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Research article

On the strong convergence of the solution of a generalized non-Newtonian fluid with Coulomb law in a thin film

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Abstract: The goal of this paper is to examine the strong convergence of the velocity of a non-Newtonian incompressible fluid whose viscosity follows the power law with Coulomb friction. We assume that the fluid coefficients of the thin layer vary with respect to the thin layer parameter ε. We give in a first step the description of the problem and basic equations. Then, we present the functional framework. The following paragraph is reserved for the main convergence results. Finally, we give the detail of the proofs of these results.

Keywords: mathematical operators; partial differential equations; Coulomb law; non-Newtonian fluid; weak generalized equation; variational inequality
Mathematics Subject Classification: 35R35, 76F10, 78M35, 35B40, 35J85

1. Introduction

Non-Newtonian fluids are defined as fluids with an extra-tension tensor that cannot be expressed as a linear, isotropic function of the components of the strain rate tensor.One of the goals of asymptotic analysis is to obtain and describe a two-dimensional problem from a three-dimensional problem, passing to the limit on the thickness of the domain assumed to be already thin. In this context, several previous studies have been conducted to deal with this problem.

The first study we mention is what the authors have done in [4], where they mainly examine the existence and behavior of weak solutions for a lubrication problem with Tresca law. In another study, the authors in [1] gave the nonlinear Reynolds equations for non-Newtonian thin-film fluid flows over a rough boundary. Suárez-Grau in [25] studied the asymptotic behavior of a non-Newtonian flow in a thin domain with Navier law on a rough boundary. The convergence stability of the solutions for the non-Newtonian fluid motion with large perturbation in $\mathbb{R}^2$ has been given in [9]. In [20], the authors presented an extension of the results related to the solutions of weakly compressible fluids with pressure-dependent viscosity. In contrast, the existence and uniqueness of stationary solutions of non-Newtonian viscous incompressible fluids were obtained in [11]. Other contexts and problems be found in the monographs such as in [21, 27], and in the literature quoted within.

The Herschel-Bulkley fluid is a generalized model of a non-Newtonian fluid. The name is related to Winslow Herschel and Ronald Bulkley [15], and it was first mentioned, in 1926, where the relationship between the stress tensor σ^ε and the symmetric deformation velocity $d(u^\varepsilon)$ is given by:

$$\sigma_{ij}^\varepsilon = -\pi^\varepsilon \delta_{ij} + \mu \left|d(u^\varepsilon)\right|^{r-2} d(u^\varepsilon) + \delta^\varepsilon \frac{d(u^\varepsilon)}{|d(u^\varepsilon)|},$$

where, $d(u^\varepsilon) = \dfrac{1}{2}(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T)$, u^ε is the velocity field, $\mu > 0$ is the viscosity constant, π^ε is the pressure, $\delta^\varepsilon \geq 0$ is the yield stress, $1 < r \leq 2$ is the power law exponent of the material and δ_{ij} is the Kronecker symbol.

In this paper, we will adopt the constitutive law by considering that a Herschel-Bulkley incompressible fluid whose viscosity will follow the power law with a liquid-solid friction condition of Coulomb in three-dimensional domain $\mathbb{Q}^\varepsilon \subset \mathbb{R}^3$.

The Herschel-Bulkley fluid has been studied intensively by mathematicians, physicists, and engineers as intensively as the Navier-Stokes. For example, we mention the studies carried out in the fields of metal fluxes, plastic solids and some polymers. The literature concerning this topic is extensive; see e.g. [24, 26] and many others references. More recently, the authors in [17], have studied the two-dimensional slow flow of non-Newtonian fluids of the Herschel-Bulkley type an inclined plane. In the context of the Bingham fluid, $r = 2$, the authors in [8, 22] proved the asymptotic convergence of this fluid in the isothermal and non-isothermal case with non linear friction law. In the case $\delta^\varepsilon = 0$, with the particular conditions of Tresca, this problem has been studied by [5, 6] respectively in both non-isothermal and isothermal study cases. Benseridi et al. in [3] studied theasymptotic analysis of a contact between two general Bingham fluids, however Saadallah et al. in [23] studied the analog of the problem presented in this work but in thevery particular case where the velocity on the surface Γ_b is null with the friction of the Tresca type. We can also mention others studies where authors gave the numerical solutions of the Herschel-Bulkley fluid but in other particular cases (see [14, 16, 18, 19]).

In this study, the objective is to make an extension of our previous works [8, 22, 23] and to improve the result obtained in [5, 6].

The novelty of our study can be summarized in following two major points. First, we take into account a generalized model of a non-Newtonian fluid ($1 < r \leq 2$ and $\delta^\varepsilon \neq 0$). Second, we choose the Coulomb friction with the velocity of the lower surface Γ_b different to zero, since all previously mentioned works were restricted only to the particular friction of Tresca.

From our side, this choice will cause different difficulties in other parts of the study, especially with regard to Lemma 5.1, Theorems 4.2–4.4 and the uniqueness theorem.

Accordingly, this work makes the following new contributions by finding solutions to these problems:

The first contribution consists of finding the solution for the first difficulty coming from the fact that the integral on Γ_b has no clear meaning. In our study, we will replace the normal stress by some regularization as in [10]. The second contribution consists of dealing with the problem of choosing the test functions. In fact, we cannot choose the test functions as it was done in [5, 6], their work does not contain the yield stress δ^{ε}.

This remaining of our paper is organized as follows: Section 2 will summarize the description of the problem and the basic equations. Moreover,we introduce some notations and preliminaries that will be used in other sections. Section 3 will be reserved to the proof of the related weak formulation. We will also discussing the problem in transpose form. The corresponding main convergence results will be stated in Theorems $(4.j)$, $j = 1$ to 5 of Section 4. The mathematical proofs will be presented in Section 5.

2. Description of the problem and basic equations

We start by introducing some notations used in the paper. Motivated by lubrication problems, we consider:

$$\mathbb{Q}^{\varepsilon} = \left\{ y = (y',\, y_3) \in \mathbb{R}^3 \,:\, y' = (y_1, y_2) \in \Gamma_b \text{ and } 0 < y_3/\varepsilon < h(y') \right\},$$

the domain of the flow, where Γ_b is a non-empty bounded domain of $\mathbb{R}^2$ with a Lipschitz continuous boundary, $h\,(.)$ is a Lipschitz continuous function defined on Γ_b such that $0 < h_{\star} \leq h(y') \leq h^{\star}$, for all $(y', 0)$ in Γ_b and ε is a small parameter that will tend to zero.

We decompose the boundary of $\mathbb{Q}^{\varepsilon}$ as $\Gamma^{\varepsilon} = \overline{\Gamma}^{\varepsilon}_{u} \cup \overline{\Gamma}^{\varepsilon}_{l} \cup \overline{\Gamma}_{b}$ with

$$\begin{aligned} \overline{\Gamma}_b &= \{(y', y_3) \in \bar{\mathbb{Q}}^{\varepsilon} : y_3 = 0\}, \\ \overline{\Gamma}^{\varepsilon}_u &= \{(y', y_3) \in \bar{\mathbb{Q}}^{\varepsilon} : (y', 0) \in \Gamma_b,\ y_3/\varepsilon = h\,(y')\}, \\ \overline{\Gamma}^{\varepsilon}_l &= \{(y', y_3) \in \bar{\mathbb{Q}}^{\varepsilon} : y' \in \partial\Gamma_b,\, 0 < y_3 < \varepsilon h(y')\}, \end{aligned}$$

where Γ_b is the bottom of the domain, Γ^{ε}_u is the upper surface and Γ^{ε}_l the lateral part of Γ^{ε}. Let $u^{\varepsilon}(y) : \mathbb{Q}^{\varepsilon} \to \mathbb{R}^3$ be the velocity and $\pi^{\varepsilon}(y) : \mathbb{Q}^{\varepsilon} \to \mathbb{R}$ the pressure of the fluid. We denote by $\eta = (\eta_1, \eta_2, \eta_3)$ the unit outward normal to the boundary Γ^{ε}, and we define the normal and tangential velocities of u^{ε} on Γ^{ε} as:

$$u^{\varepsilon}_{\eta} = u^{\varepsilon}.\eta,\ u^{\varepsilon}_{\tau} = u^{\varepsilon} - u^{\varepsilon}.\eta.$$

Similarly, for a regular tensor field σ^{ε}, we denote by $\sigma^{\varepsilon}_{\eta}$ and $\sigma^{\varepsilon}_{\tau}$ the normal and tangential components of σ^{ε} given by

$$\sigma^{\varepsilon}_{\eta} = \sum_{i=1}^{3} \left(\sigma^{\varepsilon}_{ij}.\eta_j\right).\eta_i,\ \ \sigma^{\varepsilon}_{\tau} = \left(\sum_{i=1}^{3} \sigma^{\varepsilon}_{ij}.\eta_j - \left(\sigma^{\varepsilon}_{\eta}\right).\eta_i\right)_{1\leq i\leq 3}.$$

Let $\mathbb{S}$ be denotes the set of all symmetric 3×3 matrices and for $\eta, \zeta \in \mathbb{S}$, we define the scalar product and the corresponding norm by

$$(\eta : \zeta) = \sum_{i,j=1}^{3} \eta_{ij}\zeta_{ij} \ \text{ and } \ |\eta| = (\eta : \eta)^{\frac{1}{2}}.$$

The boundary-value problem describing the stationary flow for generalized non-Newtonian and incompressible fluid is described by:

Problem $\mathcal{P}^{\varepsilon}$. Find the pressure $\pi^{\varepsilon}:\mathbb{Q}^{\varepsilon}\to\mathbb{R}$ and a velocity field $u^{\varepsilon}:\mathbb{Q}^{\varepsilon}\to\mathbb{R}^3$ such that

$$-\operatorname{div}(\sigma^{\varepsilon}) = f^{\varepsilon} \quad \text{in } \mathbb{Q}^{\varepsilon}, \tag{2.1}$$

$$\left.\begin{array}{c} \sigma^{\varepsilon}_{ij} = \tilde{\sigma}^{\varepsilon}_{ij} - \pi^{\varepsilon}\delta_{ij}, \\ \tilde{\sigma}^{\varepsilon} = \delta^{\varepsilon}\dfrac{d(u^{\varepsilon})}{|d(u^{\varepsilon})|} + \mu\,|d(u^{\varepsilon})|^{r-2}\,d(u^{\varepsilon}) \text{ if } d(u^{\varepsilon}) \neq 0, \\ |\tilde{\sigma}^{\varepsilon}| \leq \delta^{\varepsilon} \text{ if } d(u^{\varepsilon}) = 0, \end{array}\right\} \text{ in } \mathbb{Q}^{\varepsilon}, \tag{2.2}$$

$$div(u^{\varepsilon}) = 0 \quad \text{in } \mathbb{Q}^{\varepsilon}, \tag{2.3}$$

$$u^{\varepsilon} = 0 \quad \text{on } \Gamma^{\varepsilon}_u, \tag{2.4}$$

$$u^{\varepsilon} = g \text{ with } g_3 = 0 \quad \text{on } \Gamma^{\varepsilon}_l \tag{2.5}$$

$$u^{\varepsilon}.\eta = 0 \quad \text{on } \Gamma_b, \tag{2.6}$$

$$\left.\begin{array}{c} \left|\sigma^{\varepsilon}_{\tau}\right| < k^{\varepsilon}\left|\sigma^{\varepsilon}_{\eta}\right| \Rightarrow u^{\varepsilon}_{\tau} = s \\ \left|\sigma^{\varepsilon}_{\tau}\right| = k^{\varepsilon}\left|\sigma^{\varepsilon}_{\eta}\right| \Rightarrow \exists\beta \geq 0 : u^{\varepsilon}_{\tau} = s - \beta\sigma^{\varepsilon}_{\tau} \end{array}\right\} \text{ on } \Gamma_b, \tag{2.7}$$

where, $f^{\varepsilon} = (f^{\varepsilon}_i)_{1\leq i\leq 3}$ is the body forces, s is the velocity of the bottom boundary Γ_b. Furthermore, the Eq (2.1) represents the law of conservation of momentum. Relation (2.2) gives the law of behavior of the Herschel-Bulkley fluid. The formula (2.3) represents the incompressibility equation. Equations (2.4) and (2.5) represent the velocity on Γ^{ε}_u and Γ^{ε}_l respectively. On the other hand, Eq (2.6) justified the no-flux through on Γ_b. However, assuming that the friction is sufficiently large, the tangential velocity is unknown and satisfies the Coulomb boundary condition (2.7) on the part Γ_b, with k^{ε} is the friction coefficient. This law introduced by [2] is one of the most spread laws in mathematics and it is more realistic than the law of Tresca.

Suppose that the function $g = (g_i)_{1\leq i\leq 3}$ is in $\left(W^{1-1/r,r}(\Gamma^{\varepsilon})\right)^3$, the space of traces of functions from $\left(W^{1,r}(\mathbb{Q}^{\varepsilon})\right)^3$ on Γ^{ε} which will define in the next section. Due to $\int_{\Gamma^{\varepsilon}} g.nd\sigma = 0$ that there exists a function G^{ε} ([10]):

$$G^{\varepsilon} \in \left(W^{1,r}(\mathbb{Q}^{\varepsilon})\right)^3 \text{ with } \operatorname{div}(G^{\varepsilon}) = 0 \text{ in } \mathbb{Q}^{\varepsilon},\ G^{\varepsilon} = g \text{ on } \Gamma^{\varepsilon}.$$

Also, we suppose that $g_3 = 0$ on Γ^{ε} and $g = s$ on Γ_b.

3. Study of the weak solution

3.1. Functional framework and regularization of $\sigma^{\varepsilon}_{\eta}$

Before starting this study, we need to introduce the functional framework and the functional spaces that we use in the rest of this work: Let $L^r(\mathbb{Q}^{\varepsilon})$ represents the Lebesgue space for the norm $\|.\|_{L^r(\mathbb{Q}^{\varepsilon})}$ and $W^{1,r}(\mathbb{Q}^{\varepsilon})$ are the standard Sobolev spaces given by

$$\left(W^{1,r}(\mathbb{Q}^{\varepsilon})\right)^3 = \left\{v \in (L^r(\mathbb{Q}^{\varepsilon}))^3 : \frac{\partial v_i}{\partial y_j} \in L^r(\mathbb{Q}^{\varepsilon}) \text{ for } i,j = 1,2,3\right\},$$

for $1 < r < \infty$, and $W^{1,r}_0(\mathbb{Q}^{\varepsilon})$ is the closure of $\mathcal{D}(\mathbb{Q}^{\varepsilon})$ in $W^{1,r}(\mathbb{Q}^{\varepsilon})$. We denoted by $W^{-1,q}(\mathbb{Q}^{\varepsilon})$ the dual space of $W^{1,r}_0(\mathbb{Q}^{\varepsilon})$, where $r^{-1} + q^{-1} = 1$.

Moreover, we need the following functional spaces

$$E^{\varepsilon} = \left\{ v \in \left(W^{1,r}(\mathbb{Q}^{\varepsilon}) \right)^3 : v = G^{\varepsilon} \text{ on } \Gamma_l^{\varepsilon},\ v = 0 \text{ on } \Gamma_u^{\varepsilon}, v.\eta = 0 \text{ on } \Gamma_b \right\},$$
$$E_{div}^{\varepsilon} = \{ v \in E^{\varepsilon} : div(v) = 0 \},$$
$$E_0^q(\mathbb{Q}^{\varepsilon}) = \left\{ v \in L^q(\mathbb{Q}^{\varepsilon}) : \int_{\mathbb{Q}^{\varepsilon}} v\, dy' dy_3 = 0 \right\}.$$

Assume that the problem $(\mathcal{P}^{\varepsilon})$ admits a solution denoted by $(u^{\varepsilon}, \pi^{\varepsilon})$, with sufficient regularity. Multiplying (2.1) by $(v - u^{\varepsilon}) \in E^{\varepsilon}$ and then using Green's formula, along with the boundary conditions (2.4)–(2.7), we obtain:

Problem $\mathcal{P}_{\mathcal{K}}^{\varepsilon}$. We are looking for the velocity $u^{\varepsilon} \in E_{div}^{\varepsilon}$ and $\pi^{\varepsilon} \in E_0^q(\mathbb{Q}^{\varepsilon})$, which verify:

$$F(u^{\varepsilon}, v - u^{\varepsilon}) - (\pi^{\varepsilon}, div\, v) + \widetilde{j}(u^{\varepsilon}, v) - \widetilde{j}(u^{\varepsilon}, u^{\varepsilon}) \geq (f^{\varepsilon}, v - u^{\varepsilon}),\ \forall v \in E^{\varepsilon} \tag{3.1}$$

where

$$F(u^{\varepsilon}, v) = \mu \int_{\mathbb{Q}^{\varepsilon}} |d(u^{\varepsilon})|^{r-2}\, d(u^{\varepsilon}) d(v)\, dy' dy_3, \tag{3.2}$$

$$(\pi^{\varepsilon}, div\, v) = \int_{\mathbb{Q}^{\varepsilon}} \pi^{\varepsilon}\, div\, v dy' dy_3, \tag{3.3}$$

$$\widetilde{j}(u^{\varepsilon}, v) = \int_{\Gamma_b} k^{\varepsilon} \left| \sigma_{\eta}^{\varepsilon} \right| |v - s|\ dy' + \sqrt{2}\delta^{\varepsilon} \int_{\mathbb{Q}^{\varepsilon}} |d(v)|\, dy' dy_3, \tag{3.4}$$

$$(f^{\varepsilon}, v) = \sum_{i=1}^{3} \int_{\mathbb{Q}^{\varepsilon}} f_i^{\varepsilon} v_i\, dy' dy_3. \tag{3.5}$$

The integral $\widetilde{j}(u^{\varepsilon}, v)$ has no meaning for $u^{\varepsilon} \in E^{\varepsilon}$. Indeed, $\sigma_{\eta}^{\varepsilon}$ is defined by duality as an element of $W^{-\frac{1}{2},r}(\Gamma_b)$ and $|\sigma_{\eta}^{\varepsilon}|$ is not well defined on Γ_b. So following [10], we replace $\sigma_{\eta}^{\varepsilon}$ by some regularization $R(\sigma_{\eta}^{\varepsilon})$, where R is a regularization operator from $W^{-\frac{1}{2},r}(\Gamma_b)$ into $L^r(\Gamma_b)$ can be obtained by convolution with a positive regular function and defined by

$$\forall \tau \in W^{-\frac{1}{2},r}(\Gamma_b),\ R(\tau) \in L^2(\Gamma_b),\ R(\tau)(x) = \langle \tau, \phi(x - t) \rangle_{W^{-\frac{1}{2},r}(\Gamma_b), W_{00}^{\frac{1}{2},r}(\Gamma_b)}\quad \forall x \in \Gamma_b, \tag{3.6}$$

ϕ is a given positive function of class C^{∞} with compact support in Γ_b and $W^{-\frac{1}{2},r}(\Gamma_b)$ is the dual space to $W_{00}^{\frac{1}{2},r}(\Gamma_b) = \{ v_{|_{\Gamma_b}} : v \in W^{1,r}(\mathbb{Q}^{\varepsilon}), v = 0 \text{ on } \Gamma_u^{\varepsilon} \cup \Gamma_l^{\varepsilon} \}$.

After the regularization, we get the new problem:

Problem $\mathcal{P}_{\mathcal{K}}^{\varepsilon,r}$. Find $(u^{\varepsilon}, \pi^{\varepsilon}) \in E_{div}^{\varepsilon} \times E_0^q(\mathbb{Q}^{\varepsilon})$, provided it verifies the problem:

$$F(u^{\varepsilon}, v - u^{\varepsilon}) - (\pi^{\varepsilon}, div\, v) + j(u^{\varepsilon}, v) - j(u^{\varepsilon}, u^{\varepsilon}) \geq (f^{\varepsilon}, v - u^{\varepsilon}),\ \forall v \in E^{\varepsilon} \tag{3.7}$$

where

$$j(u^{\varepsilon}, v) = \int_{\Gamma_b} k^{\varepsilon} \left| R(\sigma_{\eta}^{\varepsilon}) \right| |v - s|\ dy' + \sqrt{2}\delta^{\varepsilon} \int_{\mathbb{Q}^{\varepsilon}} |d(v)|\, dy' dy_3.$$

Remark 3.1. If $v \in E^{\varepsilon}_{div}$ the inequality (3.7) becomes

$$F(u^{\varepsilon}, v - u^{\varepsilon}) + j(u^{\varepsilon}, v) - j(u^{\varepsilon}, u^{\varepsilon}) \geq (f^{\varepsilon}, v - u^{\varepsilon}), \forall v \in E^{\varepsilon}. \tag{3.8}$$

Theorem 3.1. *For $f^{\varepsilon} \in L^{q}(\mathbb{Q}^{\varepsilon})^{3}$ and $k^{\varepsilon} > 0$ in $L^{\infty}(\Gamma_b)$; then the problem $\mathcal{P}^{\varepsilon,r}_{\mathcal{K}}$* admits a unique pair $(u^{\varepsilon}, \pi^{\varepsilon}) \in E^{\varepsilon}_{div} \times E^{q}_{0}(\mathbb{Q}^{\varepsilon})$ *verifying* (3.7). *Moreover, for a small value of the friction threshold k^{ε}, this solution becomes unique.*

Proof. To show the existence and uniqueness result of (3.7), we define the following intermediate problem:

$$\begin{aligned} F(u^{\varepsilon}, v - u^{\varepsilon}) + \int_{\Gamma_b} Y\left(|v - s| - |u^{\varepsilon} - s|\right) dy' + \delta_{E^{\varepsilon}_{\text{div}}}(v) - \delta_{E^{\varepsilon}_{\text{div}}}(u^{\varepsilon}) \\ \geq (f^{\varepsilon}, v - u^{\varepsilon}),\ \forall v \in W^{1,r}_{div}(\mathbb{Q}^{\varepsilon})^{3} \end{aligned} \tag{3.9}$$

where, Y defined from $L^{r}(\Gamma_b)$ into $L^{r}(\Gamma_b)$ as: $Y \to -k^{\varepsilon} R(\sigma^{\varepsilon}_{\eta})$ and

$$W^{1,r}_{div}(\mathbb{Q}^{\varepsilon})^{3} = \left\{v \in W^{1,r}(\mathbb{Q}^{\varepsilon})^{3} : div(v) = 0\right\},$$

$$\delta_{E^{\varepsilon}_{\text{div}}} = \begin{cases} 0 \ \text{ for } v \in E^{\varepsilon}_{\text{div}}, \\ +\infty \text{ otherwise}. \end{cases}$$

By the analog of the techniques used in [16], it is easy to see that $F(u^{\varepsilon}, v - u^{\varepsilon})$ is bounded coercive hemicontinuous and strictly monotone.

$Y + \delta_{E^{\varepsilon}_{\text{div}}}$ is a proper, convex and continuous function on $L^{r}(\Gamma_b)$, then by Tichovo's fixed point theorem (as in [7]), we ensure the existence of a unique $u^{\varepsilon} \in E^{\varepsilon}_{\text{div}}$ verifying the variational inequality (3.9). The existence of the pressure $\pi^{\varepsilon} \in E^{q}_{0}(\mathbb{Q}^{\varepsilon})$ such that $(u^{\varepsilon}, \pi^{\varepsilon})$ satisfy (3.7) is found in [12]. □

3.2. Problem in transpose form

In this subsection, we use the dilatation in the variable y_3 given by $y_3 = z\varepsilon$, then our problem will be defined on a domain $\mathbb{Q}$ does not depend on ε given by:

$$\mathbb{Q} = \left\{(y', z) \in \mathbb{R}^{3} : (y', 0) \in \Gamma_b, \quad 0 < z < h(y')\right\},$$

and its boundary $\Gamma = \overline{\Gamma}_u \cup \overline{\Gamma}_l \cup \overline{\Gamma}_b$.

After this change of scale following the third component, it is normal to give the new functions and the new data defined on the new fixed domain $\mathbb{Q}$:

$$\hat{u}^{\varepsilon}_{i}(y', z) = u^{\varepsilon}_{i}(y', y_3), i = 1, 2,\ \hat{u}^{\varepsilon}_{3}(y', z) = \varepsilon^{-1} u^{\varepsilon}_{3}(y', y_3) \text{ and } \widehat{\pi}^{\varepsilon}(y', z) = \varepsilon^{r} \pi^{\varepsilon}(y', y_3). \tag{3.10}$$

$$\hat{f}(y', z) = \varepsilon^{r} f^{\varepsilon}(y', y_3),\ \hat{\delta} = \varepsilon^{r-1}\delta^{\varepsilon},\ \hat{k} = \varepsilon^{r-1} k^{\varepsilon}, \tag{3.11}$$

$$\begin{aligned} \widehat{g}(y', z) = g(y', y_3), \widehat{G_i}(y', z) = G^{\varepsilon}_{i}(y', y_3),\ i = 1, 2, \\ \widehat{G_3}(y', y_3) = \varepsilon^{-1} G^{\varepsilon}_{3}(y', y_3) \text{ also } \operatorname{div}(\widehat{G}) = 0 \text{ and } \widehat{G} = \widehat{g} \text{ on } \Gamma \end{aligned} \tag{3.12}$$

with all the new notations given in (3.11) and (3.12) do not depend on ε.

Also, we denote by:

$$E = \left\{\hat{v} \in \left(W^{1,r}(\mathbb{Q})\right)^{3} : \hat{v} = \widehat{G} \text{ on } \Gamma_l, \hat{v} = 0 \text{ on } \Gamma_u;\ \hat{v}.n = 0 \text{ on } \Gamma_b\right\},$$

$$E_{\text{div}} = \{\hat{v} \in E(\mathbb{Q}) : \ \operatorname{div} \hat{v} = 0\},$$
$$\Xi(E) = \left\{\hat{v} \in \left(W^{1,r}(\mathbb{Q})\right)^2 : \hat{v}_i = \widehat{G}_i \ \text{ on } \Gamma_l, \hat{v}_i = 0 \text{ on } \Gamma_u, i = 1,2\right\},$$
$$\widetilde{\Xi}(E) = \{\hat{v} \in \Xi(E) : \ \hat{v} \text{ satisfy (3.13)}\},$$

where the condition (3.13) is given by

$$\int_{\mathbb{Q}} \left(\hat{v}_1 \frac{\partial \omega}{\partial y_1} + \hat{v}_2 \frac{\partial \omega}{\partial y_2}\right) dy' dz = 0, \text{ for all } \hat{v} \in (L^r(\mathbb{Q}))^2 \text{ and } \omega \in C_0^\infty(\mathbb{Q}). \tag{3.13}$$

Finally, the Banach space Θ_z and its linear subspace $\widetilde{\Theta}_z$ are denoted by:

$$\Theta_z = \left\{\hat{v} \in (L^r(\mathbb{Q}))^2 ; \frac{\partial \hat{v}_i}{\partial z} \in L^r(\mathbb{Q}), i = 1,2 : \hat{v} = 0 \ \text{ on } \Gamma_u\right\},$$

$$\widetilde{\Theta}_z = \{\hat{v} \in \Theta_z : \ \hat{v} \text{ satisfy the condition (3.13)}\},$$

with the norm of Θ_z is given as follows:

$$\|\hat{v}\|^r_{\Theta_z} = \sum_{i=1}^{2} \left(\|\hat{v}_i\|^r_{L^r(\mathbb{Q})} + \left\|\frac{\partial \hat{v}_i}{\partial z}\right\|^r_{L^r(\mathbb{Q})}\right).$$

By introducing all these new notations into the variational inequality (3.7), and then multiplying all the terms deduced by ε^{r-1} after this scaling, then the problem $\mathcal{P}^{\varepsilon,r}_{\mathcal{K}}$ takes the following form:

Problem $\mathcal{P}_{\mathcal{K}}$. Find $(\hat{u}^\varepsilon, \hat{\pi}^\varepsilon) \in E_{div} \times E_0^q(\mathbb{Q})$, such that

$$\widehat{F}(\hat{u}^\varepsilon, \hat{v} - \hat{u}^\varepsilon) - (\widehat{\pi}^\varepsilon, div\ (\hat{v} - \hat{u}^\varepsilon)) + \widehat{j}(\hat{u}^\varepsilon, \hat{v}) - \widehat{j}(\hat{u}^\varepsilon, \hat{u}^\varepsilon) \geq (\hat{f}, \hat{v} - \hat{u}^\varepsilon), \ \forall \hat{v} \in E \tag{3.14}$$

where

$$\begin{aligned}
\widehat{F}(\hat{u}^\varepsilon, \hat{v} - \hat{u}^\varepsilon) = &\sum_{i,j=1}^{2} \int_{\mathbb{Q}} \left[\varepsilon^2 \mu \left|\tilde{d}(\hat{u}^\varepsilon)\right|^{r-2} \left(\frac{1}{2}\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial y_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial y_i}\right)\right)\right] \frac{\partial(\hat{v}_i - \hat{u}_i^\varepsilon)}{\partial y_j} dy' dz \\
&+ \sum_{i=1}^{2} \int_{\mathbb{Q}} \mu \left|\tilde{d}(\hat{u}^\varepsilon)\right|^{r-2} \left(\frac{1}{2}\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial y_i}\right)\right) \frac{\partial(\hat{v}_i - \hat{u}_i^\varepsilon)}{\partial z} dy' dz \\
&+ \int_{\mathbb{Q}} \left(\mu \left|\tilde{d}(\hat{u}^\varepsilon)\right|^{r-2} \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial z}\right) \frac{\partial(\hat{v}_3 - \hat{u}_3^\varepsilon)}{\partial z} dy' dz \\
&+ \sum_{j=1}^{2} \int_{\mathbb{Q}} \varepsilon^2 \mu \left|\tilde{d}(\hat{u}^\varepsilon)\right|^{r-2} \left(\frac{1}{2}\left(\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial y_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial z}\right)\right) \frac{\partial(\hat{v}_3 - \hat{u}_3^\varepsilon)}{\partial y_j} dy' dz,
\end{aligned}$$

$$(\widehat{\pi}^\varepsilon, \operatorname{div}(\hat{v} - \hat{u}^\varepsilon)) = \int_{\mathbb{Q}} \widehat{\pi}^\varepsilon \operatorname{div}(\hat{v} - \hat{u}^\varepsilon)\, dy' dz,$$
$$\widehat{j}(\hat{u}^\varepsilon, \hat{v}) = \int_{\Gamma_b} \hat{k} \left|R(\widehat{\sigma}^\varepsilon_\eta)\right| |\hat{v} - s| dy' + \sqrt{2}\hat{\delta} \int_{\mathbb{Q}} \left|\tilde{d}(\hat{v})\right| dy' dz,$$

$$(\hat{f}, \hat{v} - \hat{u}^{\varepsilon}) = \sum_{i=1}^{2} \int_{\mathbb{Q}} \hat{f}_i(\hat{v}_i - \hat{u}_i^{\varepsilon}) dy' dz + \int_{\mathbb{Q}} \varepsilon \hat{f}_3(\hat{v}_3 - \hat{u}_3^{\varepsilon}) dy' dz,$$

$$\left|\widetilde{d}\,(\hat{u}^{\varepsilon})\right| = \left(\frac{1}{4}\sum_{i,j=1}^{2}\varepsilon^2\left(\frac{\partial \hat{u}_i^{\varepsilon}}{\partial y_j} + \frac{\partial \hat{u}_j^{\varepsilon}}{\partial y_i}\right)^2 + \frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial \hat{u}_i^{\varepsilon}}{\partial z} + \varepsilon^2\frac{\partial \hat{u}_3^{\varepsilon}}{\partial y_i}\right)^2 + \varepsilon^2\left(\frac{\partial \hat{u}_3^{\varepsilon}}{\partial z}\right)^2\right)^{1/2}.$$

We introduce some results found in [4] which we will need to use in the rest of this work.

$$\|\nabla v^{\varepsilon}\|_{L^r(\mathbb{Q}^{\varepsilon})} \leq C \, \|d\,(v^{\varepsilon})\|_{L^r(\mathbb{Q}^{\varepsilon})}, \tag{3.15}$$

$$\|v^{\varepsilon}\|_{L^r(\mathbb{Q}^{\varepsilon})} \leq \varepsilon h^{\star} \left\|\frac{\partial v^{\varepsilon}}{\partial z}\right\|_{L^r(\mathbb{Q}^{\varepsilon})}, \tag{3.16}$$

$$\alpha\beta \leq \frac{\alpha^r}{r} + \frac{\beta^q}{q}, \ \forall\,(\alpha,\beta) \in \mathbb{R}^2. \tag{3.17}$$

4. Main convergence results

The convergence results of $(\hat{u}^{\varepsilon}, \widehat{\pi^{\varepsilon}})$ towards $(u^{\star}, \pi^{\star})$ as well as the limit problem independently of the parameter ε will be given in the next of this subsection.

Theorem 4.1. *Assume that the assumptions of Theorem 3.1 hold, there exist $\pi^{\star} \in E_0^q(\mathbb{Q})$ and $u^{\star} = \left(u_1^{\star}, u_2^{\star}\right) \in \widetilde{\Theta}_z$ satisfy the following convergences*:

$$\widehat{u_i^{\varepsilon}} \rightharpoonup u_i^{*} \ in \ \ \tilde{\Theta}_z, \ \ 1 \leq i \leq 2, \tag{4.1}$$

$$\varepsilon\frac{\partial \hat{u}_i^{\varepsilon}}{\partial y_j} \rightharpoonup 0, \ \ in \ \ L^r(\mathbb{Q}), \ \ 1 \leq i, j \leq 2, \tag{4.2}$$

$$\varepsilon\frac{\partial \hat{u}_3^{\varepsilon}}{\partial z} \rightharpoonup 0, \ \ \ in \ \ L^r(\mathbb{Q}), \tag{4.3}$$

$$\varepsilon^2\frac{\partial \hat{u}_3^{\varepsilon}}{\partial y_i} \rightharpoonup 0, \ \ in \ \ L^r(\mathbb{Q}), \ \ 1 \leq i \leq 2, \tag{4.4}$$

$$\varepsilon\hat{u}_3^{\varepsilon} \rightharpoonup 0, \ \ \ in \ \ L^r(\mathbb{Q}), \tag{4.5}$$

$$\widehat{\pi^{\varepsilon}} \rightharpoonup \pi^{\star}, \ in \ \ E_0^q(\mathbb{Q}), \ with \ \ \pi^{\star} \ depend \ only \ of \ y'. \tag{4.6}$$

Theorem 4.2. *With the same assumptions as Theorem 4.1, the pair $(u^{\star}, \pi^{\star})$ satisfies:*

$$\hat{u}_i^{\varepsilon} \to u_i^{\star}, \ \ strongly \ in \ \ \tilde{\Theta}_z \ , \ i = 1,2, \forall \ 1 < r \leq 2, \tag{4.7}$$

$$\begin{aligned} &\mu\sum_{i=1}^{2}\int_{\mathbb{Q}}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^{\star}}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial(u_i^{\star})}{\partial z}\frac{\partial(\hat{v}_i - u_i^{\star})}{\partial z}dy'dz - \int_{\mathbb{Q}}\pi^{\star}\,(y')\,\left(\frac{\partial \hat{v}_1}{\partial y_1} + \frac{\partial \hat{v}_2}{\partial y_2}\right)dy'dz \\ &+\hat{\delta}\tfrac{\sqrt{2}}{2}\int_{\mathbb{Q}}\left(\left|\frac{\partial \hat{v}}{\partial z}\right| - \left|\frac{\partial u^{\star}}{\partial z}\right|\right)dy'dz + \int_{\Gamma_b}\hat{k}\left|R(-\pi^{\star})\right|\left(|\hat{v} - s| - |u^{\star} - s|\right)dy' \\ &\geq \sum_{i=1}^{2}\int_{\mathbb{Q}}\hat{f}_i(\hat{v}_i - u_i^{\star})dy'dz, \ \ \forall \hat{v} \in \Xi(E). \end{aligned} \tag{4.8}$$

Theorem 4.3. *Suppose that the assumptions of the previous theorem hold, and if* $\left|\frac{\partial u^\star}{\partial z}\right| \neq 0$, the solution $(u^\star, \pi^\star)$ satisfies

$$\pi^\star \in W^{1,q}(\Gamma_b) \tag{4.9}$$

$$-\frac{\partial}{\partial z}\left[\frac{1}{2}\mu\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^\star}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u^\star}{\partial z}+\hat{\delta}\frac{\sqrt{2}}{2}\frac{\partial u^\star/\partial z}{|\partial u^\star/\partial z|}\right]=\hat{f}-\nabla\pi^\star\ ,\ in\ L^q(\mathbb{Q})^2. \tag{4.10}$$

Theorem 4.4. *Suppose that the assumptions of Theorem 4.2 hold, then* $\tau^\star$, $s^\star$ *satisfy the inequality:*

$$\sum_{i=1}^{2}\int_{\Gamma_b}\widehat{k}\left|R(\widehat{\sigma}_\eta(-\pi^\star))\right|\phi_i(s_i^\star-s_i)dy'-\int_{\Gamma_b}\widehat{\mu}\tau^\star\phi\left|s^\star-s\right|dy'\geq 0,\ \forall\phi\in L^r(\Gamma_b)^2, \tag{4.11}$$

and the limit form of Coulomb law:

$$\left.\begin{array}{r}\mu\left|\tau^\star\right|<\widehat{k}\left|R(\widehat{\sigma}_\eta(-\pi^\star))\right|\Longrightarrow s^\star=s\\ \mu\left|\tau^\star\right|=\widehat{k}\left|R(\widehat{\sigma}_\eta(-\pi^\star))\right|\Longrightarrow\exists\beta\geq 0: s^\star=s+\beta\tau^\star\end{array}\right\}\ a.e.\ in\ \Gamma_b. \tag{4.12}$$

Also, the solution $(u^\star, \pi^\star)$ *satisfies the weak generalized form:*

$$\begin{aligned}&\int_{\Gamma_b}\left[\frac{h^3}{12}\nabla\pi^\star+\tilde{H}+\mu\int_0^h\int_0^y B^\star(y',\xi)\frac{\partial u^\star(y',\xi)}{\partial\xi}d\xi dy+\hat{\delta}\int_0^h\int_0^y\frac{\partial u^\star/\partial z}{|\partial u^\star/\partial z|}(y',\xi)\,d\xi dy\right].\nabla v(y')\,dy'\\&+\int_{\Gamma_b}\left[-\frac{h\mu}{2}\int_0^h B^\star(y',\xi)\frac{\partial u^\star(y',\xi)}{\partial\xi}d\xi-\frac{\hat{\delta}h}{2}\int_0^h\frac{\partial u^\star/\partial z}{|\partial u^\star/\partial z|}(y',\xi)\,d\xi\right]\nabla v(y')\,dy',\forall v\in W^{1,r}(\Gamma_b),\end{aligned} \tag{4.13}$$

where

$$\tau^\star=B^\star(y',0)\frac{\partial u^\star}{\partial z}(y',0),\ s^\star=\frac{\partial u^\star}{\partial z}(y',0),\ B^\star(y',\xi)=\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u^\star}{\partial z}(y',\xi)\right)^2\right)^{\frac{r-2}{2}}$$

$$\tilde{H}(y',h)=\int_0^h H(y',y)\,dy-\frac{h}{2}H(y',h),\ H(y',y)=\int_0^y\int_0^\xi\hat{f}(y',t)\,dtd\xi.$$

Theorem 4.5. *For* $\hat{f}\in L^q(\mathbb{Q})^3$ *and* $\hat{k}>0$ *in* $L^\infty(\Gamma_b)$*; there exists* $\overline{k}>0$ *sufficiently small such that for* $\|\widehat{k}\|_{L^\infty(\Gamma_b)}\leq\overline{k}$, *the solution* $(u^\star,\pi^\star)$ *of the limiting problem* (4.8) *is unique in* $\widetilde{\Theta}_z\times\left(E_0^q(\Gamma_b)\cap W^{1,q}(\Gamma_b)\right)^2$.

5. Proofs of the main results

Proof of Theorem 4.1. Before starting the proof of this theorem, we need the following estimates which can be considered as the key that allows us to make a passage to the limit when ε tends to zero.

Lemma 5.1. *Assume that* $f^\varepsilon\in L^q(\mathbb{Q}^\varepsilon)^3$ *and let* $(u^\varepsilon,\pi^\varepsilon)\in E^\varepsilon_{\text{div}}\times E_0^q(\mathbb{Q}^\varepsilon)$ *be a solution of* $\mathcal{P}^{\varepsilon,r}_{\mathcal{K}}$, *where the friction coefficient* $k^\varepsilon>0$ *in* $L^\infty(\Gamma_b)$. *Then there exists a constant C independent of* ε *such that*

$$\sum_{i,j=1}^{2}\left\|\varepsilon\frac{\partial\hat{u}_i^\varepsilon}{\partial y_j}\right\|^r_{L^r(\mathbb{Q})}+\left\|\varepsilon\frac{\partial\hat{u}_3^\varepsilon}{\partial z}\right\|^r_{L^r(\mathbb{Q})}+\sum_{i=1}^{2}\left(\left\|\frac{\partial\hat{u}_i^\varepsilon}{\partial z}\right\|^r_{L^r(\mathbb{Q})}+\left\|\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial y_i}\right\|^r_{L^r(\mathbb{Q})}\right)\leq C, \tag{5.1}$$

$$\left\| \frac{\partial \widehat{\pi^\varepsilon}}{\partial y_i} \right\|_{W^{-1,q}(\mathbb{Q})} \leq C, \quad \textit{for } i = 1, 2, \tag{5.2}$$

$$\left\| \frac{\partial \widehat{\pi^\varepsilon}}{\partial z} \right\|_{W^{-1,q}(\mathbb{Q})} \leq \varepsilon C. \tag{5.3}$$

Proof of Lemma 5.1. Choosing $v = G^\varepsilon$ in (3.8) and using the fact that $G^\varepsilon = s$ on Γ_b, we find

$$F(u^\varepsilon, u^\varepsilon) \leq F(u^\varepsilon, G^\varepsilon) + (f^\varepsilon, u^\varepsilon) - (f^\varepsilon, G^\varepsilon). \tag{5.4}$$

By applying Korn's inequality, we ensure the existence of a constant $C_K > 0$ that does not depend on ε with:

$$F(u^\varepsilon, u^\varepsilon) \geq 2\mu C_K \|\nabla u^\varepsilon\|^r_{L^r(\mathbb{Q})}. \tag{5.5}$$

Now we apply Hölder's inequality and then Young's, the increase of the first term of (5.4) is given by

$$F(u^\varepsilon, G^\varepsilon) \leq \frac{\mu C_K}{2} \int_{\mathbb{Q}^\varepsilon} \mu \left| d(u^\varepsilon) \right|^{q(r-1)} dy' dy_3 + \frac{2^{(r-1)}\mu}{r(qC_K)^{r/q}} \int_{\mathbb{Q}^\varepsilon} \left| d(G^\varepsilon) \right|^r dy' dy_3. \tag{5.6}$$

By (3.15), the inequality (5.6) becomes

$$F(u^\varepsilon, v) \leq \|\nabla u^\varepsilon\|^r_{L^r(\mathbb{Q}^\varepsilon)} + \frac{2^{(r-1)}\mu}{r(qC_K)^{r/q}} \|\nabla G^\varepsilon\|^r_{L^r(\mathbb{Q}^\varepsilon)}. \tag{5.7}$$

We apply (3.16) and (3.17), we obtain the analogue of (5.7)

$$|(f^\varepsilon, u^\varepsilon)| \leq \frac{\mu C_K}{2} \|\nabla u^\varepsilon\|^r_{L^r(\mathbb{Q}^\varepsilon)} + \frac{(\varepsilon h^\star)^q}{q(\frac{1}{2}\mu r C_K)^{q/r}} \|f^\varepsilon\|^q_{L^q(\mathbb{Q}^\varepsilon)}, \tag{5.8}$$

$$|(f^\varepsilon, G^\varepsilon)| \leq \frac{\mu C_K}{2} \|\nabla G^\varepsilon\|^r_{L^r(\mathbb{Q}^\varepsilon)} + \frac{(\varepsilon h^\star)^q}{q(\frac{\mu}{2} r C_K)^{q/r}} \|f^\varepsilon\|^q_{L^q(\mathbb{Q}^\varepsilon)}. \tag{5.9}$$

Now, from (5.4)–(5.9), we obtain

$$\mu C_K \|\nabla u^\varepsilon\|^r_{L^r(\mathbb{Q})} \leq \left(\frac{2^{(r-1)}\mu}{r(qC_K)^{r/q}} + \frac{\mu C_K}{2} \right) \|\nabla G^\varepsilon\|^r_{L^r(\mathbb{Q}^\varepsilon)} + \frac{2(\varepsilon h^\star)^q}{q(\frac{\mu}{2} r C_K)^{q/r}} \|f^\varepsilon\|^q_{L^q(\mathbb{Q}^\varepsilon)}. \tag{5.10}$$

We multiply (5.10) by ε^{r-1} then using the fact that

$$\varepsilon^q \|f^\varepsilon\|^q_{L^q(\mathbb{Q}^\varepsilon)} = \varepsilon^{1-r} \left\| \hat{f} \right\|^q_{L^q(\mathbb{Q})}$$

and

$$\left\| \frac{\partial u_i^\varepsilon}{\partial x_3} \right\|^r_{L^r(\mathbb{Q}^\varepsilon)} = \varepsilon^{1-r} \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|^r_{L^r(\mathbb{Q})},$$

for $i = 1, 2$, we deduce (5.1) with

$$C = \frac{1}{\mu C_K} \left[\left(\frac{2^{(r-1)}\mu}{r(qC_K)^{r/q}} + \frac{\mu C_K}{2} \right) \left\| \nabla \widehat{G} \right\|^r_{L^r(\mathbb{Q})} + \frac{2(\varepsilon h^\star)^q}{q(\frac{\mu}{2} r C_K)^{q/r}} \left\| \hat{f} \right\|^q_{L^q(\mathbb{Q})} \right].$$

For get the estimate (5.2), we choose in (3.14), $\hat{v} = \hat{u}^\varepsilon + \phi$, with $\phi \in W_0^{1,r}(\mathbb{Q})^3$, we find

$$F(\hat{u}^\varepsilon, \phi) - (\widehat{\pi}^\varepsilon, div\,\phi) + \hat{\delta}\int_{\mathbb{Q}} \left|\widetilde{d}(\hat{u}^\varepsilon + \phi)\right| dy'dz - \hat{\delta}\int_{\mathbb{Q}} \left|\widetilde{d}(\hat{u}^\varepsilon)\right| dy'dz \geq (\hat{f}^\varepsilon, \phi),$$

then

$$(\widehat{\pi}^\varepsilon, div\,\phi) \leq a(\hat{u}^\varepsilon, \phi) + \sqrt{2}\hat{\delta}\int_{\mathbb{Q}} \left|\widetilde{d}(\hat{u}^\varepsilon + \phi)\right| dy'dz - \sqrt{2}\hat{\delta}\int_{\mathbb{Q}} \left|\widetilde{d}(\hat{u}^\varepsilon)\right| dy'dz - (\hat{f}^\varepsilon, \phi),$$

as

$$\left|\widetilde{d}(\hat{u}^\varepsilon + \phi)\right| \leq \sqrt{2}\left|\widetilde{d}(\hat{u}^\varepsilon)\right| + \sqrt{2}\left|\widetilde{d}(\phi)\right|,$$

we obtain

$$(\widehat{\pi}^\varepsilon, div\,\phi) \leq a(\hat{u}^\varepsilon, \phi) + 2\hat{\delta}\int_{\mathbb{Q}} \left|\widetilde{d}(\phi)\right| dy'dz + \left(2 - \sqrt{2}\right)\hat{\delta}\int_{\mathbb{Q}} \left|\widetilde{d}(\hat{u}^\varepsilon)\right| dy'dz - \int_{\mathbb{Q}} \hat{f}\phi dy'dz.$$

As

$$\left\|\tilde{d}(\phi)\right\|_{L^r(\mathbb{Q})} \leq \|\phi\|_{W^{1,r}(\mathbb{Q})^3}, \forall\varepsilon \in]0,1[\,.$$

By Hölder's inequality, we get

$$\begin{aligned}(\widehat{\pi}^\varepsilon, div\,\phi) \leq \mu \|d(\hat{u}^\varepsilon)\|_{L^r(\mathbb{Q})}^{\frac{r}{q}} \|\phi\|_{W^{1,r}(\mathbb{Q})^3} + 2\hat{\delta}\,|\mathbb{Q}|^{\frac{1}{q}} \|\phi\|_{W^{1,r}(\mathbb{Q})^3} \\ + \left(2 - \sqrt{2}\right)\hat{\delta}\,|\mathbb{Q}|^{\frac{1}{q}} \|\hat{u}^\varepsilon\|_{W^{1,r}(\mathbb{Q})^3} + \left\|\hat{f}\right\|_{L^q(\mathbb{Q})^3} \|\phi\|_{W^{1,r}(\mathbb{Q})^3}.\end{aligned} \tag{5.11}$$

We apply the results of (5.1), we have:

$$\int_{\mathbb{Q}} \frac{\partial\widehat{\pi}^\varepsilon}{\partial y_i}\phi dy'dz \leq \mu C \|\phi\|_{W^{1,r}(\mathbb{Q})^3} + 2\hat{\delta}\,|\mathbb{Q}|^{\frac{1}{q}} \|\phi\|_{W^{1,r}(\mathbb{Q})^3} + \left(2 - \sqrt{2}\right)\hat{\delta}\,|\mathbb{Q}|^{\frac{1}{q}} C + \left\|\hat{f}\right\|_{L^q(\mathbb{Q})^3} \|\phi\|_{W^{1,r}(\mathbb{Q})^3}. \tag{5.12}$$

The same, we choose in (3.14): $\hat{v} = \hat{u}^\varepsilon - \phi, \phi \in W_0^{1,r}(\mathbb{Q})^3$, we obtain

$$\begin{aligned}-\int_{\mathbb{Q}} \frac{\partial\widehat{\pi}^\varepsilon}{\partial y_i}\phi dy'dz \leq \mu \|d(\hat{u}^\varepsilon)\|_{L^r(\mathbb{Q})}^{\frac{r}{q}} \|\phi\|_{W^{1,r}(\mathbb{Q})^3} + 2\hat{\delta}\,|\mathbb{Q}|^{\frac{1}{q}} \|\phi\|_{W^{1,r}(\mathbb{Q})^3} \\ + \left(2 - \sqrt{2}\right)\hat{\delta}\,|\mathbb{Q}|^{\frac{1}{q}} C + \left\|\hat{f}\right\|_{L^q(\mathbb{Q})^3} \|\phi\|_{W^{1,r}(\mathbb{Q})^3}.\end{aligned} \tag{5.13}$$

From (5.12) and (5.13), we deduce

$$\begin{aligned}\left|\int_{\mathbb{Q}} \frac{\partial\widehat{\pi}^\varepsilon}{\partial x_i}\phi dy'dz\right| \leq \mu \|d(\hat{u}^\varepsilon)\|_{L^r(\mathbb{Q})}^{\frac{r}{q}} \|\phi\|_{W^{1,r}(\mathbb{Q})^3} + 2\hat{\delta}\,|\mathbb{Q}|^{\frac{1}{q}} \|\phi\|_{W^{1,r}(\mathbb{Q})^3} \\ + \left(2 - \sqrt{2}\right)\hat{\delta}\,|\mathbb{Q}|^{\frac{1}{q}} C + \left\|\hat{f}\right\|_{L^q(\mathbb{Q})^3} \|\phi\|_{W^{1,r}(\mathbb{Q})^3}.\end{aligned} \tag{5.14}$$

Choosing $\phi = (\phi_1, 0, 0)$ then $\phi = (0, \phi_2, 0)$, in (5.14), we find

$$\left|\int_{\mathbb{Q}} \frac{\partial\widehat{\pi}^\varepsilon}{\partial x_i}\phi dy'dz\right| \leq \left(\mu C + 2\hat{\delta}\,|\mathbb{Q}|^{\frac{1}{q}} + \left\|\hat{f}_i\right\|_{L^q(\mathbb{Q})}\right)\|\phi\|_{W^{1,r}(\mathbb{Q})^3} + \left(2 - \sqrt{2}\right)\hat{\delta}\,|\mathbb{Q}|^{\frac{1}{q}} C.$$

Then (5.2) follows for $i = 1, 2$.

For get (5.3), we take in the inequality (5.14), $\phi = (0, 0, \phi_3)$, we find

$$\frac{1}{\varepsilon}\left|\int_{\mathbb{Q}} \frac{\partial \widehat{p^\varepsilon}}{\partial z}\ \phi dy' dz\right|$$
$$\leq \left(C + 2\hat{\delta}\left|\mathbb{Q}\right|^{\frac{1}{q}} + \left\|\hat{f}_3\right\|_{L^q(\mathbb{Q})}\right) \|\phi\|_{W^{1,r}(\mathbb{Q})^3} + \left(\sqrt{2} - 1\right)\hat{\delta}\left|\mathbb{Q}\right|^{\frac{1}{q}} C.$$

Which completes the proof of Lemma 5.1.

Now, the convergence (4.1)–(4.6) of Theorem 4.1 are a direct result of inequalities (5.1)–(5.3). Indeed, by (5.1), $\exists C > 0$ not related to ε, and verifying

$$\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial z}\right\|_{L^r(\mathbb{Q})} \leq C, \text{ for } i = 1, 2. \tag{5.15}$$

It is clear that (4.1) deduces directly from (5.15) and the using of the Poincaré's inequality in the fixed domain $\mathbb{Q}$. Also (4.2)–(4.4) follows from (5.1). The obtaining of (4.5) is done as in [6]. Finally, it is easy (4.6) follows from (5.2) and (5.3).

In order to proceed to the proof of strong convergence (4.7) of Theorem 4.2, it suffices to demonstrate the strong convergence of the integral term defined on Γ_b.

Lemma 5.2. *Let R is a regularization operator from $W^{-\frac{1}{2},r}(\Gamma_b)$ into $L^r(\Gamma_b)$, then the choice of R ensures the existence of a subsequence of $R(\widehat{\sigma}^\varepsilon_\eta(\hat{u}^\varepsilon, \widehat{\pi^\varepsilon}))$ strongly converges to $R(-\pi^\star)$ in $L^r(\Gamma_b)$.*

Proof of Lemma 5.2. From the equilibrium Eq (2.1), we have

$$-\operatorname{div}(\sigma^\varepsilon) = f^\varepsilon \quad \text{in } \mathbb{Q}^\varepsilon,$$

with $f^\varepsilon \in (L^q(\mathbb{Q}))^3$. By the results of Theorem 4.1, we deduce that $(\hat{u}^\varepsilon, \widehat{\pi^\varepsilon})$ are bounded in $\widetilde{\Theta}_z \times E_0^q(\mathbb{Q})$, then $\widehat{\sigma^\varepsilon}$ is bounded in

$$H_{\text{div}} = \left\{v \in (L^r(\mathbb{Q}))^3 : \operatorname{div}(v) \in L^q(\mathbb{Q})\right\},$$

which shows that there exists a subsequence converging weakly towards $\sigma^\star$. Now, we show that $\widehat{\sigma}_\eta(\widehat{u^\varepsilon}, \widehat{\pi^\varepsilon})$ converges weakly to $(-\pi^\star)$ in $W^{\frac{-1}{2},r}(\Gamma_b)$.

Indeed, as $\sigma_\eta^\varepsilon = \sigma_{ij}^\varepsilon \eta_i \eta_j$, $1 \leq i, j \leq 3$, we have

$$\begin{aligned}\widehat{\sigma}_\eta(\widehat{u^\varepsilon}, \widehat{\pi^\varepsilon}) = \sum_{i=1}^{2}&\left(\varepsilon^2\mu\left|\tilde{d}(\hat{u}^\varepsilon)\right|^{r-2}\frac{\partial \hat{u}_i^\varepsilon}{\partial y_i} + \varepsilon\hat{\delta}\left(\left|\tilde{d}(\hat{u}^\varepsilon)\right|\right)^{-1}\frac{\partial \hat{u}_i^\varepsilon}{\partial x_i} - \pi^\varepsilon\right)\\ &+\left(\varepsilon^2\mu\left|\tilde{d}(\hat{u}^\varepsilon)\right|^{r-2}\frac{\partial \hat{u}_3^\varepsilon}{\partial z} + \varepsilon\hat{\delta}\left(\left|\tilde{d}(\hat{u}^\varepsilon)\right|\right)^{-1}\frac{\partial \hat{u}_3^\varepsilon}{\partial z} - \pi^\varepsilon\right).\end{aligned}$$

Since $\widehat{\sigma^\varepsilon}$ is bounded in $H_{div}(\mathbb{Q})$, then there exists a subsequence converging weakly towards $\sigma^\star$ in $H_{div}(\mathbb{Q})$. Using the fact that the trace operator is continuous from $H_{div}(\mathbb{Q})$ into $W^{\frac{-1}{2},r}(\Gamma_b)$, we therefore obtain the weakly convergence of $\widehat{\sigma}_\eta(\widehat{u^\varepsilon}, \widehat{\pi^\varepsilon})$ to $\widehat{\sigma}_\eta(u^\star, \pi^\star)$ in $W^{\frac{-1}{2},r}(\Gamma_b)$. We apply now the results of Theorem 4.1 in the formula of $\widehat{\sigma}_\eta(\widehat{u^\varepsilon}, \widehat{\pi^\varepsilon})$, we obtain the desired result.

For the rest of proof, using the same techniques as in [2, 13], we get the result.

Proof of Theorem 4.2. For u^ε the solution on (3.8), we obtain for $v \in E_{div}^\varepsilon$

$$
\begin{aligned}
&F(u^\varepsilon, u^\varepsilon - v) - F(v, u^\varepsilon - v) - j(u^\varepsilon, v) + j(u^\varepsilon, u^\varepsilon)\\
\leq &(f^\varepsilon, v - u^\varepsilon) + F(v, u^\varepsilon - v).
\end{aligned}
$$

Using the inequality as ([24])

$$
\left(|a|^{r-2} a - |b|^{r-2} b, a - b\right) \geq (r-1)\left(|a| + |b|\right)^{r-2} |a-b|^2, \text{ for } a, b \in \mathbb{R}^n \text{ and } r \in]1,2[\tag{5.16}
$$

and by using the Korn's inequality, we find

$$
\begin{aligned}
(r-1)\,\mu C_K \sum_{i,j=1}^{3} \int_{\mathbb{Q}^\varepsilon} \left(\left|\frac{\partial u_i^\varepsilon}{\partial y_j}\right|^{r-2} + \left|\frac{\partial v_i}{\partial y_j}\right|^{r-2}\right)\left(\left|\frac{\partial}{\partial y_j}(u_i^\varepsilon - v_i)\right|^2\right) dy' dy_3\\
- j(u^\varepsilon, v) + j(u^\varepsilon, u^\varepsilon) \leq (f^\varepsilon, u^\varepsilon - v) + F(v, u^\varepsilon - v).
\end{aligned}
$$

We multiply the last formula by ε^{r-1}, as well as the convergence of Theorem 4.1, we get in the fixed domain $\mathbb{Q}$

$$
\begin{aligned}
(r-1)\,\mu C_K \sum_{i=1}^{2} \left\| \frac{\partial}{\partial z}\left(\hat{u}_i^\varepsilon - \widehat{v_i}\right)\right\|_{L^r(\mathbb{Q})}^r dy' dz - \widehat{j}(\hat{u}^\varepsilon, \widehat{v}) + \widehat{j}(\hat{u}^\varepsilon, \hat{u}^\varepsilon)\\
\leq \sum_{i=1}^{2} \int_{\mathbb{Q}} \hat{f}_i(\hat{u}_i^\varepsilon - \hat{v}_i) dy' dz + a(\widehat{v}, \hat{u}^\varepsilon - \widehat{v}).
\end{aligned}
$$

We pose, $\overline{u}^\varepsilon = (\hat{u}_1^\varepsilon, \hat{u}_2^\varepsilon)$, $u^\star = (u_1^\star, u_2^\star)$, $\overline{v} = (\widehat{v_1}, \widehat{v_2})$, so $\overline{v} \in \widetilde{\Xi}(E)$ and

$$
\begin{aligned}
&\limsup_{\varepsilon \to 0} \left[(r-1)\,\mu C_K \left\| \frac{\partial}{\partial z}\left(\overline{u}^\varepsilon - \widehat{v_i}\right)\right\|_{L^r(\mathbb{Q})}^r dy' dz - \widehat{j}(\overline{u}^\varepsilon, \overline{v}) + \widehat{j}(\overline{u}^\varepsilon, \overline{u}^\varepsilon)\right]\\
&\leq \mu \int_{\mathbb{Q}} \left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial \widehat{v_i}}{\partial z}\right)^2\right)^{\frac{r-2}{2}} \frac{\partial \overline{v}}{\partial z}\frac{\partial}{\partial z}\left(\overline{v} - u^\star\right) dy' dz + \sum_{i=1}^{2} \int_{\mathbb{Q}} \hat{f}_i(u_i^\star - \hat{v}_i) dy' dz.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&(r-1)\,\mu C_K \left\| \frac{\partial}{\partial z}\left(\overline{u}^\varepsilon - \overline{v}\right)\right\|_{L^r(\mathbb{Q})}^r dy' dz - \widehat{j}(\overline{u}^\varepsilon, \overline{v}) + \widehat{j}(\overline{u}^\varepsilon, \overline{u}^\varepsilon)\\
&\leq \mu \int_{\mathbb{Q}} \left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial \widehat{v_i}}{\partial z}\right)^2\right)^{\frac{r-2}{2}} \frac{\partial \overline{v}}{\partial z}\frac{\partial}{\partial z}\left(\overline{v} - u^\star\right) dy' dz + \sum_{i=1}^{2} \int_{\mathbb{Q}} \hat{f}_i(u_i^\star - \hat{v}_i) dy' dz + \eth,
\end{aligned}
$$

for $\varepsilon < \varepsilon(\eth)$, where $\eth > 0$ is arbitrary.

Therefore, $\exists \overline{v} \in \widetilde{\Xi}(E) : \overline{v} \to u^\star$ in $\widetilde{\Theta}_z$, which gives

$$
(r-1)\,\mu C_K \left\| \frac{\partial}{\partial z}\left(\overline{u}^\varepsilon - u^\star\right)\right\|_{L^r(\mathbb{Q})}^r dy' dz + \widehat{j}(\overline{u}^\varepsilon, \overline{u}^\varepsilon) - \widehat{j}(\overline{u}^\varepsilon, u^\star) \leq \eth, \forall \varepsilon < \varepsilon(\eth).
$$

Now, since $\liminf \widehat{j}(\overline{u}^\varepsilon) \geq \widehat{j}(u^\star)$, we deduce: $\overline{u}^\varepsilon \rightharpoonup u^\star$ in $\widetilde{\Theta}_z$. Furthermore, $\widehat{j}(\overline{u}^\varepsilon, \overline{u}^\varepsilon) \to \widehat{j}(\overline{u}^\varepsilon, u^\star)$ for $\varepsilon \to 0$, which gives the convergence (4.7).

If $r = 2$, we follow the same techniques but (5.16) we will be replaced by

$$\left(|a|^{r-2}\, a - |b|^{r-2}\, b, a - b\right) \geq (1/2)^{r-1}\, |a - b|^{r}\,,\ \text{for}\ \ a, b \in \mathbb{R}^{n}. \tag{5.17}$$

For the proof of the inequality (4.8), we introduce in (3.14) the condition of incompressibility of the fluid ($\operatorname{div}(\hat{u}^{\varepsilon}) = 0$ in $\mathbb{Q}$), then by the application of Minty's Lemma, we deduce:

$$\begin{aligned} & F(\hat{v}, \hat{v} - \hat{u}^{\varepsilon}) - \sum_{i=1}^{2} \left(\widehat{\pi}^{\varepsilon}, \frac{\partial \hat{v}_i}{\partial y_i}\right) - \left(\widehat{\pi}^{\varepsilon}, \frac{\partial \hat{v}_3}{\partial z}\right) + \widehat{j}(\hat{u}^{\varepsilon}, \hat{v}) - \widehat{j}(\hat{u}^{\varepsilon}, \hat{u}^{\varepsilon}) \\ & \geq \sum_{i=1}^{2} \int_{\mathbb{Q}} \hat{f}_i (\hat{v}_i - \hat{u}_i^{\varepsilon}) dy' dz + \int_{\mathbb{Q}} \varepsilon \hat{f}_3 (\hat{v}_3 - \hat{u}_3^{\varepsilon}) dy' dz, \forall \hat{v} \in E. \end{aligned}$$

We apply the convergence of Theorem 4.1, Lemma 5.2 and the fact $\widehat{j}$ is convex and lower semi-continuous, we obtain

$$\begin{aligned} & \mu \sum_{i=1}^{2} \int_{\mathbb{Q}} \frac{1}{2} \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial \hat{v}_i}{\partial z} \right)^2 \right)^{\frac{r-2}{2}} \frac{\partial \hat{v}_i}{\partial z} \frac{\partial (\hat{v}_i - u_i^{\star})}{\partial z} dy' dz \\ & - \int_{\mathbb{Q}} \pi^{\star} \left(\frac{\partial \hat{v}_1}{\partial y_1} + \frac{\partial \hat{v}_2}{\partial y_2} \right) dy' dz + \hat{\delta} \frac{\sqrt{2}}{2} \int_{\mathbb{Q}} \left(\left| \frac{\partial \hat{v}}{\partial z} \right| - \left| \frac{\partial u^{\star}}{\partial z} \right| \right) dy' dz \\ & + \int_{\Gamma_b} \hat{k} \left| R(-\pi^{\star}) \right| \left(|\hat{v} - s| - |u^{\star} - s| \right) dy' \\ & \geq \sum_{i=1}^{2} \int_{\mathbb{Q}} \hat{f}_i (\hat{v}_i - u_i^{\star}) dy' dz\,. \end{aligned}$$

From [5, Lemma 5.1], $\pi^{\star}$ independent of z, then applying Minty's lemma for the second time, we deduce (4.8).

Proof of Theorem 4.3. Choosing $\hat{v}$ in (4.8) (as in [3]) by: $\hat{v}_i = u_i^{\star} + \phi_i$, $i = 1, 2$, with $\phi_i \in W_0^{1,r}(\mathbb{Q})$, we find

$$\begin{aligned} & \mu \sum_{i=1}^{2} \int_{\mathbb{Q}} \frac{1}{2} \left(\frac{1}{2} \left(\frac{\partial u_1^{\star}}{\partial z} + \frac{\partial u_2^{\star}}{\partial z} \right)^2 \right)^{\frac{r-2}{2}} \frac{\partial u_i^{\star}}{\partial z} \frac{\partial \phi_i}{\partial z} dy' dz - \int_{\mathbb{Q}} \pi^{\star} \left(\frac{\partial \phi_1}{\partial y_1} + \frac{\partial \phi_2}{\partial y_2} \right) dy' dz \\ & = \sum_{i=1}^{2} \int_{\mathbb{Q}} \hat{f}_i \phi_i dy' dz. \end{aligned}$$

By using Green's formula, and choosing in the first step $\phi_1 = 0$ and $\phi_2 \in W_0^{1,r}(\mathbb{Q})$ then reversing this choice in the second step, we find (4.9).

Now, for the prove of (4.10), we cannot choose the test function as in [5, 6], since their works do not contain the term $\hat{\delta} \frac{\sqrt{2}}{2} \int_{\mathbb{Q}} \partial \hat{v} / \partial z \, dy' dz$. For this, we use the following techniques. Firstly, we choose $\hat{v}$ in (4.8) by $v = u^{\star} + \lambda \phi$ then $v = u^{\star} - \lambda \phi$, $\phi \in W_0^{1,r}(\mathbb{Q})^2$, we obtain

$$\mu \sum_{i=1}^{2} \int_{\mathbb{Q}} \frac{1}{2} \left(\frac{1}{2} \left(\frac{\partial u_1^{\star}}{\partial z} + \frac{\partial u_2^{\star}}{\partial z} \right)^2 \right)^{\frac{r-2}{2}} \frac{\partial u_i^{\star}}{\partial z} \frac{\partial (\lambda \phi_i)}{\partial z} dy' dz - \sum_{i=1}^{2} \int_{\mathbb{Q}} \pi^{\star}(y') \frac{\partial (\lambda \phi_i)}{\partial y_i} dy' dz \tag{5.18}$$

$$+\hat{\delta}\frac{\sqrt{2}}{2}\int_{\mathbb{Q}}\left(\left|\frac{\partial\left(u^{\star}+\lambda\phi\right)}{\partial z}\right|-\left|\frac{\partial u^{\star}}{\partial z}\right|\right)dy'dz \geq \sum_{i=1}^{2}\int_{\mathbb{Q}}\hat{f}_i(\lambda\phi_i)dy'dz, \quad \forall\phi\in W_0^{1,r}(\mathbb{Q})^2.$$

$$\begin{aligned}&\mu\sum_{i=1}^{2}\int_{\mathbb{Q}}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^{\star}}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^{\star}}{\partial z}\frac{\partial\phi_i}{\partial z}dy'dz-\sum_{i=1}^{2}\int_{\mathbb{Q}}\pi^{\star}(y')\frac{\partial(\lambda\phi_i)}{\partial y_i}dy'dz\\&-\hat{\delta}\frac{\sqrt{2}}{2}\int_{\mathbb{Q}}\left(\left|\frac{\partial(u^{\star}-\lambda\phi)}{\partial z}\right|-\left|\frac{\partial u^{\star}}{\partial z}\right|\right)dy'dz \leq \sum_{i=1}^{2}\int_{\mathbb{Q}}\hat{f}_i(\lambda\phi_i)dy'dz, \quad \forall\phi\in W_0^{1,r}(\mathbb{Q})^2.\end{aligned} \tag{5.19}$$

Secondly, dividing (5.18) and (5.19) by λ and the passage to the limit when λ tends to zero, we find

$$\begin{aligned}&\mu\sum_{i=1}^{2}\int_{\mathbb{Q}}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^{\star}}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^{\star}}{\partial z}\frac{\partial\phi_i}{\partial z}dy'dz-\sum_{i=1}^{2}\int_{\mathbb{Q}}\pi^{\star}(y')\frac{\partial\phi_i}{\partial y_i}dy'dz\\&+\hat{\delta}\tfrac{\sqrt{2}}{2}\sum_{i=1}^{2}\int_{\mathbb{Q}}\left(\left|\frac{\partial u^{\star}}{\partial z}\right|\right)^{-1}\frac{\partial u^{\star}}{\partial z}\frac{\partial\phi_i}{\partial z}dy'dz \geq \sum_{i=1}^{2}\int_{\mathbb{Q}}\hat{f}_i\phi_i dy'dz, \quad \forall\phi\in W_0^{1,r}(\mathbb{Q})^2,\end{aligned} \tag{5.20}$$

$$\begin{aligned}&\mu\sum_{i=1}^{2}\int_{\mathbb{Q}}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^{\star}}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^{\star}}{\partial z}\frac{\partial\phi_i}{\partial z}dy'dz-\sum_{i=1}^{2}\int_{\mathbb{Q}}\pi^{\star}(y')\frac{\partial\phi_i}{\partial y_i}dy'dz\\&+\hat{\delta}\tfrac{\sqrt{2}}{2}\sum_{i=1}^{2}\int_{\mathbb{Q}}\left(\left|\frac{\partial u^{\star}}{\partial z}\right|\right)^{-1}\frac{\partial u^{\star}}{\partial z}\frac{\partial\phi_i}{\partial z}dy'dz \leq \sum_{i=1}^{2}\int_{\mathbb{Q}}\hat{f}_i\phi_i dy'dz, \quad \forall\phi\in W_0^{1,r}(\mathbb{Q})^2.\end{aligned} \tag{5.21}$$

So the last two formulas, we give:

$$\begin{aligned}&\mu\sum_{i=1}^{2}\int_{\mathbb{Q}}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^{\star}}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^{\star}}{\partial z}\frac{\partial\phi_i}{\partial z}dy'dz-\sum_{i=1}^{2}\int_{\mathbb{Q}}\pi^{\star}(y')\frac{\partial\phi_i}{\partial y_i}dy'dz\\&+\hat{\delta}\tfrac{\sqrt{2}}{2}\sum_{i=1}^{2}\int_{\mathbb{Q}}\left(\left|\frac{\partial u^{\star}}{\partial z}\right|\right)^{-1}\frac{\partial u^{\star}}{\partial z}\frac{\partial\phi_i}{\partial z}dy'dz = \sum_{i=1}^{2}\int_{\mathbb{Q}}\hat{f}_i\phi_i dy'dz, \quad \forall\phi\in W_0^{1,r}(\mathbb{Q})^2.\end{aligned} \tag{5.22}$$

By the Green's formula, we get (4.10).

Proof of Theorem 4.4. We take in (4.8), $\hat{v}_i = u_i^{\star} + \lambda\phi_i$ for $i = 1, 2$, where $\phi_i \in W^{1,r}_{\Gamma_u\cup\Gamma_l}(\mathbb{Q})$ and

$$W^{1,r}_{\Gamma_u\cup\Gamma_l}(\mathbb{Q}) = \{\phi \in W^{1,r}(\mathbb{Q}) : \phi_i = 0 \text{ on } \Gamma_u \cup \Gamma_l\},$$

then

$$\begin{aligned}&\mu\sum_{i=1}^{2}\int_{\mathbb{Q}}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^{\star}}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^{\star}}{\partial z}\frac{\partial(\lambda\phi_i)}{\partial z}dy'dz-\sum_{i=1}^{2}\int_{\mathbb{Q}}\pi^{\star}(y')\frac{\partial(\lambda\phi_i)}{\partial y_i}dy'dz\\&+\hat{\delta}\frac{\sqrt{2}}{2}\sum_{i=1}^{2}\int_{\mathbb{Q}}\left(\left|\frac{\partial\left(\lambda\phi+u^{\star}\right)}{\partial z}\right|-\left|\frac{\partial u^{\star}}{\partial z}\right|\right)dy'dz+\int_{\Gamma_b}\hat{k}\left|R(-\pi^{\star})\right|\left(|\lambda\phi+s^{\star}-s|-|s^{\star}-s|\right)dy'\\&\geq \sum_{i=1}^{2}\int_{\mathbb{Q}}\hat{f}_i(\hat{v}_i-u_i^{\star})dy'dz.\end{aligned}$$

Dividing the last inequality by λ and the passage to the limit when λ tends to zero, we find

$$\begin{aligned} &\mu\sum_{i=1}^{2}\int_{\mathbb{Q}}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^\star}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^\star}{\partial z}\frac{\partial \phi_i}{\partial z}dy'dz \\ &-\sum_{i=1}^{2}\int_{\mathbb{Q}}\pi^\star(y')\frac{\partial \phi_i}{\partial y_i}dy'dz+\hat{\delta}\tfrac{\sqrt{2}}{2}\sum_{i=1}^{2}\int_{\mathbb{Q}}\left(\left|\frac{\partial u^\star}{\partial z}\right|\right)^{-1}\frac{\partial u^\star}{\partial z}\frac{\partial \phi_i}{\partial z}dy'dz \\ &+\sum_{i=1}^{2}\int_{\Gamma_b}\hat{k}\left|R(-p^\star)\right|\frac{\phi_i\left(s_i^\star-s_i\right)}{|s^\star-s|}dy' \geq \sum_{i=1}^{2}\int_{\mathbb{Q}}\hat{f}_i(\hat{v}_i-u_i^\star)dy'dz. \end{aligned} \tag{5.23}$$

Finally, using the Green formula in (5.23) and from (4.10), we find

$$\sum_{i=1}^{2}\int_{\Gamma_b}\widehat{k}\left|R(\widehat{\sigma}_\eta(-\pi^\star))\right|\phi_i(s_i^\star-s_i)dy'-\int_{\Gamma_b}\widehat{\mu}\tau^\star\phi\left|s^\star-s\right|dy'\geq 0,\ \forall\phi\in\left(W^{1,r}_{\Gamma_u\cup\Gamma_l}(\mathbb{Q})\right)^2.$$

This last formula holds for any $\phi\in D(\Gamma_b)^2$, but given the density of $D(\Gamma_b)$ in $L^r(\Gamma_b)$, we find the desired result (4.11). For the proof of (4.12), we follow the same techniques as in [4].

To establish (4.13), we integrate twice (4.10) from 0 to z, we get

$$\begin{aligned} &-\int_0^z\mu B^\star(y',\xi)\frac{\partial u^\star}{\partial\xi}(y';\xi)\,d\xi-\hat{\delta}\tfrac{\sqrt{2}}{2}\int_0^z\frac{\partial u^\star/\partial z}{|\partial u^\star/\partial z|}d\xi+\mu\tau^\star(y')\,z \\ &+\hat{\delta}\tfrac{\sqrt{2}}{2}\frac{s^\star(y')}{|s^\star(y')|}\,z=\int_0^z\int_0^\xi\hat{f}(y',t)\,dtd\xi-\frac{z^2}{2}\nabla\pi^\star(y'). \end{aligned} \tag{5.24}$$

Substituting z by h in (4.24), we get

$$\begin{aligned} &-\int_0^h\mu B^\star(y',\xi)\frac{\partial u^\star}{\partial\xi}(y';\xi)\,d\xi-\hat{\delta}\tfrac{\sqrt{2}}{2}\int_0^h\frac{\partial u^\star/\partial z}{|\partial u^\star/\partial z|}d\xi+\mu\tau^\star(y')\,h \\ &+\hat{\delta}\tfrac{\sqrt{2}}{2}\frac{s^\star(y')}{|s^\star(y')|}\,h=\int_0^h\int_0^\xi\hat{f}(y',t)\,dtd\xi-\frac{h^2}{2}\nabla\pi^\star(y'). \end{aligned} \tag{5.25}$$

We integrate (5.24) from 0 to z, it comes:

$$\begin{aligned} &-\int_0^h\int_0^y\mu B^\star(y',\xi)\frac{\partial u^\star}{\partial z}(y';\xi)\,d\xi dy-\hat{\delta}\tfrac{\sqrt{2}}{2}\int_0^h\int_0^y\frac{\partial u^\star/\partial z}{|\partial u^\star/\partial z|}d\xi dy+\mu\tau^\star(y')\frac{h^2}{2} \\ &+\hat{\delta}\tfrac{\sqrt{2}}{4}\frac{s^\star(y')}{|s^\star(y')|}h^2=\int_0^h\int_0^y\int_0^\xi\hat{f}(y',t)\,dtd\xi dy-\frac{h^3}{6}\nabla\pi^\star(y'). \end{aligned} \tag{5.26}$$

From (5.25), we deduce

$$\begin{aligned} &\left[\mu\tau^\star(y')+\hat{\delta}\tfrac{\sqrt{2}}{4}\frac{s^\star(y')}{|s^\star(y')|}\right]\frac{h^2}{2}=\frac{\mu h}{2}\int_0^h B^\star(y',\xi)\frac{\partial u^\star}{\partial\xi}(y';\xi)\,d\xi \\ &+\hat{\delta}h\frac{\sqrt{2}}{4}\int_0^h\tfrac{\partial u^\star/\partial z}{|\partial u^\star/\partial z|}d\xi+\frac{h}{2}\int_0^h\int_0^y\hat{f}(y',\xi)\,d\xi dy-\frac{h^3}{4}\nabla\pi^\star(y'). \end{aligned} \tag{5.27}$$

By (5.26) and (5.27), we deduce (4.13).

Proof of Theorem 4.5. Suppose that the boundary value problem (4.8) admits two solutions which we denote by $\left(u^{\star,1},\pi^{\star,1}\right)$ and $\left(u^{\star,2},\pi^{\star,2}\right)$. Taking $\hat{v}=u^{\star,2}$ and $\hat{v}=u^{\star,1}$ respectively, as test function in (4.8) then by summing two inequalities, we get

$$\begin{aligned}&\mu\sum_{i=1}^{2}\int_{\mathbb{Q}}\left(\frac{1}{2}\right)^{\frac{r}{2}}\left(\sum_{i=1}^{2}\left(\frac{\partial u_i^{\star,1}}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^{\star,1}}{\partial z}\frac{\partial}{\partial z}(u_i^{\star,1}-u_i^{\star,2})dy'dz\\&-\mu\sum_{i=1}^{2}\int_{\mathbb{Q}}\left(\frac{1}{2}\right)^{\frac{r}{2}}\left(\sum_{i=1}^{2}\left(\frac{\partial u_i^{\star,2}}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^{\star,2}}{\partial z}\frac{\partial}{\partial z}(u_i^{\star,1}-u_i^{\star,2})dy'dz\\&-\int_{\Gamma_b}\hat{k}\left|R(-\pi^{\star,1})-R(-\pi^{\star,2})\right||u_i^{\star,1}-u_i^{\star,2}|dy'\leq 0.\end{aligned}\tag{5.28}$$

We apply (5.16) and (5.17), we obtain

$$\mu\left\|\frac{\partial}{\partial z}\left(u^{\star,1}-u^{\star,2}\right)\right\|_{(L^r(\mathbb{Q}))^2}^{r}\leq\|\widehat{k}\|_{L^\infty(\Gamma_b)}\int_{\Gamma_b}\left|R(-\pi^{\star,1})-R(-\pi^{\star,2})\right||u^{\star,1}-u^{\star,2}|dy'.\tag{5.29}$$

By the inequality (3.16), then we apply the Hölder inequality on the second term of (5.29), we have

$$\begin{aligned}&\left\|\frac{\partial}{\partial z}\left(u^{\star,1}-u^{\star,2}\right)\right\|_{(L^r(\mathbb{Q}))^2}^{r}\\&\leq h^\star\|\widehat{k}\|_{L^\infty(\Gamma_b)}C_0\left(\int_{\Gamma_b}\left|R(-\pi^{\star,1})-R(-\pi^{\star,2})\right|^q dy'\right)^{1/q}\left\|\frac{\partial}{\partial z}\left(u^{\star,1}-u^{\star,2}\right)\right\|_{(L^r(\mathbb{Q}))^2}\end{aligned}$$

whence

$$\left\|\frac{\partial}{\partial z}\left(u^{\star,1}-u^{\star,2}\right)\right\|_{(L^r(\mathbb{Q}))^2}^{r-1}\leq\frac{h^\star\|\widehat{k}\|_{L^\infty(\Gamma_b)}C_0}{\mu}\left\|R(-\pi^{\star,1})-R(-\pi^{\star,2})\right\|_{L^q(\Gamma_b)}.\tag{5.30}$$

Using the fact that R is a linear continuous operator $W^{-\frac{1}{2},r}(\Gamma_b)$ into $L^r(\Gamma_b)$, there exists a constant C_1 depending on R, such that

$$\left\|R(-\pi^{\star,1})-R(-\pi^{\star,2})\right\|_{L^q(\Gamma_b)}\leq C_1\left\|\pi^{\star,1}-\pi^{\star,2}\right\|_{L^q(\Gamma_b)}.\tag{5.31}$$

Combining (5.30) and (5.31) we deduce that if $\|\widehat{k}\|_{L^\infty(\Gamma_b)}\leq\overline{k}$ for sufficiently small $\overline{k}$, then we have

$$\left\|\frac{\partial}{\partial z}\left(u^{\star,1}-u^{\star,2}\right)\right\|_{(L^r(\mathbb{Q}))^2}=0.$$

Using Poincaré's inequality, we get

$$\left\|u^{\star,1}-u^{\star,2}\right\|_{\widetilde{\Theta}_z}=0.$$

The uniqueness of the π^* in the $E_0^q(\Gamma_b)$ follows from (4.13), in fact we take first in the Reynolds equation (4.13) the pressure value $\pi^*=\pi^{\star,1}$ then $\pi^*=\pi^{\star,2}$ respectively, at the end by subtracting the equations obtained, it becomes:

$$\int_{\Gamma_b}\frac{h^3}{12}\nabla\left(\pi^{\star,1}-\pi^{\star,2}\right)\nabla v dy'=0.$$

Choosing $v=\pi^{\star,1}-\pi^{\star,2}$, and by Poincaré's inequality, we find

$$\pi^{\star,1}=\pi^{\star,2},\text{ almost everywhere in }\Gamma_b.$$

This ends the proof of the Theorem 4.5.

6. Conclusions

The aim of this study is to examine the strong convergence of the velocity of a non-Newtonian incompressible fluid whose viscosity follows the power law with Coulomb friction, where we give in a first step the description of the problem and basic equations. Then, we present the functional framework. The following paragraph is reserved for the main convergence results. Finally, we give the detail of the proofs of these results. In the future work we will extend and develop our work to new space.

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Conflict of interest

The authors declares that they have no conflicts of interest.

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