \begin{cases} \frac{1}{\Gamma(\xi)}(u(1-v)^{(\xi-1)} - (u-v)^{(\xi-1)}, & \text{if } 0 \leq v \leq u \leq 1, \\ \frac{u(1-v)^{(\xi-1)}}{\Gamma(\xi)}, & \text{if } 0 \leq u \leq v \leq 1. \end{cases}$$

The main result in this part is provided in the following theorem:

Theorem 3.1. *Let $\mathcal{Y} = C[0, 1]$ and $\mathcal{J} : \mathcal{Y} \to \mathcal{Y}$ be an operator defined by*

$$\mathcal{J}(h(u)) = \int_0^1 G(a, v)\Upsilon(v, h(v))dv, \text{ for each } h(u) \in \mathcal{Y}.$$

If

$$|\Upsilon(v,h(v)) - \Upsilon(v,g(v))| \leq \frac{1}{2}(|h(v) - \mathcal{J}(g(v))| + |g(v) - \mathcal{J}(h(v))|),$$

then the M−iteration scheme (1.2) associated with $\mathcal{J}$ *converges to some solution* S *of (3.1) provided that* $\liminf_{r\to\infty} d(a_r, S) = 0$.

Proof. It is known that an element h of $\mathcal{Y}$ is a solution to (3.1) if and only if it is a solution to the following equation:

$$h(u) = \int_0^1 G(u,v)\Upsilon(v,h(v))dv.$$

Now, for arbitrary $h, g \in \mathcal{Y}$ and $0 \leq u \leq 1$, it follows that

$$\begin{aligned}
\|\mathcal{J}(h(u)) - \mathcal{J}(g(u))\| &\leq \left|\int_0^1 G(u,v)\Upsilon(v,h(v)))dv - \int_0^1 G(u,v)\Upsilon(v,g(v))dv\right| \\
&= \left|\int_0^1 G(u,v)[\Upsilon(v,h(v)) - \Upsilon(v,g(v))]dv\right| \\
&\leq \int_0^1 G(u,v)\,|\Upsilon(v,h(v)) - \Upsilon(v,g(v))|\,dv \\
&\leq \int_0^1 G(u,v)(\frac{1}{2}\,|h(v)) - \mathcal{J}(g(v))| + \frac{1}{2}\,|g(v) - \mathcal{J}(h(v))|)dv \\
&\leq (\frac{1}{2}\|h(v)) - \mathcal{J}(g(v))\| + \frac{1}{2}\|g(v) - \mathcal{J}(h(v))\|) \\
&\quad \left(\int_0^1 G(u,v)dv\right) \\
&\leq \frac{1}{2}\|h(v)) - \mathcal{J}(g(v))\| + \frac{1}{2}\|g(v) - \mathcal{J}(h(v))\| \\
&= \frac{1}{2}\left(\|h(v)) - \mathcal{J}(g(v))\| + \|g(v) - \mathcal{J}(h(v))\|\right).
\end{aligned}$$

Consequently, we get

$$\|\mathcal{J}(h) - \mathcal{J}(g)\| \leq \frac{1}{2}\left(\|h - \mathcal{J}(g)\| + \|g - \mathcal{J}(h)\|\right).$$

Hence, $\mathcal{J}$ satisfies the (CSC) condition. In view of Theorem 2.5, the sequence generated by (1.2) converges to a fixed point of $\mathcal{J}$, and this point is the solution of the considered equation. □

4. Numerical example

Now, we essentially suggest a numerical example that is enriched with the (CSC) condition. An M-iteration of this example converges at a rate better than the other schemes. The observations are provided in tables and a graph.

Example 4.1. *Let* $\mathcal{G} = [8, 14]$ *be endowed with the norm* $\|.\| = |.|$ *and* $\mathcal{J} : \mathcal{G} \to \mathcal{G}$ *be a function defined by the formula*

$$\mathcal{J}x = \begin{cases} 8, & \text{if } x = 14, \\ \frac{x+8}{2}, & \text{otherwise.} \end{cases}$$

We prove the following.

(i) $F_{\mathcal{J}} \neq \emptyset$*;*
(ii) $\mathcal{J}$ *does not satisfy the* (C) *condition;*
(iii) $\mathcal{J}$ *satisfies the* (CSC) *condition.*

Proof. (i) Since $F_{\mathcal{J}} = \{8\}$, that is, $\mathcal{J}$ has a unique fixed point, and $F_{\mathcal{J}} \neq \emptyset$.
(ii) Choose $x = 12$ and $y = 13$ and then $\mathcal{J}$ does not satisfy the (C) condition.
(iii) We proceed as follows:

(I): If $x = 14 = y$, then, $|\mathcal{J}x - \mathcal{J}y| = 0$. Hence,

$$\frac{1}{2}\left(|x - \mathcal{J}y| + |y - \mathcal{J}x|\right) \geq 0 = |\mathcal{J}x - \mathcal{J}y|.$$

(II): If $8 \leq x, y < 14$, then, $|\mathcal{J}x - \mathcal{J}y| = |\frac{x-y}{2}|$. Hence,

$$\begin{aligned}
\frac{1}{2}\left(|x - \mathcal{J}y| + |y - \mathcal{J}x|\right) &= \left|\frac{x - (\frac{y+8}{2})}{2}\right| + \left|\frac{y - (\frac{x+8}{2})}{2}\right| \\
&\geq \left|\frac{\left(x - (\frac{y+8}{2})\right) - \left(y - (\frac{x+8}{2})\right)}{2}\right| \\
&= \left|\frac{3x - 3y}{4}\right| \\
&\geq |\frac{x - y}{2}| = |\mathcal{J}x - \mathcal{J}y|.
\end{aligned}$$

(III): If $x = 14$ and $8 \leq y < 14$, then, $|\mathcal{J}x - \mathcal{J}y| = |\frac{x-8}{2}|$.

$$\begin{aligned}
\frac{1}{2}\left(|x - \mathcal{J}y| + |y - \mathcal{J}x|\right) &= |\frac{x - 8}{2}| + \left|\frac{y - (\frac{x+8}{2})}{2}\right| \\
&\geq \left|\frac{x - 8}{2}\right| \\
&= |\mathcal{J}x - \mathcal{J}y|.
\end{aligned}$$

(IV): If $y = 14$ and $8 \leq x < 14$, then, $|\mathcal{J}x - \mathcal{J}y| = |\frac{y-8}{2}|$.

$$\begin{aligned}
\frac{1}{2}\left(|x - \mathcal{J}y| + |y - \mathcal{J}x|\right) &= \left|\frac{x - (\frac{y+8}{2})}{2}\right| + |\frac{y - 8}{2}| \\
&\geq \left|\frac{y - 8}{2}\right| \\
&= |\mathcal{J}x - \mathcal{J}y|.
\end{aligned}$$

□

Now, ***(I)**–**(IV)*** *completes the proof of (iii).*

Now, we connect Mann [11], Ishikawa [12], Noor [13], Agarwal [14], Abbas [15] and M [8] with this example. The observations are listed in Table 1 and Figure 1, where $\alpha_r = 0.85$, $\beta_r = 0.65$, $\gamma_r = 0.90$, and $a_1 = 10.5$.

Table 1. Numerical results produced by M, Thakur, Abbas, Agarwal, Noor, Ishikawa and Mann approximation schemes for $\mathcal{J}$ of Example 4.1.

r	**M**	**Thakur**	**Abbas**	**Agarwal**	**Noor**	**Ishikawa**	**Mann**
1	10.50000	10.50000	10.50000	10.50000	10.50000	10.50000	10.50000
2	8.359375	8.452344	8.650703	8.904680	8.936797	9.092188	9.437000
3	8.051660	8.081846	8.169366	8.327384	8.351035	8.477149	8.826563
4	8.007426	8.014809	8.044083	8.118472	8.131540	8.208455	8.475273
5	8.001068	8.002680	8.011474	8.042871	8.049290	8.091069	8.273282
6	8.000153	8.000485	8.002986	8.015514	8.018470	8.039788	8.157137
7	8.000022	8.000088	8.000777	8.005614	8.006921	8.017381	8.090354
8	8.000003	8.000016	8.000202	8.002032	8.002593	8.007593	8.051954
9	8.000000	8.000003	8.000053	8.000735	8.000972	8.003317	8.029873
10	8.000000	8.000001	8.000014	8.000266	8.000364	8.001449	8.017177
11	8.000000	8.000000	8.000004	8.000096	8.000136	8.000633	8.009877
12	8.000000	8.000000	8.000001	8.000035	8.000051	8.000277	8.005679

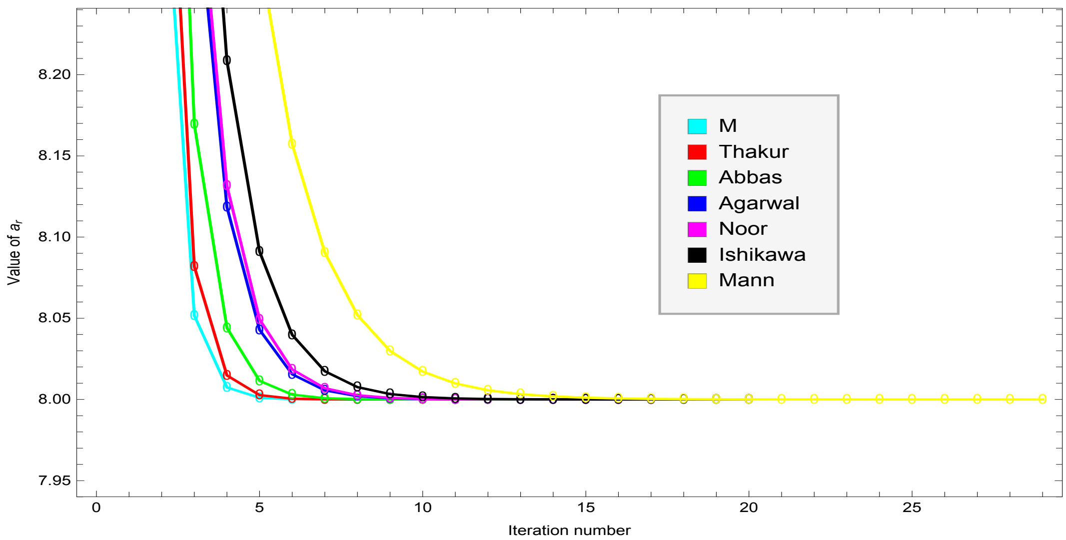

Figure 1. Graphical analysis of iteration schemes towards the fixed point of $\mathcal{J}$ in Example 4.1.

Finally, we set $\|a_r - s^*\| < 10^{-15}$ to be the stopping criterion, and the leading iterative schemes are once again compared under the different choice of starting and set of parameters. Bold values in Table 2 suggests the high accuracy of the M iterative scheme.

Table 2. Comparison of numbers of iterates to get fixed point under different starting points and parameters.

Iterations	8.5	9.5	10.5	11.5	12.5	13.5
	Initial Points					
for $\alpha_r = \frac{r}{(r+3)^{\frac{11}{10}}}$, $\beta_r = \frac{1}{(r+5)^{\frac{3}{2}}}$, $\gamma_r = \frac{r}{(4r+2)}$						
Agarwal	47	48	49	49	50	50
Abbas	32	33	34	35	35	35
Thakur	24	25	25	25	25	25
M	**19**	**20**	**20**	**21**	**21**	**21**
for $\alpha_r = \frac{r}{(r+8)^{\frac{18}{15}}}$, $\beta_r = \frac{r}{(r+4)}$, $\gamma_r = \sqrt{\frac{1}{r}}$						
Agarwal	39	40	41	42	42	42
Abbas	26	27	28	28	28	28
Thakur	22	23	23	23	24	24
M	**21**	**22**	**22**	**23**	**23**	**23**
for $\alpha_r = 1 - \sqrt{\frac{1}{6r+4}}, \beta_r = r^{-3}, \gamma_r = 1 - (\frac{1}{r})$						
Agarwal	46	48	49	49	50	50
Abbas	26	27	27	27	27	27
Thakur	23	24	25	25	25	25
M	**17**	**18**	**18**	**18**	**18**	**18**
for $\alpha_r = \sqrt[25]{\frac{2r}{(10r+1)}}$, $\beta_r = 1 - \sqrt[5]{\frac{1}{(r+1)}}$, $\gamma_r = 1 - \frac{6r}{(7r+5)^4}$						
Agarwal	36	37	38	38	38	38
Abbas	24	24	24	25	25	25
Thakur	21	22	22	22	22	23
M	**16**	**17**	**17**	**17**	**18**	**18**

Now, we provide some comment comparing the advantages of the proposed iterative scheme as follows:

(i) Our proposed iterative scheme is more convergent to a fixed point than other iterative schemes in the literature.

(ii) Instead of three scalars sequences $\{\alpha_r\}$, $\{\beta_r\}$ and $\{\gamma_r\}$, our proposed iterative scheme uses only one sequence of scalars $\{\alpha_r\}$ and converges better than the iterative schemes which use three sequences of scalars (e.g., Noor and Abbas iterative schemes).

(iii) The suggested iterative scheme is stable with respect to initial points and sequences of scalars, as shown in the tables and graph.

5. Conclusions

The paper examined an iterative $M-$scheme with the connection of operators enriched with the (CSC) condition. It has been shown that this scheme converges weakly and strongly towards a fixed point of an operator enriched with the (CSC) condition when suitable conditions are applied to the operator or the domain. Also, we solved a FDE in the setting of operators enriched with the (CSC) condition. In addition, a new numerical example was given to show that a (CSC) operator does not have to be continuous in its domain. Finally, some tables and one graph are obtained to illustrate the high accuracy of the M-iterative scheme when compared with the other available schemes in the literature.

Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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