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Research article

On coupled snap system with integral boundary conditions in the $\mathbb{G}$-Caputo sense

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Abstract: In this paper, we consider a coupled snap system in a fractional $\mathbb{G}$-Caputo derivative sense with integral boundary conditions. Hyers-Ulam stability criterion is investigated, and a numerical simulation will be supplied to some applications. Some numerical simulations are presented to guarantee the theoretical results.

Keywords: snap problem; $\mathbb{G}$-Caputo fractional differential equation; boundary value problem; Ulam-Hyers-Rassias stability
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1. Introduction

Over the past decades, the study of nonlinear problems has been the interest of many researchers [5, 10, 11, 14, 19, 24–26]. Also, study of fractional calculus has recently gained great momentum, and has emerged as a significant research area [5, 7, 15, 20, 21, 30]. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes; see, for instance, the contribution [2–4, 6, 16, 22, 23, 28, 29] and references therein.

The authors in [8] focused on the study of nonlinear jerk problems due to its various physical applications, as form

$$\frac{d^3y}{dt^3} = \mathfrak{T}\left(y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right).$$

In 2020, the authors investigated the existence and uniqueness of solutions for the following nonlocal generalized fractional Sturm-Liouville and Langevin equations:

$$\begin{cases} {}^c\mathcal{D}^{\alpha}_{\iota_1^+}\left(\left[p(t)^c\mathcal{D}^{\gamma}_{\iota_1^+} + q(t)\right]y(t)\right) = \mathfrak{T}(t, y(t)), \quad t \in [0, T], \alpha, \gamma \in (0, 1], \\ y(0) + \varkappa_1(y) = y_1 \in \mathbb{R}, \\ {}^c\mathcal{D}^{\gamma}_{\iota_1^+} y(T) + \varkappa_2(y) = y_2 \in \mathbb{R}, \end{cases}$$

where ${}^c\mathcal{D}^{\alpha}_{\iota_1^+}$, ${}^c\mathcal{D}^{\gamma}_{\iota_1^+}$ are the Caputo fractional derivatives, $p, q \in C([0, T])$ with $|p| \geq K > 0$, $\varkappa_1, \varkappa_2 : C(J) \to \mathbb{R}$ are continuous functions and $\mathfrak{T} \in C([0, T] \times \mathbb{R})$ [27]. The second derivative of the accelaration (fourth derivative of position) is a physical quantity called a snap or jounce, which can be modeled as

$$\begin{cases} \frac{dy_1}{dt} = y_2(t), \quad \frac{dv_2}{dt} = y_3(t), \quad \frac{dy_3}{dt} = y_4(t), \\ \frac{dy_4}{dt} = \mathfrak{T}(y_1, y_2, y_3, y_4). \end{cases} \tag{1.1}$$

It is obvious that the model (1.1) can be reduced to the following equation:

$$\frac{d^4y_1}{dt^4} = \mathfrak{T}\left(y_1, \frac{dy_1}{dt}, \frac{d^2y_1}{dt^2}, \frac{d^3y_1}{dt^3}\right). \tag{1.2}$$

Scientifically, jerk and snap are the third and fourth derivatives of our position with regard to time, respectively. The Eq (1.1) contains a 4^{th}-order derivative of the variable y_1, and it describes a 4^{th}-order dynamical vibration model.

The corresponding fractional model is achieved by using the fractional derivative (of order less than or equal 1) instead of the standard deivative $\frac{d}{dt}$. Many types of fractional derivatives can be used here, such as Riemann-Liouville, Caputo, Hadamard, etc. We prefer to use the generalized fractional derivative ($\mathbb{GFD}$), with respect to differentiable increasing function $\mathbb{G}$. In 2020, Liu et al., developed two iterative algorithms to determine the periods, and then the periodic solutions of nonlinear jerk equations for two possible cases with initial values unknown and initial values given [13]. The authors in a recent article [23] considered the $\mathbb{G}$-fractional snap model ($\mathbb{GFSM}$) with constant, initial conditions

$$\begin{cases} {}^c\mathcal{D}^{\alpha;\mathbb{G}}_{\iota_1^+} y(t) = y_1(t), & y(\iota_1) = v_0, \\ {}^c\mathcal{D}^{\beta;\mathbb{G}}_{\iota_1^+} y_1(t) = y_2(t), & y_1(\iota_1) = v_1, \\ {}^c\mathcal{D}^{\gamma;\mathbb{G}}_{\iota_1^+} y_2(t) = y_3(t), & y_2(\iota_1) = v_2, \\ {}^c\mathcal{D}^{\delta;\mathbb{G}}_{\iota_1^+} y_3(t) = \mathcal{T}(t, y, y_1, y_2, y_3), & y_3(\iota_1) = v_3, \end{cases} \tag{1.3}$$

where the $\mathbb{G}$-Caputo derivatives are illustrated by the symbol

$$ {}^{c}\mathcal{D}_{\iota_1^+}^{\eta;\mathbb{G}}, \qquad \eta \in \left\{\alpha,\beta,\gamma,\delta\right\}, \, 0<\eta<1, $$

here and the increasing function $\mathbb{G} \in C^1([\iota_1,\iota_2])$ is such that $\mathbb{G}'(\mathrm{t}) \neq 0$, for each $\mathrm{t} \in [\iota_1,\iota_2]$ and continuous function $\mathcal{T}$ belongs to $C([\iota_1,\iota_2]\times\mathbb{R}^4)$ and $\mathrm{y}_0,\mathrm{y}_1,\mathrm{y}_2,\mathrm{y}_3 \in \mathbb{R}$. Abbas et al. studied the following coupled system of fractional differential equations:

$$ \begin{cases} {}^{RL}\mathcal{D}_{\iota_1^+}^{\alpha_1;\varrho}\mathrm{y}_1(\mathrm{t}) = \mathfrak{T}_1(\mathrm{t},\mathrm{y}_1(\mathrm{t}),\mathrm{y}_2(\mathrm{t})), \\ {}^{RL}\mathcal{D}_{\iota_1^+}^{\alpha_2;\varrho}\mathrm{y}_2(\mathrm{t}) = \mathfrak{T}_2(\mathrm{t},\mathrm{y}_1(\mathrm{t}),\mathrm{y}_2(\mathrm{t})), \end{cases} $$

for $\mathrm{t} \in [\iota_1,\iota_2]$ equipped with the generalized fractional integral boundary conditions

$$ \begin{cases} \mathrm{y}_1(\tau_1) = 0, \quad \mathrm{y}_1(\iota_2) = \mathcal{I}_{\iota_1^+}^{\zeta_1;\varrho}\mathrm{y}_1(\eta_1), \\ \mathrm{y}_2(\tau_2) = 0, \quad \mathrm{y}_2(\iota_2) = \mathcal{I}_{\iota_1^+}^{\zeta_2;\varrho}\mathrm{y}_2(\eta_2), \end{cases} $$

where $\varrho \in (0,1]$, ${}^{RL}\mathcal{D}_{\iota_1^+}^{\alpha_i;\varrho}$ denotes the generalized proportional fractional derivatives of Riemann-Liouville type of order $1<\alpha_i\leq 2$, $\mathcal{I}_{\iota_1^+}^{\zeta_i;\varrho}$ that denotes the generalized proportional fractional integrals of order $0<\zeta_i<1$ and $\tau_i,\eta_i \in (\iota_1,\iota_2)$ and $\mathfrak{T}_i \in C([\iota_1,\iota_2]\times\mathbb{R}^2)$ [1].

We center our consideration on the problem of the existence and uniqueness along with the Hyers-Ulam stability (U-H-$\mathbb{S}$) of solutions for fractional nonlinear couple snap system ($\mathbb{CSS}$) in the $\mathbb{G}$-Caputo sense ($\mathbb{G}$C) with initial conditions

$$ \begin{cases} {}^{c}\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t}) = \mathrm{u}_1(\mathrm{t}), \quad {}^{c}\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t}) = \mathrm{u}_2(\mathrm{t}), \\ {}^{c}\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\mathrm{u}_1(\mathrm{t}) = \mathrm{w}_1(\mathrm{t}), \quad {}^{c}\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\mathrm{u}_2(\mathrm{t}) = \mathrm{w}_2(\mathrm{t}), \\ {}^{c}\mathcal{D}_{\iota_1^+}^{r_1;\mathbb{G}}\mathrm{w}_1(\mathrm{t}) = \mathrm{x}_1(\mathrm{t}), \quad {}^{c}\mathcal{D}_{\iota_1^+}^{r_2;\mathbb{G}}\mathrm{w}_2(\mathrm{t}) = \mathrm{x}_2(\mathrm{t}), \\ {}^{c}\mathcal{D}_{\iota_1^+}^{s_1;\mathbb{G}}\mathrm{x}_1(\mathrm{t}) = h_1(\mathrm{t},\mathrm{v}_1,\mathrm{v}_2,\mathrm{u}_1,\mathrm{u}_2,\mathrm{w}_1,\mathrm{w}_2,\mathrm{x}_1,\mathrm{x}_2), \\ {}^{c}\mathcal{D}_{\iota_1^+}^{s_2;\mathbb{G}}\mathrm{x}_2(\mathrm{t}) = h_2(\mathrm{t},\mathrm{v}_1,\mathrm{v}_2,\mathrm{u}_1,\mathrm{u}_2,\mathrm{w}_1,\mathrm{w}_2,\mathrm{x}_1,\mathrm{x}_2), \end{cases} \tag{1.4} $$

subject to the following integral boundary conditions

$$ \begin{aligned} \mathrm{v}_1(\iota_1) &= \int_{\iota_1}^{\iota_2} g_{10}(s)\,\mathrm{d}s, & \mathrm{v}_2(\iota_1) &= \int_{\iota_1}^{\iota_2} g_{20}(s)\,\mathrm{d}s, \\ \mathrm{u}_1(\iota_1) &= \int_{\iota_1}^{\iota_2} g_{11}(s)\,\mathrm{d}s, & \mathrm{u}_2(\iota_1) &= \int_{\iota_1}^{\iota_2} g_{21}(s)\,\mathrm{d}s, \\ \mathrm{w}_1(\iota_1) &= \int_{\iota_1}^{\iota_2} g_{12}(s)\,\mathrm{d}s, & \mathrm{w}_2(\iota_1) &= \int_{\iota_1}^{\iota_2} g_{22}(s)\,\mathrm{d}s, \\ \mathrm{x}_1(\iota_1) &= \int_{\iota_1}^{\iota_2} g_{13}(s)\,\mathrm{d}s, & \mathrm{x}_2(\iota_1) &= \int_{\iota_1}^{\iota_2} g_{23}(s)\,\mathrm{d}s, \end{aligned} \tag{1.5} $$

where the $\mathbb{G}$C derivatives are illustrated by symbol

$$ {}^{c}\mathcal{D}_{\iota_1^+}^{\eta;\mathbb{G}}, \qquad \eta \in \left\{q_k, p_k, r_k, s_k\right\}, \, 0<q_k,p_k,r_k,s_k\leq 1, $$

here the function $\mathbb{G} \in C^1(\Sigma)$ is increasing with $\mathbb{G}'(\mathrm{t}) \neq 0$, for all $\mathrm{t} \in \Sigma = [\iota_1, \iota_2]$ and the functions $h_k \in C(\Sigma \times \mathbb{R}^8), (k = 1, 2)$ and $\mathrm{g}_{kj} \in C(\Sigma, \mathbb{R}), (j = 0, 1, 2, 3;\ k = 1, 2)$ are continuous functions. It is obvious that the $\mathbb{CSS}$ (1.4) and (1.5) can be rewritten as

$$\begin{cases} {}^c\mathcal{D}^{s_k;\mathbb{G}}_{\iota_1^+}\left({}^c\mathcal{D}^{r_k;\mathbb{G}}_{\iota_1^+}\left({}^c\mathcal{D}^{p_k;\mathbb{G}}_{\iota_1^+}\left({}^c\mathcal{D}^{q_k;\mathbb{G}}_{\iota_1^+}\mathrm{v}_k(\mathrm{t})\right)\right)\right) = h_k(\mathrm{t}) \\ \mathrm{v}_k(\iota_1) = \displaystyle\int_{\iota_1}^{\iota_2} g_{k0}(s)\,\mathrm{d}s, \qquad {}^c\mathcal{D}^{q_k;\mathbb{G}}_{\iota_1^+}\mathrm{v}_k(\mathrm{t})\Big|_{\mathrm{t}=\iota_1} = \displaystyle\int_{\iota_1}^{\iota_2} g_{k1}(s)\,\mathrm{d}s, \\ {}^c\mathcal{D}^{p_k;\mathbb{G}}_{\iota_1^+}\left({}^c\mathcal{D}^{q_k;\mathbb{G}}_{\iota_1^+}\mathrm{v}_k(\mathrm{t})\right)\Big|_{\mathrm{t}=\iota_1} = \displaystyle\int_{\iota_1}^{\iota_2} g_{k2}(s)\,\mathrm{d}s, \\ {}^c\mathcal{D}^{r_k;\mathbb{G}}_{\iota_1^+}\left({}^c\mathcal{D}^{p_k;\mathbb{G}}_{\iota_1^+}\left({}^c\mathcal{D}^{q_k;\mathbb{G}}_{\iota_1^+}\mathrm{v}_k(\mathrm{t})\right)\right)\Big|_{\mathrm{t}=\iota_1} = \displaystyle\int_{\iota_1}^{\iota_2} g_{k3}(s)\,\mathrm{d}s, \qquad k = 1, 2, \end{cases} \tag{1.6}$$

where

$$\begin{aligned} h_{\mathrm{v}_1,\mathrm{v}_2,k}(\mathrm{t}) = h_k\Big(&\mathrm{t}, \mathrm{v}_1(\mathrm{t}), \mathrm{v}_2(\mathrm{t}), {}^c\mathcal{D}^{q_1;\mathbb{G}}_{\iota_1^+}\mathrm{v}_1(\mathrm{t}), {}^c\mathcal{D}^{q_2;\mathbb{G}}_{\iota_1^+}\mathrm{v}_2(\mathrm{t}), \\ &{}^c\mathcal{D}^{p_1;\mathbb{G}}_{\iota_1^+}\left({}^c\mathcal{D}^{q_1;\mathbb{G}}_{\iota_1^+}\mathrm{v}_1(\mathrm{t})\right), {}^c\mathcal{D}^{p_2;\mathbb{G}}_{\iota_1^+}\left({}^c\mathcal{D}^{q_2;\mathbb{G}}_{\iota_1^+}\mathrm{v}_2(\mathrm{t})\right), \\ &{}^c\mathcal{D}^{r_1;\mathbb{G}}_{\iota_1^+}\left({}^c\mathcal{D}^{p_1;\mathbb{G}}_{\iota_1^+}\left({}^c\mathcal{D}^{q_1;\mathbb{G}}_{\iota_1^+}\mathrm{v}_1(\mathrm{t})\right)\right), {}^c\mathcal{D}^{r_2;\mathbb{G}}_{\iota_1^+}\left({}^c\mathcal{D}^{p_2;\mathbb{G}}_{\iota_1^+}\left({}^c\mathcal{D}^{q_2;\mathbb{G}}_{\iota_1^+}\mathrm{v}_2(\mathrm{t})\right)\right)\Big). \end{aligned}$$

The main novelty of this work is that we establish our results with the help of the technique of fixed point theorems for a fractional nonlinear $\mathbb{CSS}$ furnished with generalized operators, which leads to some general theoretical findings involving the following special cases: $\mathbb{G}$ as $\mathbb{G}_1(\iota) = 2^\iota$, $\mathbb{G}_2(\iota) = \iota$ (Caputo derivative), $\mathbb{G}_3(\iota) = \ln\iota$ (Caputo-Hadamard derivative), $\mathbb{G}_4(\iota) = \sqrt{\iota}$ (Katugampola derivative).

This paper is organized as follows: In Section 2, we present some necessary definitions and lemmas that are needed in the subsequent sections. In Section 3, we adopt some fixed point theorems to prove the existence and uniqueness of solutions for problem (1.4). The stability results are extensively discussed in Section 3.2. An illustrative example is presented in Section 4.

2. Preliminaries

Some primitive notions, definitions and notations, which will be utilized throughout the manuscript, are recalled here. Consider the function $\mathbb{G}$ with assumptions in system (1.4). We start this part by defining $\mathbb{G}$-Riemann-Liouville fractional ($\mathbb{GF}$-RL) integrals and derivatives [17]. For $\eta > 0$, the η^{th}-$\mathbb{GF}$-RL integral for an integrable function $\mathrm{v} : \Sigma \to \mathbb{R}$ w.r.t $\mathbb{G}$ is illustrated as follows

$$\mathcal{I}^{\eta;\mathbb{G}}_{\iota_1^+}\mathrm{v}(\mathrm{t}) = \frac{1}{\Gamma(\eta)}\int_{\iota_1}^{\mathrm{t}} (\mathbb{G}(\mathrm{t}) - \mathbb{G}(\sigma))^{\eta-1}\mathbb{G}'(\sigma)\mathrm{v}(\sigma)\,\mathrm{d}\sigma, \tag{2.1}$$

where $\Gamma(\eta) = \int_0^{+\infty} e^{-\mathrm{t}}\mathrm{t}^{\eta-1}\,\mathrm{dt}$, $\eta > 0$. Let $n \in \mathbb{N}$ and $\mathbb{G}, \mathrm{v} \in C^n(\Sigma)$ be such that $\mathbb{G}$ has the same properties mentioned above. The η^{th}-$\mathbb{GF}$-RL derivative of v is defined by

$$\begin{aligned} \mathcal{D}^{\eta;\mathbb{G}}_{\iota_1^+}\mathrm{v}(\mathrm{t}) &= A^{(n)}\mathcal{I}^{n-\eta;\mathbb{G}}_{\iota_1^+}\mathrm{v}(\mathrm{t}) \\ &= \frac{1}{\Gamma(n-\eta)}A^{(n)}\int_{\iota_1}^{\mathrm{t}} (\mathbb{G}(\mathrm{t}) - \mathbb{G}(\sigma))^{n-\eta-1}\mathbb{G}'(\sigma)\mathrm{v}(\sigma)\,\mathrm{d}\sigma, \end{aligned}$$

in which $n = [\eta] + 1$, where $A = \frac{1}{\mathbb{G}'(\mathrm{t})}\frac{\mathrm{d}}{\mathrm{dt}}$. The η^{th}-$\mathbb{G}$-fractional-Caputo derivative of v is defined by ${}^{c}\mathcal{D}_{\iota_1^+}^{\eta;\mathbb{G}}\mathrm{v}(\mathrm{t}) = \mathcal{I}_{\iota_1^+}^{n-\eta;\mathbb{G}} A^{(n)}\mathrm{v}(\mathrm{t})$, in which $n = [\eta] + 1$, $(\eta \notin \mathbb{N})$, $n = \eta$ for $\eta \in \mathbb{N}$ [17]. In other words,

$$ {}^{c}\mathcal{D}_{\iota_1^+}^{\eta;\mathbb{G}}\mathrm{v}(\mathrm{t}) = \begin{cases} \displaystyle\int_{\iota_1}^{\mathrm{t}} \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\xi))^{n-\eta-1}}{\Gamma(n-\eta)} \mathbb{G}'^{(n)}\mathrm{v}(\xi)\,\mathrm{d}\xi, & \eta \notin \mathbb{N}, \\ A^n\mathrm{v}(\mathrm{t}), & \eta = n \in \mathbb{N}. \end{cases} \tag{2.2} $$

This extension (2.2) gives the Caputo derivative when $\mathbb{G}(\mathrm{t}) = \mathrm{t}$ [17]. Also, in the case $\mathbb{G}(\mathrm{t}) = \ln \mathrm{t}$, it yields the Caputo-Hadamard derivative. If $\mathrm{v} \in C^n(\Sigma)$, the η^{th}-$\mathbb{G}$-fractional-Caputo derivative of v is specified as [18]

$$ {}^{c}\mathcal{D}_{\iota_1^+}^{\eta;\mathbb{G}}\mathrm{v}(\mathrm{t}) = \mathcal{D}_{\iota_1^+}^{\eta;\mathbb{G}}\left(\mathrm{v}(\mathrm{t}) - \sum_{j=0}^{n-1} \frac{A^{(j)}\mathrm{v}(\iota_1)}{j!}(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\iota_1))^j\right). $$

The composition rules for above $\mathbb{G}$-operators are recalled in this lemma.

Lemma 2.1. *[18] Let $n - 1 < \eta < n$ and $\mathrm{v} \in C^n(\Sigma)$. Then the following holds*

$$ \mathcal{I}_{\iota_1^+}^{\eta;\mathbb{G}}\, {}^{c}\mathcal{D}_{\iota_1^+}^{\eta;\mathbb{G}}\mathrm{v}(\mathrm{t}) = \mathrm{v}(\mathrm{t}) - \sum_{j=0}^{n-1} \frac{A^{(j)}\mathrm{v}(\iota_1)}{j!}\left[\mathbb{G}(\mathrm{t}) - \mathbb{G}(\iota_1)\right]^j, $$

for all $\mathrm{t} \in \Sigma$. Moreover, if $m \in \mathbb{N}$ and $\mathrm{v} \in C^{n+m}(\Sigma)$, then, the following holds

$$ A^{(m)}\left({}^{c}\mathcal{D}_{\iota_1^+}^{\eta;\mathbb{G}}\mathrm{v}\right)(\mathrm{t}) = {}^{c}\mathcal{D}_{\iota_1^+}^{\eta+m;\mathbb{G}}\mathrm{v}(\mathrm{t}) + \sum_{j=0}^{m-1} \frac{[\mathbb{G}(\mathrm{t}) - \mathbb{G}(\iota_1)]^{j+n-\eta-m}}{\Gamma(j+n-\eta-m+1)} A^{(j+n)}\mathrm{v}(\iota_1). \tag{2.3} $$

Observe that from Eq (2.3) if $A^{(j)}\mathrm{v}(\iota_1) = 0$, for $j = n, n+1, \ldots, n+m-1$, we can get the following relation

$$ A^{(m)}\left({}^{c}\mathcal{D}_{\iota_1^+}^{\eta;\mathbb{G}}\mathrm{v}\right)(\mathrm{t}) = {}^{c}\mathcal{D}_{\iota_1^+}^{\eta+m;\mathbb{G}}\mathrm{v}(\mathrm{t}), \qquad \mathrm{t} \in \Sigma. $$

Lemma 2.2. *[12] Let $\eta, \nu > 0$, and $\mathrm{v} \in C(\Sigma)$. Then for each $\mathrm{t} \in \Sigma$ and by assuming*

$$ F_{\iota_1}(\mathrm{t}) = \mathbb{G}(\mathrm{t}) - \mathbb{G}(\iota_1), \tag{2.4} $$

we have

(1) $\mathcal{I}_{\iota_1^+}^{\eta;\mathbb{G}}\left(\mathcal{I}_{\iota_1^+}^{\nu;\mathbb{G}}\mathrm{v}\right)(\mathrm{t}) = \mathcal{I}_{\iota_1^+}^{\eta+\nu;\mathbb{G}}\mathrm{v}(\mathrm{t})$*;*

(2) ${}^{c}\mathcal{D}_{\iota_1^+}^{\eta;\mathbb{G}}\left(\mathcal{I}_{\iota_1^+}^{\eta;\mathbb{G}}\mathrm{v}\right)(\mathrm{t}) = \mathrm{v}(\mathrm{t})$*;*

(3) $\mathcal{I}_{\iota_1^+}^{\eta;\mathbb{G}}(F_{\iota_1}(\mathrm{t}))^{\nu-1} = \dfrac{\Gamma(\nu)}{\Gamma(\nu+\eta)}(F_{\iota_1}(\mathrm{t}))^{\nu+\eta-1}$*;*

(4) ${}^{c}\mathcal{D}_{\iota_1^+}^{\eta;\mathbb{G}}(F_a(\mathrm{t}))^{\nu-1} = \dfrac{\Gamma(\nu)}{\Gamma(\nu-\eta)}(F_{\iota_1}(\mathrm{t}))^{\nu-\eta-1}$*;*

(5) ${}^{c}\mathcal{D}^{\eta;\mathbb{G}}_{\iota_1^+}(F_{\iota_1}(\mathrm{t}))^j = 0,\ n-1 \le \eta \le n, n \in \mathbb{N},\ j = 0, 1, \ldots, n-1.$

Theorem 2.3. *[9] (Banach's fixed point theorem) Consider $\Pi : \mathcal{Y} \to \mathcal{Y}$ to be a contraction operator, such that $\mathcal{Y}$ is a Banach space. Then, there are only one $\mathrm{y}^* \in \mathcal{Y}$, such that $\Pi(\mathrm{y}^*) = \mathrm{y}^*$.*

Lemma 2.4. *[9] (Krasnoselskii's fixed point theorem) Assume that $\mathcal{B} \subset X$ is a closed convex and nonempty, and $\mathfrak{L}_1, \mathfrak{L}_2 : \mathcal{B} \to X$ nonlinear operators, such that:*
(i) $\mathfrak{L}_1 u + \mathfrak{L}_2 v \in \mathcal{B}$ *whenever $u, v \in \mathcal{B}$;*
(ii) $\mathfrak{L}_1$ *is a contraction mapping;*
(iii) $\mathfrak{L}_2$ *is compact and continuous.*
Then, there exists $w \in \mathcal{B}$, such that $w = \mathfrak{L}_1 w + \mathfrak{L}_2 w$.

Definition 2.5. *[29] Let X_1, X_2 be Banach spaces and $\Lambda_1, \Lambda_2 : X_1 \times X_2 \to X_1 \times X_2$ be two operators. Then, the operational equations system provided by*

$$\begin{cases} \mathrm{u}_1(\mathrm{t}) = \Lambda_1(\mathrm{u}_1, \mathrm{u}_2)(\mathrm{t}), \\ \mathrm{u}_2(\mathrm{t}) = \Lambda_2(\mathrm{u}_1, \mathrm{u}_2)(\mathrm{t}), \end{cases} \tag{2.5}$$

is called U-H-$\mathbb{S}$*, if there exist $\alpha_i > 0$, $(i = 1, \ldots, 4)$, such that, $\forall\, \rho_1, \rho_2 > 0$, and each solution $(\mathrm{u}_1^*, \mathrm{u}_2^*) \in X_1 \times X_2$ of the identities*

$$\begin{cases} \left\|\mathrm{u}_1^* - \Lambda_1(\mathrm{u}_1^*, \mathrm{u}_2^*)\right\| \le \rho_1, \\ \left\|\mathrm{u}_2^* - \Lambda_2(\mathrm{u}_1^*, \mathrm{u}_2^*)\right\| \le \rho_2, \end{cases}$$

there exists $(\mathrm{v}_1^*, \mathrm{v}_2^*) \in X_1 \times X_2$ a solution of system (2.5)*, such that*

$$\begin{cases} \left\|\mathrm{u}_1^* - \mathrm{v}_1^*\right\| \le \alpha_1\rho_1 + \alpha_2\rho_2, \\ \left\|\mathrm{u}_2^* - \mathrm{v}_2^*\right\| \le \alpha_3\rho_1 + \alpha_4\rho_2. \end{cases}$$

Theorem 2.6. *[29] Let X_1, X_2 be Banach spaces and $\Lambda_1, \Lambda_2 : X_1 \times X_2 \to X_1 \times X_2$ be two operators that satisfy*

$$\begin{cases} \left\|\Lambda_1(\mathrm{u}_1, \mathrm{u}_2) - \Lambda_1(\mathrm{u}_1^*, \mathrm{u}_2^*)\right\| \le \alpha_1 \left\|\mathrm{u}_1 - \mathrm{u}_1^*\right\| + \alpha_2 \left\|\mathrm{u}_2 - \mathrm{u}_2^*\right\|, \\ \left\|\Lambda_2(\mathrm{u}_1, \mathrm{u}_2) - \Lambda_2(\mathrm{u}_1^*, \mathrm{u}_2^*)\right\| \le \alpha_3 \left\|\mathrm{u}_1 - \mathrm{u}_1^*\right\| + \alpha_4 \left\|\mathrm{u}_2 - \mathrm{u}_2^*\right\|, \end{cases} \tag{2.6}$$

for each $(\mathrm{u}_1, \mathrm{u}_2), (\mathrm{u}_1^*, \mathrm{u}_2^*) \in X_1 \times X_2$ and if the matrix

$$\Xi = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix},$$

it converges to zero. Then, the system (2.6) *is* U-H-$\mathbb{S}$.

3. Main results

Here, we analyze the existence properties of solutions, and their uniqueness for the proposed fractional $\mathbb{G}$-$\mathbb{CSS}$ (1.6) using Krasnoselskii and Banach fixed point theorems. We need after lemma, which indicate the corresponding integral equation.

Lemma 3.1. *For given continuous mappings h, g_k $(k = 0,1,2,3)$ belongs to $C(\Sigma)$, and the solution of the linear $\mathbb{G}$-snap problem is*

$$\begin{cases} {}^{c}\mathcal{D}_{\iota_1^+}^{s;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\right)\right)\right) = h(\mathrm{t}), \\ \mathrm{v}(\iota_1) = \int_{\iota_1}^{b} g_0(\xi)\,\mathrm{d}\xi, \quad {}^{c}\mathcal{D}_{\iota_1^+}^{q;\mathbb{G}}\mathrm{v}(\iota_1) = \int_{\iota_1}^{b} g_1(\xi)\,\mathrm{d}\xi, \\ {}^{c}\mathcal{D}_{\iota_1^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q;\mathbb{G}}\mathrm{v}(\iota_1)\right) = \int_{\iota_1}^{\iota_2} g_2(\xi)\,\mathrm{d}\xi, \\ {}^{c}\mathcal{D}_{\iota_1^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q;\mathbb{G}}\mathrm{v}(\iota_1)\right)\right) = \int_{\iota_1}^{\iota_2} g_3(\xi)\,\mathrm{d}\xi, \end{cases} \tag{3.1}$$

where $q, p, r, s \in (0,1]$, are formulated by

$$\mathrm{v}(\mathrm{t}) = \int_{\iota_1}^{\iota_2} g_0(\xi)\,\mathrm{d}\xi + \int_{\iota_1}^{\iota_2} \frac{(F_{\iota_1}(\mathrm{t}))^{q} g_1(\xi)}{\Gamma(q+1)}\,\mathrm{d}\xi + \int_{\iota_1}^{\iota_2} \frac{(F_{\iota_1}(\mathrm{t}))^{q+p} g_2(\xi)}{\Gamma(q+p+1)}\,\mathrm{d}\xi$$

$$+ \int_{\iota_1}^{\iota_2} \frac{(F_{\iota_1}(\mathrm{t}))^{q+p+r} g_3(\xi)}{\Gamma(q+p+r+1)}\,\mathrm{d}\xi + \int_{\iota_1}^{\mathrm{t}} \mathbb{G}'(\xi)\frac{(F_{\xi}(\mathrm{t}))^{q+p+r+s-1}}{\Gamma(q+p+r+k)} h(\xi)\,\mathrm{d}\xi.$$

Define the vector space

$$X_k = \left\{ \mathrm{v}_k \in C(\Sigma,\mathbb{R}) \,:\, {}^{c}\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\mathrm{v}_\mathrm{k},\, {}^{c}\mathcal{D}_{\iota_1^+}^{p_k;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\mathrm{v}_k\right),\ {}^{c}\mathcal{D}_{\iota_1^+}^{r_k;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{p_k;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\mathrm{v}_k(\mathrm{t})\right)\right) \in C(\Sigma,\mathbb{R}) \right\}.$$

Then, $X_k,\ k = 1,2$, are Banach spaces via the norm

$$\|\mathrm{v}_k\| = \sup_{\mathrm{t}\in\Sigma} |\,\mathrm{v}_k(\mathrm{t})| + \sup_{\mathrm{t}\in\Sigma}\left|{}^{c}\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\mathrm{v}_k(\mathrm{t})\right| + \sup_{\mathrm{t}\in\Sigma}\left|{}^{c}\mathcal{D}_{\iota_1^+}^{p_k;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\mathrm{v}_k(\mathrm{t})\right)\right|$$

$$+ \sup_{\mathrm{t}\in\Sigma}\left|{}^{c}\mathcal{D}_{\iota_1^+}^{r_k;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{p_k;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\mathrm{v}_k(\mathrm{t})\right)\right)\right|.$$

Hence, the product space $X_1 \times X_2$ is a Banach space with the norm

$$\|(v_1, v_2)\| = \max\left\{\|\mathrm{v}_1\|, \|\mathrm{v}_2\|\right\}.$$

3.1. Existence and uniqueness

In view of Lemma 3.1, the solution of the coupled system (1.6) can be given as

$$\mathrm{v}_k(\mathrm{t}) = \int_{\iota_1}^{\iota_2} g_{k0}(\xi)\,\mathrm{d}\xi + \int_{\iota_1}^{\iota_2} \frac{(F_{\iota_1}(\mathrm{t}))^{q_k} g_{k1}(\xi)}{\Gamma(q_k+1)}\,\mathrm{d}\xi$$
$$+ \int_{\iota_1}^{\iota_2} \frac{(F_{\iota_1}(\mathrm{t}))^{q_k+p_k} g_{k2}(\xi)}{\Gamma(q_k+p_k+1)}\,\mathrm{d}\xi + \int_{\iota_1}^{\iota_2} \frac{\mathrm{v}_{k3}(F_{\iota_1}(\mathrm{t}))^{q_k+p_k+r_k} g_{k3}(\xi)}{\Gamma(q_k+p_k+r_k+1)}\,\mathrm{d}\xi$$

$$+ \int_{\iota_1}^{\mathrm{t}} \mathbb{G}'(\xi)\frac{(F_{\xi}(\mathrm{t}))^{q_k+p_k+r_k+s_k-1}}{\Gamma(q_k+p_k+r_k+s_k)} h_{\mathrm{v}_1,\mathrm{v}_2,k}(\xi)\mathrm{d}\xi.$$

Define the functional $\Lambda_k : X_k \to \mathbb{R}$, such that

$$(\Lambda_k \mathrm{v}_k)(\mathrm{t}) = \int_{\iota_1}^{\iota_2} g_{k0}(\xi)\,\mathrm{d}\xi + \int_{\iota_1}^{\iota_2} \frac{(F_{\iota_1}(\mathrm{t}))^{q_k} g_{k1}(\xi)}{\Gamma(q_k+1)}\,\mathrm{d}\xi + \int_{\iota_1}^{\iota_2} \frac{(F_{\iota_1}(\mathrm{t}))^{q_k+p_k} g_{k2}(\xi)}{\Gamma(q_k+p_k+1)}\,\mathrm{d}\xi$$
$$+ \int_{\iota_1}^{\iota_2} \frac{(F_{\iota_1}(\mathrm{t}))^{q_k+p_k+r_k} g_{k3}(\xi)}{\Gamma(q_k+p_k+r_k+1)}\,\mathrm{d}\xi + \int_{\iota_1}^{\mathrm{t}} \mathbb{G}'(\xi) \frac{(F_{\xi}(\mathrm{t}))^{q_k+p_k+r_k+s_k-1} h_{\mathrm{v}_1,\mathrm{v}_2,k}(\xi)}{\Gamma(q_k+p_k+r_k+s_k)}\,\mathrm{d}\xi. \tag{3.2}$$

Under some conditions, we show next that the functional $\Lambda : X_1 \times X_2 \to \mathbb{R}^2$ is a contraction, where Λ is given as

$$\Lambda(v_1, v_2) = (\Lambda_1(v_1, v_2), \Lambda_2(v_1, v_2)).$$

Theorem 3.2. *Let $h_k \in C(\Sigma \times \mathbb{R}^8)$, $(k = 1, 2)$ be continuous functions. Moreover, assume that*

(H1) *there exist real constants $\ell_k > 0$, $(k = 1, 2)$, so that*

$$|h_k(\mathrm{t}, \mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_8) - h_k(\mathrm{t}, \mathrm{v}_1^*, \mathrm{v}_2^*, \ldots, \mathrm{v}_8^*)| \leq \ell_k \sum_{i=1}^{8} |\mathrm{v}_i - \mathrm{v}_i^*|, \tag{3.3}$$

for any $\mathrm{t} \in \Sigma$, $\mathrm{v}_i, \mathrm{v}_i^* \in C([a, b])$ and $i = 1, 2, \ldots, 8$.

Then, the fractional $\mathbb{G}$-$\mathbb{CSS}$ (1.6) admits a unique solution on Σ if $\Phi\ell < 1$, whenever $\ell = \max\{\ell_1, \ell_2\}$, $\Phi = \max\{\Phi_1, \Phi_2\}$ and

$$\Phi_k = \frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k+p_k+r_k+s_k+1)} + \frac{(F_{\iota_1}(\iota_2))^{p_k+r_k+s_k}}{\Gamma(p_k+r_k+s_k+1)}$$
$$+ \frac{(F_{\iota_1}(\iota_2))^{r_k+s_k}}{\Gamma(r_k+s_k+1)} + \frac{(F_{\iota_1}(\iota_2))^{s_k}}{\Gamma(s_k+1)}, \tag{3.4}$$

with $\Phi_k \ell_k < 1$.

Proof. First of all, we define a closed bounded ball

$$\mathbb{B}_\varepsilon = \Big\{(\mathrm{v}_1, \mathrm{v}_2) \in X_1 \times X_2 : \quad \|(\mathrm{v}_1, \mathrm{v}_2)\| \leq \varepsilon\Big\},$$

satisfying

$$\varepsilon \geq \max\left\{\frac{\Delta_1 + h_1^0 \Phi_1}{(1-\ell_1\Phi_1)}, \frac{\Delta_2 + h_2^0 \Phi_2}{(1-\ell_2\Phi_2)}\right\}, \tag{3.5}$$

where

$$\Delta_k = M_{k0} + M_{k1}\left(1 + \frac{(F_{\iota_1}(\iota_2))^{q_k}}{\Gamma(q_k+1)}\right)$$
$$+ M_{k2}\left(1 + \frac{(F_{\iota_1}(\iota_2))^{p_k}}{\Gamma(p_k+1)} + \frac{(F_{\iota_1}(\iota_2))^{q_k+p_k}}{\Gamma(q_k+p_k+1)}\right)$$
$$+ M_{k3}\left(1 + \frac{(F_{\iota_1}(\iota_2))^{r_k}}{\Gamma(r_k+1)} + \frac{(F_{\iota_1}(\iota_2))^{p_k+r_k}}{\Gamma(p_k+r_k+1)} + \frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k}}{\Gamma(q_k+p_k+r_k+1)}\right), \tag{3.6}$$

and

$$M_{kj} = \sup_{\mathrm{t}\in\Sigma} \int_{\iota_1}^{\iota_2} |g_{kj}(\xi)|\,\mathrm{d}\xi, \qquad (j = 0, 1, 2, 3),$$

$$h_k^0=\sup_{\mathrm{t}\in\Sigma}|h_k(\mathrm{t},0,0,0,0,0,0,0,0)|,\qquad (k=1,2). \tag{3.7}$$

Now, define the operator

$$\Lambda(v_1,v_2)=(\Lambda_1(v_1,v_2),\Lambda_2(v_1,v_2)),\qquad \forall\,(v_1,v_2)\in X_1\times X_2,$$

where Λ_k is given in (3.2). To show that $\Lambda(\mathbb{B}_\varepsilon)\subset\mathbb{B}_\varepsilon$, by using hypotheses (H1), for $(\mathrm{v}_1,\mathrm{v}_2)\in\mathbb{B}_\varepsilon$ and $\mathrm{t}\in\Sigma$, we get

$$\begin{aligned}
|\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)(\mathrm{t})| \le{}& \int_{\iota_1}^{\iota_2}|g_{k0}(\xi)|\,\mathrm{d}\xi+\int_{\iota_1}^{\iota_2}\frac{(F_{\iota_1}(\mathrm{t}))^{q_k}|g_{k1}(\xi)|}{\Gamma(q_k+1)}\,\mathrm{d}\xi\\
&+\int_{\iota_1}^{\iota_2}\frac{(F_{\iota_1}(\mathrm{t}))^{q_k+p_k}|g_{k2}(\xi)|}{\Gamma(q_k+p_k+1)}\,\mathrm{d}\xi+\int_{\iota_1}^{\iota_2}\frac{(F_{\iota_1}(\mathrm{t}))^{q_k+p_k+r_k}|g_{k3}(\xi)|}{\Gamma(q_k+p_k+r_k+1)}\,\mathrm{d}\xi\\
&+\mathcal{I}_{\iota_1^+}^{q_k+p_k+r_k+s_k;\mathbb{G}}\Big(|h_{\mathrm{v}_1,\mathrm{v}_2,k}(\mathrm{t})-h_k(\mathrm{t},0,0,0,0,0,0,0,0)|\\
&+|h_k(\mathrm{t},0,0,0,0,0,0,0,0)|\Big)\\
\le{}& \int_a^b|g_{k0}(\xi)|\,\mathrm{d}\xi+\int_{\iota_1}^{\iota_2}\frac{(F_{\iota_1}(\mathrm{t}))^{q_k}|g_{k1}(\xi)|}{\Gamma(q_k+1)}\,\mathrm{d}\xi\\
&+\int_{\iota_1}^{\iota_2}\frac{(F_{\iota_1}(\mathrm{t}))^{q_k+p_k}|g_{k2}(\xi)|}{\Gamma(q_k+p_k+1)}\mathrm{d}+\int_{\iota_1}^{\iota_2}\frac{(F_{\iota_1}(\mathrm{t}))^{q_k+p_k+r_k}|g_{k3}(\xi)|}{\Gamma(q_k+p_k+r_k+1)}\,\mathrm{d}\xi\\
&+\mathcal{I}_{\iota_1^+}^{q_k+p_k+r_k+s_k;\mathbb{G}}\Big(\ell_k\Big(|\mathrm{v}_1(\mathrm{t})|+|\mathrm{v}_2(\mathrm{t})|\\
&+\left|{}^c\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right|+\left|{}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t})\right|\\
&+\left|{}^c\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right)\right|+\left|{}^c\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t})\right)\right|\\
&+\left|{}^c\mathcal{D}_{\iota_1^+}^{r_1;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right)\right)\right|\\
&+\left|{}^c\mathcal{D}_{\iota_1^+}^{r_2;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t})\right)\right)\right|\Big)\\
&+|h_k(\mathrm{t},0,0,0,0,0,0,0,0)|\Big)\\
\le{}& M_{k0}+M_{k1}\frac{(F_{\iota_1}(\iota_2))^{q_k}}{\Gamma(q_k+1)}+M_{k2}\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k}}{\Gamma(q_k+p_k+1)}\\
&+M_{k3}\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k}}{\Gamma(q_k+p_k+r_k+1)}\\
&+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k+p_k+r_k+s_k+1)}\Big(\ell_k\left(\|\mathrm{v}_1\|+\|\mathrm{v}_2\|\right)+h_k^0\Big).
\end{aligned}\tag{3.8}$$

Also,

$$\begin{aligned}
|{}^c\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\left(\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)(\mathrm{t})\right)| \le{}& \int_{\iota_1}^{\iota_2}|g_{k1}(\xi)|\,\mathrm{d}\xi+\int_{\iota_1}^{\iota_2}\frac{(F_{\iota_1}(\mathrm{t}))^{p_k}|g_{k2}(\xi)|}{\Gamma(p_k+1)}\,\mathrm{d}\xi\\
&+\int_{\iota_1}^{\iota_2}\frac{(F_{\iota_1}(\mathrm{t}))^{p_k+r_k}|g_{k3}(\xi)|}{\Gamma(p_k+r_k+1)}\,\mathrm{d}\xi
\end{aligned}$$

$$
\begin{aligned}
&+ \mathcal{I}_{\iota_1^+}^{p_k+r_k+s_k;\mathbb{G}}\Big(|h_{\mathrm{v}_1,\mathrm{v}_2,k}(\mathrm{t}) - h_k(\mathrm{t},0,0,0,0,0,0,0,0)| \\
&+ |h_k(\mathrm{t},0,0,0,0,0,0,0,0)|\Big) \\
\leq & \int_{\iota_1}^{\iota_2} |g_{k1}(\xi)|\,\mathrm{d}\xi + \int_{\iota_1}^{\iota_2} \frac{(F_{\iota_1}(\mathrm{t}))^{p_k}|g_{k2}(\xi)|}{\Gamma(p_k+1)}\mathrm{d}\xi + \int_{\iota_1}^{\iota_2} \frac{(F_{\iota_1}(\mathrm{t}))^{p_k+r_k}|g_{k3}(\xi)|}{\Gamma(p_k+r_k+1)}\,\mathrm{d}\xi \\
&+ \mathcal{I}_{\iota_1^+}^{p_k+r_k+s_k;\mathbb{G}}\Big(\ell_k\Big(|\mathrm{v}_1(\mathrm{t})| + |\mathrm{v}_2(\mathrm{t})| + |{}^c\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t})| + |{}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t})| \\
&+ \left|{}^c\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right)\right| + \left|{}^c\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t})\right)\right| \\
&+ \left|{}^c\mathcal{D}_{\iota_1^+}^{r_1;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right)\right)\right| \\
&+ \left|{}^c\mathcal{D}_{\iota_1^+}^{r_2;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t})\right)\right)\right|\Big) \\
&+ |h_k(\mathrm{t},0,0,0,0,0,0,0,0)|\Big) \\
\leq & M_{k1} + M_{k2}\frac{(F_{\iota_1}(\iota_2))^{p_k}}{\Gamma(p_k+1)} + M_{k3}\frac{(F_{\iota_1}(\iota_2))^{p_k+r_k}}{\Gamma(p_k+r_k+1)} \\
&+ \mathcal{I}_{\iota_1^+}^{p_k+r_k+s_k;\mathbb{G}}\left(\ell_k\left(\|\mathrm{v}_1\| + \|\mathrm{v}_2\|\right) + h_k^0\right) \\
\leq & M_{k1} + M_{k2}\frac{(F_{\iota_1}(\iota_2))^{p_k}}{\Gamma(p_k+1)} + M_{k3}\frac{(F_{\iota_1}(\iota_2))^{p_k+r_k}}{\Gamma(p_k+r_k+1)} \\
&+ \frac{(F_{\iota_1}(\iota_2))^{p_k+r_k+s_k}}{\Gamma(p_k+r_k+s_k+1)}\left(\ell_k\left(\|\mathrm{v}_1\| + \|\mathrm{v}_2\|\right) + h_k^0\right), \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
\left|{}^c\mathcal{D}_{\iota_1^+}^{p_k;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\left(\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)(\mathrm{t})\right)\right)\right| \leq & M_{k2} + M_{k3}\frac{(F_{\iota_1}(\iota_2))^{r_k}}{\Gamma(r_k+1)} \\
&+ \frac{(F_{\iota_1}(\iota_2))^{r_k+s_k}}{\Gamma(r_k+s_k+1)}\left(\ell_k\left(\|\mathrm{v}_1\| + \|\mathrm{v}_2\|\right) + h_k^0\right), \qquad (3.10)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|{}^c\mathcal{D}_{\iota_1^+}^{r_k;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{p_k;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\left(\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)(\mathrm{t})\right)\right)\right)\right| \\
&\qquad \leq M_{k3} + \frac{(F_{\iota_1}(\iota_2))^{s_k}}{\Gamma(s_k+1)}\left(\ell_k\left(\|\mathrm{v}_1\| + \|\mathrm{v}_2\|\right) + h_k^0\right). \qquad (3.11)
\end{aligned}
$$

Thus, due to (3.8)–(3.11) and (3.5), we obtain

$$
\begin{aligned}
\|\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)\| = & \sup_{\mathrm{t}\in\Sigma}|\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)(\mathrm{t})| + \sup_{\mathrm{t}\in\Sigma}\left|{}^c\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\left(\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)\right)(\mathrm{t})\right| \\
&+ \sup_{\mathrm{t}\in\Sigma}\left|{}^c\mathcal{D}_{\iota_1^+}^{p_k;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\left(\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)\right)(\mathrm{t})\right)\right| \\
&+ \sup_{\mathrm{t}\in\Sigma}\left|{}^c\mathcal{D}_{\iota_1^+}^{r_k;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{p_k;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\left(\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)\right)(\mathrm{t})\right)\right)\right| \\
\leq & \Bigg[M_{k0} + M_{k1}\left(1 + \frac{(F_{\iota_1}(\iota_2))^{q_k}}{\Gamma(q_k+1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ M_{k2}\left(1+\frac{(F_{\iota_1}(\iota_2))^{p_k}}{\Gamma(p_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k}}{\Gamma(q_k+p_k+1)}\right)\\
&+M_{k3}\left(1+\frac{(F_{\iota_1}(\iota_2))^{r_k}}{\Gamma(r_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{p_k+r_k}}{\Gamma(p_k+r_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k}}{\Gamma(q_k+p_k+r_k+1)}\right)\Bigg]\\
&+\left(\ell_k\|(\mathrm{v}_1,\mathrm{v}_2)\|+h_k^0\right)\Bigg[\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k+p_k+r_k+s_k+1)}\\
&+\frac{(F_{\iota_1}(\iota_2))^{p_k+r_k+s_k}}{\Gamma(p_k+r_k+s_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{r_k+s_k}}{\Gamma(r_k+s_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{s_k}}{\Gamma(s_k+1)}\Bigg]\\
\le&\Bigg[M_{k0}+M_{k1}\left(1+\frac{(F_{\iota_1}(\iota_2))^{q_k}}{\Gamma(q_k+1)}\right)\\
&+ M_{k2}\left(1+\frac{(F_{\iota_1}(\iota_2))^{p_k}}{\Gamma(p_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k}}{\Gamma(q_k+p_k+1)}\right)\\
&+M_{k3}\left(1+\frac{(F_{\iota_1}(\iota_2))^{r_k}}{\Gamma(r_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{p_k+r_k}}{\Gamma(p_k+r_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k}}{\Gamma(q_k+p_k+r_k+1)}\right)\Bigg]\\
&+\left(\ell_k\varepsilon+h_k^0\right)\Bigg[\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k+p_k+r_k+s_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{p_k+r_k+s_k}}{\Gamma(p_k+r_k+s_k+1)}\\
&+\frac{(F_{\iota_1}(\iota_2))^{r_k+s_k}}{\Gamma(r_k+s_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{s_k}}{\Gamma(s_k+1)}\Bigg]\\
\le&\ \Delta_k+\left(\ell_k\varepsilon+h_k^0\right)\Phi_k\le\varepsilon.
\end{aligned}
$$

Hence, we deduce that $\|\Lambda(\mathrm{v}_1,\mathrm{v}_2)\|\le\varepsilon$, for $(\mathrm{v}_1,\mathrm{v}_2)\in\mathbb{B}_\varepsilon$, so $\Lambda(\mathbb{B}_\varepsilon)\subset\mathbb{B}_\varepsilon$. Next, we prove that Λ is a contraction operator, by using (H1), for $(\mathrm{v}_1,\mathrm{v}_2),(\mathrm{u}_1,\mathrm{u}_2)\in\mathbb{B}_\varepsilon$ and $\mathrm{t}\in\Sigma$, we have

$$
\begin{aligned}
&|\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)(\mathrm{t})-\Lambda_k(\mathrm{u}_1,\mathrm{u}_2)(\mathrm{t})|\\
&\le\mathcal{I}_{\iota_1^+}^{q_k+p_k+r_k+s_k;\mathbb{G}}|h_{\mathrm{v}_1,\mathrm{v}_2,k}(\mathrm{t})-h_{\mathrm{u}_1,\mathrm{u}_2,k}(\mathrm{t})|\\
&\le\mathcal{I}_{\iota_1^+}^{q_k+p_k+r_k+s_k;\mathbb{G}}\Big(\ell_k\Big(|\mathrm{v}_1(\mathrm{t})-\mathrm{u}_1(\mathrm{t})|+|\mathrm{v}_2(\mathrm{t})-\mathrm{u}_2(\mathrm{t})|\\
&\quad+\left|{}^c\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t})-{}^c\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{u}_1(\mathrm{t})\right|+\left|{}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t})-{}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{u}_2(\mathrm{t})\right|\\
&\quad+\left|{}^c\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right)-{}^c\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{u}_1(\mathrm{t})\right)\right|\\
&\quad+\left|{}^c\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t})\right)-{}^c\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{u}_2(\mathrm{t})\right)\right|\\
&\quad+\Big|{}^c\mathcal{D}_{\iota_1^+}^{r_1;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right)\right)\\
&\quad-{}^c\mathcal{D}_{\iota_1^+}^{r_1;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{u}_1(\mathrm{t})\right)\right)\Big|\\
&\quad+\Big|{}^c\mathcal{D}_{\iota_1^+}^{r_2;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t})\right)\right)\\
&\quad-{}^c\mathcal{D}_{a^+}^{r_2;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\left({}^c\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{u}_2(\mathrm{t})\right)\right)\Big|\Big)\Big)\\
&\le\mathcal{I}_{\iota_1^+}^{q_k+p_k+r_k+s_k;\mathbb{G}}\left(\ell_k\left(\|\mathrm{v}_1-\mathrm{u}_1\|+\|\mathrm{v}_2-\mathrm{u}_2\|\right)\right)
\end{aligned}
$$

$$\leq \frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k+p_k+r_k+s_k+1)}\left(\ell_k\left(\|\mathrm{v}_1-\mathrm{u}_1\|+\|\mathrm{v}_2-\mathrm{u}_2\|\right)\right), \tag{3.12}$$

$$\begin{aligned}
&|{}^{c}\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\left(\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)\right)(\mathrm{t})-{}^{c}\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\left(\Lambda_k(\mathrm{u}_1,\mathrm{u}_2)\right)(\mathrm{t})|\\
&\qquad\leq \mathcal{I}_{\iota_1^+}^{p_k+r_k+s_k;\mathbb{G}}\left|h_{\mathrm{v}_1,\mathrm{v}_2,k}(\mathrm{t})-h_{\mathrm{u}_1,\mathrm{u}_2,k}(\mathrm{t})\right|\\
&\qquad\leq \mathcal{I}_{\iota_1^+}^{p_k+r_k+s_k;\mathbb{G}}\Big(\ell_k\Big(|\mathrm{v}_1(\mathrm{t})-\mathrm{u}_1(\mathrm{t})|+|\mathrm{v}_2(\mathrm{t})-\mathrm{u}_2(\mathrm{t})|\\
&\qquad\quad+\left|{}^{c}\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t})-{}^{c}\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{u}_1(\mathrm{t})\right|\\
&\qquad\quad+\left|{}^{c}\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t})-{}^{c}\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{u}_2(\mathrm{t})\right|\\
&\qquad\quad+\left|{}^{c}\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right)-{}^{c}\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{u}_1(\mathrm{t})\right)\right|\\
&\qquad\quad+\left|{}^{c}\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t})\right)-{}^{c}\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{u}_2(\mathrm{t})\right)\right|\\
&\qquad\quad+\Big|{}^{c}\mathcal{D}_{\iota_1^+}^{r_1;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right)\right)\\
&\qquad\quad-{}^{c}\mathcal{D}_{\iota_1^+}^{r_1;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{p_1;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_1;\mathbb{G}}\mathrm{u}_1(\mathrm{t})\right)\right)\Big|\\
&\qquad\quad+\Big|{}^{c}\mathcal{D}_{\iota_1^+}^{r_2;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{v}_2(\mathrm{t})\right)\right)\\
&\qquad\quad-{}^{c}\mathcal{D}_{\iota_1^+}^{r_2;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{p_2;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_2;\mathbb{G}}\mathrm{u}_2(\mathrm{t})\right)\right)\Big|\Big)\Big)\\
&\qquad\leq \mathcal{I}_{\iota_1^+}^{p_k+r_k+s_k;\mathbb{G}}\left(\ell_k\left(\|\mathrm{v}_1-\mathrm{u}_1\|+\|\mathrm{v}_2-\mathrm{u}_2\|\right)\right)\\
&\qquad\leq \frac{(F_{\iota_1}(\iota_2))^{p_k+r_k+s_k}}{\Gamma(p_k+r_k+s_k+1)}\left(\ell_k\left(\|\mathrm{v}_1-\mathrm{u}_1\|+\|\mathrm{v}_2-\mathrm{u}_2\|\right)\right),
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
&\left|{}^{c}\mathcal{D}_{\iota_1^+}^{p_k;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\left(\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)\right)\right)(\mathrm{t})-{}^{c}\mathcal{D}_{\iota_1^+}^{p_k;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\left(\Lambda_k(\mathrm{u}_1,\mathrm{u}_2)\right)\right)(\mathrm{t})\right|\\
&\qquad\leq \frac{(F_{\iota_1}(\iota_2))^{r_k+s_k}}{\Gamma(r_k+s_k+1)}\left(\ell_k\left(\|\mathrm{v}_1-\mathrm{u}_1\|+\|\mathrm{v}_2-\mathrm{u}_2\|\right)\right),
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
&\left|{}^{c}\mathcal{D}_{\iota_1^+}^{r_k;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{p_k;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\left(\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)\right)\right)\right)(\mathrm{t})-{}^{c}\mathcal{D}_{\iota_1^+}^{r_k;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{p_k;\mathbb{G}}\left({}^{c}\mathcal{D}_{\iota_1^+}^{q_k;\mathbb{G}}\left(\Lambda_k(\mathrm{u}_1,\mathrm{u}_2)\right)\right)\right)(\mathrm{t})\right|\\
&\qquad\leq \frac{(F_{\iota_1}(\iota_2))^{s_k}}{\Gamma(s_k+1)}\left(\ell_k\left(\|\mathrm{v}_1-\mathrm{u}_1\|+\|\mathrm{v}_2-\mathrm{u}_2\|\right)\right).
\end{aligned} \tag{3.15}$$

Therefore, due to (3.12)–(3.15), we get

$$\begin{aligned}
&\|\Lambda_k(\mathrm{v}_1,\mathrm{v}_2)-\Lambda_k(\mathrm{u}_1,\mathrm{u}_2)\|\\
&\qquad\leq \Bigg[\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k+p_k+r_k+s_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{p_k+r_k+s_k}}{\Gamma(p_k+r_k+s_k+1)}
\end{aligned}$$

$$+\frac{(F_{\iota_1}(\iota_2))^{r_k+s_k}}{\Gamma(r_k+s_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{s_k}}{\Gamma(s_k+1)}\Bigg](\ell_k(\|\mathrm{v}_1-\mathrm{u}_1\|+\|\mathrm{v}_2-\mathrm{u}_2\|))$$
$$\leq \Phi_k\ell_k(\|\mathrm{v}_1-\mathrm{u}_1\|+\|\mathrm{v}_2-\mathrm{u}_2\|).$$

Consequently,

$$\|\Lambda(\mathrm{v}_1,\mathrm{v}_2)-\Lambda(\mathrm{u}_1,\mathrm{u}_2)\|\leq \Phi\ell\|(\mathrm{v}_1,\mathrm{u}_1)-(\mathrm{v}_2,\mathrm{u}_2)\|.$$

Since $\Phi\ell<1$, therefore, Λ is a contraction operator. Thus, by Banach's fixed point Theorem 2.3, the operator Λ has a unique fixed point, which is the unique solution of fractional G-snap system (1.6) and the proof is finished. □

Next, we are ready to study the existence of solution of fractional $\mathbb{G}$-($\mathbb{CSS}$) (1.6). For this regaed, we define the operators $\Omega,\Pi: X_1\times X_2\to\mathbb{R}^2, \Omega=(\Omega_1,\Omega_2), \Pi=(\Pi_1,\Pi_2)$, such that $\Lambda_k=\Omega_k+\Pi_k$, where

$$(\Omega_k\mathrm{v}_k)(\mathrm{t})=\int_{\iota_1}^{\mathrm{t}}\mathbb{G}'(\xi)\frac{(F_{\iota_1}(\mathrm{t}))^{q_k+p_k+r_k+s_k-1}}{\Gamma(q_k+p_k+r_k+s_k)}h_{\mathrm{v}_1,\mathrm{v}_2,k}(\xi)\mathrm{d}\xi, \tag{3.16}$$

and

$$\begin{aligned}(\Pi_k\mathrm{v}_k)(\mathrm{t})=&\int_{\iota_1}^{\iota_2}g_{k0}(\xi)\,\mathrm{d}\xi+\frac{(F_{\iota_1}(\mathrm{t}))^{q_k}}{\Gamma(q_k+1)}\int_{\iota_1}^{\iota_2}g_{k1}(\xi)\,\mathrm{d}\xi\\&+\int_{\iota_1}^{\iota_2}\frac{(F_{\iota_1}(\mathrm{t}))^{q_k+p_k}g_{k2}(\xi)}{\Gamma(q_k+p_k+1)}\,\mathrm{d}\xi+\int_{\iota_1}^{\iota_2}\frac{(F_{\iota_1}(\mathrm{t}))^{q_k+p_k+r_k}g_{k3}(\xi)}{\Gamma(q_k+p_k+r_k+1)}\,\mathrm{d}\xi.\end{aligned} \tag{3.17}$$

Theorem 3.3. *Let $h_k\in C(\Sigma\times\mathbb{R}^8), (k=1,2)$ be continuous functions. Moreover, assume that*

(H2) *there exist real constants $\lambda_k>0$, $(k=1,2)$, so that*

$$|h_k(\mathrm{t},\mathrm{v}_1,\mathrm{v}_2,\dots,\mathrm{v}_8)|\leq\lambda_k,\qquad \forall\mathrm{t}\in\Sigma\,\&\,\mathrm{v}_i\in C(\Sigma),\ (i=1,2,\dots,8).$$

Then, the fractional $\mathbb{G}$-$\mathbb{CSS}$ (1.6) has at least one solution on Σ.

Proof. At the beginning, we define a closed bounded ball

$$\mathbb{B}_r=\Big\{(\mathrm{v}_1,\mathrm{v}_2)\in X_1\times X_2\ :\ \|(\mathrm{v}_1,\mathrm{v}_2)\|\leq r\Big\},$$

which satisfying

$$r\geq\max\{r_1,r_2\}, \tag{3.18}$$

where

$$\begin{aligned}r_k\geq\ & M_{k0}+M_{k1}\left(1+\frac{(F_{\iota_1}(\iota_2))^{q_k}}{\Gamma(q_k+1)}\right)\\&+M_{k2}\left(1+\frac{(F_{\iota_1}(\iota_2))^{p_k}}{\Gamma(p_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k}}{\Gamma(q_k+p_k+1)}\right)\\&+M_{k3}\left(1+\frac{(F_{\iota_1}(\iota_2))^{r_k}}{\Gamma(r_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{p_k+r_k}}{\Gamma(p_k+r_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k}}{\Gamma(q_k+p_k+r_k+1)}\right)\end{aligned}$$

$$+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k+p_k+r_k+s_k+1)}\lambda_k.$$

Firstly, we will prove $\Omega\mathrm{v}+\Pi\mathrm{u}\in\mathbb{B}_r$. By using (H2), for $\mathrm{v}=(\mathrm{v}_1,\mathrm{v}_2),\mathrm{u}=(\mathrm{u}_1,\mathrm{u}_2)\in\mathbb{B}_r$ and $\mathrm{t}\in[a,b]$, we have

$$\|\Omega_k(\mathrm{v}_1,\mathrm{v}_2)\|\le\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k+p_k+r_k+s_k+1)}\lambda_k, \tag{3.19}$$

and

$$\begin{aligned}\|\Pi_k(\mathrm{u}_1,\mathrm{u}_2)\|\le M_{k0}&+M_{k1}\left(1+\frac{(F_{\iota_1}(\iota_2))^{q_k}}{\Gamma(q_k+1)}\right)\\&+M_{k2}\left(1+\frac{(F_{\iota_1}(\iota_2))^{p_k}}{\Gamma(p_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k}}{\Gamma(q_k+p_k+1)}\right)\\&+M_{k3}\left(1+\frac{(F_{\iota_1}(\iota_2))^{r_k}}{\Gamma(r_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{p_k+r_k}}{\Gamma(p_k+r_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k}}{\Gamma(q_k+p_k+r_k+1)}\right).\end{aligned}\tag{3.20}$$

Hence, from (3.19) and (3.20), we have

$$\begin{aligned}&\|\Omega_k(\mathrm{v}_1,\mathrm{v}_2)+\Pi_k(\mathrm{u}_1,\mathrm{u}_2)\|\\&\quad\le M_{k0}+M_{k1}\left(1+\frac{(F_{\iota_1}(\iota_2))^{q_k}}{\Gamma(q_k+1)}\right)\\&\quad+M_{k2}\left(1+\frac{(F_{\iota_1}(\iota_2))^{p_k}}{\Gamma(p_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k}}{\Gamma(q_k+p_k+1)}\right)\\&\quad+M_{k3}\left(1+\frac{(F_{\iota_1}(\iota_2))^{r_k}}{\Gamma(r_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{p_k+r_k}}{\Gamma(p_k+r_k+1)}+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k}}{\Gamma(q_k+p_k+r_k+1)}\right)\\&\quad+\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k+p_k+r_k+s_k+1)}\lambda_k\le r.\end{aligned}$$

Then,

$$\|\Omega(\mathrm{v}_1,\mathrm{v}_2)+\Pi(\mathrm{u}_1,\mathrm{u}_2)\|\le r,$$

this implying that $\Omega\mathrm{v}+\Pi\mathrm{u}\in\mathbb{B}_r$.

Secondly, we will prove that the operator Π is a contraction mapping. It is clearly that Π_k is a contraction with the constant zero. Thus, Π is a contraction operator.

Third, we will prove that the operator Ω is a continuous. Let $(\mathrm{v}_{n,1},\mathrm{v}_{n,2})$ be a sequence of a bounded ball $\mathbb{B}_r$, such that $(\mathrm{v}_{n,1},\mathrm{v}_{n,2})\to(\mathrm{v}_1,\mathrm{v}_2)$ as $n\to\infty$ in $\mathbb{B}_r$, we find that

$$\begin{aligned}&|(\Omega_k(\mathrm{v}_{\mathrm{n},1},\mathrm{v}_{\mathrm{n},2}))(\mathrm{t})-(\Omega_k(\mathrm{v}_1,\mathrm{v}_2))(\mathrm{t})|\\&\qquad\le\int_{\iota_1}^{\mathrm{t}}\frac{\mathbb{G}'(\xi)(F_{\iota_1}(\mathrm{t}))^{q_k+p_k+r_k+s_k-1}}{\Gamma(q_k+p_k+r_k+s_k)}\left|h_{\mathrm{v}_{n,1},\mathrm{v}_{n,2},k}(\xi)-h_{\mathrm{v}_1,\mathrm{v}_2,k}(\xi)\right|\mathrm{d}\xi\\&\qquad\le\frac{(F_{\iota_1}(\iota_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k+p_k+r_k+s_k+1)}\left\|h_{\mathrm{v}_{n,1},\mathrm{v}_{n,2},k}(.)-h_{\mathrm{v}_1,\mathrm{v}_2,k}(.)\right\|.\end{aligned}$$

By continuity of $h_{\mathrm{v}_1,\mathrm{v}_2,k}$, we have

$$\|\Omega_k(\mathrm{v}_{\mathrm{n},1},\mathrm{v}_{\mathrm{n},2})-\Omega_k(\mathrm{v}_1,\mathrm{v}_2)\|\to 0,$$

as $n \to \infty$. So, Ω is a continuous operator.

Fourth, we will prove that the operator Ω is a compact operator. By using (H2), for $\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2), \in \mathbb{B}_r$ and $\mathrm{t}_g, \mathrm{t}_y \in \Sigma$ with $\mathrm{t}_g < \mathrm{t}_y$, we have

$$
\begin{aligned}
|(\Omega_k(\mathrm{v}_1, \mathrm{v}_2))(\mathrm{t}_y) - & (\Omega_k(\mathrm{v}_1, \mathrm{v}_2))(\mathrm{t}_g)| \\
& \leq \left| \int_{\iota_1}^{\mathrm{t}_y} \frac{\mathbb{G}'(\xi)(F_{\iota_1}(\mathrm{t}_2))^{q_k+p_k+r_k+s_k-1}}{\Gamma(q_k + p_k + r_k + s_k)} h_{\mathrm{v}_1,\mathrm{v}_2,k}(\xi) \mathrm{d}\xi \right. \\
& \quad \left. - \int_{\iota_1}^{\mathrm{t}_g} \frac{\mathbb{G}'(\xi)(F_{\iota_1}(\mathrm{t}_1))^{q_k+p_k+r_k+s_k-1}}{\Gamma(q_k + p_k + r_k + s_k)} h_{\mathrm{v}_1,\mathrm{v}_2,k}(\xi)\, \mathrm{d}\xi \right| \\
& \leq \left| \int_{\iota_1}^{\mathrm{t}_g} \frac{\mathbb{G}'(\xi)(F_{\iota_1}(\mathrm{t})_2)^{q_k+p_k+r_k+s_k-1}}{\Gamma(q_k + p_k + r_k + s_k)} h_{\mathrm{v}_1,\mathrm{v}_2,k}(\xi)\, \mathrm{d}\xi \right. \\
& \quad \left. - \int_{\iota_1}^{\mathrm{t}_g} \frac{\mathbb{G}'(\xi)(F_{\iota_1}(\mathrm{t})_2)^{q_k+p_k+r_k+s_k-1}}{\Gamma(q_k + p_k + r_k + s_k)} h_{\mathrm{v}_1,\mathrm{v}_2,k}(\xi)\, \mathrm{d}\xi \right| \\
& \quad + \left| \int_{\mathrm{t}_g}^{\mathrm{t}_y} \frac{\mathbb{G}'(\xi)(F_{\iota_1}(\mathrm{t}_2))^{q_k+p_k+r_k+s_k-1}}{\Gamma(q_k + p_k + r_k + s_k)} h_{\mathrm{v}_1,\mathrm{v}_2,k}(\xi)\, \mathrm{d}\xi \right| \\
& \leq \lambda_k \left[\frac{(F_{\mathrm{t}_1}(\mathrm{t}_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k + p_k + r_k + s_k + 1)} - \frac{(F_{\mathrm{t}_1}(\mathrm{t}_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k + p_k + r_k + s_k + 1)} \right] \\
& \quad + \lambda_k \frac{(F_{\mathrm{t}_1}(\mathrm{t}_2))^{q_k+p_k+r_k+s_k}}{\Gamma(q_k + p_k + r_k + s_k + 1)}.
\end{aligned}
$$

As $\mathrm{t}_g \to \mathrm{t}_y$, we obtain

$$
|(\Omega_k(\mathrm{v}_1, \mathrm{v}_2))(\mathrm{t}_y) - (\Omega_k(\mathrm{v}_1, \mathrm{v}_2))(\mathrm{t}_g)| \to 0,
$$

impling that Ω is equicontinuous. Furthermore, in view of (3.19), Ω is uniformly bounded. Hence, due to the Arzelá-Ascoli theorem, we deduce that Ω is a compact operator. Then, all the conditions of Theorem 2.4 are holding. Thus, fractional $\mathbb{G}$-$\mathbb{CSS}$ (1.6) has at least one solution $(\mathrm{v}_1, \mathrm{v}_2) \in \mathbb{B}_r$. The proof is completed. □

3.2. Ulam-Hyers stability

In this part, we review the stability criterion in the context of the U-H-$\mathbb{S}$ for solutions of the fractional $\mathbb{G}$-$\mathbb{CSS}$ (1.6).

Theorem 3.4. *Let* (H1) *and* $\Phi_k \ell_k < 1, (k = 1, 2)$ *hold. Then, the fractional* $\mathbb{G}$-$\mathbb{CSS}$ (1.6) *is* U-H-$\mathbb{S}$.

Proof. According to Theorem 3.2, we have

$$
\|\Lambda_k(\mathrm{v}_1, \mathrm{v}_2) - \Lambda_k(\mathrm{u}_1, \mathrm{u}_2)\| \leq \Phi_k \ell_k \|\mathrm{v}_1 - \mathrm{u}_1\| + \Phi_k \ell_k \|\mathrm{v}_2 - \mathrm{u}_2\|,
$$

which yields that

$$
\|\Lambda(\mathrm{v}_1, \mathrm{v}_2) - \Lambda(\mathrm{u}_1, \mathrm{u}_2)\| \leq \varXi \times \begin{pmatrix} \|\mathrm{v}_1 - \mathrm{u}_1\| \\ \|\mathrm{v}_2 - \mathrm{u}_2\| \end{pmatrix},
$$

where

$$
\varXi = \begin{pmatrix} \Phi_1 \ell_1 & \Phi_1 \ell_1 \\ \Phi_2 \ell_2 & \Phi_2 \ell_2 \end{pmatrix}.
$$

Since $\Phi_k \ell_k < 1$ and each geometric sequences $(\Phi_k \ell_k)^n \to 0$, hence $\Xi^n \to 0$ as $n \to \infty$. Therefore, due to Theorem 2.6, the fractional $\mathbb{G}$-$\mathbb{CSS}$ (1.6) is U-H $-$ $\mathbb{S}$. □

4. Application

We allow here a few illustrations of the fractional $\mathbb{G}$-$\mathbb{CSS}$, based on numerical recreation to analyze their solutions. In these cases, we consider distinctive cases of the function $\mathbb{G}$ to cover the Caputo, Caputo-Hadamard and Katugampola adaptations.

Example 4.1. *Based on the system* (1.6)*, by assuming* $\Sigma = [0.05, 0.95]$*,*

$$q_1 = 0.73 \in (0,1], \quad q_2 = 0.36 \in (0,1], \quad p_1 = 0.92 \in (0,1], \quad p_2 = 0.45 \in (0,1],$$

$$r_1 = 0.12 \in (0,1], \quad r_2 = 0.87 \in (0,1], \quad s_1 = 0.54 \in (0,1], \quad s_2 = 0.27 \in (0,1],$$

we consider a fractional $\mathbb{CSS}$ *as*

$$\begin{cases} {}^c\mathcal{D}_{\iota_1^+}^{0.73;\mathbb{G}} \mathrm{v}_1(\mathrm{t}) = \mathrm{u}_1(\mathrm{t}), \quad {}^c\mathcal{D}_{\iota_1^+}^{0.36;\mathbb{G}} \mathrm{v}_2(\mathrm{t}) = \mathrm{u}_2(\mathrm{t}), \\ {}^c\mathcal{D}_{\iota_1^+}^{0.92;\mathbb{G}} \mathrm{u}_1(\mathrm{t}) = \mathrm{w}_1(\mathrm{t}), \quad {}^c\mathcal{D}_{\iota_1^+}^{0.45;\mathbb{G}} \mathrm{u}_2(\mathrm{t}) = \mathrm{w}_2(\mathrm{t}), \\ {}^c\mathcal{D}_{\iota_1^+}^{0.12;\mathbb{G}} \mathrm{w}_1(\mathrm{t}) = \mathrm{x}_1(\mathrm{t}), \quad {}^c\mathcal{D}_{\iota_1^+}^{0.87;\mathbb{G}} \mathrm{w}_2(\mathrm{t}) = \mathrm{x}_2(\mathrm{t}), \\ {}^c\mathcal{D}_{\iota_1^+}^{0.54;\mathbb{G}} \mathrm{x}_1(\mathrm{t}) = \dfrac{5\mathrm{t}}{36(\sqrt{15}+\mathrm{t}^2)} + \dfrac{\arctan^2(\mathrm{v}_1)}{36\,(7+\arctan^2(\mathrm{v}_1))} \\ \qquad + \dfrac{\exp(|\mathrm{v}_2|+1)}{36\left(\sqrt{15}+\exp(|\mathrm{v}_2|)\right)} + \dfrac{1}{72}\arcsin\dfrac{\mathrm{u}_1}{\sqrt[3]{36+\mathrm{u}_1}} \\ \qquad + \dfrac{\mathrm{t}}{36}\dfrac{\sin|\mathrm{u}_2|}{15+\sin|\mathrm{u}_2|} + \dfrac{\exp(\mathrm{w}_1)}{36(\sqrt{7}+\exp(\mathrm{w}_1))} + \dfrac{\mathrm{w}_2^2}{108\,(\mathrm{w}_2+3)^2} \\ \qquad + \dfrac{(\mathrm{x}_1\mathrm{x}_2)^2}{54\,(\mathrm{x}_1\mathrm{x}_2+21)^2} \\ {}^c\mathcal{D}_{\iota_1^+}^{0.27;\mathbb{G}} \mathrm{x}_2(\mathrm{t}) = \dfrac{\mathrm{t}^2}{2\sqrt{3}(\mathrm{t}^2+49)} + \dfrac{\tan^2(\mathrm{v}_1)}{1.5\,(7+\tan^2(\mathrm{v}_1))} \\ \qquad + \dfrac{\cos^2(|\mathrm{v}_2|+1)}{36\left(\sqrt{15}+\cos^2(|\mathrm{v}_2|)\right)} + \dfrac{1}{25}\arctan\dfrac{3\mathrm{u}_1}{\sqrt[5]{6+\mathrm{u}_1}} \\ \qquad + \dfrac{\arctan|\mathrm{u}_2|}{45+\arctan|\mathrm{u}_2|} + \dfrac{|\mathrm{w}_1|+3}{48\,(|\mathrm{w}_1|+5)^2} + \dfrac{\exp(\mathrm{w}_2)}{45(\sqrt{12}+\exp(\mathrm{w}_2))} \\ \qquad + \dfrac{(\mathrm{x}_1+\mathrm{x}_2)^2}{\sqrt{10}\,(\mathrm{x}_1+\mathrm{x}_2+21)^2}, \end{cases} \tag{4.1}$$

for $\mathrm{t} \in \Sigma$ *and*

$$\mathrm{v}_1(\iota_1) = \int_{\iota_1}^{\iota_2} \frac{s}{2}\,\mathrm{d}s = 0.2250, \qquad \mathrm{v}_2(\iota_1) = \int_{\iota_1}^{\iota_2} \sqrt{s}\,\mathrm{d}s = 0.6098,$$

$$\mathrm{u}_1(\iota_1) = \int_{\iota_1}^{\iota_2} \frac{s^2}{5}\,\mathrm{d}s = 0.0571, \qquad \mathrm{u}_2(\iota_1) = \int_{\iota_1}^{\iota_2} \frac{3}{2}s\,\mathrm{d}s = 0.6750,$$

$$\mathrm{w}_1(\iota_1)=\int_{\iota_1}^{\iota_2}\frac{s}{\sqrt{2}}\,\mathrm{d}s=0.3181,\qquad \mathrm{w}_2(\iota_1)=\int_{\iota_1}^{\iota_2}\frac{\sqrt{s}}{7}\,\mathrm{d}s=0.0871,$$
$$\mathrm{x}_1(\iota_1)=\int_{\iota_1}^{\iota_2}\sin(\pi s)\,\mathrm{d}s=0.6287,\qquad \mathrm{x}_2(\iota_1)=\int_{\iota_1}^{\iota_2}\cos(\pi s)\,\mathrm{d}s=0.$$

Clearly,

$$\begin{aligned}
&h_1(\mathrm{t},\mathrm{v}_1,\mathrm{v}_2,\mathrm{u}_1,\mathrm{u}_2,\mathrm{w}_1,\mathrm{w}_2,\mathrm{x}_1,\mathrm{x}_2)\\
&=\frac{5\mathrm{t}}{36(\sqrt{15}+\mathrm{t}^2)}+\frac{\arctan^2(\mathrm{v}_1)}{36\,(7+\arctan^2(\mathrm{v}_1))}+\frac{\exp(|\mathrm{v}_2|+1)}{36\left(\sqrt{15}+\exp(|\mathrm{v}_2|)\right)}\\
&\quad+\frac{1}{72}\arcsin\frac{\mathrm{u}_1}{\sqrt[3]{36+\mathrm{u}_1}}+\frac{\mathrm{t}}{36}\frac{\sin|\mathrm{u}_2|}{15+\sin|\mathrm{u}_2|}+\frac{\exp(\mathrm{w}_1)}{36(\sqrt{7}+\exp(\mathrm{w}_1))}\\
&\quad+\frac{\mathrm{w}_1^2}{108\,(\mathrm{w}_1+3)^2}+\frac{(\mathrm{x}_1\mathrm{x}_2)^2}{54\,(\mathrm{x}_1\mathrm{x}_2+21)^2},
\end{aligned}$$

and

$$\begin{aligned}
&h_2(\mathrm{t},\mathrm{v}_1,\mathrm{v}_2,\mathrm{u}_1,\mathrm{u}_2,\mathrm{w}_1,\mathrm{w}_2,\mathrm{x}_1,\mathrm{x}_2)\\
&=\frac{\mathrm{t}^2}{2\sqrt{3}(\mathrm{t}^2+49)}+\frac{\tan^2(\mathrm{v}_1)}{1.5\,(7+\tan^2(\mathrm{v}_1))}\frac{\cos^2(|\mathrm{v}_2|+1)}{36\left(\sqrt{15}+\cos^2(|\mathrm{v}_2|)\right)}\\
&\quad++\frac{1}{25}\arctan\frac{3\mathrm{u}_1}{\sqrt[5]{6+\mathrm{u}_1}}+\frac{\arctan|\mathrm{u}_2|}{45+\arctan|\mathrm{u}_2|}+\frac{|\mathrm{w}_1|+3}{48\,(|\mathrm{w}_1|+5)^2}\\
&\quad+\frac{\exp(\mathrm{w}_2)}{45(\sqrt{12}+\exp(\mathrm{w}_2))}+\frac{(\mathrm{x}_1+\mathrm{x}_2)^2}{\sqrt{10}\,(\mathrm{x}_1+\mathrm{x}_2+21)^2}.
\end{aligned}$$

Thus, we can rewrite the above system as Eq (1.6)*. At present, we will have*

$$\begin{aligned}
&\left|h_1(\mathrm{t},\mathrm{v}_1,\mathrm{v}_2,\ldots,\mathrm{v}_8)-h_1(\mathrm{t},\mathrm{v}_1^*,\mathrm{v}_2^*,\ldots,\mathrm{v}_8^*)\right|\\
&=\Bigg|\frac{5\mathrm{t}}{36(\sqrt{15}+\mathrm{t}^2)}+\frac{\arctan^2(\mathrm{v}_1)}{36\,(7+\arctan^2(\mathrm{v}_1))}+\frac{\exp(|\mathrm{v}_2|+1)}{36\left(\sqrt{15}+\exp(|\mathrm{v}_2|)\right)}\\
&\quad+\frac{1}{72}\arcsin\frac{\mathrm{v}_3}{\sqrt[3]{36+\mathrm{v}_3}}+\frac{\mathrm{t}}{36}\frac{\sin|\mathrm{v}_4|}{15+\sin|\mathrm{v}_4|}+\frac{\exp(\mathrm{v}_5)}{36(\sqrt{7}+\exp(\mathrm{v}_5))}\\
&\quad+\frac{\mathrm{v}_6^2}{108\,(\mathrm{v}_6+3)^2}+\frac{(\mathrm{v}_7\mathrm{v}_8)^2}{54\,(\mathrm{v}_7\mathrm{v}_8+21)^2}\\
&\quad-\Bigg(\frac{5\mathrm{t}}{36(\sqrt{15}+\mathrm{t}^2)}+\frac{\arctan^2(\mathrm{v}_1^*)}{36\left(7+\arctan^2(\mathrm{v}_1^*)\right)}+\frac{\exp(|\mathrm{v}_2^*|+1)}{36\left(\sqrt{15}+\exp(|\mathrm{v}_2^*|)\right)}\\
&\quad+\frac{1}{72}\arcsin\frac{\mathrm{v}_3^*}{\sqrt[3]{36+\mathrm{v}_3^*}}+\frac{\mathrm{t}}{36}\frac{\sin|\mathrm{v}_4^*|}{15+\sin|\mathrm{v}_4^*|}+\frac{\exp(\mathrm{v}_5^*)}{36(\sqrt{7}+\exp(\mathrm{v}_5^*))}\\
&\quad+\frac{(\mathrm{v}_6^*)^2}{108\left(\mathrm{v}_6^*+3\right)^2}+\frac{\left(\mathrm{v}_7^*\mathrm{v}_8^*\right)^2}{54\left(\mathrm{v}_7^*\mathrm{v}_8^*+21\right)^2}\Bigg)\Bigg|\le\frac{5}{36}\sum_{i=1}^{8}|\mathrm{v}_i-\mathrm{v}_i^*|,
\end{aligned}$$

with $\ell_1 = \frac{5}{36}$ *and*

$$\begin{aligned}
&\left|h_2(\mathrm{t}, \mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_8) - h_2(\mathrm{t}, \mathrm{v}_1^*, \mathrm{v}_2^*, \ldots, \mathrm{v}_8^*)\right| \\
&\quad = \Bigg| \frac{\mathrm{t}^2}{2\sqrt{3}(\mathrm{t}^2 + 49)} + \frac{\tan^2(\mathrm{v}_1)}{1.5\,(7 + \tan^2(\mathrm{v}_1))} + \frac{\cos^2(|\mathrm{v}_2| + 1)}{36\left(\sqrt{15} + \cos^2(|\mathrm{v}_2|)\right)} \\
&\qquad + \frac{1}{25}\arctan\frac{3\mathrm{v}_3}{\sqrt[5]{6 + \mathrm{v}_3}} + \frac{\arctan|\mathrm{v}_4|}{45 + \arctan|\mathrm{v}_4|} + \frac{|\mathrm{v}_5| + 3}{48\,(|\mathrm{v}_5| + 5)^2} \\
&\qquad + \frac{\exp(\mathrm{v}_6)}{45(\sqrt{12} + \exp(\mathrm{v}_6))} + \frac{(\mathrm{v}_7 + \mathrm{v}_8)^2}{\sqrt{10}\,(\mathrm{v}_7 + \mathrm{v}_8 + 21)^2} \\
&\qquad - \Bigg(\frac{\mathrm{t}^2}{2\sqrt{3}(\mathrm{t}^2 + 49)} + \frac{\tan^2(\mathrm{v}_1^*)}{1.5\left(7 + \tan^2(\mathrm{v}_1^*)\right)} + \frac{\cos^2(|\mathrm{v}_2^*| + 1)}{36\left(\sqrt{15} + \cos^2(|\mathrm{v}_2^*|)\right)} \\
&\qquad + \frac{1}{25}\arctan\frac{3\mathrm{v}_3^*}{\sqrt[5]{6 + \mathrm{v}_3^*}} + \frac{\arctan|\mathrm{v}_4^*|}{45 + \arctan|\mathrm{v}_4^*|} + \frac{|\mathrm{v}_5^*| + 3}{48\left(|\mathrm{v}_5^*| + 5\right)^2} \\
&\qquad + \frac{\exp(\mathrm{v}_6^*)}{45(\sqrt{12} + \exp(\mathrm{v}_6^*))} + \frac{\left(\mathrm{v}_7^* + \mathrm{v}_8^*\right)^2}{\sqrt{10}\left(\mathrm{v}_8^* + \mathrm{v}_8^* + 21\right)^2} \Bigg) \Bigg| \leq \frac{1}{2\sqrt{3}} \sum_{i=1}^{8} |\mathrm{v}_i - \mathrm{v}_i^*|,
\end{aligned}$$

with $\ell_2 = \frac{1}{2\sqrt{3}}$. *So* $\ell = \frac{1}{2\sqrt{3}}$. *Now, from* (3.6), *we consider four cases for* $\mathbb{G}$ *as:*

- $\mathbb{G}_1(\iota) = 2^\iota$,
- $\mathbb{G}_2(\iota) = \iota$ *(Caputo derivative)*,
- $\mathbb{G}_3(\iota) = \ln\iota$ *(Caputo-Hadamard derivative)*,
- $\mathbb{G}_4(\iota) = \sqrt{\iota}$ *(Katugampola derivative)*.

Thus,

$$\begin{aligned}
\Phi_1 &= \frac{(F_{\iota_1}(\iota_2))^{q_1+p_1+r_1+s_1}}{\Gamma(q_1 + p_1 + r_1 + s_1 + 1)} + \frac{(F_{\iota_1}(\iota_2))^{p_1+r_1+s_1}}{\Gamma(p_k + r_1 + s_1 + 1)} \\
&\quad + \frac{(F_{\iota_1}(\iota_2))^{r_1+s_1}}{\Gamma(r_1 + s_1 + 1)} + \frac{(F_{\iota_1}(\iota_2))^{s_1}}{\Gamma(s_1 + 1)} \simeq \begin{cases} 0.4091, & \mathbb{G}_1(\iota) = 2^\iota, \\ 0.4106, & \mathbb{G}_2(\iota) = \iota, \\ 1.7597, & \mathbb{G}_3(\iota) = \ln\iota, \\ 0.3473, & \mathbb{G}_4(\iota) = \sqrt{\iota}, \end{cases}
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
\Phi_2 &= \frac{(F_{\iota_1}(\iota_2))^{q_2+p_2+r_2+s_2}}{\Gamma(q_2 + p_2 + r_2 + s_2 + 1)} + \frac{(F_{\iota_1}(\iota_2))^{p_2+r_2+s_2}}{\Gamma(p_k + r_2 + s_2 + 1)} \\
&\quad + \frac{(F_{\iota_1}(\iota_2))^{r_2+s_2}}{\Gamma(r_2 + s_2 + 1)} + \frac{(F_{\iota_1}(\iota_2))^{s_2}}{\Gamma(s_2 + 1)} \simeq \begin{cases} 0.3370, & \mathbb{G}_1(\iota) = 2^\iota, \\ 0.3383, & \mathbb{G}_2(\iota) = \iota, \\ 1.4912, & \mathbb{G}_3(\iota) = \ln\iota, \\ 0.2826, & \mathbb{G}_4(\iota) = \sqrt{\iota}. \end{cases}
\end{aligned} \tag{4.3}$$

Hence,

$$\Phi \simeq \begin{cases} 2.9461, & \mathbb{G}_1(\iota) = 2^\iota, \\ 2.9567, & \mathbb{G}_2(\iota) = \iota, \\ 12.9144, & \mathbb{G}_3(\iota) = \ln \iota, \\ 2.5009, & \mathbb{G}_4(\iota) = \sqrt{\iota}, \end{cases}$$

and we have

$$\Phi\ell \simeq \begin{cases} 0.4091 < 1, & \mathbb{G}_1(\iota) = 2^\iota, \\ 0.4106 < 1, & \mathbb{G}_2(\iota) = \iota, \\ 1.7936 \nleq 1, & \mathbb{G}_3(\iota) = \ln \iota, \\ 0.3473 < 1, & \mathbb{G}_4(\iota) = \sqrt{\iota}. \end{cases}$$

On the other hand, by using equations in (3.7)*, we get*

$$M_{1j} = \sup_{\mathrm{t}\in\Sigma} \int_{\iota_1}^{\mathrm{t}} |g_{1j}(\xi)| \,\mathrm{d}\xi = 0.6328, \qquad M_{2j} = \sup_{\mathrm{t}\in\Sigma} \int_{\iota_1}^{\iota_2} |g_{1j}(\xi)| \,\mathrm{d}\xi = 0.7632,$$

for $j = 0, 1, 2, 3$ *and*

$$h_1^0 = \sup_{\mathrm{t}\in\Sigma} |h_1(\mathrm{t}, 0, 0, 0, 0, 0, 0, 0, 0)| \simeq \frac{5}{36} + \frac{1}{36(1 + \sqrt{7})},$$
$$h_2^0 = \sup_{\mathrm{t}\in\Sigma} |h_2(\mathrm{t}, 0, 0, 0, 0, 0, 0, 0, 0)| \simeq \frac{1}{2\sqrt{3}} + \frac{1}{18(1 + \sqrt{15})},$$

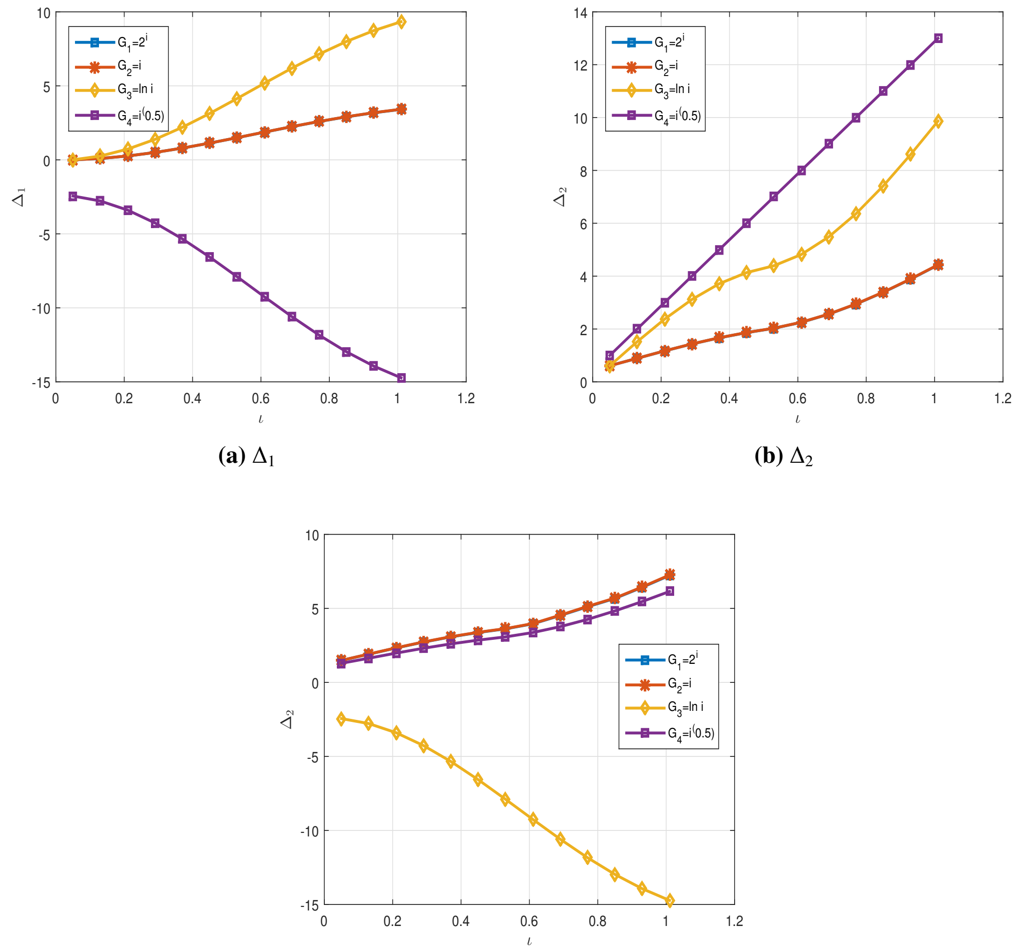
for $k = 1, 2$*. By employing Eq* (3.6)*, we obtain*

$$\begin{aligned} \Delta_1 &= M_{10} + M_{11}\left(1 + \frac{(F_{\iota_1}(\iota_2))^{q_1}}{\Gamma(q_1+1)}\right) + M_{12}\left(1 + \frac{(F_{\iota_1}(\iota_2))^{p_1}}{\Gamma(p_1+1)} + \frac{(F_{\iota_1}(\iota_2))^{q_1+p_1}}{\Gamma(q_1+p_1+1)}\right) \\ &\quad + M_{13}\left(1 + \frac{(F_{\iota_1}(\iota_2))^{r_1}}{\Gamma(r_1+1)} + \frac{(F_{\iota_1}(\iota_2))^{p_1+r_1}}{\Gamma(p_1+r_1+1)} + \frac{(F_{\iota_1}(\iota_2))^{q_1+p_1+r_1}}{\Gamma(q_1+p_1+r_1+1)}\right), \\ &\simeq \begin{cases} 3.4102, & \mathbb{G}_1(\iota) = 2^\iota, \\ 3.4173, & \mathbb{G}_2(\iota) = \iota, \\ 9.3326, & \mathbb{G}_3(\iota) = \ln \iota, \\ 3.1122, & \mathbb{G}_4(\iota) = \sqrt{\iota}, \end{cases} \\ \Delta_2 &= M_{20} + M_{21}\left(1 + \frac{(F_{\iota_1}(\iota_2))^{q_2}}{\Gamma(q_2+1)}\right) + M_{22}\left(1 + \frac{(F_{\iota_1}(\iota_2))^{p_2}}{\Gamma(p_2+1)} + \frac{(F_{\iota_1}(\iota_2))^{q_2+p_2}}{\Gamma(q_2+p_2+1)}\right) \\ &\quad + M_{23}\left(1 + \frac{(F_{\iota_1}(\iota_2))^{r_2}}{\Gamma(r_2+1)} + \frac{(F_{\iota_1}(\iota_2))^{p_2+r_2}}{\Gamma(p_2+r_2+1)} + \frac{(F_{\iota_1}(\iota_2))^{q_2+p_2+r_2}}{\Gamma(q_2+p_2+r_2+1)}\right) \\ &\simeq \begin{cases} 4.4208, & \mathbb{G}_1(\iota) = 2^\iota, \\ 4.4285, & \mathbb{G}_2(\iota) = \iota, \\ 9.8667, & \mathbb{G}_3(\iota) = \ln \iota, \\ 4.0927, & \mathbb{G}_4(\iota) = \sqrt{\iota}, \end{cases} \end{aligned}$$

and so we can choose

$$\varepsilon \geq \max\left\{\frac{\Delta_1 + h_1^0\Phi_1}{(1-\ell_1\Phi_1)}, \frac{\Delta_2 + h_2^0\Phi_2}{(1-\ell_2\Phi_2)}\right\} \simeq \begin{cases} 7.2317, & \mathbb{G}_1(\iota) = 2^\iota, \\ 7.2597, & \mathbb{G}_2(\iota) = \iota, \\ -14.7276, & \mathbb{G}_3(\iota) = \ln \iota, \\ 6.1479, & \mathbb{G}_4(\iota) = \sqrt{\iota}. \end{cases}$$

We define the Algorithm 1 for obtaining the values of $\Phi\ell$, Δ_i *and* ε, *which is shown in the MATLAB commands. One can check numerical results of* $\Phi\ell$, Δ_i *and* ε *in Tables 1 and 2 for* $\iota \in [0.05, 0.95]$, *and in Figure 1. Accordingly, all requirements of Theorem 3.2 hold, and so the fractional nonlinear couple snap system* ($\mathbb{CSS}$) *in the* $\mathbb{G}$*-Caputo sense* ($\mathbb{GC}$) *with initial conditions* (4.1) *has one unique solution on the* $[0.05, 0.95]$.



(a) Δ_1 **(b)** Δ_2 **(c)** $\varepsilon \geq$

Figure 1. Graphical representation of Δ_1, Δ_1 and ε for $\iota \in [0.05, 0.95]$ in Example 4.1.

Table 1. Numerical values of $\Phi\ell$, Δ_1, Δ_1 and ε in Example 4.1 $\forall\iota \in [0.05, 0.95]$ when $\mathbb{G}_1 = 2^\iota$ and $\mathbb{G}_2 = \iota$.

	$\mathbb{G}_1(\iota) = 2^\iota$				$\mathbb{G}_2(\iota) = \iota$			
ι	$\Phi\ell$	Δ_1	Δ_2	$\varepsilon \geq$	$\Phi\ell$	Δ_1	Δ_2	$\varepsilon \geq$
0.05	0.4092	0.0000	0.6098	1.4836	0.4107	0.0000	0.6098	1.4886
0.13	0.4092	0.0916	0.8907	1.9071	0.4107	0.0918	0.8915	1.9142
0.21	0.4092	0.2602	1.1697	2.3280	0.4107	0.2608	1.1713	2.3371
0.29	0.4092	0.4976	1.4321	2.7238	0.4107	0.4986	1.4344	2.7347
0.37	0.4092	0.7921	1.6660	3.0766	0.4107	0.7938	1.6687	3.0888
0.45	0.4092	1.1297	1.8627	3.3733	0.4107	1.1321	1.8658	3.3867
0.53	0.4092	1.4944	2.0267	3.6206	0.4107	1.4976	2.0300	3.6348
0.61	0.4092	1.8697	2.2517	3.9599	0.4107	1.8736	2.2552	3.9752
0.69	0.4092	2.2394	2.5593	4.5209	0.4107	2.2441	2.5634	4.5429
0.77	0.4092	2.5889	2.9421	5.1124	0.4107	2.5943	2.9469	5.1371
0.85	0.4092	2.9058	3.3892	5.6758	0.4107	2.9119	3.3949	5.6977
0.93	0.4092	3.1810	3.8873	6.4269	0.4107	3.1877	3.8940	6.4518

Table 2. Numerical values of $\Phi\ell$, Δ_1, Δ_1 and ε in Example 4.1 $\forall\iota \in [0.05, 0.95]$ when $\mathbb{G}_3(\iota) = \ln\iota$ and $\mathbb{G}_4(\iota) = \sqrt{\iota}$.

	$\mathbb{G}_3 = \ln\iota$				$\mathbb{G}_4 = \sqrt{\iota}$			
ι	$\Phi\ell$	Δ_1	Δ_2	$\varepsilon \geq$	$\Phi\ell$	Δ_1	Δ_2	$\varepsilon \geq$
0.05	1.7937	0.0000	0.6098	−2.4433	0.3474	0.0000	0.6098	1.2925
0.13	1.7937	0.2549	1.5187	−2.7789	0.3474	0.0835	0.8562	1.6360
0.21	1.7937	0.7237	2.3707	−3.3959	0.3474	0.2373	1.1034	1.9805
0.29	1.7937	1.3828	3.1158	−4.2635	0.3474	0.4537	1.3383	2.3081
0.37	1.7937	2.1996	3.7133	−5.3387	0.3474	0.7224	1.5507	2.6041
0.45	1.7937	3.1339	4.1339	−6.5685	0.3474	1.0303	1.7330	2.8582
0.53	1.7937	4.1410	4.3933	−7.8940	0.3474	1.3629	1.8890	3.0758
0.61	1.7937	5.1740	4.8174	−9.2538	0.3474	1.7053	2.1001	3.3700
0.69	1.7937	6.1876	5.4819	−10.5880	0.3474	2.0427	2.3850	3.7672
0.77	1.7937	7.1402	6.3609	−11.8418	0.3474	2.3617	2.7371	4.2581
0.85	1.7937	7.9969	7.4169	−12.9695	0.3474	2.6511	3.1472	4.8298
0.93	1.7937	8.7318	8.6033	−13.9368	0.3474	2.9026	3.6035	5.4660

5. Conclusions

In this paper, we defined a new fractional mathematical model of a BVP consisting of a coupled snap equation with integral boundary conditions in the framework of the generalized sequential $\mathbb{G}$-operators, and turned to the investigation of the qualitative behaviors of its solutions, including existence, uniqueness, stability and inclusion version. To confirm the existence criterion, we used the Krasnoselskii theorem, and to confirm the uniqueness criterion, we utilized the Banach theorem. Different kinds of stability criteria were studied based on the standard definitions of these notions. In the final step, we designed examples, and, by assuming different cases for the function $\mathbb{G}$ and order q, we obtained numerical results of these two suggested fractional coupled snap systems in some versions, such as Caputo, Caputo-Hadamard and Katugampola.

Conflict of interest

We declare that no competing interests.

References

1. M. I. Abbas, M. Ghaderi, S. Rezapour, S. T. M. Thabet, On a coupled system of fractional differential equations via the generalized proportional fractional derivatives, *J. Funct. Spaces*, **2022** (2022), 4779213. https://doi.org/10.1155/2022/4779213
2. Y. Adjabi, M. E. Samei, M. M. Matar, J. Alzabut, Langevin differential equation in frame of ordinary and hadamard fractional derivatives under three point boundary conditions, *AIMS Math.*, **6** (2021), 2796–2843. https://doi.org/10.3934/math.2021171
3. H. Afshari, S. Kalantari, E. Karapinar, Solution of fractional differential equations via coupled fixed point, *Electron. J. Differ. Equ.*, **2015** (2015), 1–12.
4. S. T. M. Thabet, B. Ahmad, R. P. Agarwal, On abstract Hilfer fractional integrodifferential equations with boundary conditions, *Arab J. Math. Sci.*, **26** (2020), 107–125. https://doi.org/10.1016/j.ajmsc.2019.03.001
5. J. Alzabut, A. G. M. Selvam, R. A. El-Nabulsi, V. Dhakshinamoorthy, M. E. Samei, Asymptotic stability of nonlinear discrete fractional pantograph equations with non-local initial conditions, *Symmetry*, **13** (2021), 473. https://doi.org/10.3390/sym13030473
6. P. Amiri, M. E. Samei, Existence of urysohn and atangana-baleanu fractional integral inclusion systems solutions via common fixed point of multi-valued operators, *Chaos Solitons Fract.*, **165** (2022), 112822. https://doi.org/10.1016/j.chaos.2022.112822
7. A. Boutiara, K. Guerbati, M. Benbachir, Caputo-Hadamard fractional differential equation with three-point boundary conditions in Banach spaces, *AIMS Math.*, **5** (2020), 259–272. https://doi.org/10.3934/math.2020017
8. J. V. d. C. Sousa, E. C. de Oliveira, On the Ulam-Hyers-Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer operator, *J. Fixed Point Theory Appl.*, **20** (2018), 96. https://doi.org/10.1007/s11784-018-0587-56
9. K. Deimling, *Nonlinear functional analysis*, Springer Berlin, Heidelberg, 1985. https://doi.org/10.1007/978-3-662-00547-7
10. A. R. Elsonbaty, A. M. El-Sayed, Further nonlinear dynamical analysis of simple jerk system with multiple attractors, *Nonlinear Dyn.*, **87** (2017), 1169–1186. https://doi.org/10.1007/s11071-016-3108-3

11. H. P. W. Gottlieb, Harmonic balance approach to periodic solutions of nonlinear jerk equations, *J. Sound Vib.*, **271** (2004), 671–683. https://doi.org/10.1016/S0022-460X(03)00299-2

12. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Vol. 204, North-Holland Mathematics Studies, Elsevier, 2006.

13. C. S. Liu, J. R. Chang, The periods and periodic solutions of nonlinear jerk equations solved by an iterative algorithm based on a shape function method, *Appl. Math. Lett.*, **102** (2019), 106151. https://doi.org/10.1016/j.aml.2019.106151

14. N. Mahmudov, M. M. Matar, Existence of mild solution for hybrid differential equations with arbitrary fractional order, *TWMS J. Pure Appl. Math.*, **8** (2017), 160–169.

15. M. M. Matar, Qualitative properties of solution for hybrid nonlinear fractional differential equations, *Afr. Mat.*, **30** (2019), 1169–1179. https://doi.org/10.1007/s13370-019-00710-2

16. M. M. Matar, M. Jarad, M. Ahmad, A. Zada, S. Etemad, S. Rezapour, On the existence and stability of two positive solutions of a hybrid differential system of arbitrary fractional order via Avery-Anderson-Henderson criterion on cones, *Adv. Differ. Equ.*, **2021** (2021), 423. https://doi.org/10.1186/s13662-021-03576-6

17. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: Wiley, 1993.

18. I. Podlubny, *Fractional differential equations*, New York: Academic Press, 1993.

19. M. S. Rahman, A. S. M. Z. Hasan, Modified harmonic balance method for the solution of nonlinear jerk equations, *Results Phys.*, **8** (2018), 893–897. https://doi.org/10.1016/j.rinp.2018.01.030

20. S. Rezapour, M. E. Samei, On the existence of solutions for a multi-singular point-wise defined fractional q-integro-differential equation, *Bound. Value Probl.*, **2020** (2020), 38. https://doi.org/10.1186/s13661-020-01342-3

21. S. Rezapour, S. T. M. Thabet, M. M. Matar, J. Alzabut, S. Etemad, Some existence and stability criteria to a generalized FBVP having fractional composite p-Laplacian operator, *J. Funct. Spaces*, **2021** (2021), 9554076. https://doi.org/10.1155/2021/9554076

22. M. E. Samei, V. Hedayati, S. Rezapour, Existence results for a fraction hybrid differential inclusion with Caputo-Hadamard type fractional derivative, *Adv. Differ. Equ.*, **2019** (2019), 163. https://doi.org/10.1186/s13662-019-2090-8

23. M. E. Samei, M. M. Matar, S. Etemad, S. Rezapour, On the generalized fractional snap boundary problems via G-Caputo operators: existence and stability analysis, *Adv. Differ. Equ.*, **2021** (2021), 498. https://doi.org/10.1186/s13662-021-03654-9

24. S. T. M. Thabet, M. B. Dhakne, On abstract fractional integro-differential equations via measure of noncompactness, *Adv. Fixed Point Theory*, **6** (2016), 175–193.

25. S. T. M. Thabet, M. B. Dhakne, On nonlinear fractional integro-differential equations with two boundary conditions, *Adv. Stud. Contemp. Math.*, **26** (2016), 513–526.

26. S. T. M. Thabet, M. B. Dhakne, On positive solutions of higher order nonlinear fractional integro-differential equations with boundary conditions, *Malaya J. Mat.*, **7** (2019), 20–26. https://doi.org/10.26637/MJM0701/0005

27. S. T. M. Thabet, M. B. Dhakne, M. A. Salman, R. Gubran, Generalized fractional Sturm-Liouville and langevin equations involving Caputo derivative with nonlocal conditions, *Progr. Fract. Differ. Appl.*, **6** (2020), 225–237. https://doi.org/10.18576/pfda/060306

28. S. T. M. Thabet, S. Etemad, S. Rezapour, On a coupled Caputo conformable system of pantograph problems, *Turk. J. Math.*, **45** (2021), 496–519. https://doi.org/10.3906/mat-2010-70

29. C. Urs, Coupled fixed point theorems and applications to periodic boundary value problems, *Miskolc Math. Notes*, **14** (2013), 323–333. https://doi.org/10.18514/MMN.2013.598

30. H. Zhou, J. Alzabut, S. Rezapour, M. E. Samei, Uniform persistence and almost periodic solutions of a non-autonomous patch occupancy model, *Adv. Differ. Equ.*, **2020** (2020), 143. https://doi.org/10.1186/s13662-020-02603-2

Appendix

Algorithm A1: MATLAB lines for Example 4.1.

```
clear;
format short;
syms v e;
q_1=0.83; q_2=0.36; p_1=0.92; p_2=0.45;
r_1=0.12; r_2=0.87; s_1=0.54; s_2=0.27;
iota_1=0.05; iota_2=0.95;
G1=2^v; G2=v; G3=log(v); G4=sqrt(v);
g_10=v/2; g_20=sqrt(v);
g_11=v^2/5; g_21=3*v/2;
g_12=v/sqrt(2); g_22=sqrt(v)/7;
g_13=sin(v*pi); g_23=cos(v*pi);
mathrmv_1=int(g_10, v, iota_1, iota_2);
mathrmv_2=int(g_20, v, iota_1, iota_2);
mathrmu_1=int(g_11, v, iota_1, iota_2);
mathrmu_2=int(g_21, v, iota_1, iota_2);
mathrmw_1=int(g_12, v, iota_1, iota_2);
mathrmw_2=int(g_22, v, iota_1, iota_2);
mathrmx_1=int(g_13, v, iota_1, iota_2);
mathrmx_2=int(g_23, v, iota_1, iota_2);
ell_1=5/36; ell_2=1/(5*sqrt(3));
ell=max(ell_1,ell_2);
h_1_0=5/36+1/(36*(1+sqrt(7)));
h_2_0=1/(5*sqrt(3))+1/(18*(1+sqrt(15)));
%G1
t=iota_1;
column=1;
nn=1;
while t≤iota_2+0.08
    MI(nn,column) = nn;
    MI(nn,column+1) = t;
    Phi_1=(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(q_1+p_1+r_1+s_1)...
        /gamma(q_1+p_1+r_1+s_1+1)+(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(p_1+r_1+s_1)...
        /gamma(p_1+r_1+s_1+1)+(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(r_1+s_1)...
```

```
        /gamma(r_1+s_1+1)+(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(s_1)/gamma(s_1+1);
    MI(nn,column+2)=Phi_1*ell_1;
    MI(nn,column+3)=Phi_1*ell_1<1;
    Phi_2=(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(q_2+p_2+r_2+s_2)...
        /gamma(q_2+p_2+r_2+s_2+1)+(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(p_2+r_2+s_2)...
        /gamma(p_2+r_2+s_2+1)+(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(r_2+s_2)...
        /gamma(r_2+s_2+1)+(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(s_2)/gamma(s_2+1);
    MI(nn,column+4)=Phi_2*ell_2;
    MI(nn,column+5)=Phi_2*ell_2<1;
    Phi=max(Phi_1, Phi_2);
    MI(nn,column+6)=Phi;
    MI(nn,column+7)=Phi*ell;
    MI(nn,column+8)=Phi*ell<1;
    M_10=int(abs(g_10), v, iota_1, t);
    MI(nn,column+9)=M_10;
    M_11=int(abs(g_11), v, iota_1, t);
    MI(nn,column+10)=M_11;
    M_12=int(abs(g_12), v, iota_1, t);
    MI(nn,column+11)=M_12;
    M_13=int(abs(g_13), v, iota_1, t);
    MI(nn,column+12)=M_13;
    M_20=int(abs(g_20), v, iota_1, iota_2);
    MI(nn,column+13)=M_20;
    M_21=int(abs(g_21), v, iota_1, t);
    MI(nn,column+14)=M_21;
    M_22=int(abs(g_22), v, iota_1, t);
    MI(nn,column+15)=M_22;
    M_23=int(abs(g_23), v, iota_1, t);
    MI(nn,column+16)=M_23;
    M_1j=max(max(max(M_10, M_11),M_12),M_13);
    MI(nn,column+17)=M_1j;
    M_2j=max(max(max(M_20, M_21),M_22),M_23);
    MI(nn,column+18)=M_2j;
    Delta_1=M_10+M_11*(1+(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(q_1)/gamma(q_1+1))...
        +M_12*(1+(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(p_1)/gamma(p_1+1)...
        +(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(q_1+p_1)/gamma(q_1+p_1+1))...
        +M_13*(1+(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(r_1)/gamma(r_1+1)...
        +(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(p_1+r_1)/gamma(p_1+r_1+1)...
        +(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(q_1+p_1+r_1)...
        /gamma(q_1+p_1+r_1+1));
    MI(nn,column+19)=Delta_1;
    Delta_2=M_20+M_21*(1+(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(q_2)/gamma(q_2+1))...
        +M_22*(1+(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(p_2)/gamma(p_2+1)...
        +(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(q_2+p_2)/gamma(q_2+p_2+1))...
        +M_23*(1+(eval(subs(G1, {v}, {iota_2}))...
```

```
        -eval(subs(G1, {v}, {iota_1})))^(r_2)/gamma(r_2+1)...
        +(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(p_2+r_2)/gamma(p_2+r_2+1)...
        +(eval(subs(G1, {v}, {iota_2}))...
        -eval(subs(G1, {v}, {iota_1})))^(q_2+p_2+r_2)...
        /gamma(q_2+p_2+r_2+1));
    MI(nn,column+20)=Delta_2;
    D1=(Delta_1+h_1_0*Phi_1)/(1-ell_1*Phi_1);
    MI(nn,column+21)=D1;
    D2=(Delta_2+h_2_0*Phi_1)/(1-ell_2*Phi_2);
    MI(nn,column+22)=D2;
    MI(nn,column+23)=max(D1, D2);
    t=t+0.08;
    nn=nn+1;
end;
%G2
t=iota_1;
column=25;
nn=1;
while t≤iota_2+0.08
    MI(nn,column) = nn;
    MI(nn,column+1) = t;
    Phi_1=(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(q_1+p_1+r_1+s_1)...
        /gamma(q_1+p_1+r_1+s_1+1)+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(p_1+r_1+s_1)...
        /gamma(p_1+r_1+s_1+1)+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(r_1+s_1)...
        /gamma(r_1+s_1+1)+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(s_1)/gamma(s_1+1);
    MI(nn,column+2)=Phi_1*ell_1;
    MI(nn,column+3)=Phi_1*ell_1<1;
    Phi_2=(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(q_2+p_2+r_2+s_2)...
        /gamma(q_2+p_2+r_2+s_2+1)+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(p_2+r_2+s_2)...
        /gamma(p_2+r_2+s_2+1)+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(r_2+s_2)...
        /gamma(r_2+s_2+1)+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(s_2)/gamma(s_2+1);
    MI(nn,column+4)=Phi_2*ell_2;
    MI(nn,column+5)=Phi_2*ell_2<1;
    Phi=max(Phi_1, Phi_2);
    MI(nn,column+6)=Phi;
    MI(nn,column+7)=Phi*ell;
    MI(nn,column+8)=Phi*ell<1;
    M_10=int(abs(g_10), v, iota_1, t);
    MI(nn,column+9)=M_10;
    M_11=int(abs(g_11), v, iota_1, t);
    MI(nn,column+10)=M_11;
    M_12=int(abs(g_12), v, iota_1, t);
    MI(nn,column+11)=M_12;
    M_13=int(abs(g_13), v, iota_1, t);
    MI(nn,column+12)=M_13;
    M_20=int(abs(g_20), v, iota_1, iota_2);
    MI(nn,column+13)=M_20;
    M_21=int(abs(g_21), v, iota_1, t);
    MI(nn,column+14)=M_21;
    M_22=int(abs(g_22), v, iota_1, t);
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```
    MI(nn,column+15)=M_22;
    M_23=int(abs(g_23), v, iota_1, t);
    MI(nn,column+16)=M_23;
    M_1j=max(max(max(M_10, M_11),M_12),M_13);
    MI(nn,column+17)=M_1j;
    M_2j=max(max(max(M_20, M_21),M_22),M_23);
    MI(nn,column+18)=M_2j;
    Delta_1=M_10+M_11*(1+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(q_1)/gamma(q_1+1))...
        +M_12*(1+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2,{v}, {iota_1})))^(p_1)/gamma(p_1+1)...
        +(eval(subs(G2,{v}, {iota_2}))-eval(subs(G2,{v}, {iota_1})))^(q_1+p_1)...
        /gamma(q_1+p_1+1))+M_13*(1+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2,{v}, {iota_1})))^(r_1)/gamma(r_1+1)...
        +(eval(subs(G2,{v}, {iota_2}))-eval(subs(G2, {v}, {iota_1})))^(p_1+r_1)...
        /gamma(p_1+r_1+1)+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2,{v}, {iota_1})))^(q_1+p_1+r_1)/gamma(q_1+p_1+r_1+1));
    MI(nn,column+19)=Delta_1;
    Delta_2=M_20+M_21*(1+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(q_2)/gamma(q_2+1))...
        +M_22*(1+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(p_2)/gamma(p_2+1)...
        +(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(q_2+p_2)/gamma(q_2+p_2+1))...
        +M_23*(1+(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(r_2)/gamma(r_2+1)...
        +(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(p_2+r_2)/gamma(p_2+r_2+1)...
        +(eval(subs(G2, {v}, {iota_2}))...
        -eval(subs(G2, {v}, {iota_1})))^(q_2+p_2+r_2)...
        /gamma(q_2+p_2+r_2+1));
    MI(nn,column+20)=Delta_2;
    D1=(Delta_1+h_1_0*Phi_1)/(1-ell_1*Phi_1);
    MI(nn,column+21)=D1;
    D2=(Delta_2+h_2_0*Phi_1)/(1-ell_2*Phi_2);
    MI(nn,column+22)=D2;
    MI(nn,column+23)=max(D1, D2);
    t=t+0.08;
    nn=nn+1;
end;
%G3
t=iota_1;
column=49;
nn=1;
while t≤iota_2+0.08
    MI(nn,column) = nn;
    MI(nn,column+1) = t;
    Phi_1=(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(q_1+p_1+r_1+s_1)...
        /gamma(q_1+p_1+r_1+s_1+1)+(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(p_1+r_1+s_1)...
        /gamma(p_1+r_1+s_1+1)+(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(r_1+s_1)...
        /gamma(r_1+s_1+1)+(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(s_1)/gamma(s_1+1);
    MI(nn,column+2)=Phi_1*ell_1;
    MI(nn,column+3)=Phi_1*ell_1<1;
    Phi_2=(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(q_2+p_2+r_2+s_2)...
```

```
        /gamma(q_2+p_2+r_2+s_2+1)+(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(p_2+r_2+s_2)...
        /gamma(p_2+r_2+s_2+1)+(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(r_2+s_2)...
        /gamma(r_2+s_2+1)+(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(s_2)/gamma(s_2+1);
    MI(nn,column+4)=Phi_2*ell_2;
    MI(nn,column+5)=Phi_2*ell_2<1;
    Phi=max(Phi_1, Phi_2);
    MI(nn,column+6)=Phi;
    MI(nn,column+7)=Phi*ell;
    MI(nn,column+8)=Phi*ell<1;
    M_10=int(abs(g_10), v, iota_1, t);
    MI(nn,column+9)=M_10;
    M_11=int(abs(g_11), v, iota_1, t);
    MI(nn,column+10)=M_11;
    M_12=int(abs(g_12), v, iota_1, t);
    MI(nn,column+11)=M_12;
    M_13=int(abs(g_13), v, iota_1, t);
    MI(nn,column+12)=M_13;
    M_20=int(abs(g_20), v, iota_1, iota_2);
    MI(nn,column+13)=M_20;
    M_21=int(abs(g_21), v, iota_1, t);
    MI(nn,column+14)=M_21;
    M_22=int(abs(g_22), v, iota_1, t);
    MI(nn,column+15)=M_22;
    M_23=int(abs(g_23), v, iota_1, t);
    MI(nn,column+16)=M_23;
    M_1j=max(max(max(M_10, M_11),M_12),M_13);
    MI(nn,column+17)=M_1j;
    M_2j=max(max(max(M_20, M_21),M_22),M_23);
    MI(nn,column+18)=M_2j;
    Delta_1=M_10+M_11*(1+(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(q_1)/gamma(q_1+1))...
        +M_12*(1+(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(p_1)/gamma(p_1+1)...
        +(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(q_1+p_1)/gamma(q_1+p_1+1))...
        +M_13*(1+(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(r_1)/gamma(r_1+1)...
        +(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(p_1+r_1)/gamma(p_1+r_1+1)...
        +(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(q_1+p_1+r_1)...
        /gamma(q_1+p_1+r_1+1));
    MI(nn,column+19)=Delta_1;
    Delta_2=M_20+M_21*(1+(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(q_2)/gamma(q_2+1))...
        +M_22*(1+(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(p_2)/gamma(p_2+1)...
        +(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(q_2+p_2)/gamma(q_2+p_2+1))...
        +M_23*(1+(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(r_2)/gamma(r_2+1)...
        +(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(p_2+r_2)/gamma(p_2+r_2+1)...
        +(eval(subs(G3, {v}, {iota_2}))...
        -eval(subs(G3, {v}, {iota_1})))^(q_2+p_2+r_2)...
        /gamma(q_2+p_2+r_2+1));
```

```
    MI(nn,column+20)=Delta_2;
    D1=(Delta_1+h_1_0*Phi_1)/(1-ell_1*Phi_1);
    MI(nn,column+21)=D1;
    D2=(Delta_2+h_2_0*Phi_1)/(1-ell_2*Phi_2);
    MI(nn,column+22)=D2;
    MI(nn,column+23)=max(D1, D2);
    t=t+0.08;
    nn=nn+1;
end;
%G4
t=iota_1;
column=73;
nn=1;
while t≤iota_2+0.08
    MI(nn,column) = nn;
    MI(nn,column+1) = t;
    Phi_1=(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(q_1+p_1+r_1+s_1)...
        /gamma(q_1+p_1+r_1+s_1+1)+(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(p_1+r_1+s_1)...
        /gamma(p_1+r_1+s_1+1)+(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(r_1+s_1)...
        /gamma(r_1+s_1+1)+(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(s_1)/gamma(s_1+1);
    MI(nn,column+2)=Phi_1*ell_1;
    MI(nn,column+3)=Phi_1*ell_1<1;
    Phi_2=(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(q_2+p_2+r_2+s_2)...
        /gamma(q_2+p_2+r_2+s_2+1)+(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(p_2+r_2+s_2)...
        /gamma(p_2+r_2+s_2+1)+(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(r_2+s_2)...
        /gamma(r_2+s_2+1)+(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(s_2)/gamma(s_2+1);
    MI(nn,column+4)=Phi_2*ell_2;
    MI(nn,column+5)=Phi_2*ell_2<1;
    Phi=max(Phi_1, Phi_2);
    MI(nn,column+6)=Phi;
    MI(nn,column+7)=Phi*ell;
    MI(nn,column+8)=Phi*ell<1;
    M_10=int(abs(g_10), v, iota_1, t);
    MI(nn,column+9)=M_10;
    M_11=int(abs(g_11), v, iota_1, t);
    MI(nn,column+10)=M_11;
    M_12=int(abs(g_12), v, iota_1, t);
    MI(nn,column+11)=M_12;
    M_13=int(abs(g_13), v, iota_1, t);
    MI(nn,column+12)=M_13;
    M_20=int(abs(g_20), v, iota_1, iota_2);
    MI(nn,column+13)=M_20;
    M_21=int(abs(g_21), v, iota_1, t);
    MI(nn,column+14)=M_21;
    M_22=int(abs(g_22), v, iota_1, t);
    MI(nn,column+15)=M_22;
    M_23=int(abs(g_23), v, iota_1, t);
    MI(nn,column+16)=M_23;
    M_1j=max(max(max(M_10, M_11),M_12),M_13);
    MI(nn,column+17)=M_1j;
    M_2j=max(max(max(M_20, M_21),M_22),M_23);
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```
    MI(nn,column+18)=M_2j;
    Delta_1=M_10+M_11*(1+(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(q_1)/gamma(q_1+1))...
        +M_12*(1+(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(p_1)/gamma(p_1+1)...
        +(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(q_1+p_1)/gamma(q_1+p_1+1))...
        +M_13*(1+(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(r_1)/gamma(r_1+1)...
        +(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(p_1+r_1)/gamma(p_1+r_1+1)...
        +(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(q_1+p_1+r_1)...
        /gamma(q_1+p_1+r_1+1));
    MI(nn,column+19)=Delta_1;
    Delta_2=M_20+M_21*(1+(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(q_2)/gamma(q_2+1))...
        +M_22*(1+(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(p_2)/gamma(p_2+1)...
        +(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(q_2+p_2)/gamma(q_2+p_2+1))...
        +M_23*(1+(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(r_2)/gamma(r_2+1)...
        +(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(p_2+r_2)/gamma(p_2+r_2+1)...
        +(eval(subs(G4, {v}, {iota_2}))...
        -eval(subs(G4, {v}, {iota_1})))^(q_2+p_2+r_2)...
        /gamma(q_2+p_2+r_2+1));
    MI(nn,column+20)=Delta_2;
    D1=(Delta_1+h_1_0*Phi_1)/(1-ell_1*Phi_1);
    MI(nn,column+21)=D1;
    D2=(Delta_2+h_2_0*Phi_1)/(1-ell_2*Phi_2);
    MI(nn,column+22)=D2;
    MI(nn,column+23)=max(D1, D2);
    t=t+0.08;
    nn=nn+1;
end;
```

AIMS Press