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Research article

On the oscillation of certain class of conformable Emden-Fowler type elliptic partial differential equations

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Abstract: This article examines the oscillatory behaviour of solutions to a particular class of conformable elliptic partial differential equations of the Emden-Fowler type. Using the Riccati method, we create some new necessary conditions for the oscillation of all solutions. The previously discovered conclusions for the integer order equations are expanded upon by these additional findings. We provide an example to demonstrate the usefulness of our new finding.

Keywords: oscillation; conformable partial differential equations; elliptic partial differential equations

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1. Introduction

An essential area of applied theory of differential equations that deals with the study of oscillatory processes in the social and technological sciences is the theory of oscillations, see [10, 18–23, 25, 34, 39, 41].

Fractional differential equations have popularity and gained importance during the past few years. Several researchers trying to develop the earlier work with definitions of fractional derivatives like Riemann-Liouville and Caputo derivative, see [17, 24, 28, 32, 35]. Khalil et al. [27] introduced a new fractional derivative called the conformable derivative. Oscillation of theory conformable were done by several authors [1, 3, 8, 16, 26, 29, 38]. The conformable fractional derivative seeks to extend the common derivative while satisfying the properties of nature and provides a fresh approach.

The Emden-Fowler equations have been considered one of the important classical objects in the theory of differential equations. This type of equation has a variety of interesting physical applications occurring in astrophysics and atomic physics. The oscillation for the Emden-Fowler equation has interest over the last 50 years, many results appeared in the oscillatory behavior of the Emden-Fowler differential equations, see [5, 9, 15, 40, 42] and references cited therein. The references [4, 13, 14] lists numerous studies have been conducted on Emden-Fowler generalization when the gradient term is utilized.

Elliptic partial differential equations have various uses in physics as well as practically all branches of mathematics, including harmonic analysis, geometry, and Lie theory. The Laplacian equation and Poisson equation serve as the fundamental illustration of an elliptic PDE. The theory of elliptic PDE have application in electrostatics, heat and mass diffusion and hydrodynamics, see [2, 6, 11, 12, 30, 33, 36, 43, 44].

Throughout the past few decades, the issue of oscillation and nonoscillation of elliptic partial differential equation solutions has drawn a lot of attention. To the present time, there exists almost no literature in conformable Emden-Fowler type elliptic partial differential equations is of the form

$$\Delta_x^\alpha u + c(x) \mid u \mid^{\beta-1} u \mid D^\alpha(u) \mid^{1-\beta} = 0 \qquad \Delta_x^\alpha u = \sum_{i=1}^{n} \frac{\partial^\alpha}{\partial x_i^\alpha} (D^\alpha u) \tag{1.1}$$

where $\alpha, \beta \in (0, 1)$, Δ_x^α is the conformable nabla operator, $D^\alpha(u) = (u_{x_1}, u_{x_2}, ..., u_{x_n})$ for the conformable gradient of order α of u with respect to the spatial variable x, $\| \,.\, \|$ is the usual Euclidean norm on $\mathbb{R}^n$. $c(x) : \mathbb{R}^n \to \mathbb{R}$ is potential function $c \in L^1_{loc}(\Omega)$, with $\Omega : \{x \in \mathbb{R}^n :\| x \|\geq 1\}$.

By a solution of (1.1), we mean a function $u(x) : \mathbb{R}^n \to \mathbb{R}$ which is absolutely continuous with first α-fractional derivative in every compact subset of Ω and satisfies the Eq (1.1) a.e. in Ω.

A bounded domain $G \subset \Omega$ is said to be a *nodal domain* for (1.1) if there exists a nontrival function $u \in C^{2\alpha}(G; \mathbb{R}) \cap C(\overline{G}; \mathbb{R})$ such that u is a solution of (1.1) with $u = 0$ on ∂G. Equation (1.1) is said to be *nodally oscillatory* in Ω if for any $r > 0$ Eq (1.1) has a nodal domain contained in $\Omega_r = \Omega \cap \{x \in \mathbb{R}^n : |x| > r\}$. Some different approach in the oscillation theory form the nodal domains. A bounded domain is said to be the nodal domain of a nontrivial solution and $u|_{\partial\Omega} = 0$.

By the Hartman-Winter Theorem, we get the function $C(r)$

$$C(r) = \frac{\beta}{r^\alpha} \int_1^r \int_{\Omega(1,r)} r^{1-n+\lambda} c(x) d_\alpha x d_\alpha r \tag{1.2}$$

where

$$\Omega(1,r) = \{x \in \mathbb{R}^n : 1 \le |x| \le r\}.$$

The motivation for this work comes from the papers [7, 31, 37]. In this paper, we studied the α-fractional partial differential equations for Emden-Fowler type elliptic equations (1.1) using the Riccati technique by considering the two cases

i) there exists a finite limit $\lim_{r\to\infty} C(r) = C_0$;
ii) the case *i*) fails to hold and $\liminf_{r\to\infty} C(r) > -\infty$.

We introduce several fundamental definitions, properties, and lemmas that are helpful throughout the rest of this study in the following part, Section 2. We established the main findings in Section 3. We provide an example to emphasize the key findings in Section 4.

2. Preliminaries

This part introduces several fundamental terms, characteristics, and lemmas that will be helpful throughout the rest of the paper.

Definition 2.1. *[27] Given $f : [0,\infty) \to \mathbb{R}$. Then, the conformable fractional derivative of f of order α is defined by*

$$T_\alpha(f)(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for every $t > 0$, $\alpha \in (0,1)$. If f is α-differentiable in some $(0,a)$, $a > 0$ and $\lim_{t\to o^+} f^\alpha(t)$ exists, then we define

$$f^\alpha(0) = \lim_{t\to o^+} f^\alpha(t).$$

Definition 2.2. *[27] Let $a \ge 0$ and $t \ge a$. Also, let f be a functiondefined on $(a,t]$ and $\alpha \in (0,1)$. Then, the α - fractional integral of f is given by*

$$I^a_\alpha(f)(t) = I^\alpha_1(t^{\alpha-1})(f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral and $\alpha \in (0,1)$.

Properties 2.1. *[27] Let $\alpha \in (0,1]$ and f, g be α-differentiable at some point $t > 0$. Then,*

(1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$.

(2) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

(3) $T_\alpha(t^p) = pt^{p-\alpha}$.

(4) $T_\alpha(C) = 0$, $C \in \mathbb{R}$.

(5) $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

(6) If f is differential, then $T_\alpha(f(t)) = t^{1-\alpha}\frac{df(t)}{dt}$.

Definition 2.3. *[3] Let f be a function with m variable $x_1, ..., x_m$, the conformable partial derivative of f of order $0 < \alpha \le 1$ in x_i is defined as follows*

$$\frac{\partial^\alpha}{\partial x_i^\alpha} f(x_1, ..., x_m) = \lim_{\epsilon \to 0} \frac{f(x_1, ..x_{i-1}, x_i + \epsilon x_i^{1-\alpha} ...., x_m) - f(x_1, ..., x_m)}{\epsilon}.$$

Definition 2.4. *[3] Consider the scalar field $f(x)$ and the vector field $F(x)$ that are assumed to possess partial conformable derivative of order α with respect to all the Cartesian coordinates $x_i, i = 1, 2, 3$. We define the conformable gradient of order α of the scalar field f as follows*

$$\nabla_x^\alpha f = \textstyle\sum_{i=1}^3 (\partial_{x_i}^\alpha f) e_i,$$

where e_i is the unit vector in the i direction. The conformable gradient of order α of the vector fieldF is defined as follows

$$\nabla_x^\alpha F = \textstyle\sum_{i=1}^3 (\partial_{x_i}^\alpha F_i).$$

Definition 2.5. *A solution of Eq (1.1) is called oscilatory if it has arbitrarily large zeros in G, and is called nonoscillatory otherwise. Equation (1.1) are said to be oscillatory if all their solutions are oscillatory.*

Lemma 2.1. *If $\overrightarrow{\mathrm{r}} = \overrightarrow{\mathrm{x}}_\mathrm{i} + \overrightarrow{\mathrm{y}}_\mathrm{j} + \overrightarrow{\mathrm{z}}_\mathrm{k}$ and $r = |\overrightarrow{\mathrm{r}}|$ then $D^\alpha f(r) = r^{1-\alpha} D(f(r))$.*

Proof. If f is differentiable, then using the Properties 2.1, we get

$$D^\alpha f(r) = r^{1-\alpha} D(f(r)).$$

The proof of Lemma 2.1 is completed. □

Lemma 2.2. *If $f(t) = At - Bt^{\frac{\beta+1}{\beta}}$, then*

$$At - Bt^{\frac{\beta+1}{\beta}} \le \left(\frac{A\beta}{B(\beta+1)}\right)^\beta \frac{A}{\beta+1}$$

where $\beta \in (0, 1)$.

Proof. If f differentiable, then

$$f'(t) = A - \frac{\beta+1}{\beta} Bt^{\left(\frac{\beta+1}{\beta}-1\right)} = A - \frac{\beta+1}{\beta} Bt^{\frac{1}{\beta}}$$

so that the maximum point of f is realized in

$$t = \left(\frac{A\beta}{B(\beta+1)}\right)^\beta.$$

Consequently,

$$f(t) \le A\left(\frac{A\beta}{B(\beta+1)}\right)^\beta - B\left(\frac{A\beta}{B(\beta+1)}\right)^{\beta+1} = \left(\frac{A\beta}{B(\beta+1)}\right)^\beta \frac{A}{\beta+1}.$$

Hence, the proof is completed. □

First we introduce the Riccati technique. There exists a $\Omega_r = \{x \in \mathbb{R}^n :\| x \|\geq r\}$ and a solution u of (1.1) which is positive on Ω_r. Let vector function $W = \left(\frac{|D^\alpha u|^{\beta-1} D^\alpha u}{|u|^{\beta-1} u}\right)$ be the solution of Riccati equation defined on the set Ω_r.

The i - component of gradient is

$$\frac{\partial^\alpha W_i}{\partial x_i^\alpha} = \frac{\frac{\partial^\alpha}{\partial x_i^\alpha}\left(|D^\alpha u|^{\beta-1} D^\alpha u\right)}{(|u|^{\beta-1} u)} - \frac{\beta\left(|D^\alpha u|^{\beta-1} D^\alpha u\right)\frac{\partial^\alpha u}{\partial x_i^\alpha}}{|u|^{\beta-1} u^2}.$$

Taking the summation, we get

$$div_\alpha W + \beta c(x) + \beta \| W \|^{\frac{\beta+1}{\beta}} = 0, \tag{2.1}$$

where $div_\alpha W = \sum_{i=1}^n \dfrac{\partial^\alpha W_i}{\partial x_i^\alpha}$.

For the next, we define

$$\begin{aligned} \Omega(a) &= \{x \in \mathbb{R}^n : a \leq r\}; \\ \Omega(a,b) &= \{x \in \mathbb{R}^n : a \leq r \leq b\}; \\ S(a) &= \{x \in \mathbb{R}^n : r = a\}. \end{aligned}$$

Lemma 2.3. *Let the Eq* (1.1) *be nonoscillatory, i.e.,* (1.1) *has a positive solution on* Ω_a *for some* $a \leq 1$. *The following statements are equivalent:*

i)

$$\int_{\Omega(a,\infty)} r^{1-n+\lambda} \| W \|^{\frac{\beta+1}{\beta}} \, d_\alpha x < \infty; \tag{2.2}$$

ii) there exists a finite limit

$$\lim_{r\to\infty} C(r) = C_0; \tag{2.3}$$

iii) there exists a infinite limit

$$\liminf_{r\to\infty} C(r) > -\infty. \tag{2.4}$$

Proof. Let the Eq (1.1) be nonoscillatory. There exists a number $a \in \mathbb{R}^+$ and a solution u of (1.1) which is positive on Ω_a. Let, vector function $W = \left(\frac{|D^\alpha u|^{\beta-1} D^\alpha u}{|u|^{\beta-1} u}\right)$ is the solution of Riccati equation defined on the set Ω_a.

$$div_\alpha W + \beta c(x) + \beta \| W \|^{\frac{\beta+1}{\beta}} = 0.$$

By the Gauss divergence theorem [37],

$$\begin{aligned} div_\alpha(r^{\alpha-n+\lambda} W) =& r^{\alpha-n+\lambda} div_\alpha W + D^\alpha(r^{\alpha-n+\lambda})\overrightarrow{W} \\ =& r^{\alpha-n+\lambda} div_\alpha W + r^{1-\alpha} D(r^{\alpha-n+\lambda})\langle W, e_i\rangle, \end{aligned}$$

that is,

$$div_\alpha(r^{\alpha-n+\lambda}W) = r^{\alpha-n+\lambda}div_\alpha W + (\alpha - n + \lambda)r^{-n+\lambda}\langle W, e_i\rangle,$$

where e_i is the unit vector in the i direction and $\langle W, e_i\rangle$ is the usual scalar product in $\mathbb{R}$, implies that W satisfies the equality

$$\begin{aligned}
&\int_{S(r)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS - \int_{S(a)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS + \\
&\beta\int_{\Omega(a,r)} r^{\alpha-n+\lambda}c(x)dx + \beta\int_{\Omega(a,r)} r^{\alpha-n+\lambda} \parallel W \parallel^{\frac{\beta+1}{\beta}} dx - \\
&(\alpha - n + \lambda)\int_{\Omega(a,r)} r^{-n+\lambda}\langle W, e_i\rangle dx = 0. \qquad (2.5)
\end{aligned}$$

Therefore, $i) \Rightarrow ii)$. Next, we suppose that (2.2) holds, then the Cauchy inequality gives

$$\begin{aligned}
\int_{\Omega(a,r)} r^{-n+\lambda}\langle W, e_i\rangle dx &\leq \left(\int_{\Omega(a,r)} r^{\alpha-n+\lambda} \parallel W \parallel^{\frac{\beta+1}{\beta}} dx\right)^{\frac{1}{2}} \left(\int_{\Omega(a,r)} r^{-\alpha-n+\lambda} \parallel W \parallel^{\frac{\beta-1}{\beta}} dx\right)^{\frac{1}{2}} \\
&= \left(\int_{\Omega(a,t)} r^{\alpha-n+\lambda} \parallel W \parallel^{\frac{\beta+1}{\beta}} dx\right)^{\frac{1}{2}} \left(\omega_n \parallel W \parallel^{\frac{\beta-1}{\beta}} \int_a^r r^{-\alpha-1+\lambda}dr\right)^{\frac{1}{2}},
\end{aligned}$$

where ω_n is the measure of the unit sphere in $\mathbb{R}^n$ and $\omega_n = \frac{2\Pi^{\frac{n}{2}}}{\Gamma\frac{n}{2}}$. Therefore,

$$\int_{\Omega(a,\infty)} r^{-n+\lambda}\langle W, e_i\rangle dx < \infty. \qquad (2.6)$$

Combining (2.5) and (2.6), we get

$$\begin{aligned}
\widehat{C} - \beta\int_{\Omega(1,r)} r^{\alpha-n+\lambda}c(x)dx &= \int_{s(r)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS \\
&+ (\alpha - n + \lambda)\int_{\Omega(r,\infty)} r^{-n+\lambda}\langle W, e_i\rangle dx - \beta\int_{\Omega(r,\infty)} r^{\alpha-n+\lambda} \parallel W \parallel^{\frac{\beta+1}{\beta}} dx \qquad (2.7)
\end{aligned}$$

where

$$\begin{aligned}
\widehat{C} &= \int_{s(a)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS + (\alpha - n + \lambda)\int_{\Omega(a,\infty)} r^{-n+\lambda}\langle W, e_i\rangle dx \\
&+ \beta\int_{\Omega(1,a)} r^{\alpha-n+\lambda}c(x)dx - \beta\int_{\Omega(a,\infty)} r^{\alpha-n+\lambda} \parallel W \parallel^{\frac{\beta+1}{\beta}} dx
\end{aligned}$$

is a finite number. Next, we will show that

$$\widehat{C} = \alpha C_0. \qquad (2.8)$$

One can prove this results as in the proof of the Lemma 2.1 in [33]. This implies $ii) \Rightarrow iii)$ is trivial. To show, $iii) \Rightarrow i)$, we suppose that (2.4) holds and (2.2) does not hold. Let us define the function

$$\Phi(r) := \int_a^r \beta\int_{\Omega(a,r)} r^{1-n+\lambda} \parallel W \parallel^{\frac{\beta+1}{\beta}} d_\alpha x d_\alpha r.$$

This function satisfies

$$\lim_{r\to\infty}\frac{\Phi(r)}{r}\to\infty \quad for \quad r\to\infty \tag{2.9}$$

and

$$\beta\int_{\Omega(a,\infty)} r^{1-n+\lambda}\parallel W\parallel^{\frac{\beta+1}{\beta}} d_\alpha x=\infty.$$

Similarly, the rest of the proof follows from the Lemma 2.1 [33]. □

For the next, we define

$$\sigma(r)=\int_{S(r)} r^{\alpha-n+\lambda}\langle W,e_i\rangle ds, \tag{2.10}$$

$$Q(r)=r^\beta\left(\alpha C_o-\beta\int_{\Omega(1,r)} r^{1-n+\lambda}c(x)d_\alpha x\right),$$

and

$$H(r)=\frac{1}{r^\beta}\int_{\Omega(1,r)} r^{3-n+\lambda}c(x)d_\alpha x$$

where $\alpha,\beta,\lambda\in(0,1)$.

Lemma 2.4. *Consider* (2.3) *holds and the Eq* (1.1) *have a nonoscillatory solution. Then the equation*

$$Q(r)-\left(\frac{(\alpha-n+\lambda)r^{1-\alpha}}{\beta+\alpha-1}+1\right)l+\frac{\beta r^{(1-(\alpha+\lambda))}}{\omega_n(\alpha+\beta+\lambda-1)}l^{\frac{\beta+1}{\beta}}\leq 0,$$

and

$$r^{\beta-2}\left(\beta r^\beta H(r)-\beta\tau_\epsilon^\beta H(\tau_\epsilon)-\tau_\epsilon^2\sigma(\tau_\epsilon)\right)-\left(\frac{2r^{\alpha-1}}{1+\alpha-\beta}+\frac{(\alpha-n+\lambda)r^{1-\alpha}}{3-\alpha-\beta}-1\right)L$$
$$+\frac{\beta r^{(1-(\alpha+\lambda))}}{\omega_n(3-\alpha-\lambda-\beta)}L^{\frac{\beta+1}{\beta}}\leq 0$$

are solvable.

Proof. Let W be the solution of the Riccati Eq (2.1) defined on W_a for some $a\in\mathbb{R}$. From Cauchy inequality gives

$$\sigma^{\frac{\beta+1}{\beta}}(r)=\omega_n r^{\alpha-1+\lambda}\int_{S(r)} r^{\alpha-n+\lambda}\parallel W\parallel^{\frac{\beta+1}{\beta}} ds. \tag{2.11}$$

The equalities (2.7) and (2.8) gives

$$r^\beta\sigma(r)=Q(r)+\frac{\beta r^\beta}{\omega_n}\int_r^\infty s^{1-(\alpha+\lambda)}\sigma^{\frac{\beta+1}{\beta}}(s)ds-(\alpha-n+\lambda)r^\beta\int_r^\infty s^{-\alpha}\sigma(s)ds. \tag{2.12}$$

Differentiate (2.5) with respect to r, multiply by r^2 and integrating over τ to R, we get

$$R^{\beta}\sigma(R) = R^{\beta-2}(\tau^2\sigma(\tau) + \beta\tau^{\beta}H(\tau) - \beta R^{\beta}H(R)) + 2R^{\beta-2}\int_{\tau}^{R} s^{-\alpha}\sigma(s)ds +$$
$$R^{\beta-2}\int_{\tau}^{R}(\alpha - n + \lambda)s^{2-\alpha}\sigma(s)ds - \frac{R^{\beta-2}}{\omega_n}\int_{\tau}^{R}\beta s^{(3-(\alpha+\lambda))}\sigma^{\frac{\beta+1}{\beta}}(s)ds.$$

Now, substituting $R = r$ and $\tau = \tau_\epsilon$

$$r^{\beta}\sigma(r) = r^{\beta-2}(\tau_\epsilon^2\sigma(\tau_\epsilon) + \beta\tau_\epsilon^{\beta}H(\tau) - \beta r^{\beta}H(r)) + 2r^{\beta-2}\int_{\tau}^{r} s^{-\alpha}\sigma(s)ds +$$
$$r^{\beta-2}\int_{\tau}^{r}(\alpha - n + \lambda)s^{2-\alpha}\sigma(s)ds - \frac{r^{\beta-2}}{\omega_n}\int_{\tau}^{r}\beta s^{(3-(\alpha+\lambda))}\sigma^{\frac{\beta+1}{\beta}}(s)ds. \tag{2.13}$$

Let us introduce the notation

$$l = \liminf r^{\beta}\sigma(r). \qquad L = \limsup r^{\beta}\sigma(r).$$

Obviously, for any $0 < \epsilon < \min\{l, 1 - L\}$ there exists $\tau_\epsilon > r_0$ and $r_\epsilon > \tau_\epsilon$ such that

$$l - \epsilon < r^{\beta}\sigma(r) < L + \epsilon. \tag{2.14}$$

Due to this fact we have from (2.12) and (2.13) gives

$$l - \epsilon > Q(r) - \frac{(\alpha - n + \lambda)r^{1-\alpha}}{(\alpha + \beta - 1)}(l - \epsilon) + \frac{\beta r^{1-(\alpha+\lambda)}}{\omega_n(\alpha + \beta + \lambda - 1)}(l - \epsilon)^{\frac{\beta+1}{\beta}},$$

that is,

$$L + \epsilon < r^{\beta-2}(\tau_\epsilon^2\rho(\tau_\epsilon) + \beta\tau_\epsilon^{\beta}H(\tau_\epsilon) - \beta r^{\beta}H(r)) +$$
$$\left(\frac{2r^{\alpha-1}}{(1 + \alpha - \beta)} + \frac{(\alpha - n + \lambda)r^{1-\alpha}}{3 - \alpha - \beta}\right)(L + \epsilon) - \left(\frac{\beta r^{(1-(\alpha+\lambda))}}{\omega_n(3 - \alpha - \beta - \lambda)}\right)(L + \epsilon)^{\frac{\beta+1}{\beta}}.$$

Therefore,

$$Q(r) - \left(\frac{(\alpha - n + \lambda)r^{1-\alpha}}{\alpha + \beta - 1} + 1\right)l + \frac{\beta r^{1-(\alpha+\lambda)}}{\omega_n(\alpha + \beta + \lambda - 1)}l^{\frac{\beta+1}{\beta}} \leq 0$$

and

$$r^{\beta-2}\left(\beta r^{\beta}H(r) - \beta\tau_\epsilon^{\beta}H(\tau_\epsilon) - \tau_\epsilon^2\sigma(\tau_\epsilon)\right) - \left(\frac{2r^{\alpha-1}}{1 + \alpha - \beta} + \frac{(\alpha - n + \lambda)r^{1-\alpha}}{3 - \alpha - \beta} - 1\right)L$$
$$+ \frac{\beta r^{(1-(\alpha+\lambda))}}{\omega_n(3 - \alpha - \lambda - \beta)}L^{\frac{\beta+1}{\beta}} \leq 0.$$

Hence, the Lemma is proved. □

Lemma 2.5. *Let Eq* (1.1) *has oscillatory solution u. Then all the solutions of Eq* (1.1) *are oscillatory.*

Proof. Assume the contrary. Let u be nonoscillatory solution of Eq (1.1). There exists $r_0 > 0$ such that $uD^{\alpha}u > 0$. Let us introduce the notation

$$\sigma(r) = \int_{S(r)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS,$$
$$\rho(r) = \int_{S(r)} r^{\alpha-n+\lambda}\langle W, e_i\rangle dS.$$

There exists $r_3 > r_2$ such that

$$\sigma(r) < \rho(r) \quad r_3 < r < r_3 + \epsilon, \quad \sigma(r_3) = \rho(r_3). \tag{2.15}$$

Because of this fact we have from (2.13) that

$$\sigma(r) = \sigma(r_3) + (\alpha - n + \lambda)\int_{r_3}^{r} \frac{\sigma(s)}{s^{\alpha}} ds - \frac{\beta}{\omega_n}\int_{r_3}^{r} s^{(1-(\alpha+\lambda))}\sigma^{\frac{\beta+1}{\beta}}(s)ds \geq \rho(r_3)+$$
$$(\alpha - n + \lambda)\int_{r_3}^{r} \frac{\rho(s)}{s^{\alpha}} ds - \frac{\beta}{\omega_n}\int_{r_3}^{r} s^{(1-(\alpha+\lambda))}\rho^{\frac{\beta+1}{\beta}}(s)ds = \rho(r) \quad for \;\; r_3 < r < r_3 + \epsilon,$$

but this contradicts the Eq (2.15) and hence it is proved. □

Lemma 2.6. *Let there exists the function v which is locally absolutely continuous together with its first derivative and satisfying the inequalities*

$$\Delta_x^{\alpha} v + c(x) \mid v \mid^{\beta-1} v \mid D^{\alpha} v \mid^{1-\beta} \leq 0. \quad for \;\; r > r_0. \tag{2.16}$$

almost everywhere. Then Eq (1.1) *is nonoscillatory.*

3. Main results

In this paper, the following main results has been established.

Theorem 3.1. *(Hartman-Wintner Type Oscillation Criteria) If*

$$-\infty < \liminf_{t\to\infty} C(r) < \limsup_{r\to\infty} C(r) \leq \infty. \tag{3.1}$$

or if

$$\lim_{r\to\infty} C(r) = \infty, \tag{3.2}$$

then the Eq (1.1) *is oscillatory*

Proof. Suppose on the contrary that (3.1) holds and there exists number a such that positive solution of (1.1) on Ω_a exists. Then, Lemma 2.3 would implies that there exists a finite limit $\lim_{r\to\infty} C(r)$, this contradicts our assumption. □

Corollary 3.1. *(Leighton-Wintner Type Criteria) If*

$$\lim_{r\to\infty} \int_{\Omega(1,r)} r^{1-n+\lambda} c(x) d_{\alpha}x = \infty. \tag{3.3}$$

then Eq (1.1) *has no positive solution on Ω_a for any $a > 1$.*

Theorem 3.2. *Let Eq* (1.1) *has oscillatory solution u. Then*

$$Q(r) \geq \frac{\beta r^{1-(\alpha+\lambda)}}{\omega_n(\alpha+\beta+\lambda-1)} N^{\frac{\beta+1}{\beta}} - \frac{(\alpha-n+\lambda)r^{1-\alpha}}{\alpha+\beta-1} N - M, \tag{3.4}$$

and

$$H(r) \geq \frac{1}{\beta r^{2(\beta-1)}} \left[r^{\beta-2}(\tau_\epsilon^2 \sigma(\tau_\epsilon) + \beta\tau_\epsilon^\beta H(\tau_\epsilon)) + r^{\beta-1} h(\epsilon) - M \right]. \tag{3.5}$$

are oscillatory. Moreover,

$$\liminf r^\beta \sigma(r) \geq M, \quad \liminf r^\beta \sigma(r) \leq N. \tag{3.6}$$

where M is the least nonnegative root of equation and N is the largest root of equation.

Proof. On the contrary we assume the Eq (1.1) have the nonoscillatory solution. The Eq (2.1) has the solution of $\sigma : [r_0, \infty) \to (0, \infty)$ satisfying the condition

$$\liminf r^\beta \sigma(r) \geq M, \quad \liminf r^\beta \sigma(r) \leq N.$$

Clearly, for any $0 < \epsilon < 1 - N(0 < \epsilon < 1)$ there exists r_ϵ such that

$$\begin{aligned} &M - \epsilon < r^\beta \sigma(r) < N + \epsilon, \quad for \;\; r > r_\epsilon. \\ &t^\beta \sigma(r) \left(2r^{\alpha-\beta} + (\alpha-n+\lambda) r^{2-\alpha-\beta} - \frac{\beta r^{2-(\alpha+\beta+\lambda)}}{\omega_n} (r^\beta \sigma(r))^{\frac{1}{\beta}} \right) \leq h(\epsilon), \end{aligned}$$

where $h(\epsilon) = (N+\epsilon)\left(2r^{\alpha-\beta} + (\alpha-n+\lambda)r^{2-\alpha-\beta} - \frac{\beta r^{2-(\alpha+\beta+\lambda)}}{\omega_n}(N+\epsilon)^{\frac{1}{\beta}}\right)(h(\epsilon) = 1)$. Finally, from (2.12) and (2.13) we get

$$Q(r) \leq \frac{\beta r^{1-(\alpha+\lambda)}}{\omega_n(\alpha+\beta+\lambda-1)} N^{\frac{\beta+1}{\beta}} - \frac{(\alpha-n+\lambda)r^{1-\alpha}}{(\alpha+\beta-1)} N - M, \quad for \;\; r > r_\epsilon,$$

and

$$H(r) \leq \frac{1}{\beta r^{2(\beta-1)}} \left[r^{\beta-2}(\tau_\epsilon^2 \sigma(\tau_\epsilon) + \beta\tau_\epsilon^\beta H(\tau_\epsilon)) + r^{\beta-1} h(\epsilon) - M \right], \quad for \;\; r > r_\epsilon,$$

but this contradicts (3.4) and (3.5) and hence the theorem is proved. □

Theorem 3.3. *Let* (2.3) *holds and*

$$\limsup_{r\to\infty} \frac{r^\alpha (C_0 - C(r))}{\log r} > \left(\frac{1+\alpha-n+\lambda}{\beta+1} \right)^{\beta+1} \omega_n^\beta. \tag{3.7}$$

Then the Eq (1.1) *is oscillatory.*

Proof. By contradiction, a solution W of the Riccati equation defined on Ω_a and there exists number $a \geq 1$. We combine the Eqs (2.7) and (2.8) and by using integration by parts, we obtain

$$r^{\alpha}(C_0 - C(r)) = \int_a^r \int_{S(r)} r^{(\alpha-n+\lambda)} \langle W, e_i \rangle dS\, dr - \beta r \int_r^{\infty} \int_{S(r)} r^{(\alpha-n+\lambda)} \parallel W \parallel^{\frac{\beta+1}{\beta}} dS\, dr$$
$$-\beta \int_a^r r \int_{S(r)} r^{(\alpha-n+\lambda)} \parallel W \parallel^{\frac{\beta+1}{\beta}} dS\, d_{\alpha} r + (\alpha - n + \lambda) r \int_r^{\infty} \int_{S(r)} r^{-n+\lambda} \langle W, e_i \rangle dS\, dr$$
$$+(\alpha - n + \lambda) \int_a^r r \int_{S(r)} r^{-n+\lambda} \langle W, e_i \rangle dS\, d_{\alpha} r + constant,$$

that is,

$$r^{\alpha}(C_0 - C(r)) = \int_a^t \sigma(s) ds - \frac{\beta r}{\omega_n} \int_r^{\infty} s^{1-(\alpha+\lambda)} \sigma^{\frac{\beta+1}{\beta}}(s) ds - \frac{\beta}{\omega_n} \int_a^r s^{1-\lambda} \sigma^{\frac{\beta+1}{\beta}}(s) ds$$
$$+ (\alpha - n + \lambda) r \int_r^{\infty} s^{-\alpha} \sigma(s) ds + (\alpha - n + \lambda) \int_a^r \sigma(s) ds + constant$$

$$= \int_a^r \left((1 + \alpha - n + \lambda) \sigma(s) - \frac{\beta s^{1-\lambda}}{\omega_n} \sigma^{\frac{\beta+1}{\beta}}(s) \right) ds$$
$$+ r \int_r^{\infty} \left((\alpha - n + \lambda) s^{-\alpha} \sigma(s) - \frac{\beta s^{1-(\alpha+\lambda)}}{\omega_n} \sigma^{\frac{\beta+1}{\beta}}(s) \right) ds + constant$$
$$\leq \int_a^r \left((1 + \alpha - n + \lambda) s^{\beta} \sigma(s) - \frac{\beta s^{-\lambda}}{\omega_n} (s^{\beta} \sigma(s))^{\frac{\beta+1}{\beta}} \right) \frac{ds}{s^{\beta}} + constant.$$

Using Lemma 2.2, we get

$$\leq \left(\frac{1 + \alpha - n + \lambda}{\beta + 1} \right)^{\beta+1} (\omega_n)^{\beta} \log r + constant.$$

Hence,

$$\frac{r^{\alpha}(C_0 - C(r))}{\log r} \leq \left(\frac{1 + \alpha - n + \lambda}{\beta + 1} \right)^{\beta+1} \omega_n^{\beta} + \frac{const}{\log r},$$

a contradiction. Hence, the proof is complete. □

Corollary 3.2. *Assume* (2.3) *hold, and*

$$\limsup_{r \to \infty} \frac{1}{\log r} \int_{\Omega(1,r)} r^{1+\alpha-n+\lambda} c(x) d_{\alpha} x > \left(\frac{1 + \alpha - n + \lambda}{\beta + 1} \right)^{\beta+1} \omega_n^{\beta}.$$

Then the Eq (1.1) *is oscillatory.*

Corollary 3.3. *Assume* (2.3) *hold, and*

$$\liminf_{r \to \infty} [Q(r) + H(r)] > \frac{(1 + \alpha - n + \lambda)^{\beta+1}}{\beta + 1} \omega_n^{\beta}.$$

Then the Eq (1.1) *is oscillatory.*

Corollary 3.4. *Assume* (2.3) *hold, and each conditions are guarantees the oscillation of the Eq* (1.1)

$$\liminf_{r\to\infty} Q(r) > \left(\frac{1+\alpha-n+\lambda}{\beta+1}\right)^{\beta+1} \omega_n^{\beta},$$

$$\liminf_{r\to\infty} H(r) > \left(\frac{1+\alpha-n+\lambda}{\beta+1}\right)^{\beta+1} \omega_n^{\beta}.$$

4. Example

In this section, we give an example to illustrate the main results.

Example 4.1. *Consider the conformable partial differential equations in Emden-Fowler type Laplacian equation*

$$\Delta_x^{\alpha} u + \frac{(n-1-\alpha-\lambda)^{\beta+1}}{(\beta+1)^{(\beta+1)} r^{2\alpha}} \mid u \mid^{\beta-1} u \mid D^{\alpha}(u) \mid^{1-\beta} = 0.$$

In paper [27, Lemma 4], for a linear equation $\beta = 1, \lambda < \beta$ *and here* $\beta = 1, \lambda = 0$ *and* $\alpha = 1$. *Then*

$$\frac{(2-n)}{2} r^{\left(\frac{-n}{2}-1\right)} \left(\frac{-n}{2} + n - 1\right) + \frac{(n-2)^2}{4} r^{\left(\frac{-n}{2}-1\right)} = 0.$$

By apply the result of Theorem 3.3, Corollaries 3.2–3.4 in the right hand side, the solution is satisfied.

5. Conclusions

In this paper, the authors have obtained some new oscillation criteria for certain class of conformable Emden-Fowler type elliptic partial differential equations by using Riccati technique. These newly derived results extend and complements the known results in the existing literature for the integer-order equations. To prove the effectiveness of our result we have illustrate with an example.

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Conflict of interest

The authors declare no conflict of interest.

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