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Research article

T-spherical linear Diophantine fuzzy aggregation operators for multiple attribute decision-making

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Abstract: This paper aims to amalgamate the notion of a T-spherical fuzzy set (T-SFS) and a linear Diophantine fuzzy set (LDFS) to elaborate on the notion of the T-spherical linear Diophantine fuzzy set (T-SLDFS). The new concept is very effective and is more dominant as compared to T-SFS and LDFS. Then, we advance the basic operations of T-SLDFS and examine their properties. To effectively aggregate the T-spherical linear Diophantine fuzzy data, a T-spherical linear Diophantine fuzzy weighted averaging (T-SLDFWA) operator and a T-spherical linear Diophantine fuzzy weighted geometric (T-SLDFWG) operator are proposed. Then, the properties of these operators are also provided. Furthermore, the notions of the T-spherical linear Diophantine fuzzy-ordered weighted averaging (T-SLDFOWA) operator; T-spherical linear Diophantine fuzzy hybrid weighted averaging (T-SLDFHWA) operator; T-spherical linear Diophantine fuzzy-ordered weighted geometric (T-SLDFOWG) operator; and T-spherical linear Diophantine fuzzy hybrid weighted geometric (T-SLDFHWG) operator are proposed. To compare T-spherical linear Diophantine fuzzy numbers (T-SLDFNs), different types of score and accuracy functions are defined. On the basis of the T-SLDFWA and T-SLDFWG operators, a multiple attribute decision-making (MADM) method within the framework of T-SLDFNs is designed, and the ranking results are examined by different types of score functions. A numerical example is provided to depict the practicality and ascendancy of the proposed method. Finally, to demonstrate the excellence and accessibility of the proposed method, a comparison analysis with other methods is conducted.

Keywords: aggregation operators; decision-making; fuzzy sets; linear Diophantine fuzzy sets; score function; T-spherical fuzzy sets
Mathematics Subject Classification: 94D05, 03B52

Abbreviations

The following abbreviations are used in this manuscript:

AF	Accuracy Function
ESF	Expected Score Function
FS	Fuzzy Set
F-grade	Falsity-Membership Grade
I-grade	Indeterminacy Grade
IFS	Intuitionistic Fuzzy Set
LDFWG	Linear Diophantine Fuzzy Weighted Geometric
LDFS	Linear Diophantine Fuzzy Set
MADM	Multiple Attribute Decision-Making
PFS	Picture Fuzzy Set
PyFS	Pythagorean Fuzzy Set
q-LDFWA	q-Linear Diophantine Fuzzy Weighted Averaging
q-LDFWG	q-Linear Diophantine Fuzzy Weighted Geometric
q-LDFS	q-Rung Linear Diophantine Fuzzy Set
q-ROFS	q-Rung Orthopair Fuzzy Set
QAF	Quadratic Accuracy Function
QSF	Quadratic Score Function
RPs	Reference Parameters
SF	Score Function
SLDFWA	Spherical Linear Diophantine Fuzzy Weighted Averaging
SLDFWG	Spherical Linear Diophantine Fuzzy Weighted Geometric
SFS	Spherical Fuzzy Set
SLDFS	Spherical Linear Diophantine Fuzzy Set
T-grade	Truth-Membership Grade
T-SFS	T-spherical fuzzy set
T-SLDFHWA	T-spherical Linear Diophantine Fuzzy Hybrid Weighted Averaging
T-SLDFHWG	T-spherical Linear Diophantine Fuzzy Hybrid Weighted Geometric
T-SLDFNs	T-spherical Linear Diophantine Fuzzy Numbers
T-SLDFOWA	T-spherical Linear Diophantine Fuzzy-Ordered Weighted Averaging
T-SLDFOWG	T-spherical Linear Diophantine Fuzzy-Ordered Weighted Geometric
T-SLDFS	T-spherical Linear Diophantine Fuzzy Set
T-SLDFWA	T-spherical Linear Diophantine Fuzzy Weighted Averaging
T-SLDFWG	T-spherical Linear Diophantine Fuzzy Weighted Geometric

1. Introduction

The fuzzy set (FS), coined by Zadeh [1], is a pioneering notion that has a great strides on the classical decision-making (DM) theory. It has been successfully utilized to address ambiguous and misleading information in complicated decision-making processes. However, the modeling parameters of FS have been limited to tackling fuzzy and obscure data where two or more sources of uncertainty arise simultaneously. Consequently, many variations and generalizations of FS have been proposed. One of the uttermost significant and acclaimed generalizations is the intuitionistic FS (IFS) [2]. IFS is constructed by a truth-membership grade (T-grade) and falsity-membership grade (F-grade) fulfilling the constraint that their sum is limited to one, which confines the options to the satisfaction and dissatisfaction classes. To eradicate such problem, the principle of the Pythagorean FS (PyFS) has been settled by Yager [3], where the sum of the squares of its T-grade and F-grade is limited to one. Yager [4] again put forward that for some cases, the constraint of the PyFS structure may be violated and thus elaborated the q-rung orthopair FS (q-ROFS) characterized by a T-grade and an F-grade, providing the following condition: The sum of the qth power of the T-grade and F-grade is not exceed one.

The above-mentioned uncertainty sets have seen a great deal of investigations and applications in real-life [5–10]. However, these models have various strict restrictions on their membership grades. To overcome this difficulty, Riaz and Hashmi [11] initiated the new framework of the linear Diophantine fuzzy (LDF) set (LDFS). The LDFS extends IFS and PyFS by assigning reference parameters (RPs) associated to the T-grades and F-grades. This paradigm is the most appropriate structure for DM where the users can freely choose the grades. In this direction, Almagrabi et al. [12] designed the flexible model of q-rung LDFS (q-LDFS), that involves the addition of RPs to the structure of the q-ROFS to make it more effectual and adaptable than other methods. Some aggregation operators have been developed based on LDFSs and q-LDFS [11–20]. In [11], the authors presented the LDF weighted geometric (LDFWG) operator and proposed a new method for MADM based on LDF topological space and LDFWG operator. Almagrabi et al. [12] defined a series of averaging and geometric aggregation operators under q-LDFS. Riaz et al. presented LDF prioritized weighted (LDFPW) average (LDFPWA) and LDFPW geometric (LDFPWG) operators in [13]. Iampan et al. [14] developed the LDF Einstein weighted averaging (LDFEWA) and LDF Einstein weighted geometric (LDFEWG) operators. Mahmood et al. [15] introduced the interval LDF power Muirhead mean (IV-LDFMM) and interval LDF weighted power Muirhead mean (IV-LDFWPMM) operators. Several types of linear Diophantine, uncertain linguistic power Einstein (LDULPE) operator were elaborated in [16], including the LDULPE averaging (LDULPEA) operator, LDULPE weighted averaging (LDULPEWA) operator, LDULPE geometric (LDULPEG) operator and LDULPE weighted geometric (LDULPEWG) operator. Izatmand et al. [17] proposed LD uncertain linguistic generalized Hamacher (LDULGH) averaging and LDULGH hybrid averaging operators. Qiyas et al. [18] proposed some new q-LDFS distances and similarity measures, including Jaccard, exponential, and cosine similarity measures. In the complex space, Ali et al. [19] explained the complex LDULPEA, complex LDULPEWA, complex LDULPEG, and complex LDULPEWG operators. Kamaci [20] introduced the cosine distances and similarity measures among the complex LDFSs. The proposed cosine measures for complex LDFSs are used in medical diagnoses. Differently, Kamaci [21] investigated the theoretical LDFS approaches to algebraic structures, and Ayub et al. [22] formulated the LDF relation and its application in DM. Riaz et al. [23] also developed the LDF rough sets, followed by LDF graph by Prakash et al. [24] and the Dijkstra algorithm to solve an LDF environment by Parimala et al. [25].

By incorporating the indeterminacy grade (I-grade) into the above-mentioned sets consisting of T-grades and F-grades, various versions and generalizations were proposed. Apart from these generalizations, the principle of the picture FS (PFS) is introduced, which has been presented by Cuong [26]. A PFS is expressed by a T-grade ($\mathfrak{L}$), an I-grade ($\mathfrak{M}$) and a F-grade ($\mathfrak{N}$) such that $0 \leq \mathfrak{L} + \mathfrak{M} + \mathfrak{N} \leq 1$. Many researchers have created PFS methods and applications in various disciplines based on this idea [27–29]. However, it is noticed that this condition of the PFS does not enable us to determine grades for $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{N}$ freely. In other words, the domain of a PFS is restricted. Consequently, the novel structure of the spherical FS (SFS) [30,31] is suggested, which gives flexibility to the structure of the PFS by expanding the space of its grades in the interval $[0,1]$ with a constraint $0 \leq \mathfrak{L}^2 + \mathfrak{M}^2 + \mathfrak{N}^2 \leq 1$. Further, Mahmood et al. [31] empowered the concept of the SFS, and he developed the framework of T-SFS, including a new constraint $0 \leq \mathfrak{L}^q + \mathfrak{M}^q + \mathfrak{N}^q \leq 1$, provided that $q \in \mathbb{Z}$ and $q \geq 1$. The SFS and T-SFS have diverse applications in decision making problems [32–44], but they have some strict restrictions for their T-grades, I-grades and F-grades. The conditions of SFSs and T-SFSs reveal that their grades are not independent. To eliminate this limitation, Riaz et al. [45] launched the notion of spherical LDFS (SLDFS) under the prominent constraints $0 \leq \mathfrak{a}\mathfrak{L} + \mathfrak{b}\mathfrak{M} + \mathfrak{c}\mathfrak{N} \leq 1$ and $0 \leq \mathfrak{a} + \mathfrak{b} + \mathfrak{c} \leq 1$, such that $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are RPs associated with the T-grade, I-grade and F-grade, respectively, and picked from $[0,1]$. The beauty of this new thought is that all grades can be taken independently from $[0,1]$. This paper aims to continue investigating SLDFS, T-SFS and their combinations by introducing the novel theory of T-SLDFS and its aggregation operators.

1.1. Motivation and interpretation for T-SLDFS

Zadeh [1] assigned T-grade say $\mathfrak{L}$ to the crisp set by coining the theory of FSs. Atanassov [2] expanded the notion of FSs to IFSs by adding F-grade $\mathfrak{N}$ to a T-grade $\mathfrak{L}$ with a condition $0 \leq \mathfrak{L} + \mathfrak{N} \leq 1$. Under some circumstances, the sum of T-grade and F-grade is sometimes greater than 1, for example $0.5 + 0.6 \geq 1$. To handle this situation, Yager [3] come up with a theory of PyFS under condition $0 \leq \mathfrak{L}^2 + \mathfrak{N}^2 \leq 1$. Thus, according to this condition $0.5^2 + 0.6^2 = 0.25 + 0.36 = 0.61 \leq 1$. However, in the situation $0.7^2 + 0.8^2 = 0.49 + 0.64 = 1.13 \geq 1$, IFS and PyFS are unusable. To remove this limitation, q-ROFS [4] on the basis of PyFS was developed. The main advantage of q-ROFS is that it replaces the constraints of PyFS with requirement $0 \leq \mathfrak{L}^q + \mathfrak{N}^q \leq 1$. For this, if we choose $q = 3$, then by using the condition of q-ROFS, we obtain $0.7^3 + 0.8^3 = 0.343 + 0.512 = 0.855 \leq 1$. Nevertheless, in the actual process of solving the DM problem, it may happen that the opinion of the decision-maker does not satisfy the q-ROFS constraint. By way of illustration, if $\mathfrak{L}$ given by the decision maker is 1 and $\mathfrak{N}$ is 0.3. It is clearly that pair $(1, 0.3)$ cannot be expressed by q-ROFS since $1^q + 0.3^q \geq 1$ for any value of q. In this situation, the theories of IFS, PyFS and q-ROFS fail to work when solving DM problems under these circumstances. To model such situations, Riaz and Hashmi [11] introduced LDFS under conditions $0 \leq \mathfrak{a}\mathfrak{L} + \mathfrak{b}\mathfrak{N} \leq 1$, and $0 \leq \mathfrak{a} + \mathfrak{b} \leq 1$. For this, if we choose RPs $\mathfrak{a} = 0.1$ and $\mathfrak{b} = 0.2$, then the LDFS technique delivers $(0.1)(1) + (0.2)(0.3) = 0.16 \leq 1$ and $0.1 + 0.2 = 0.3 \leq 1$. But here again, the sum of the RPs in LDFS is sometimes larger than 1, i.e., $\mathfrak{a} + \mathfrak{b} \geq 1$. To remove such restriction, Almagrabi et al. [12] came with q-LDFS, in which $0 \leq (\mathfrak{a})^q\mathfrak{L} + (\mathfrak{b})^q\mathfrak{N} \leq 1$, and $0 \leq (\mathfrak{a})^q + (\mathfrak{b})^q \leq 1$. However, The neutrality is not given consideration in the above mentioned uncertainty models. To keep up with this situation, the notion of PFS [26] was introduced in the form of $(\mathfrak{L}, \mathfrak{M}, \mathfrak{N})$ under the condition $0 \leq \mathfrak{L} + \mathfrak{M} + \mathfrak{N} \leq 1$. In a PFS, when $\mathfrak{L}, \mathfrak{M}$ and $\mathfrak{N}$ are assigned as $0.5, 0.4$ and 0.3, respectively, since $0.5 + 0.4 + 0.3 = 1.2 \geq 1$, condition $0 \leq \mathfrak{L} + \mathfrak{M} + \mathfrak{N} \leq 1$ does not hold. To manage such a situation, a SFS was proposed by [30,31] as a generalization of the PFS under condition $0 \leq \mathfrak{L}^2 + \mathfrak{M}^2 + \mathfrak{N}^2 \leq 1$. Thus, according to this condition $0.5^2 + 0.4^2 + 0.3^2 = 0.5 \leq 1$. Thus, SFS extends PFS but only to some range, for example when $\mathfrak{L}, \mathfrak{M}$ and $\mathfrak{N}$ are taken as $0.7, 0.6$ and 0.8, then even squaring is not enough as $0.7^2 + 0.5^2 + 0.6^2 = 1.1 \geq 1$. Therefore, T-SFS [31] is constructed under the restriction $0 \leq \mathfrak{L}^q + \mathfrak{M}^q + \mathfrak{N}^q \leq 1$. For example, in our case when $\mathfrak{L} = 0.7, \mathfrak{M} = 0.5$ and $\mathfrak{N} = 0.6$, then for $q = 3$, we have $0.7^3 + 0.5^3 + 0.6^3 = 0.684 \leq 1$. In some particular cases, the PFS, SFS and T-SFS are failed, if a decision maker provides $(\mathfrak{L}, \mathfrak{M}, \mathfrak{N}) = (1, 0.5, 0.3)$, i.e., $1^q + 0.5^q + 0.3^q \geq 1$ for any value of q, the PFS, SFS and T-SFS cannot characterize effectively such kinds of information. To precisely cope with such kind of problems, Riaz et al. [45] established the idea of SLDFS whose restriction is that $0 \leq \mathfrak{a}\mathfrak{L} + \mathfrak{b}\mathfrak{M} + \mathfrak{c}\mathfrak{N} \leq 1$, with $0 \leq \mathfrak{a} + \mathfrak{b} + \mathfrak{c} \leq 1$. Obviously, the SLDFS can present effectively such kinds of information, i.e., $0 \leq (0.2)(1) + (0.4)(0.5) + (0.1)(0.3) \leq 1$, with $0 \leq 0.2 + 0.4 + 0.1 = 0.7 \leq 1$, where $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = (0.2, 0.4, 0.1)$ are RPs associated with the grades $\mathfrak{L}, \mathfrak{M}$ and $\mathfrak{N}$, respectively. But here again, the sum of RPs provided by decision maker is often larger than one i.e., $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} \geq 1$, which violates the restriction of SLDFS related to RPs. The SLDFS has its own limitations related to the RPs. In order to remove such limitation and motivated by the idea from LDFS to q-LDFS in two-dimensional space, it is necessary to extend SLDF to T-SLDFS in three-dimensional space. In T-SLDFS we introduce the qth power of RP which cover the space of existing structure and cover the space of the membership grades with the help of qth power of RPs. The PFS, SFS, T-SFS and SLDFS all are the special cases of T-SLDFS. For example, in the environment of T-SLDFS, if $q = 1$ and each RP equals 1, then T-SLDFS is converted to PFS. If $q = 2$ and each RP equals 1, then T-SLDFS is converted to SFS. If each RP equals 1 for any q, then T-SLDFS is converted to T-SFS. If $q = 1$ and each RP is freely chosen, then T-SLDFS is converted to SLDFS. From the above discussions, it is clear that the T-SLDFS is more versatile and more superior to PFS, SFS, T-SFS and SLDFS to describe awkward and complication information in real-decision. The advantages of the proposed method and the drawbacks of the existing methods discussed above served as the motivation for this paper. Therefore, the contributions of this paper are shown as follows. Firstly, we establish the notion of the T-SLDFS, which generalizes the theories of the PFS, SFS, T-SFS, and SLDFS. Secondly, we explore the notions of T-SLDFWA and T-SLDFWG operators on the basis of the operational laws of T-SLDFNs. Thirdly, we define several types of SFs and AFs for the ranking process. Fourthly, we solve a MADM problem based on T-SLDFNs by using T-SLDFWA and T-SLDFWG operators. Lastly, a comprehensive comparative analysis and geometrical interpretations are presented to reveal the advantages of the suggested methods. Figure 1 represents the contributions graphically.

The rest of this manuscript is summarized as follows: In Section 2, we review some background on the IFS, PyFS, LDFS, q-ROFS, qRLDFS, SFS, SLDFS, and T-SFS. Throughout Section 3, we give the definition of the T-SLDFS and study its operations. Several types of SFs and AFs are also introduced in this section. In Section 4, we conceptualize the T-SLDFWA and T-SLDFWG operators and discuss their properties. Section 5 exhibits a MADM method using the proposed operators. In Section 6, We present an illustrative example to show the application of the proposed models. We also analyze the results of the proposed method in this section. Section 7 provides a comprehensive comparative analysis to depict the superiority of the proposed methods.

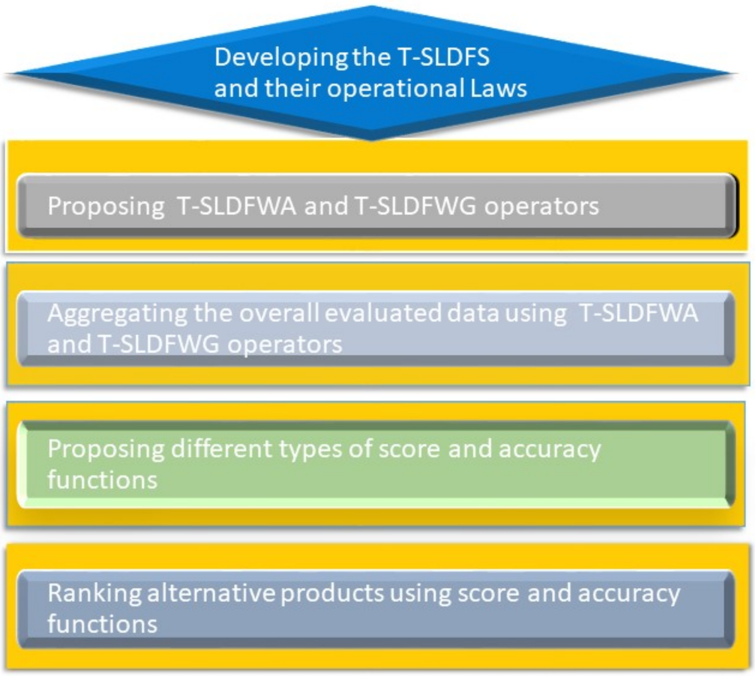


Figure 1. Graphical representation of the contributions.

2. Preliminaries

This section, briefly provides the preliminary knowledge of the IFS, PyFS, LDFS, q-ROFS, q-LDFS, SFS, SLDFS, and T-SFS, before defining T-SLDFS in the next section.

Definition 2.1. [2] Suppose a universe $\mathfrak{D}$. An IFS $\mathfrak{B}$ is defined on $\mathfrak{D}$ as

$$\mathfrak{B} = \{(t, \langle \mathfrak{T}_{\mathfrak{B}}(t), \mathfrak{F}_{\mathfrak{B}}(t)\rangle) : t \in \mathfrak{D}\},$$

where $\mathfrak{T}_{\mathfrak{B}}$ and $\mathfrak{F}_{\mathfrak{B}} \in [0, 1]$ are, respectively, the T-grade and F-grade, such that $0 \leq \mathfrak{T}_{\mathfrak{B}}(t) + \mathfrak{F}_{\mathfrak{B}}(t) \leq 1$, $\forall t \in \mathfrak{D}$.

The IFS fails in situations when the sum of the T-grade and F-grade is larger than 1. Thus, Yager [3] generalized the IFS to PyFS, whose main characteristic is that the square sum of the T-grade and F-grade cannot exceed 1.

Definition 2.2. [3] A PyFS $\mathfrak{K}$ in a universe of discourse $\mathfrak{D}$ is given as

$$\mathfrak{K} = \{(t, \langle \mathfrak{T}_{\mathfrak{K}}(t), \mathfrak{F}_{\mathfrak{K}}(t)\rangle) : t \in \mathfrak{D}\},$$

where $\mathfrak{T}_{\mathfrak{K}} : \mathfrak{D} \longrightarrow [0, 1]$ denotes the T-grade and $\mathfrak{F}_{\mathfrak{K}} : \mathfrak{D} \longrightarrow [0, 1]$ denotes the F-grade with the condition that $0 \leq (\mathfrak{T}_{\mathfrak{K}}(t))^2 + (\mathfrak{F}_{\mathfrak{K}}(t))^2 \leq 1$.

Riaz and Hashmi [11] developed the LDFS by combining the grades of RPs with the T-grade and F-grade in the definitions of the IFS and PyFS.

Definition 2.3. [11] A LDFS $\mathfrak{B}_H$ on the reference set $\mathfrak{C}$ is defined as

$$\mathfrak{B}_H = \{(s, \langle \mathfrak{R}_H(s), \mathfrak{T}_H(s)\rangle, \langle \mathfrak{a}(s), \mathfrak{b}(s)\rangle) : s \in \mathfrak{C}\},$$

where $\mathfrak{R}_H(s), \mathfrak{T}_H(s), \mathfrak{a}(s), \mathfrak{b}(s) \in [0,1]$ are, respectively, the T-grade, F-grade and reference parameters. These functions fulfill the constraint $0 \leq \mathfrak{a}\mathfrak{R}_H(s) + \mathfrak{b}\mathfrak{T}_H(s) \leq 1$, $\forall s \in \mathfrak{C}$, with $0 \leq \mathfrak{a} + \mathfrak{b} \leq 1$.

Definition 2.4. [11] Let $\mathfrak{E}_H = (\langle \mathfrak{R}_H, \mathfrak{T}_H \rangle, \langle \mathfrak{a}, \mathfrak{b} \rangle)$ be a LDFN on the reference set $\mathfrak{C}$ and $LDFN(\mathfrak{C})$ be the collection of LDFNs on $\mathfrak{C}$. Then:

(1) The score function (SF) is characterized through the transformation $\mathfrak{L} : LDFN(\mathfrak{C}) \longrightarrow [-1,1]$ which formalized as $\mathfrak{L}_{\mathfrak{E}_H} = \mathfrak{L}(\mathfrak{E}_H) = \frac{1}{2}[(\mathfrak{R}_H - \mathfrak{T}_H) + (\mathfrak{a} - \mathfrak{b})]$.

(2) The accuracy function (AF) is determined by the transformation $\mathfrak{M} : LDFN(\mathfrak{C}) \longrightarrow [0,1]$ and defined by $\mathfrak{M}_{\mathfrak{E}_H} = \mathfrak{M}(\mathfrak{E}_H) = \frac{1}{2}[\frac{\mathfrak{R}_H - \mathfrak{T}_H}{2} + (\mathfrak{a} + \mathfrak{b})]$.

(3) The quadratic score function (QSF) is a transformation $\mathfrak{N} : LDFN(\mathfrak{C}) \longrightarrow [-1,1]$ which is written as $\mathfrak{N}_{\mathfrak{E}_H} = \mathfrak{N}(\mathfrak{E}_H) = \frac{1}{2}[(\mathfrak{R}_H^2 - \mathfrak{T}_H^2) + (\mathfrak{a}^2 - \mathfrak{b}^2)]$.

(4) The quadratic accuracy function (QAF) is a transformation $\mathfrak{O} : LDFN(\mathfrak{C}) \longrightarrow [0,1]$ which is formalized by $\mathfrak{O}_{\mathfrak{E}_H} = \mathfrak{O}(\mathfrak{E}_H) = \frac{1}{2}[\frac{\mathfrak{R}_H^2 - \mathfrak{T}_H^2}{2} + (\mathfrak{a}^2 + \mathfrak{b}^2)]$.

(5) The expected score function (ESF) determined by $\mathfrak{P} : LDFN(\mathfrak{C}) \longrightarrow [0,1]$ and defined by $\mathfrak{P}_{\mathfrak{E}_H} = \mathfrak{P}(\mathfrak{E}_H) = \frac{1}{2}[\frac{(\mathfrak{R}_H - \mathfrak{T}_H + 1)}{2} + \frac{(\mathfrak{a} - \mathfrak{b} + 1)}{2}]$.

Definition 2.5. [11] Let $\mathfrak{E}_{H_\kappa} = \{(\langle {}^\kappa\mathfrak{R}_H, {}^\kappa\mathfrak{T}_H \rangle, \langle {}^\kappa\mathfrak{a}, {}^\kappa\mathfrak{b} \rangle) : \kappa = 1, 2, ..., r\}$ be a assembling of LDFNs on the universe $\mathfrak{C}$ and $\tau = (\tau_1, \tau_2, ..., \tau_r)$ be the weight vector such that $\sum_{\kappa=1}^{r} \tau_\kappa = 1$. Then, the mapping $\varphi : LDFN(\mathfrak{C}) \longrightarrow LDFN(\mathfrak{C})$ is called the LDFWG operator and portrayed as LDFWG $(\mathfrak{E}_{H_1}, \mathfrak{E}_{H_2}, ..., \mathfrak{E}_{H_r}) = \prod_{\kappa=1}^{r} \mathfrak{E}_{H_\kappa}^{\varphi_\kappa} = (\langle \prod_{\kappa=1}^{r} {}^\kappa\mathfrak{R}_H^{\varphi_\kappa}, 1 - \prod_{\kappa=1}^{r} (1 - {}^\kappa\mathfrak{T}_H)^{\varphi_\kappa} \rangle, \langle \prod_{\kappa=1}^{r} {}^\kappa\mathfrak{a}^{\varphi_\kappa}, 1 - \prod_{\kappa=1}^{r} (1 - {}^\kappa\mathfrak{b})^{\varphi_\kappa} \rangle)$.

The IFS and PFS have been developed also into the q-ROFS [4], which is more adaptable than the IFS and PYFS since the sum of the qth power of the T-grade and F-grade is less than 1.

Definition 2.6. [4] Let $\mathfrak{D}$ be a universal set. Then, the q-ROFS $\mathfrak{R}$ on $\mathfrak{D}$ is formalized as

$$\mathfrak{R} = \{(t, \mathfrak{T}_{\mathfrak{Q}}(t), \mathfrak{F}_{\mathfrak{Q}}(t)) : t \in \mathfrak{D}\},$$

where $\mathfrak{T}_{\mathfrak{Q}}(t)$ and $\mathfrak{F}_{\mathfrak{Q}}(t)$) stand for the T-grade and F-grade, respectively, where $\mathfrak{T}_{\mathfrak{Q}}(t)$ and $\mathfrak{F}_{\mathfrak{Q}}(t)$) lie in $[0,1]$ and $0 \leq (\mathfrak{T}_{\mathfrak{Q}}(t))^q + (\mathfrak{F}_{\mathfrak{Q}}(t))^q \leq 1$ $(q \geq 1)$, $\forall t \in \mathfrak{D}$. The refusal part is given as:

$$\mathfrak{B}_{\mathfrak{Q}}(t) = \Big((\mathfrak{T}_{\mathfrak{Q}}(t))^q + (\mathfrak{F}_{\mathfrak{Q}}(t))^q - (\mathfrak{T}_{\mathfrak{Q}}(t))^q(\mathfrak{F}_{\mathfrak{Q}}(t))^q\Big)^{1/q}.$$

Almagrabi et al. [12] introduced the q-LDFS with the addition of RPs to the construction of the q-ROFS to be more effective and versatile than other approaches.

Definition 2.7. [12] A q-LDFS $\mathfrak{Q}_M$ on the reference $\mathfrak{U}$ is determined by

$$\mathfrak{Q}_M = \{(t, \langle \mathfrak{Y}_M(t), \mathfrak{X}_M(t) \rangle, \langle \mathfrak{f}, \mathfrak{g} \rangle) : t \in \mathfrak{U}\},$$

where $\mathfrak{Y}_M(t), \mathfrak{X}_M(t), \mathfrak{f}, \mathfrak{g} \in [0,1]$ are the T-grade, F-grade and RPs, respectively. These grades satisfy the restriction $0 \leq (\mathfrak{f})^q \mathfrak{Y}_M(t) + (\mathfrak{g})^q \mathfrak{X}_M(t) \leq 1$, $\forall t \in \mathfrak{U}$ $(q \geq 1)$, with $0 \leq (\mathfrak{f})^q + (\mathfrak{g})^q \leq 1$.

Apart from the above sets, the notion of the SFSs has been introduced by [30, 31], which encompasses three membership degrees where the sum of squares of all degrees is less than one.

Definition 2.8. [30, 31] A SFS $\mathfrak{K}$ on the set $\mathfrak{Z}$ is characterized by

$$\mathfrak{K} = \{(\mathfrak{h}, \mathfrak{T}_{\mathfrak{K}}(\mathfrak{h}), \mathfrak{I}_{\mathfrak{K}}(\mathfrak{h}), \mathfrak{F}_{\mathfrak{K}}(\mathfrak{h})) : \mathfrak{h} \in \mathfrak{Z}\},$$

where $\mathfrak{T}_{\mathfrak{K}}(\mathfrak{h}), \mathfrak{I}_{\mathfrak{K}}(\mathfrak{h})$ and $\mathfrak{F}_{\mathfrak{K}}(\mathfrak{h}) \in [0, 1]$ represent , T-grade, I-grade and F-grade, respectively, and $0 \leq (\mathfrak{T}_{\mathfrak{K}}(\mathfrak{h}))^2 + (\mathfrak{I}_{\mathfrak{K}}(\mathfrak{h}))^2 + (\mathfrak{F}_{\mathfrak{K}}(\mathfrak{h}))^2 \leq 1$, for all $\mathfrak{h} \in \mathfrak{Z}$. The refusal degree of $\mathfrak{h}$ to $\mathfrak{Z}$ is determined by

$$\mathfrak{B}_{\mathfrak{K}}(\mathfrak{h}) = \left(1 - [(\mathfrak{T}_{\mathfrak{K}}(\mathfrak{h}))^2 + (\mathfrak{I}_{\mathfrak{K}}(\mathfrak{h}))^2 + (\mathfrak{F}_{\mathfrak{K}}(\mathfrak{h}))^2]\right)^{1/2}.$$

Riaz et al. [45] characterized the SLDFS by taking the RPs into account in SFSs.

Definition 2.9. [45] A SLDFS $\mathfrak{W}_D$ on the reference set $\mathfrak{Z}$ is defined as

$$\mathfrak{W}_D = \{(\varepsilon, \langle \mathfrak{L}_D(\varepsilon), \mathfrak{M}_D(\varepsilon), \mathfrak{N}_D(\varepsilon)\rangle, \langle \mathfrak{a}, \mathfrak{b}, \mathfrak{c}\rangle) : \varepsilon \in \mathfrak{Z}\},$$

such that $\mathfrak{L}_D(\varepsilon), \mathfrak{M}_D(\varepsilon), \mathfrak{N}_D(\varepsilon), \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in [0, 1]$ are the T-grade, I-grade, F-grade and RPs, respectively. These grades satisfy the condition $0 \leq \mathfrak{a}\mathfrak{L}_D(\varepsilon) + \mathfrak{b}\mathfrak{M}_D(\varepsilon) + \mathfrak{c}\mathfrak{N}_D(\varepsilon) \leq 1$, $\forall \varepsilon \in \mathfrak{Z}$, with $0 \leq \mathfrak{a} + \mathfrak{b} + \mathfrak{c} \leq 1$.

Mahmood et al. [31] extended the SFSs to the T-SFSs, where there are no restrictions on their constraints.

Definition 2.10. [31] A T-SFS $\mathfrak{L}$ on the finite set $\mathfrak{Q}$ is portrayed as follows.

$$\mathfrak{L} = \{(\mathfrak{a}, \mathfrak{T}_{\mathfrak{L}}(\mathfrak{a}), \mathfrak{I}_{\mathfrak{L}}(\mathfrak{a}), \mathfrak{F}_{\mathfrak{L}}(\mathfrak{a})) : \mathfrak{a} \in \mathfrak{Q}\},$$

where $\mathfrak{T}_{\mathfrak{L}}(\mathfrak{a}), \mathfrak{I}_{\mathfrak{L}}(\mathfrak{a})$ and $\mathfrak{F}_{\mathfrak{L}}(\mathfrak{a}) \in [0, 1]$ are T-grade, I-grade, F-grade, respectively, and $0 \leq (\mathfrak{T}_{\mathfrak{L}}(\mathfrak{a}))^q + (\mathfrak{I}_{\mathfrak{L}}(\mathfrak{a}))^q + (\mathfrak{F}_{\mathfrak{L}}(\mathfrak{a}))^q \leq 1$ $(q \geq 1)$, $\forall \mathfrak{a} \in \mathfrak{Q}$. The refusal part is determined by

$$\mathfrak{B}_{\mathfrak{L}}(\mathfrak{a}) = \left(1 - [(\mathfrak{T}_{\mathfrak{L}}(\mathfrak{a}))^q + (\mathfrak{I}_{\mathfrak{L}}(\mathfrak{a}))^q + (\mathfrak{F}_{\mathfrak{L}}(\mathfrak{a}))^q]\right)^{1/q}.$$

In this paper, we formalize the T-SLDFS by combining the grades of RPs to the T-grade, I-grade, and F-grade in the constructions of TSFS.

3. T-spherical linear Diophantine fuzzy set

In this section, we propose the concept of T-SLDFS, SFs and AFs of T-SLDFNs and some operations on T-SLDFS.

Definition 3.1. Let X be a fixed non-empty reference set. The T-SLDFS S_Λ over X can be defined as

$$S_\Lambda = \{(t, \langle T_\Lambda(t), I_\Lambda(t), F_\Lambda(t)\rangle, \langle \mu(t), \nu(t), \omega(t)\rangle) : t \in X\},$$

where $T_\Lambda(t), I_\Lambda(t), F_\Lambda(t) \in [0, 1]$ denote respectively, the reality grades, abstinence grades and falsity grades. $\mu(t), \nu(t), \omega(t) \in [0, 1]$ are RPs associated with the grades $T_\Lambda(t), I_\Lambda(t)$ and $F_\Lambda(t)$, respectively, and they satisfy the following conditions:

$0 \leq \mu^q(t)T_\Lambda(t) + \nu^q(t)I_\Lambda(t) + \omega^q(t)F_\Lambda(t) \leq 1$, $\forall t \in X,\ q \geq 1$, with
$0 \leq \mu^q(t) + \nu^q(t) + \omega^q(t) \leq 1$. The refusal part is given by

$$\theta M_\Lambda = (1 - (\mu^q(t)T_\Lambda(t) + \nu^q(t)I_\Lambda(t) + \omega^q(t)F_\Lambda(t)))^{\frac{1}{q}},$$

where θ represents the RP associated with the refusal degree.

Definition 3.2. A collection of $\Omega = (\langle T_\Lambda, I_\Lambda, F_\Lambda\rangle, \langle \mu, \nu, \omega\rangle)$ is called T-SLDFN with

$$0 \le \mu^q T_\Lambda + \nu^q I_\Lambda + \omega^q F_\Lambda \le 1 \textit{ and } 0 \le \mu^q + \nu^q + \omega^q \le 1,\ (q \ge 1).$$

Remark 3.3. In the above definition,

(1) If $q = 1$, then the T-SLDFN is reduced to SLDFN.

(2) If $I_\Lambda = \nu = 0$, then the T-SLDFN is reduced to q-LDFN.

Remark 3.4. (1) Any PFS is an SLDFS and any SLDFS is a T-SLDFS for $q = 1$, but the converse is not true.

(2) Any SFS is a SLDFS and any SLDFS is a T-SLDFS for $q = 1$, but the converse is not true.

Now we put forward the definition of absolute T-SLDFS and the definition of null T-SLDFS.

Definition 3.5. Let $S_\Lambda = \{(t, \langle T_\Lambda(t), I_\Lambda(t), F_\Lambda(t)\rangle, \langle \mu(t), \nu(t), \omega(t)\rangle) : t \in X\}$ be a T-SLDFS over X. Then, S_Λ is said to be an absolute T-SLDFS denoted by $\tilde{\Psi}$ if $T_\Lambda(t) = \mu(t) = 1$ and $F_\Lambda(t) = I_\Lambda(t) = \nu(t) = \omega(t) = 0,\ \forall t \in X$, i.e., $\tilde{\Psi} = (\langle 1, 0, 0\rangle, \langle 1, 0, 0\rangle)$.

Definition 3.6. Let $S_\Lambda = \{(t, \langle T_\Lambda(t), I_\Lambda(t), F_\Lambda(t)\rangle, \langle \mu(t), \nu(t), \omega(t)\rangle) : t \in X\}$ be a T-SLDFS over X. Then, S_Λ is said to be a null T-SLDFS denoted by $\tilde{\Phi}$ if $T_\Lambda(t) = \mu(t) = 0$ and $F_\Lambda(t) = I_\Lambda(t) = \nu(t) = \omega(t) = 1,\ \forall t \in X$, i.e., $\tilde{\Phi} = (\langle 0, 1, 1\rangle, \langle 0, 1, 1\rangle)$.

3.1. Score functions

In this section, we define the SF, QSF, ESF, AF, and QAF.

Definition 3.1.1. Let $\Omega = (\langle T_\Lambda, I_\Lambda, F_\Lambda\rangle, \langle \mu, \nu, \omega\rangle)$ be a T-SLDFN; then the SF on Ω is determined by the transformation $\kappa : T - SLDFN(X) \longrightarrow [-1, 1]$ and defined as

$$\kappa_\Omega = \kappa(\Omega) = \frac{1}{2}[(T_\Lambda - I_\Lambda - F_\Lambda) + (\mu^q - \nu^q - \omega^q)], q \ge 1,$$

where $T - SLDFN(X)$ is the collection of T-SLDFNs on the reference set X.

Definition 3.1.2. The AF ξ is determined by the transformation $\xi : T - SLDFN(X) \longrightarrow [0, 1]$ and defined as

$$\xi_\Omega = \xi(\Omega) = \frac{1}{2}\Big[\frac{(T_\Lambda + I_\Lambda + F_\Lambda)}{3} + (\mu^q + \nu^q + \omega^q)\Big], q \ge 1,$$

where $T - SLDFN(X)$ is the collection of T-SLDFNs on the reference set X.

Definition 3.1.3. Let Ω_1 and Ω_2 be two T-SLDFNs. By using Definitions 3.1.1 and 3.1.2, we can compare the T-SLDFNs Ω_1 and Ω_2 as follows.

(1) If $\kappa_{\Omega_1} < \kappa_{\Omega_2}$, then $\Omega_1 < \Omega_2$.

(2) If $\kappa_{\Omega_1} > \kappa_{\Omega_2}$, then $\Omega_1 > \Omega_2$.

(3) If $\kappa_{\Omega_1} = \kappa_{\Omega_2}$, and $\xi_{\Omega_1} < \xi_{\Omega_2}$, then $\Omega_1 < \Omega_2$.

(4) If $\kappa_{\Omega_1} = \kappa_{\Omega_2}$, and $\xi_{\Omega_1} > \xi_{\Omega_2}$, then $\Omega_1 > \Omega_2$.

(5) If $\kappa_{\Omega_1} = \kappa_{\Omega_2}$, and $\xi_{\Omega_1} = \xi_{\Omega_2}$, then $\Omega_1 = \Omega_2$.

The next definition is QSF.

Definition 3.1.4. The mapping $\pi : T - SLDFN(X) \longrightarrow [-1, 1]$ represents the QSF for the T-SLDFN Ω, which can be given as

$$\pi_\Omega = \pi(\Omega) = \frac{1}{2}\Big[(T_\Lambda^2 - I_\Lambda^2 - F_\Lambda^2) + ((\mu^q)^2 - (\nu^q)^2 - (\omega^q)^2)\Big], q \geq 1.$$

Definition 3.1.5. The mapping $\Phi : T - SLDFN(X) \longrightarrow [0, 1]$ represents the QAF for the T-SLDFN Ω, which can be given as

$$\Phi_\Omega = \Phi(\Omega) = \frac{1}{2}\Big[\frac{(T_\Lambda^2 + I_\Lambda^2 + F_\Lambda^2)}{3} + ((\mu^q)^2 + (\nu^q)^2 + (\omega^q)^2)\Big], q \geq 1.$$

The QSF and QAF are used to compare the T-SLDFNs as follows.

Definition 3.1.6. If Ω_1 and Ω_2 are two T-SLDFNs. By using Definitions 3.1.4 and 3.1.5, we can compare the T-SLDFNs Ω_1 and Ω_2 as follows.

(1) If $\pi_{\Omega_1} < \pi_{\Omega_2}$, then $\Omega_1 < \Omega_2$.

(2) If $\pi_{\Omega_1} > \pi_{\Omega_2}$, then $\Omega_1 > \Omega_2$.

(3) If $\pi_{\Omega_1} = \pi_{\Omega_2}$, and $\Phi_{\Omega_1} < \Phi_{\Omega_2}$, then $\Omega_1 < \Omega_2$.

(4) If $\pi_{\Omega_1} = \pi_{\Omega_2}$, and $\Phi_{\Omega_1} > \Phi_{\Omega_2}$, then $\Omega_1 > \Omega_2$.

(5) If $\pi_{\Omega_1} = \pi_{\Omega_2}$, and $\Phi_{\Omega_1} = \Phi_{\Omega_2}$, then $\Omega_1 = \Omega_2$.

In the following, we present a generalized form of the SF called the ESF.

Definition 3.1.7. Let $\Omega = (\langle T_\Lambda, I_\Lambda, F_\Lambda\rangle, \langle \mu, \nu, \omega\rangle)$ be a T-SLDFN; then, a ESF on Ω is determined by the transformation $\Gamma : T - SLDFN(X) \longrightarrow [0, 1]$ and defined as

$$\Gamma_\Omega = \Gamma(\Omega) = \frac{1}{3}\Big[\frac{(T_\Lambda - I_\Lambda - F_\Lambda + 2)}{2} + \frac{(\mu^q - \nu^q - \omega^q + 2)}{2}\Big], q \geq 1.$$

3.2. Operations of T-SLDFNs

In this part, we provide some operations on T-SLDFNs.

Definition 3.2.1. Let $\Omega_1 = (\langle {}^1T_\Lambda, {}^1I_\Lambda, {}^1F_\Lambda\rangle, \langle {}^1\mu, {}^1\nu, {}^1\omega\rangle)$ and $\Omega_2 = (\langle {}^2T_\Lambda, {}^2I_\Lambda, {}^2F_\Lambda\rangle, \langle {}^2\mu, {}^2\nu, {}^2\omega\rangle)$ be two T-SLDFNs over X and $\lambda > 0$; then,

(1) $\Omega_1^c = (\langle {}^1F_\Lambda, 1 - {}^1I_\Lambda, {}^1T_\Lambda\rangle, \langle {}^1\omega, 1 - {}^1\nu, {}^1\mu\rangle)$,

(2) $\Omega_1 = \Omega_2 \Longleftrightarrow {}^1T_\Lambda = {}^2T_\Lambda, {}^1I_\Lambda = {}^2I_\Lambda, {}^1F_\Lambda = {}^2F_\Lambda, {}^1\mu = {}^2\mu, {}^1\nu = {}^2\nu, {}^1\omega = {}^2\omega$,

(3) $\Omega_1 \subseteq \Omega_2 \Longleftrightarrow {}^1T_\Lambda \leq {}^2T_\Lambda, {}^1I_\Lambda \geq {}^2I_\Lambda, {}^1F_\Lambda \geq {}^2F_\Lambda, {}^1\mu \leq {}^2\mu, {}^1\nu \geq {}^2\nu, {}^1\omega \geq {}^2\omega$,

(4) $\Omega_1 \cup \Omega_2 = (\langle \max({}^1T_\Lambda, {}^2T_\Lambda), \min({}^1I_\Lambda, {}^2I_\Lambda), \min({}^1F_\Lambda, {}^2F_\Lambda)\rangle, \langle \max({}^1\mu_\Lambda, {}^2\mu_\Lambda), \min({}^1\nu_\Lambda, {}^2\nu_\Lambda), \min({}^1\omega_\Lambda, {}^2\omega_\Lambda)\rangle)$,

(5) $\Omega_1 \cap \Omega_2 = (\langle \min({}^1T_\Lambda, {}^2T_\Lambda), \max({}^1I_\Lambda, {}^2I_\Lambda), \max({}^1F_\Lambda, {}^2F_\Lambda)\rangle, \langle \min({}^1\mu_\Lambda, {}^2\mu_\Lambda), \max({}^1\nu_\Lambda, {}^2\nu_\Lambda), \max({}^1\omega_\Lambda, {}^2\omega_\Lambda)\rangle)$,

(6) $\Omega_1 \oplus \Omega_2 = \Big(\Big\langle\Big(({}^1T_\Lambda)^q + ({}^2T_\Lambda)^q - ({}^1T_\Lambda)^q({}^2T_\Lambda)^q\Big)^{\frac{1}{q}}, ({}^1I_\Lambda\ {}^2I_\Lambda), ({}^1F_\Lambda\ {}^2F_\Lambda)\Big\rangle, \Big\langle\Big(({}^1\mu)^q + ({}^2\mu)^q - ({}^1\mu)^q({}^2\mu)^q\Big)^{\frac{1}{q}}, \Big({}^1\nu\ {}^2\nu\Big), \Big({}^1\omega\ {}^2\omega\Big)\Big\rangle\Big), q \geq 1$,

(7) $\Omega_1 \otimes \Omega_2 = \Big(\Big\langle({}^1T_\Lambda\ \ {}^2T_\Lambda), \Big(({}^1I_\Lambda)^q + ({}^2I_\Lambda)^q - ({}^1I_\Lambda)^q({}^2I_\Lambda)^q\Big)^{\frac{1}{q}}, \Big(({}^1F_\Lambda)^q + ({}^2F_\Lambda)^q - ({}^1F_\Lambda)^q({}^2F_\Lambda)^q\Big)^{\frac{1}{q}}\Big\rangle, \Big\langle({}^1\mu\ {}^2\mu), \Big(({}^1\nu)^q + ({}^2\nu)^q - ({}^1\nu)^q({}^2\nu)^q\Big)^{\frac{1}{q}}, \Big(({}^1\omega)^q + ({}^2\omega)^q - ({}^1\omega)^q({}^2\omega)^q\Big)^{\frac{1}{q}}\Big\rangle\Big)$, $q \geq 1$,

(8) $\lambda\Omega_1 = \Big(\Big\langle\Big(1 - (1 - ({}^1T_\Lambda)^q)^\lambda\Big)^{\frac{1}{q}}, \Big({}^1I_\Lambda\Big)^\lambda, \Big({}^1F_\Lambda\Big)^\lambda\Big\rangle, \Big\langle\Big(1 - (1 - ({}^1\mu)^q)^\lambda\Big)^{\frac{1}{q}}, \Big({}^1\nu\Big)^\lambda, \Big({}^1\omega\Big)^\lambda\Big\rangle\Big)$ $\lambda > 0, q \geq 1$,

(9) $\Omega_1^\lambda = \Big(\Big\langle\Big({}^1T_\Lambda\Big)^\lambda, \Big(1 - (1 - ({}^1I_\Lambda)^q)^\lambda\Big)^{\frac{1}{q}}, \Big(1 - (1 - ({}^1F_\Lambda)^q)^\lambda\Big)^{\frac{1}{q}}, \Big\rangle, \Big\langle\Big({}^1\mu\Big)^\lambda, \Big(1 - (1 - ({}^1\nu)^q)^\lambda\Big)^{\frac{1}{q}}, \Big(1 - (1 - ({}^1\omega)^q)^\lambda\Big)^{\frac{1}{q}}, \Big\rangle\Big), \lambda > 0, q \geq 1$.

Remark 3.2.2. If ${}^1I_\Lambda ={}^2 I_\Lambda ={}^1 \nu ={}^2 \nu = 0$, then all operations in Definition 3.2.1 reduce to the operations of q-LDFNs.

Proposition 3.2.3. Let $\Omega_\delta = (\langle{}^\delta T_\Lambda, {}^\delta I_\Lambda, {}^\delta F_\Lambda\rangle, \langle{}^\delta\mu, {}^\delta\nu, {}^\delta\omega\rangle)$ for $\delta \in \Delta$ (indexing Set) be a collection of T-SLDFNs over X and $\lambda > 0$. Then, $\bigcup_{\delta\in\Delta}\Omega_\delta, \bigcap_{\delta\in\Delta}\Omega_\delta, \Omega_\delta^C, \bigoplus_{\delta\in\Delta}\Omega_\delta, \bigotimes_{\delta\in\Delta}\Omega_\delta, \Omega_\delta^\lambda$ and $\lambda\Omega$ are also T-SLDFNs.

Proof. The proof is accomplished using Definition 3.2.1. □

Example 3.2.4. If $\Omega_1 = (\langle 0.6, 0.82, 0.4\rangle, \langle 0.1, 0.42, 0.55\rangle)$ and $\Omega_2 = (\langle 0.96, 0.78, 0.1\rangle, \langle 0.6, 0.3, 0.43\rangle)$ be two 3-SLDFNs. If $\lambda = 5$, then

(1) $\Omega_1^C = (\langle 0.4, 0.18, 0.6\rangle, \langle 0.55, 0.58, 0.1\rangle)$,

(2) $\Omega_1 \subseteq \Omega_2$ as $0.6 \leq 0.96, 0.82 \geq 0.78, 0.4 \geq 0.1$ and $0.1 \leq 0.6, 0.42 \geq 0.3, 0.55 \geq 0.43$,

(3) $\Omega_1 \cup \Omega_2 = (\langle 0.96, 0.78, 0.1\rangle, \langle 0.6, 0.3, 0.43\rangle) = \Omega_2$,

(4) $\Omega_1 \cap \Omega_2 = (\langle 0.6, 0.82, 0.4\rangle, \langle 0.1, 0.42, 0.55\rangle) = \Omega_1$,

(5) $\Omega_1 \oplus \Omega_2 = (\langle 0.97, 0.64, 0.04\rangle, \langle 0.6, 0.13, 0.24\rangle)$,

(6) $\Omega_1 \otimes \Omega_2 = (\langle 0.58, 0.91, 0.4\rangle, \langle 0.06, 0.46, 0.62\rangle)$,

(7) $\lambda\Omega_1 = (\langle 0.89, 0.37, 0.01\rangle, \langle 0.17, 0.01, 0.05\rangle)$,

(8) $\Omega_1^\lambda = (\langle 0.078, 0.99, 0.66\rangle, \langle 0, 0.68, 0.84\rangle)$.

Proposition 3.2.5. Let Ω_1, Ω_2 and Ω_3 be three T-SLDFNs. Then, the following properties hold.

(1) $\Omega_1 \cup \Omega_2 = \Omega_2 \cup \Omega_1$,

(2) $\Omega_1 \cap \Omega_2 = \Omega_2 \cap \Omega_1$,

(3) $\Omega_1 \cup (\Omega_2 \cap \Omega_3) = (\Omega_1 \cup \Omega_2) \cap (\Omega_1 \cup \Omega_3)$,

(4) $\Omega_1 \cap (\Omega_2 \cup \Omega_3) = (\Omega_1 \cap \Omega_2) \cup (\Omega_1 \cap \Omega_3)$,

(5) $(\Omega_1 \cup \Omega_2)^c = \Omega_1^c \cap \Omega_2^c$,

(6) $(\Omega_1 \cap \Omega_2)^c = \Omega_1^c \cup \Omega_2^c$,

(7) $\Omega_1 \oplus \Omega_2 = \Omega_2 \oplus \Omega_1$,

(8) $\Omega_1 \otimes \Omega_2 = \Omega_2 \otimes \Omega_1$,

(9) $\lambda(\Omega_1 \oplus \Omega_2) = \lambda\Omega_1 \oplus \lambda\Omega_2$,

(10) $(\Omega_1 \otimes \Omega_2)^\lambda = \Omega_1^\lambda \otimes \Omega_2^\lambda$.

Proof. We just give the proof of (5), (7) and (9), as the proof of the other items is trivial.

(5). According to (1) and (4) in Definition 3.2.1, we can obtain, for the left side of the equation, $(\Omega_1 \cup \Omega_2)^c = (\langle \min({}^1F_\Lambda, {}^2F_\Lambda), 1 - \min({}^1I_\Lambda, {}^2I_\Lambda), \max({}^1T_\Lambda, {}^2T_\Lambda)\rangle, \langle \min({}^1\omega_\Lambda, {}^2\omega_\Lambda), 1 - \min({}^1\nu_\Lambda, {}^2\nu_\Lambda), \max({}^1\mu_\Lambda, {}^2\mu_\Lambda)\rangle)$.

For the right-hand side, consider

$\Omega_1^c = (\langle {}^1F_\Lambda, 1 - {}^1I_\Lambda, {}^1T_\Lambda\rangle, \langle {}^1\omega, 1 - {}^1\nu, {}^1\mu\rangle)$, $\Omega_2^c = (\langle {}^2F_\Lambda, 1 - {}^2I_\Lambda, {}^2T_\Lambda\rangle, \langle {}^2\omega, 1 - {}^2\nu, {}^2\mu\rangle)$, then

$\Omega_1^c \cap \Omega_2^c = (\langle \min({}^1F_\Lambda, {}^2F_\Lambda), \max(1 - {}^1I_\Lambda, 1 - {}^2I_\Lambda), \max({}^1T_\Lambda, {}^2T_\Lambda)\rangle, \langle \min({}^1\omega_\Lambda, {}^2\omega_\Lambda), \max(1 - {}^1\nu_\Lambda, 1 - {}^2\nu_\Lambda), \max({}^1\mu_\Lambda, {}^2\mu_\Lambda)\rangle)$

$= (\langle \min({}^1F_\Lambda, {}^2F_\Lambda), 1 - \min({}^1I_\Lambda, {}^2I_\Lambda), \max({}^1T_\Lambda, {}^2T_\Lambda)\rangle, \langle \min({}^1\omega_\Lambda, {}^2\omega_\Lambda), 1 - \min({}^1\nu_\Lambda, {}^2\nu_\Lambda), \max({}^1\mu_\Lambda, {}^2\mu_\Lambda)\rangle) = (\Omega_1 \cup \Omega_2)^c$. This gives (5). (7). Based on Definition 3.2.1,

$\Omega_1 \oplus \Omega_2 = \left(\left\langle \left(({}^1T_\Lambda)^q + ({}^2T_\Lambda)^q - ({}^1T_\Lambda)^q({}^2T_\Lambda)^q\right)^{\frac{1}{q}}, ({}^1I_\Lambda\ {}^2I_\Lambda), ({}^1F_\Lambda\ {}^2F_\Lambda)\right\rangle, \left\langle \left(({}^1\mu)^q + ({}^2\mu)^q - ({}^1\mu)^q({}^2\mu)^q\right)^{\frac{1}{q}}, \left({}^1\nu\ {}^2\nu\right), \left({}^1\omega\ {}^2\omega\right)\right\rangle\right), q \geq 1,$

$= \left(\left\langle \left(({}^2T_\Lambda)^q + ({}^1T_\Lambda)^q - ({}^2T_\Lambda)^q({}^1T_\Lambda)^q\right)^{\frac{1}{q}}, ({}^2I_\Lambda\ {}^1I_\Lambda), ({}^2F_\Lambda\ {}^1F_\Lambda)\right\rangle, \left\langle \left(({}^2\mu)^q + ({}^1\mu)^q - ({}^2\mu)^q({}^1\mu)^q\right)^{\frac{1}{q}}, \left({}^2\nu\ {}^1\nu\right), \left({}^2\omega\ {}^1\omega\right)\right\rangle\right), q \geq 1,$

$= \Omega_2 \oplus \Omega_1$. This gives (7).

(9). According to (6) and (8) in Definition 3.2.1, we can obtain, for the left side of the equation:

$\lambda(\Omega_1 \oplus \Omega_2) = \lambda\left(\left(\left\langle \left(({}^1T_\Lambda)^q + ({}^2T_\Lambda)^q - ({}^1T_\Lambda)^q({}^2T_\Lambda)^q\right)^{\frac{1}{q}}, ({}^1I_\Lambda\ {}^2I_\Lambda), ({}^1F_\Lambda\ {}^2F_\Lambda)\right\rangle, \left\langle \left(({}^1\mu)^q + ({}^2\mu)^q - ({}^1\mu)^q({}^2\mu)^q\right)^{\frac{1}{q}}, \left({}^1\nu\ {}^2\nu\right), \left({}^1\omega\ {}^2\omega\right)\right\rangle\right)\right), q \geq 1,$

$= \left(\left\langle \left(1 - [1 - (({}^1T_\Lambda)^q + ({}^2T_\Lambda)^q - ({}^1T_\Lambda)^q({}^2T_\Lambda)^q)]^\lambda\right)^{\frac{1}{q}}, ({}^1I_\Lambda^2 I_\Lambda)^\lambda, ({}^1F_\Lambda^2 F_\Lambda)^\lambda\right\rangle, \left\langle \left(1 - [1 - (({}^1\mu_\Lambda)^q + ({}^2\mu_\Lambda)^q - ({}^1\mu_\Lambda)^q({}^2\mu_\Lambda)^q)]^\lambda\right)^{\frac{1}{q}}, ({}^1\nu_\Lambda^2 \nu_\Lambda)^\lambda, ({}^1\omega_\Lambda^2 \omega_\Lambda)^\lambda\right\rangle\right),$

$= \left[\left\langle\left(1 - (1 -^1 T_\Lambda^q)^\lambda(1 -^2 T_\Lambda^q)^\lambda\right)^{\frac{1}{q}}, (^1I_\Lambda)^\lambda(^2I_\Lambda)^\lambda, (^1F_\Lambda)^\lambda(^2F_\Lambda)^\lambda\right\rangle, \left\langle\left(1 - (1 -^1 \mu^q)^\lambda(1 -^2 \mu^q)^\lambda\right)^{\frac{1}{q}}, (^1\nu)^\lambda(^2\nu)^\lambda, (^1\omega)^\lambda(^2\omega)^\lambda\right\rangle\right]$. For the right side of the equation, we have

$\lambda\Omega_1 = \left(\left\langle\left(1 - (1 - (^1T_\Lambda)^q)^\lambda\right)^{\frac{1}{q}}, \left(^1I_\Lambda\right)^\lambda, \left(^1F_\Lambda\right)^\lambda\right\rangle, \left\langle\left(1 - (1 - (^1\mu)^q)^\lambda\right)^{\frac{1}{q}}, \left(^1\nu\right)^\lambda, \left(^1\omega\right)^\lambda\right\rangle\right)$
$\lambda > 0, q \geq 1$,

$\lambda\Omega_2 = \left(\left\langle\left(1 - (1 - (^2T_\Lambda)^q)^\lambda\right)^{\frac{1}{q}}, \left(^2I_\Lambda\right)^\lambda, \left(^2F_\Lambda\right)^\lambda\right\rangle, \left\langle\left(1 - (1 - (^2\mu)^q)^\lambda\right)^{\frac{1}{q}}, \left(^2\nu\right)^\lambda, \left(^2\omega\right)^\lambda\right\rangle\right)$
$\lambda > 0, q \geq 1$; moreover, since

$\lambda\Omega_1 \oplus \lambda\Omega_2 = \left[\left\langle\left(1 - (1 -^1 T_\Lambda^q)^\lambda + 1 - (1 -^2 T_\Lambda^q)^\lambda - [1 - (1 -^1 T_\Lambda^q)^\lambda][1 - (1 -^2 T_\Lambda^q)^\lambda]\right)^{\frac{1}{q}}, (^1I_\Lambda)^\lambda(^2I_\Lambda)^\lambda, (^1F_\Lambda)^\lambda(^2F_\Lambda)^\lambda\right\rangle, \left\langle\left(1 - (1 -^1 \mu^q)^\lambda + 1 - (1 -^2 \mu^q)^\lambda - [1 - (1 -^1 \mu^q)^\lambda][1 - (1 -^2 \mu^q)^\lambda]\right)^{\frac{1}{q}}, (^1\nu)^\lambda(^2\nu)^\lambda, (^1\omega)^\lambda(^2\omega)^\lambda\right\rangle\right]$,

$= \left[\left\langle\left(1 - (1 -^1 T_\Lambda^q)^\lambda(1 -^2 T_\Lambda^q)^\lambda\right)^{\frac{1}{q}}, (^1I_\Lambda)^\lambda(^2I_\Lambda)^\lambda, (^1F_\Lambda)^\lambda(^2F_\Lambda)^\lambda\right\rangle, \left\langle\left(1 - (1 -^1 \mu^q)^\lambda(1 -^2 \mu^q)^\lambda\right)^{\frac{1}{q}}, (^1\nu)^\lambda(^2\nu)^\lambda, (^1\omega)^\lambda(^2\omega)^\lambda\right\rangle\right]$. Thus, we have $\lambda(\Omega_1 \oplus \Omega_2) = \lambda\Omega_1 \oplus \lambda\Omega_2$.

□

4. T-SLDF aggregation operators

This section explores the notions of T-SLDFWA and T-SLDFWG operators on the basis of operational laws of T-SLDFNs.

4.1. T-SLDFWA operator

In this section, we define the T-SLDFWA perator, T-SLDFOWA operator, and T-SLDFHWA operator.

Definition 4.1.1. Let $\Omega_\delta = \{(\langle^\delta T_\Lambda,^\delta I_\Lambda,^\delta F_\Lambda\rangle, \langle^\delta\mu,^\delta\nu,^\delta\omega\rangle) : \delta = 1, ..., n\}$ be a collection of T-SLDFNs. The T-SLDFWA operator is a transformation T-SLDFWA: T-SLDFN(X)⟶ T-SLDFN(X), defined by

$$T - SLDFWA(\Omega_1, \Omega_2, ..., \Omega_n) = \varphi_1\Omega_1 \oplus \varphi_2\Omega_2 \oplus ... \oplus \varphi_n\Omega_n,$$

where $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ is the weight vector of $\Omega_\delta(\delta = 1, 2, ..., n)$, $0 \leq \varphi_\delta \leq 1$ and $\sum_{\delta=1}^{n} \varphi_\delta = 1$.

Theorem 4.1.2. Suppose that $\Omega_\delta = \{(\langle^\delta T_\Lambda,^\delta I_\Lambda,^\delta F_\Lambda\rangle, \langle^\delta\mu,^\delta\nu,^\delta\omega\rangle) : \delta = 1, ..., n\}$ is the collection of T-SLDFNs. Let us consider the weight vector $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ of Ω_δ. Then,

$$T - SLDFWA(\Omega_1, \Omega_2, ..., \Omega_n) = \left\langle\left(1 - \prod_{\delta=1}^{n}(1 -^\delta T_\Lambda^q)^{\varphi_\delta}\right)^{\frac{1}{q}}, \prod_{\delta=1}^{n}(^\delta I_\Lambda)^{\varphi_\delta}, \prod_{\delta=1}^{n}(^\delta F_\Lambda)^{\varphi_\delta}\right\rangle, \left\langle\left(1 - \prod_{\delta=1}^{n}(1 -^\delta \mu^q)^{\varphi_\delta}\right)^{\frac{1}{q}}, \prod_{\delta=1}^{n}(^\delta\nu)^{\varphi_\delta}, \prod_{\delta=1}^{n}(^\delta\omega)^{\varphi_\delta}\right\rangle, q \geq 1. \tag{1}$$

Proof. We will use the mathematical induction method to prove this theorem.

(1) For $n=2$, since $\varphi_1\Omega_1 = \left(\left\langle\left(1-(1-({}^1T_\Lambda)^q)^{\varphi_1}\right)^{\frac{1}{q}}, ({}^1I_\Lambda)^{\varphi_1}, ({}^1F_\Lambda)^{\varphi_1}\right\rangle, \left\langle\left(1-(1-({}^1\mu)^q)^{\varphi_1}\right)^{\frac{1}{q}}, ({}^1\nu)^{\varphi_1}, ({}^1\omega)^{\varphi_1}\right\rangle\right)$,
$\varphi_2\Omega_2 = \left(\left\langle\left(1-(1-({}^2T_\Lambda)^q)^{\varphi_2}\right)^{\frac{1}{q}}, ({}^2I_\Lambda)^{\varphi_2}, ({}^2F_\Lambda)^{\varphi_2}\right\rangle, \left\langle\left(1-(1-({}^2\mu)^q)^{\varphi_2}\right)^{\frac{1}{q}}, ({}^2\nu)^{\varphi_2}, ({}^2\omega)^{\varphi_2}\right\rangle\right)$,

T-SLDFWA$(\Omega_1,\Omega_2) = \varphi_1\Omega_1 \oplus \varphi_2\Omega_2 = \Big[\Big\langle\Big(1-(1-{}^1T_\Lambda^q)^{\varphi_1} + 1-(1-{}^2T_\Lambda^q)^{\varphi_2} - [1-(1-{}^1T_\Lambda^q)^{\varphi_1}][1-(1-{}^2T_\Lambda^q)^{\varphi_2}]\Big)^{\frac{1}{q}}, ({}^1I_\Lambda)^{\varphi_1}({}^2I_\Lambda)^{\varphi_2}, ({}^1F_\Lambda)^{\varphi_1}({}^2F_\Lambda)^{\varphi_2}\Big\rangle, \Big\langle\Big(1-(1-{}^1\mu^q)^{\varphi_1}+1-(1-{}^2\mu^q)^{\varphi_2} - [1-(1-{}^1\mu^q)^{\varphi_1}][1-(1-{}^2\mu^q)^{\varphi_2}]\Big)^{\frac{1}{q}}, ({}^1\nu)^{\varphi_1}({}^2\nu)^{\varphi_2}, ({}^1\omega)^{\varphi_1}({}^2\omega)^{\varphi_2}\Big\rangle\Big]$

$= \Big[\Big\langle\Big(1-(1-{}^1T_\Lambda^q)^{\varphi_1}(1-{}^2T_\Lambda^q)^{\varphi_2}\Big)^{\frac{1}{q}}, ({}^1I_\Lambda)^{\varphi_1}({}^2I_\Lambda)^{\varphi_2}, ({}^1F_\Lambda)^{\varphi_1}({}^2F_\Lambda)^{\varphi_2}\Big\rangle, \Big\langle\Big(1-(1-{}^1\mu^q)^{\varphi_1}(1-{}^2\mu^q)^{\varphi_2}\Big)^{\frac{1}{q}}, ({}^1\nu)^{\varphi_1}({}^2\nu)^{\varphi_2}, ({}^1\omega)^{\varphi_1}({}^2\omega)^{\varphi_2}\Big\rangle\Big]$

$= \Big[\Big\langle\Big(1-\prod_{\delta=1}^{2}(1-{}^\delta T_\Lambda^q)^{\varphi_\delta}\Big)^{\frac{1}{q}}, \prod_{\delta=1}^{2}({}^\delta I_\Lambda)^{\varphi_\delta}, \prod_{\delta=1}^{2}({}^\delta F_\Lambda)^{\varphi_\delta}\Big\rangle, \Big\langle\Big(1-\prod_{\delta=1}^{2}(1-{}^\delta\mu^q)^{\varphi_\delta}\Big)^{\frac{1}{q}}, \prod_{\delta=1}^{2}({}^\delta\nu)^{\varphi_\delta}, \prod_{\delta=1}^{2}({}^\delta\omega)^{\varphi_\delta}\Big\rangle\Big]$;
obviously, Eq (1) holds for $n=2$.

(2) If Eq (1) holds for $K=n$, then
$T-SLDFWA(\Omega_1,\Omega_2,...,\Omega_n) = \Big\langle\Big(1-\prod_{\delta=1}^{n}(1-{}^\delta T_\Lambda^q)^{\varphi_\delta}\Big)^{\frac{1}{q}}, \prod_{\delta=1}^{n}({}^\delta I_\Lambda)^{\varphi_\delta}, \prod_{\delta=1}^{n}({}^\delta F_\Lambda)^{\varphi_\delta}\Big\rangle, \Big\langle\Big(1-\prod_{\delta=1}^{n}(1-{}^\delta\mu^q)^{\varphi_\delta}\Big)^{\frac{1}{q}}, \prod_{\delta=1}^{n}({}^\delta\nu)^{\varphi_\delta}, \prod_{\delta=1}^{n}({}^\delta\omega)^{\varphi_\delta}\Big\rangle, q \geq 1.$

When $\delta = n+1$, and according to the operational laws of the T-SLDFNs, we have
$T-SLDFWA(\Omega_1,\Omega_2,...,\Omega_{n+1}) = T-SLDFWA(\Omega_1,\Omega_2,...,\Omega_n) \oplus \varphi_{n+1}\Omega_{n+1}$
$= \Big[\Big\langle\Big(1-\prod_{\delta=1}^{n}(1-{}^\delta T_\Lambda^q)^{\varphi_\delta}\Big)^{\frac{1}{q}}, \prod_{\delta=1}^{n}({}^\delta I_\Lambda)^{\varphi_\delta}, \prod_{\delta=1}^{n}({}^\delta F_\Lambda)^{\varphi_\delta}\Big\rangle, \Big\langle\Big(1-\prod_{\delta=1}^{n}(1-{}^\delta\mu^q)^{\varphi_\delta}\Big)^{\frac{1}{q}}, \prod_{\delta=1}^{n}({}^\delta\nu)^{\varphi_\delta}, \prod_{\delta=1}^{n}({}^\delta\omega)^{\varphi_\delta}\Big\rangle\Big] \oplus \Big[\Big\langle\Big(1-(1-({}^{n+1}T_\Lambda)^q)^{\varphi_{n+1}}\Big)^{\frac{1}{q}}, ({}^{n+1}I_\Lambda)^{\varphi_{n+1}}, ({}^{n+1}F_\Lambda)^{\varphi_{n+1}}\Big\rangle, \Big\langle\Big(1-(1-({}^{n+1}\mu)^q)^{\varphi_{n+1}}\Big)^{\frac{1}{q}}, ({}^{n+1}\nu)^{\varphi_{n+1}}, ({}^{n+1}\omega)^{\varphi_{n+1}}\Big\rangle\Big]$,

$= \Big[\Big\langle\Big(\Big(1-\prod_{\delta=1}^{n}(1-{}^\delta T_\Lambda^q)^{\varphi_\delta}\Big) + \Big(1-(1-{}^{n+1}T_\Lambda^q)^{\varphi_{n+1}}\Big) - \Big(1-\prod_{\delta=1}^{n}(1-{}^\delta T_\Lambda^q)^{\varphi_\delta}\Big)\Big(1-(1-{}^{n+1}T_\Lambda^q)^{\varphi_{n+1}}\Big)\Big)^{\frac{1}{q}}, \Big(\prod_{\delta=1}^{n}({}^\delta I_\Lambda)^{\varphi_\delta}\Big)\Big({}^{n+1}I_\Lambda^{\varphi_{n+1}}\Big), \Big(\prod_{\delta=1}^{n}({}^\delta F_\Lambda)^{\varphi_\delta}\Big)\Big({}^{n+1}F_\Lambda^{\varphi_{n+1}}\Big)\Big\rangle, \Big\langle\Big(\Big(1-\prod_{\delta=1}^{n}(1-{}^\delta\mu^q)^{\varphi_\delta}\Big) + \Big(1-(1-{}^{n+1}\mu^q)^{\varphi_{n+1}}\Big) - \Big(1-\prod_{\delta=1}^{n}(1-{}^\delta\mu^q)^{\varphi_\delta}\Big)\Big(1-(1-{}^{n+1}\mu^q)^{\varphi_{n+1}}\Big)\Big)^{\frac{1}{q}}, \Big(\prod_{\delta=1}^{n}({}^\delta\nu)^{\varphi_\delta}\Big)\Big({}^{n+1}\nu^{\varphi_{n+1}}\Big), \Big(\prod_{\delta=1}^{n}({}^\delta\omega)^{\varphi_\delta}\Big)\Big({}^{n+1}\omega^{\varphi_{n+1}}\Big)\Big\rangle\Big]$,
for $\delta = 1,2,...,n$.
$= \Big[\Big\langle\Big(1-\prod_{\delta=1}^{n}(1-{}^\delta T_\Lambda^q)^{\varphi_\delta}(1-{}^{n+1}T_\Lambda^q)^{\varphi_{n+1}}\Big)^{\frac{1}{q}}, \prod_{\delta=1}^{n+1}({}^\delta I_\Lambda)^{\varphi_\delta}, \prod_{\delta=1}^{n+1}({}^\delta F_\Lambda)^{\varphi_\delta}\Big\rangle, \Big\langle\Big(1-\prod_{\delta=1}^{n}(1-{}^\delta\mu^q)^{\varphi_\delta}(1-{}^{n+1}\mu^q)^{\varphi_{n+1}}\Big)^{\frac{1}{q}}, \prod_{\delta=1}^{n+1}({}^\delta\nu)^{\varphi_\delta}, \prod_{\delta=1}^{n+1}({}^\delta\omega)^{\varphi_\delta}\Big\rangle\Big]$

$$= \left[\left\langle\left(1-\prod_{\delta=1}^{n+1}(1-^{\delta}T_\Lambda^q)^{\varphi_\delta}\right)^{\frac{1}{q}}, \prod_{\delta=1}^{n+1}(^{\delta}I_\Lambda)^{\varphi_\delta}, \prod_{\delta=1}^{n+1}(^{\delta}F_\Lambda)^{\varphi_\delta}\right\rangle, \left\langle\left(1-\prod_{\delta=1}^{n+1}(1-^{\delta}\mu^q)^{\varphi_\delta}\right)^{\frac{1}{q}}, \prod_{\delta=1}^{n+1}(^{\delta}\nu)^{\varphi_\delta}, \prod_{\delta=1}^{n+1}(^{\delta}\omega)^{\varphi_\delta}\right\rangle\right],$$

$k = 1, 2, ..., n+1$. That is, Eq (1) holds for $k = n+1$. Based on steps (1) and (2), we have that Eq (1) holds for any k.

□

The following are the properties of the T-SLDFWA operator.

(1) Idempotency

Let $\Omega_\delta = \{(\langle^{\delta}T_\Lambda, ^{\delta}I_\Lambda, ^{\delta}F_\Lambda\rangle, \langle^{\delta}\mu, ^{\delta}\nu, ^{\delta}\omega\rangle) : \delta = 1, ..., n\}$ be a set of T-SLDFNs. If $\Omega_\delta = \Omega = (\langle T_\Lambda, I_\Lambda, F_\Lambda\rangle, \langle\mu, \nu, \omega\rangle)$, $\forall\delta = 1, ..., n$. Then, $T-SLDFWA(\Omega_1, \Omega_2, ..., \Omega_n) = \Omega = (\langle T_\Lambda, I_\Lambda, F_\Lambda\rangle, \langle\mu, \nu, \omega\rangle)$.

(2) Boundedness

Let $\Omega_\delta = \{(\langle^{\delta}T_\Lambda, ^{\delta}I_\Lambda, ^{\delta}F_\Lambda\rangle, \langle^{\delta}\mu, ^{\delta}\nu, ^{\delta}\omega\rangle) : \delta = 1, ..., n\}$ be a set of T-SLDFNs. If $\Omega^- = (\langle T_\Lambda^-, I_\Lambda^+, F_\Lambda^+\rangle, \langle\mu^-, \nu^+, \omega^+\rangle)$ and $\Omega^+ = (\langle T_\Lambda^+, I_\Lambda^-, F_\Lambda^-\rangle, \langle\mu^+, \nu^-, \omega^-\rangle)$, where, $T_\Lambda^- = \min_\delta\{^{\delta}T_\Lambda\}$, $I_\Lambda^+ = \max_\delta\{^{\delta}I_\Lambda\}$, $F_\Lambda^+ = \max_\delta\{^{\delta}F_\Lambda\}$, $\mu^- = \min_\delta\{^{\delta}\mu\}$, $\nu^+ = \max_\delta\{^{\delta}\nu\}$, $\omega^+ = \max_\delta\{^{\delta}\omega\}$ and $T_\Lambda^+ = \max_\delta\{^{\delta}T_\Lambda\}$, $I_\Lambda^- = \min_\delta\{^{\delta}I_\Lambda\}$, $F_\Lambda^- = \min_\delta\{^{\delta}F_\Lambda\}$, $\mu^+ = \max_\delta\{^{\delta}\mu\}$, $\nu^- = \min_\delta\{^{\delta}\nu\}$, $\omega^- = \min_\delta\{^{\delta}\omega\}$. Then,

$\Omega^- \leq T-SLDFWA(\Omega_1, \Omega_2, ..., \Omega_n) \leq \Omega^+$.

(3) Monotonicity

Let $\Omega_\delta = \{(\langle^{\delta}T_\Lambda, ^{\delta}I_\Lambda, ^{\delta}F_\Lambda\rangle, \langle^{\delta}\mu, ^{\delta}\nu, ^{\delta}\omega\rangle) : \delta = 1, ..., n\}$ and $\Omega_\delta^* = \{(\langle^{\delta}T_\Lambda^*, ^{\delta}I_\Lambda^*, ^{\delta}F_\Lambda^*\rangle,$ $\langle^{\delta}\mu^*, ^{\delta}\nu^*, ^{\delta}\omega^*\rangle) : \delta = 1, ..., n\}$ be two collections of T-SLDFNs. If $^{\delta}T_\Lambda \leq {}^{\delta}T_\Lambda^*$, $^{\delta}I_\Lambda \geq {}^{\delta}I_\Lambda^*$, $^{\delta}F_\Lambda \geq {}^{\delta}F_\Lambda^*$ and $^{\delta}\mu \leq {}^{\delta}\mu^*$, $^{\delta}\nu \geq {}^{\delta}\nu^*$, $^{\delta}\omega \geq {}^{\delta}\omega^*$ $\forall\delta = 1, ..., n$. Then, $T-SLDFWA(\Omega_1, \Omega_2, ..., \Omega_n) \leq T-SLDFWA(\Omega_1^*, \Omega_2^*, ..., \Omega_n^*)$.

Proof. (1) According to Theorem 4.1.2, since $\Omega_\delta = \Omega = (\langle T_\Lambda, I_\Lambda, F_\Lambda\rangle, \langle\mu, \nu, \omega\rangle)$ $\forall\delta = 1, ..., n$, then,

$$T-SLDFWA(\Omega_1, \Omega_2, ..., \Omega_n) = \left\langle\left(1-\prod_{\delta=1}^{n}(1-T_\Lambda^q)^{\varphi_\delta}\right)^{\frac{1}{q}}, \prod_{\delta=1}^{n}(I_\Lambda)^{\varphi_\delta}, \prod_{\delta=1}^{n}(F_\Lambda)^{\varphi_\delta}\right\rangle, \left\langle\left(1-\prod_{\delta=1}^{n}(1-\mu^q)^{\varphi_\delta}\right)^{\frac{1}{q}}, \prod_{\delta=1}^{n}(\nu)^{\varphi_\delta}, \prod_{\delta=1}^{n}(\omega)^{\varphi_\delta}\right\rangle$$

$$= \left\langle\left(1-(1-T_\Lambda^q)^{\sum_{\delta=1}^{n}\varphi_\delta}\right)^{\frac{1}{q}}, (I_\Lambda)^{\sum_{\delta=1}^{n}\varphi_\delta}, (F_\Lambda)^{\sum_{\delta=1}^{n}\varphi_\delta}\right\rangle, \left\langle\left(1-(1-\mu^q)^{\sum_{\delta=1}^{n}\varphi_\delta}\right)^{\frac{1}{q}}, (\nu)^{\sum_{\delta=1}^{n}\varphi_\delta}, (\omega)^{\sum_{\delta=1}^{n}\varphi_\delta}\right\rangle$$

$$= \left\langle\left(1-(1-T_\Lambda^q)\right)^{\frac{1}{q}}, I_\Lambda, F_\Lambda\right\rangle, \left\langle\left(1-(1-\mu^q)\right)^{\frac{1}{q}}, \nu, \omega\right\rangle$$

$$= (\langle T_\Lambda, I_\Lambda, F_\Lambda\rangle, \langle\mu, \nu, \omega\rangle) = \Omega.$$

(2) For $q \geq 1$, since $T_\Lambda^- \leq {}^{\delta}T_\Lambda \leq T_\Lambda^+$, then

$(T_\Lambda^-)^q \leq {}^{\delta}T_\Lambda^q \leq (T_\Lambda^+)^q$, $1-(T_\Lambda^-)^q \geq 1-{}^{\delta}T_\Lambda^q \geq 1-(T_\Lambda^+)^q$, $(1-(T_\Lambda^-)^q)^{\varphi_\delta} \geq (1-{}^{\delta}T_\Lambda^q)^{\varphi_\delta} \geq (1-(T_\Lambda^+)^q)^{\varphi_\delta}$, $\prod_{\delta=1}^{n}\left(1-(T_\Lambda^-)^q\right)^{\varphi_\delta} \geq \prod_{\delta=1}^{n}\left(1-{}^{\delta}T_\Lambda^q\right)^{\varphi_\delta} \geq \prod_{\delta=1}^{n}\left(1-(T_\Lambda^+)^q\right)^{\varphi_\delta}$, $1-\prod_{\delta=1}^{n}\left(1-(T_\Lambda^-)^q\right)^{\varphi_\delta} \leq 1-\prod_{\delta=1}^{n}\left(1-{}^{\delta}T_\Lambda^q\right)^{\varphi_\delta} \leq$

$1-\prod_{\delta=1}^{n}\left(1-(T_\Lambda^{+})^q\right)^{\varphi_\delta}$, $\left(1-\prod_{\delta=1}^{n}(1-(T_\Lambda^{-})^q)^{\varphi_\delta}\right)^{\frac{1}{q}} \leq \left(1-\prod_{\delta=1}^{n}(1-{}^{\delta}T_\Lambda^{q})^{\varphi_\delta}\right)^{\frac{1}{q}} \leq \left(1-\prod_{\delta=1}^{n}(1-(T_\Lambda^{+})^q)^{\varphi_\delta}\right)^{\frac{1}{q}}$. Thus, $T_\Lambda^{-} \leq \left(1-\prod_{\delta=1}^{n}(1-{}^{\delta}T_\Lambda^{q})^{\varphi_\delta}\right)^{\frac{1}{q}} \leq T_\Lambda^{+}$. Similarly, as $\mu^{-} \leq^{\delta} \mu \leq \mu^{+}$, we have $\mu^{-} \leq \left(1-\prod_{\delta=1}^{n}(1-\mu^q)^{\varphi_\delta}\right)^{\frac{1}{q}} \leq \mu^{+}$,

As $I_\Lambda^{-} \leq {}^{\delta}I_\Lambda \leq I_\Lambda^{+}$, then $(I_\Lambda^{-})^{\varphi_\delta} \leq ({}^{\delta}I_\Lambda)^{\varphi_\delta} \leq (I_\Lambda^{+})^{\varphi_\delta}$, $\prod_{\delta=1}^{n}(I_\Lambda^{-})^{\varphi_\delta} \leq \prod_{\delta=1}^{n}({}^{\delta}I_\Lambda)^{\varphi_\delta} \leq \prod_{\delta=1}^{n}(I_\Lambda^{+})^{\varphi_\delta}$. Thus, $I_\Lambda^{-} \leq \prod_{\delta=1}^{n}({}^{\delta}I_\Lambda)^{\varphi_\delta} \leq I_\Lambda^{+}$.

In the same way, as $F_\Lambda^{-} \leq {}^{\delta}F_\Lambda \leq F_\Lambda^{+}$, $\nu_\Lambda^{-} \leq {}^{\delta}\nu_\Lambda \leq \nu_\Lambda^{+}$ and $\omega_\Lambda^{-} \leq {}^{\delta}\omega_\Lambda \leq \omega_\Lambda^{+}$, we obtain $F_\Lambda^{-} \leq \prod_{\delta=1}^{n}({}^{\delta}F_\Lambda)^{\varphi_\delta} \leq F_\Lambda^{+}$, $\nu_\Lambda^{-} \leq \prod_{\delta=1}^{n}({}^{\delta}\nu_\Lambda)^{\varphi_\delta} \leq \nu_\Lambda^{+}$ and $\omega_\Lambda^{-} \leq \prod_{\delta=1}^{n}({}^{\delta}\omega_\Lambda)^{\varphi_\delta} \leq \omega_\Lambda^{+}$.

Now, let $T-SLDFWA(\Omega_1,\Omega_2,...,\Omega_n) = \Omega = (\langle T_\Lambda, I_\Lambda, F_\Lambda\rangle, \langle \mu,\nu,\omega\rangle)$. Then,

$\kappa(\Omega) = \frac{1}{2}[(T_\Lambda - I_\Lambda - F_\Lambda) + (\mu^q - \nu^q - \omega^q)] \geq \frac{1}{2}[(T_\Lambda^{-} - I_\Lambda^{+} - F_\Lambda^{+}) + ((\mu^{-})^q - (\nu^{+})^q - (\omega^{+})^q)] = \kappa(\Omega^{-})$ and $\kappa(\Omega) = \frac{1}{2}[(T_\Lambda - I_\Lambda - F_\Lambda) + (\mu^q - \nu^q - \omega^q)] \leq \frac{1}{2}[(T_\Lambda^{+} - I_\Lambda^{-} - F_\Lambda^{-}) + ((\mu^{+})^q - (\nu^{-})^q - (\omega^{-})^q)] = \kappa(\Omega^{+})$.
This implies $\Omega^{-} \leq T-SLDFWA(\Omega_1,\Omega_2,...,\Omega_n) \leq \Omega^{+}$.

(3) Since ${}^{\delta}T_\Lambda \leq {}^{\delta}T_\Lambda^{*}$, $\forall\delta = 1,2,...,n$. Then,

${}^{\delta}T_\Lambda^{q} \leq ({}^{\delta}T_\Lambda^{*})^q$, $1-{}^{\delta}T_\Lambda^{q} \geq 1-({}^{\delta}T_\Lambda^{*})^q$, $(1-{}^{\delta}T_\Lambda^{q})^{\varphi_\delta} \geq (1-({}^{\delta}T_\Lambda^{*})^q)^{\varphi_\delta}$, $\prod_{\delta=1}^{n}\left(1-{}^{\delta}T_\Lambda^{q}\right)^{\varphi_\delta} \geq \prod_{\delta=1}^{n}\left(1-({}^{\delta}T_\Lambda^{*})^q\right)^{\varphi_\delta}$, $1-\prod_{\delta=1}^{n}\left(1-{}^{\delta}T_\Lambda^{q}\right)^{\varphi_\delta} \leq 1-\prod_{\delta=1}^{n}\left(1-({}^{\delta}T_\Lambda^{*})^q\right)^{\varphi_\delta}$, $\left(1-\prod_{\delta=1}^{n}\left(1-{}^{\delta}T_\Lambda^{q}\right)^{\varphi_\delta}\right)^{\frac{1}{q}} \leq \left(1-\prod_{\delta=1}^{n}\left(1-({}^{\delta}T_\Lambda^{*})^q\right)^{\varphi_\delta}\right)^{\frac{1}{q}}$. Similarly, as ${}^{\delta}\mu \leq {}^{\delta}\mu^{*}$, $\forall\delta = 1,2,...,n$, we have $\left(1-\prod_{\delta=1}^{n}\left(1-{}^{\delta}\mu_\Lambda^{q}\right)^{\varphi_\delta}\right)^{\frac{1}{q}} \leq \left(1-\prod_{\delta=1}^{n}\left(1-({}^{\delta}\mu_\Lambda^{*})^q\right)^{\varphi_\delta}\right)^{\frac{1}{q}}$.

As ${}^{\delta}I_\Lambda \geq {}^{\delta}I_\Lambda^{*}$, $\forall\delta = 1,2,...,n$. Then, $({}^{\delta}I_\Lambda)^{\varphi_\delta} \geq ({}^{\delta}I_\Lambda^{*})^{\varphi_\delta}$, $\prod_{\delta=1}^{n}({}^{\delta}I_\Lambda)^{\varphi_\delta} \geq \prod_{\delta=1}^{n}({}^{\delta}I_\Lambda^{*})^{\varphi_\delta}$. In the same way as ${}^{\delta}F_\Lambda \geq {}^{\delta}F_\Lambda^{*}$, ${}^{\delta}\nu_\Lambda \geq {}^{\delta}\nu_\Lambda^{*}$, ${}^{\delta}\omega_\Lambda \geq {}^{\delta}\omega_\Lambda^{*}$, we obtain $\prod_{\delta=1}^{n}({}^{\delta}F_\Lambda)^{\varphi_\delta} \geq \prod_{\delta=1}^{n}({}^{\delta}F_\Lambda^{*})^{\varphi_\delta}$, $\prod_{\delta=1}^{n}({}^{\delta}\nu_\Lambda)^{\varphi_\delta} \geq \prod_{\delta=1}^{n}({}^{\delta}\nu_\Lambda^{*})^{\varphi_\delta}$ and $\prod_{\delta=1}^{n}({}^{\delta}\omega_\Lambda)^{\varphi_\delta} \geq \prod_{\delta=1}^{n}({}^{\delta}\omega_\Lambda^{*})^{\varphi_\delta}$.

Let $T-SLDFWA(\Omega_1,\Omega_2,...,\Omega_n) = \Omega = (\langle T_\Lambda, I_\Lambda, F_\Lambda\rangle, \langle \mu,\nu,\omega\rangle)$ and $T-SLDFWA(\Omega_1^{*},\Omega_2^{*},...,\Omega_n^{*}) = \Omega^{*} = (\langle T_\Lambda^{*}, I_\Lambda^{*}, F_\Lambda^{*}\rangle, \langle \mu^{*},\nu^{*},\omega^{*}\rangle)$. Then,

$\kappa(\Omega) = \frac{1}{2}[(T_\Lambda - I_\Lambda - F_\Lambda) + (\mu^q - \nu^q - \omega^q)] \leq \frac{1}{2}[(T_\Lambda^{*} - I_\Lambda^{*} - F_\Lambda^{*}) + ((\mu^{*})^q - (\nu^{*})^q - (\omega^{*})^q)] = \kappa(\Omega^{*})$.
This implies $T-SLDFWA(\Omega_1,\Omega_2,...,\Omega_n) \leq T-SLDFWA(\Omega_1^{*},\Omega_2^{*},...,\Omega_n^{*})$.

□

In this part, we define the T-SLDFOWA operator.

Definition 4.1.3. Let $\Omega_\delta = \{(\langle {}^\delta T_\Lambda, {}^\delta I_\Lambda, {}^\delta F_\Lambda\rangle, \langle {}^\delta\mu, {}^\delta\nu, {}^\delta\omega\rangle) : \delta = 1, ..., n\}$ be a collection of T-SLDFNs on the fixed set X and the weight vector $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ such that $\varphi_\delta \geq 0$ $(\delta \in N)$ with $\sum_{\delta=1}^{n} \varphi_\delta = 1$ and $q \geq 1$; then, the mapping $T - SLDFOWA : T - SLDFN(X) \longrightarrow T - SLDFN(X)$ is called the T-SLDFOWA operator and defined as

$$T-SLDFOWA(\Omega_1, \Omega_2, ..., \Omega_n) = \prod_{\delta=1}^{n}(\varphi_\delta \Omega_{\delta(\varepsilon)}) = \Big\langle \Big(1-\prod_{\delta=1}^{n}(1-{}^\delta T^q_{\Lambda(\varepsilon)})^{\varphi_\delta}\Big)^{\frac{1}{q}}, \prod_{\delta=1}^{n}({}^\delta I_{\Lambda(\varepsilon)})^{\varphi_\delta}, \prod_{\delta=1}^{n}({}^\delta F_{\Lambda(\varepsilon)})^{\varphi_\delta}$$
$$\Big\rangle, \Big\langle \Big(1-\prod_{\delta=1}^{n}(1-{}^\delta\mu^q_{(\varepsilon)})^{\varphi_\delta}\Big)^{\frac{1}{q}}, \prod_{\delta=1}^{n}({}^\delta\nu_{(\varepsilon)})^{\varphi_\delta}, \prod_{\delta=1}^{n}({}^\delta\omega_{(\varepsilon)})^{\varphi_\delta}\Big\rangle, q \geq 1, \tag{2}$$

where $\varepsilon(1), \varepsilon(2), ..., \varepsilon(n)$ is the arrangement of $(\delta \in N)$, for which $\Omega_{\varepsilon(\delta-1)} \geq \Omega_{\varepsilon(\delta)}, \ \forall(\delta \in N)$.

Here, we examine the traits of the T-SLDFOWA operator.

(1) Idempotency : If $\Omega_\delta = \{(\langle {}^\delta T_\Lambda, {}^\delta I_\Lambda, {}^\delta F_\Lambda\rangle, \langle {}^\delta\mu, {}^\delta\nu, {}^\delta\omega\rangle) : \delta = 1, ..., n\}$ is a set of T-SLDFNs and $\Omega_\delta = \Omega = (\langle T_\Lambda, I_\Lambda, F_\Lambda\rangle, \langle\mu, \nu, \omega\rangle)$, $\forall\delta \in N$. Then, $T - SLDFOWA(\Omega_1, \Omega_2, ..., \Omega_n) = \Omega = (\langle T_\Lambda, I_\Lambda, F_\Lambda\rangle, \langle\mu, \nu, \omega\rangle)$.

(2) Boundedness: Let $\Omega_\delta = \{(\langle {}^\delta T_\Lambda, {}^\delta I_\Lambda, {}^\delta F_\Lambda\rangle, \langle {}^\delta\mu, {}^\delta\nu, {}^\delta\omega\rangle) : \delta = 1, ..., n\}$ be a set of T-SLDFNs. If $\Omega^- = (\langle T^-_\Lambda, I^+_\Lambda, F^+_\Lambda\rangle, \langle\mu^-, \nu^+, \omega^+\rangle)$ and $\Omega^+ = (\langle T^+_\Lambda, I^-_\Lambda, F^-_\Lambda\rangle, \langle\mu^+, \nu^-, \omega^-\rangle)$, where, $T^-_\Lambda = \min_\delta\{{}^\delta T_\Lambda\}$, $I^+_\Lambda = \max_\delta\{{}^\delta I_\Lambda\}$, $F^+_\Lambda = \max_\delta\{{}^\delta F_\Lambda\}$, $\mu^- = \min_\delta\{{}^\delta\mu\}$, $\nu^+ = \max_\delta\{{}^\delta\nu\}$, $\omega^+ = \max_\delta\{{}^\delta\omega\}$ and $T^+_\Lambda = \max_\delta\{{}^\delta T_\Lambda\}$, $I^-_\Lambda = \min_\delta\{{}^\delta I_\Lambda\}$, $F^-_\Lambda = \min_\delta\{{}^\delta F_\Lambda\}$, $\mu^+ = \max_\delta\{{}^\delta\mu\}$, $\nu^- = \min_\delta\{{}^\delta\nu\}$, $\omega^- = \min_\delta\{{}^\delta\omega\}$. Then,
$\Omega^- \leq T - SLDFOWA(\Omega_1, \Omega_2, ..., \Omega_n) \leq \Omega^+$.

(3) Monotonicity: Let $\Omega_\delta = \{(\langle {}^\delta T_\Lambda, {}^\delta I_\Lambda, {}^\delta F_\Lambda\rangle, \langle {}^\delta\mu, {}^\delta\nu, {}^\delta\omega\rangle) : \delta = 1, ..., n\}$ and $\Omega^*_\delta = \{(\langle {}^\delta T^*_\Lambda, {}^\delta I^*_\Lambda, {}^\delta F^*_\Lambda\rangle,$ $\langle {}^\delta\mu^*, {}^\delta\nu^*, {}^\delta\omega^*\rangle) : \delta \in N\}$ be two collections of T-SLDFNs. If ${}^\delta T_\Lambda \leq {}^\delta T^*_\Lambda$, ${}^\delta I_\Lambda \geq {}^\delta I^*_\Lambda$, ${}^\delta F_\Lambda \geq {}^\delta F^*_\Lambda$ and ${}^\delta\mu \leq {}^\delta\mu^*$, ${}^\delta\nu \geq {}^\delta\nu^*$, ${}^\delta\omega \geq {}^\delta\omega^*$, $\forall\delta \in N$. Then, $T - SLDFOWA(\Omega_1, \Omega_2, ..., \Omega_n) \leq T - SLDFOWA(\Omega^*_1, \Omega^*_2, ..., \Omega^*_n)$.

Next, we define the T-SLDFHWA operator.

Definition 4.1.4. Let $\Omega_\delta = \{(\langle {}^\delta T_\Lambda, {}^\delta I_\Lambda, {}^\delta F_\Lambda\rangle, \langle {}^\delta\mu, {}^\delta\nu, {}^\delta\omega\rangle) : \delta = 1, ..., n\}$ be the collection of T-SLDFNs on the reference set X and the weight vector $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ such that $\varphi_\delta \geq 0$ $(\delta \in N)$ with $\sum_{\delta=1}^{n} \varphi_\delta = 1$ and $q \geq 1$; then the mapping $T - SLDFHWA : T - SLDFN(X) \longrightarrow T - SLDFN(X)$ is called the T-SLDFHWA operator and defined as

$$T - SLDFHWA(\Omega_1, \Omega_2, ..., \Omega_n) = \prod_{\delta=1}^{n}(\varphi_\delta \Omega^\diamond_{\varepsilon(\delta)}) =$$
$$\Big\langle \Big(1 - \prod_{\delta=1}^{n}(1 - {}^{\varepsilon(\delta)}T^{q^\diamond}_\Lambda)^{\varphi_\delta}\Big)^{\frac{1}{q}}, \prod_{\delta=1}^{n}({}^{\varepsilon(\delta)}I_\Lambda)^{\diamond\varphi_\delta}, \prod_{\delta=1}^{n}({}^{\varepsilon(\delta)}F_\Lambda)^{\diamond\varphi_\delta}\Big\rangle, \Big\langle \Big(1 - \prod_{\delta=1}^{n}(1 - {}^{\varepsilon(\delta)}\mu^{q^\diamond})^{\varphi_\delta}\Big)^{\frac{1}{q}}, \prod_{\delta=1}^{n}({}^{\varepsilon(\delta)}\nu)^{\diamond\varphi_\delta},$$
$$\prod_{\delta=1}^{n}({}^{\varepsilon(\delta)}\omega)^{\diamond\varphi_\delta}\Big\rangle, q \geq 1, \tag{3}$$

where $\Omega^{\diamond}_{\varepsilon(\delta)}$ represents the δ_{th} biggest weighted T-spherical linear Diophantine fuzzy values $\Omega^{\diamond}_{\delta}(\Omega^{\diamond}_{\delta} = (\Omega_{\delta})^{n\varphi_{\delta}}, \delta \in N)$ and $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ is the weight vector by mean of $\varphi_{\delta} \geq 0$ $(\delta \in N)$ with $\sum_{\delta=1}^{n} \varphi_{\delta} = 1$.

If $\varphi = (\frac{1}{\varphi}, \frac{1}{\varphi}, ..., \frac{1}{\varphi})$, then T-SLDFWA and T-SLDFOWA operators are considered to be specific cases of T-SLDFHWA. Thus, we conclude that the generalized form of T-SLDFWA and T-SLDFOWA operators is the T-SLDFHWA operator.

We next discuss the properties of the T-SLDFHWA operator.

(1) Idempotency : If Ω_{δ} is a set of T-SLDFNs and $\Omega_{\delta} = \Omega = (\langle T_{\Lambda}, I_{\Lambda}, F_{\Lambda}\rangle, \langle \mu, \nu, \omega\rangle)$, $\forall \delta \in N$. Then, $T-SLDFHWA(\Omega_1, \Omega_2, ..., \Omega_n) = \Omega$.

(2) Boundedness: Let $\Omega_{\delta} = \{(\langle {}^{\delta}T_{\Lambda}, {}^{\delta}I_{\Lambda}, {}^{\delta}F_{\Lambda}\rangle, \langle {}^{\delta}\mu, {}^{\delta}\nu, {}^{\delta}\omega\rangle) : \delta = 1, ..., n\}$ be a set of T-SLDFNs. If $\Omega^{-} = (\langle T^{-}_{\Lambda}, I^{+}_{\Lambda}, F^{+}_{\Lambda}\rangle, \langle \mu^{-}, \nu^{+}, \omega^{+}\rangle)$ and $\Omega^{+} = (\langle T^{+}_{\Lambda}, I^{-}_{\Lambda}, F^{-}_{\Lambda}\rangle, \langle \mu^{+}, \nu^{-}, \omega^{-}\rangle)$, where, $T^{-}_{\Lambda} = \min_{\delta}\{{}^{\delta}T_{\Lambda}\}$, $I^{+}_{\Lambda} = \max_{\delta}\{{}^{\delta}I_{\Lambda}\}$, $F^{+}_{\Lambda} = \max_{\delta}\{{}^{\delta}F_{\Lambda}\}$, $\mu^{-} = \min_{\delta}\{{}^{\delta}\mu\}$, $\nu^{+} = \max_{\delta}\{{}^{\delta}\nu\}$, $\omega^{+} = \max_{\delta}\{{}^{\delta}\omega\}$ and $T^{+}_{\Lambda} = \max_{\delta}\{{}^{\delta}T_{\Lambda}\}$, $I^{-}_{\Lambda} = \min_{\delta}\{{}^{\delta}I_{\Lambda}\}$, $F^{-}_{\Lambda} = \min_{\delta}\{{}^{\delta}F_{\Lambda}\}$, $\mu^{+} = \max_{\delta}\{{}^{\delta}\mu\}$, $\nu^{-} = \min_{\delta}\{{}^{\delta}\nu\}$, $\omega^{-} = \min_{\delta}\{{}^{\delta}\omega\}$. Then,
$\Omega^{-} \leq T-SLDFHWA(\Omega_1, \Omega_2, ..., \Omega_n) \leq \Omega^{+}$.

(3) Monotonicity: Let $\Omega_{\delta}$and Ω^{*}_{δ} be two collections of T-SLDFNs. If $\Omega_{\delta} \leq \Omega^{*}_{\delta}$, $\forall \delta \in N$. Then, $T-SLDFHWA(\Omega_1, \Omega_2, ..., \Omega_n) \leq T-SLDFHWA(\Omega^{*}_1, \Omega^{*}_2, ..., \Omega^{*}_n)$.

4.2. T-SLDFWG operator

We define the T-SLDFWG operator, T-SLDFOWG operator, and T-SLDFHWG operator as follows.

Definition 4.2.1. Let $\Omega_{\delta} = \{(\langle {}^{\delta}T_{\Lambda}, {}^{\delta}I_{\Lambda}, {}^{\delta}F_{\Lambda}\rangle, \langle {}^{\delta}\mu, {}^{\delta}\nu, {}^{\delta}\omega\rangle) : \delta = 1, ..., n\}$ be a collection of T-SLDFNs. The T-SLDFWG operator is a transformation T-SLDFWG: T-SLDFN(X)$\longrightarrow$ T-SLDFN(X), defined by

$$T-SLDFWG(\Omega_1, \Omega_2, ..., \Omega_n) = \Omega_1^{\varphi_1} \otimes \Omega_2^{\varphi_2} \otimes ... \otimes \Omega_n^{\varphi_n},$$

where $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ is the weight vector of $\Omega_{\delta}(\delta = 1, 2, ..., n)$, $0 \leq \varphi_{\delta} \leq 1$ and $\sum_{\delta=1}^{n} \varphi_{\delta} = 1$.

Theorem 4.2.2. Suppose that $\Omega_{\delta} = \{(\langle {}^{\delta}T_{\Lambda}, {}^{\delta}I_{\Lambda}, {}^{\delta}F_{\Lambda}\rangle, \langle {}^{\delta}\mu, {}^{\delta}\nu, {}^{\delta}\omega\rangle) : \delta = 1, ..., n\}$ is the collection of T-SLDFNs. Let us consider the weight vector $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ of Ω_{δ}. Then,

$$T-SLDFWG(\Omega_1, \Omega_2, ..., \Omega_n) = \Big\langle \prod_{\delta=1}^{n} ({}^{\delta}T_{\Lambda})^{\varphi_{\delta}}, \Big(1 - \prod_{\delta=1}^{n} (1 -{}^{\delta} I^{q}_{\Lambda})^{\varphi_{\delta}}\Big)^{\frac{1}{q}}, \Big(1 - \prod_{\delta=1}^{n} (1 -{}^{\delta} F^{q}_{\Lambda})^{\varphi_{\delta}}\Big)^{\frac{1}{q}} \Big\rangle, \Big\langle \prod_{\delta=1}^{n} ({}^{\delta}\mu)^{\varphi_{\delta}}, \Big(1 - \prod_{\delta=1}^{n} (1 -{}^{\delta} \nu^{q})^{\varphi_{\delta}}\Big)^{\frac{1}{q}}, \Big(1 - \prod_{\delta=1}^{n} (1 -{}^{\delta} \omega^{q})^{\varphi_{\delta}}\Big)^{\frac{1}{q}}, \Big\rangle, q \geq 1. \tag{4}$$

Proof. The proof is like that of Theorem 4.1.2. □

Similar to T-SLDFWA operator, the T-SLDFWG operator also possess the certain characteristics which are stated (without proof) as follows.

(1) Idempotency

Let $\Omega_{\delta} = \{(\langle {}^{\delta}T_{\Lambda}, {}^{\delta}I_{\Lambda}, {}^{\delta}F_{\Lambda}\rangle, \langle {}^{\delta}\mu, {}^{\delta}\nu, {}^{\delta}\omega\rangle) : \delta = 1, ..., n\}$ be a set of T-SLDFNs. If $\Omega_{\delta} = \Omega = (\langle T_{\Lambda}, I_{\Lambda}, F_{\Lambda}\rangle, \langle \mu, \nu, \omega\rangle)$, $\forall \delta = 1, ..., n$. Then, $T-SLDFWG(\Omega_1, \Omega_2, ..., \Omega_n) = \Omega = (\langle T_{\Lambda}, I_{\Lambda}, F_{\Lambda}\rangle, \langle \mu, \nu, \omega\rangle)$.

(2) Boundedness

Let $\Omega_\delta = \{(\langle {}^\delta T_\Lambda, {}^\delta I_\Lambda, {}^\delta F_\Lambda\rangle, \langle {}^\delta\mu, {}^\delta\nu, {}^\delta\omega\rangle) : \delta = 1, ..., n\}$ be a set of T-SLDFNs. If $\Omega^- = (\langle T_\Lambda^-, I_\Lambda^+, F_\Lambda^+\rangle, \langle \mu^-, \nu^+, \omega^+\rangle)$ and $\Omega^+ = (\langle T_\Lambda^+, I_\Lambda^-, F_\Lambda^-\rangle, \langle \mu^+, \nu^-, \omega^-\rangle)$, where, $T_\Lambda^- = \min_\delta\{{}^\delta T_\Lambda\}$, $I_\Lambda^+ = \max_\delta\{{}^\delta I_\Lambda\}$, $F_\Lambda^+ = \max_\delta\{{}^\delta F_\Lambda\}$, $\mu^- = \min_\delta\{{}^\delta\mu\}$, $\nu^+ = \max_\delta\{{}^\delta\nu\}$, $\omega^+ = \max_\delta\{{}^\delta\omega\}$ and $T_\Lambda^+ = \max_\delta\{{}^\delta T_\Lambda\}$, $I_\Lambda^- = \min_\delta\{{}^\delta I_\Lambda\}$, $F_\Lambda^- = \min_\delta\{{}^\delta F_\Lambda\}$, $\mu^+ = \max_\delta\{{}^\delta\mu\}$, $\nu^- = \min_\delta\{{}^\delta\nu\}$, $\omega^- = \min_\delta\{{}^\delta\omega\}$. Then,

$\Omega^- \leq T - SLDFWG(\Omega_1, \Omega_2, ..., \Omega_n) \leq \Omega^+$.

(3) Monotonicity

Let $\Omega_\delta = \{(\langle {}^\delta T_\Lambda, {}^\delta I_\Lambda, {}^\delta F_\Lambda\rangle, \langle {}^\delta\mu, {}^\delta\nu, {}^\delta\omega\rangle) : \delta = 1, ..., n\}$ and $\Omega_\delta^* = \{(\langle {}^\delta T_\Lambda^*, {}^\delta I_\Lambda^*, {}^\delta F_\Lambda^*\rangle, \langle {}^\delta\mu^*, {}^\delta\nu^*, {}^\delta\omega^*\rangle) : \delta = 1, ..., n\}$ be two collections of T-SLDFNs. If ${}^\delta T_\Lambda \leq {}^\delta T_\Lambda^*$, ${}^\delta I_\Lambda \geq {}^\delta I_\Lambda^*$, ${}^\delta F_\Lambda \geq {}^\delta F_\Lambda^*$ and ${}^\delta\mu \leq {}^\delta\mu^*$, ${}^\delta\nu \geq {}^\delta\nu^*$, ${}^\delta\omega \geq {}^\delta\omega^*$ $\forall\delta = 1, ..., n$. Then, $T - SLDFWG(\Omega_1, \Omega_2, ..., \Omega_n) \leq T - SLDFWG(\Omega_1^*, \Omega_2^*, ..., \Omega_n^*)$.

Now, we define the T-SLDFOWG operator.

Definition 4.2.3. Let $\Omega_\delta = \{(\langle {}^\delta T_\Lambda, {}^\delta I_\Lambda, {}^\delta F_\Lambda\rangle, \langle {}^\delta\mu, {}^\delta\nu, {}^\delta\omega\rangle) : \delta = 1, ..., n\}$ be the collection of T-SLDFNs on the reference set X and the weight vector $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ such that $\varphi_\delta \geq 0$ $(\forall\delta \in N)$ with $\sum_{\delta=1}^{n} \varphi_\delta = 1$ and $q \geq 1$; then, the mapping $T - SLDFOWG : T - SLDFN(X) \longrightarrow T - SLDFN(X)$ is called the T-SLDFOWG operator and defined as

$$T - SLDFOWG(\Omega_1, \Omega_2, ..., \Omega_n) = \prod_{\delta=1}^{n} \Omega_{\delta(\varepsilon)}^{\varphi_\delta} =$$

$$\Big\langle \prod_{\delta=1}^{n} ({}^\delta T_{\Lambda(\varepsilon)})^{\varphi_\delta}, \Big(1 - \prod_{\delta=1}^{n} (1 - {}^\delta I_{\Lambda(\varepsilon)}^q)^{\varphi_\delta}\Big)^{\frac{1}{q}}, \Big(1 - \prod_{\delta=1}^{n} (1 - {}^\delta F_{\Lambda(\varepsilon)}^q)^{\varphi_\delta}\Big)^{\frac{1}{q}} \Big\rangle, \Big\langle \prod_{\delta=1}^{n} ({}^\delta\mu_{(\varepsilon)})^{\varphi_\delta}, \Big(1 - \prod_{\delta=1}^{n} (1 - {}^\delta\nu_{(\varepsilon)}^q)^{\varphi_\delta}\Big)^{\frac{1}{q}}, \Big(1 - \prod_{\delta=1}^{n} (1 - {}^\delta\omega_{(\varepsilon)}^q)^{\varphi_\delta}\Big)^{\frac{1}{q}}, \Big\rangle, q \geq 1, \tag{5}$$

where $\varepsilon(1), \varepsilon(2), ..., \varepsilon(n)$ is the arrangement of $(\delta \in N)$, for which $\Omega_{\varepsilon(\delta-1)} \geq \Omega_{\varepsilon(\delta)}$, $\forall(\delta \in N)$.

Remark 4.2.4. The T-SLDFOWG operator satisfies the similar properties as those of T-SLDFOWA operator.

The following is the definition of the T-SLDFHWG operator.

Definition 4.2.5. Let $\Omega_\delta = \{(\langle {}^\delta T_\Lambda, {}^\delta I_\Lambda, {}^\delta F_\Lambda\rangle, \langle {}^\delta\mu, {}^\delta\nu, {}^\delta\omega\rangle) : \delta = 1, ..., n\}$ be the collection of T-SLDFNs on the reference set X and the weight vector $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ such that $\varphi_\delta \geq 0$ $(\forall\delta \in N)$ with $\sum_{\delta=1}^{n} \varphi_\delta = 1$ and $q \geq 1$; then, the mapping $T - SLDFHWG : T - SLDFN(X) \longrightarrow T - SLDFN(X)$ is called the T-SLDFHWG operator and defined as

$$T - SLDFHWG(\Omega_1, \Omega_2, ..., \Omega_n) = \prod_{\delta=1}^{n} \Omega_{\varepsilon(\delta)}^{\diamond\varphi_\delta} =$$

$$\Big\langle \prod_{\delta=1}^{n} ({}^{\varepsilon(\delta)} T_\Lambda)^{\diamond\varphi_\delta}, \Big(1 - \prod_{\delta=1}^{n} (1 - {}^{\varepsilon(\delta)} I_\Lambda^{q^\diamond})^{\varphi_\delta}\Big)^{\frac{1}{q}}, \Big(1 - \prod_{\delta=1}^{n} (1 - {}^{\varepsilon(\delta)} F_\Lambda^{q^\diamond})^{\varphi_\delta}\Big)^{\frac{1}{q}} \Big\rangle, \Big\langle \prod_{\delta=1}^{n} ({}^{\varepsilon(\delta)}\mu)^{\diamond\varphi_\delta}, \Big(1 - \prod_{\delta=1}^{n} (1 - {}^{\varepsilon(\delta)}\nu^{q^\diamond})^{\varphi_\delta}\Big)^{\frac{1}{q}}, \Big(1 - \prod_{\delta=1}^{n} (1 - {}^{\varepsilon(\delta)}\omega^{q^\diamond})^{\varphi_\delta}\Big)^{\frac{1}{q}}, \Big\rangle, q \geq 1, \tag{6}$$

where $\Omega^{\diamond}_{\varepsilon(\delta)}$ is the δ_{th} biggest weighted, T-spherical linear Diophantine fuzzy values $\Omega^{\diamond}_{\delta}(\Omega^{\diamond}_{\delta} = (\Omega_{\delta})^{n\varphi_{\delta}}, \delta \in N)$ and $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ is the weight vector by mean of $\varphi_{\delta} \geq 0$ $(\forall \delta \in N)$ with $\sum_{\delta=1}^{n} \varphi_{\delta} = 1$.

If $\varphi = (\frac{1}{\varphi}, \frac{1}{\varphi}, ..., \frac{1}{\varphi})$, then T-SLDFWG and T-SLDFOWG operators are considered to be specific cases of the T-SLDFHWG operator. Thus, we conclude that the generalized form of T-SLDFWG and T-SLDFOWG operators is the T-SLDFHWG operator.

Remark 4.2.6. The T-SLDFHWG operator satisfies the similar properties as those of T-SLDFHWA operator.

5. MADM approach in a T-SLDF environment

To adeptly highlight the reliability and usefulness of the proposed work, we display a MADM problem in terms of T-SLDFNs by using T-SLDFWA and T-SLDFWG operators. For this, let $P = \{P_1, P_2, ..., P_n\}$ be a family of n alternatives, $G = \{G_1, G_2, ..., G_k\}$ be a family of attributes and $\varphi = (\varphi_1, \varphi_2, ..., \varphi_k)$ be the weight vector of the attributes where $\sum_{\delta=1}^{n} \varphi_{\delta} = 1$ and $\varphi_{\delta} \geq 0$. The detailed process of the decision making is displayed as follows.

Algorithm 1:

Step 1: The value of the alternative P_i corresponding to attribute G_j is stated by the decision maker in terms of the T-SLDFNs and summarized as a matrix called the decision matrix.

Step 2: Normalization: Generally, there are two attribute types of MADM problems: Cost-type and benefit-type. To maintain consistency of the types, it is necessary to normalize the input data as follows.

$$\Omega_{\delta} = \begin{cases} (\langle {}^{\delta}T_{\Lambda}, {}^{\delta}I_{\Lambda}, {}^{\delta}F_{\Lambda}\rangle, \langle {}^{\delta}\mu, {}^{\delta}\nu, {}^{\delta}\omega\rangle) & \text{for beneficial types} \\ (\langle {}^{\delta}F_{\Lambda}, 1 - {}^{\delta}I_{\Lambda}, {}^{\delta}T_{\Lambda}\rangle, \langle {}^{\delta}\omega, {}^{\delta}\nu, {}^{\delta}\mu\rangle) & \text{for cost types} \end{cases} \tag{7}$$

Step 3: Aggregate all the attribute values denoted by L_i using the T-SLDFWA operator or T-SLDFWG operator.

Step 4: Calculating the score values for the aggregated alternatives values, by using Definition 3.1.1.

Step 5: Ranking and sorting the alternatives based on their scores and selecting the best choice.

6. Illustrative example

To exhibit the practicality of the proposed methods, we carry out the following study about the ranking of several kinds of a certain product.

Example 6.1. Assume that a company intends to examine four kinds of a certain product from a manufacturer to choose the most suitable one. Let $P = \{P_1, P_2, P_3, P_4\}$ be a universe consisting of four kinds of products. The proposed methods are employed to evaluate these products based on the following features: G_1 : *Price*, G_2 : *Quality*, G_3 : *Aesthetics*, G_4 : *Functionality* and G_5 : *Easy to use*. For this, the weights for these features are as follows: 0.3, 0.3, 0.2, 0.1, 0.1.

Assume that the characteristics of the alternative P_i under attribute G_j are expressed by T-SLDFNs with $q = 3$. Additionally, we consider the reference parameters μ : the degree of attractiveness, ν : the indeterminate degree of attractiveness, and ω: the degree of repulsion. Then, we can rank the products utilizing Algorithm 1 as follows.

Step 1: The input data are applied to this algorithm. The decision maker evaluates the four kinds of product, $P_i(i = 1, 2, 3, 4)$, according to five attributes, $G_j(j = 1, 2, 3, 4, 5)$, and the decision matrix is constructed as shown in Table 1.

Table 1. Original decision matrix.

Feature	P_1	P_2	P_3	P_4
G_1	$(\langle 0.6, 0.8, 0.4\rangle, \langle 0.5, 0.4, 0.1\rangle)$	$(\langle 0.5, 0.1, 0.6\rangle, \langle 0.7, 0.7, 0.6\rangle)$	$(\langle 0, 0.9, 0.8\rangle, \langle 0.2, 0.6, 0.3\rangle)$	$(\langle 0.1, 0.3, 0.9\rangle, \langle 0.9, 0.5, 0.4\rangle)$
G_2	$(\langle 0.9, 0.1, 0.6\rangle, \langle 0.6, 0.8, 0.3\rangle)$	$(\langle 0.2, 0.5, 0.4\rangle, \langle 0.6, 0.5, 0.4\rangle)$	$(\langle 0.9, 0.1, 0.2\rangle, \langle 0.8, 0.4, 0.3\rangle)$	$(\langle 0.4, 0.5, 0.7\rangle, \langle 0.2, 0.3, 0.5\rangle)$
G_3	$(\langle 0.8, 0.4, 0.4\rangle, \langle 0.2, 0.4, 0.6\rangle)$	$(\langle 0.1, 0.4, 0.2\rangle, \langle 0.7, 0.4, 0.6\rangle)$	$(\langle 0.7, 0.8, 0.9\rangle, \langle 0.8, 0.7, 0.5\rangle)$	$(\langle 0.9, 0.9, 0.8\rangle, \langle 0.5, 0.8, 0.3\rangle)$
G_4	$(\langle 0.7, 0.3, 0.5\rangle, \langle 0.7, 0.5, 0.2\rangle)$	$(\langle 0.4, 0.2, 0.9\rangle, \langle 0.2, 0.5, 0.3\rangle)$	$(\langle 0.9, 0, 0\rangle, \langle 0.9, 0.1, 0.1\rangle)$	$(\langle 0.1, 0.4, 0.9\rangle, \langle 0.1, 0.3, 0.5\rangle)$
G_5	$(\langle 0.4, 0.6, 0.5\rangle, \langle 0.3, 0.6, 0.5\rangle)$	$(\langle 0.9, 0.8, 0.7\rangle, \langle 0.8, 0.2, 0.5\rangle)$	$(\langle 0.7, 0.1, 0.3\rangle, \langle 0.4, 0.5, 0.6\rangle)$	$(\langle 0.3, 0.1, 0.7\rangle, \langle 0.5, 0.4, 0.4\rangle)$

Step 2: Obtain the normalized T-SLDF information of Table 1 by taking the complement of G_1 = *Price*, which is the cost type attribute in this example. The normalized decision matrix is shown in Table 2.

Table 2. Normalized decision matrix.

Feature	P_1	P_2	P_3	P_4
G_1	$(\langle 0.4, 0.2, 0.6\rangle, \langle 0.1, 0.4, 0.5\rangle)$	$(\langle 0.6, 0.9, 0.5\rangle, \langle 0.6, 0.7, 0.7\rangle)$	$(\langle 0.8, 0.1, 0\rangle, \langle 0.3, 0.6, 0.2\rangle)$	$(\langle 0.9, 0.7, 0.1\rangle, \langle 0.4, 0.5, 0.9\rangle)$
G_2	$(\langle 0.9, 0.1, 0.6\rangle, \langle 0.6, 0.8, 0.3\rangle)$	$(\langle 0.2, 0.5, 0.4\rangle, \langle 0.6, 0.5, 0.4\rangle)$	$(\langle 0.9, 0.1, 0.2\rangle, \langle 0.8, 0.4, 0.3\rangle)$	$(\langle 0.4, 0.5, 0.7\rangle, \langle 0.2, 0.3, 0.5\rangle)$
G_3	$(\langle 0.8, 0.4, 0.4\rangle, \langle 0.2, 0.4, 0.6\rangle)$	$(\langle 0.1, 0.4, 0.2\rangle, \langle 0.7, 0.4, 0.6\rangle)$	$(\langle 0.7, 0.8, 0.9\rangle, \langle 0.8, 0.7, 0.5\rangle)$	$(\langle 0.9, 0.9, 0.8\rangle, \langle 0.5, 0.8, 0.3\rangle)$
G_4	$(\langle 0.7, 0.3, 0.5\rangle, \langle 0.7, 0.5, 0.2\rangle)$	$(\langle 0.4, 0.2, 0.9\rangle, \langle 0.2, 0.5, 0.3\rangle)$	$(\langle 0.9, 0, 0\rangle, \langle 0.9, 0.1, 0.1\rangle)$	$(\langle 0.1, 0.4, 0.9\rangle, \langle 0.1, 0.3, 0.5\rangle)$
G_5	$(\langle 0.4, 0.6, 0.5\rangle, \langle 0.3, 0.6, 0.5\rangle)$	$(\langle 0.9, 0.8, 0.7\rangle, \langle 0.8, 0.2, 0.5\rangle)$	$(\langle 0.7, 0.1, 0.3\rangle, \langle 0.4, 0.5, 0.6\rangle)$	$(\langle 0.3, 0.1, 0.7\rangle, \langle 0.5, 0.4, 0.4\rangle)$

Step 3: Using the T-SLDFWA operator, aggregate all the attribute values for each alternative. The aggregated attributes' values are given below.

$L_1 = (\langle 0.77, 0.22, 0.53\rangle, \langle 0.48, 0.52, 0.41\rangle)$, $L_2 = (\langle 0.58, 0.55, 0.43\rangle, \langle 0.64, 0.48, 0.51\rangle)$,$L_3 = (\langle 0.78, 0, 0\rangle, \langle 0.63, 0, 0.64\rangle)$, $L_4 = (\langle 0.79, 0.52, 0.41\rangle, \langle 0.39, 0.44, 0.53\rangle)$.

Step 4: Find the score value $\kappa(L_i)$ of each of the aggregated attribute values. We obtained $\kappa(L_1) = -0.0395$, $\kappa(L_2) = -0.1905$, $\kappa(L_3) = 0.384$ and $\kappa(L_4) = -0.3147$.

Step 5: We obtained the ranks of the four alternatives as $P_3 > P_1 > P_2 > P_4$.

6.1. Result and discussion

In this part, we will analyze the results of the proposed method by setting different value q to show the sensitivity of parameter q. We will also solve the same problem by using different types of the proposed aggregation operators . Furthermore, we will show the influences of the proposed SFs on the results.

6.1.1. Sensitivity analysis

In order to better discuss the influence of different q values, in this part, we take different values for the parameter q. The ranking results are shown in Table 3.

Table 3. Ranking results based on T-SLDFWA operator by using the different q.

q	The Score Function	Ranking Results
$q = 2$	$\kappa(P_1) = -0.1188, \kappa(P_2) = -0.2677, \kappa(P_3) = 0.5557, \kappa(P_4) = -0.2444$	$P_3 > P_1 > P_4 > P_2$
$q = 3$	$\kappa(P_1) = -0.0395, \kappa(P_2) = -0.1905, \kappa(P_3) = 0.5662, \kappa(P_4) = -0.3147$	$P_3 > P_1 > P_2 > P_4$
$q = 5$	$\kappa(P_1) = 0.0159, \kappa(P_2) = -0.1372, \kappa(P_3) = 0.5352, \kappa(P_4) = -0.0784$	$P_3 > P_1 > P_4 > P_2$
$q = 8$	$\kappa(P_1) = 0.0336, \kappa(P_2) = -0.1188, \kappa(P_3) = 0.4901, \kappa(P_4) = -0.048$	$P_3 > P_1 > P_4 > P_2$
$q = 10$	$\kappa(P_1) = 0.0379, \kappa(P_2) = -0.1113, \kappa(P_3) = 0.4714, \kappa(P_4) = -0.0412$	$P_3 > P_1 > P_4 > P_2$
$q = 12$	$\kappa(P_1) = 0.0412, \kappa(P_2) = -0.1044, \kappa(P_3) = 0.4592, \kappa(P_4) = -0.0372$	$P_3 > P_1 > P_4 > P_2$
$q = 15$	$\kappa(P_1) = 0.0454, \kappa(P_2) = -0.0951, \kappa(P_3) = 0.4487, \kappa(P_4) = -0.0333$	$P_3 > P_1 > P_4 > P_2$
$q = 20$	$\kappa(P_1) = 0.0507, \kappa(P_2) = -0.0833, \kappa(P_3) = 0.4414, \kappa(P_4) = -0.0292$	$P_3 > P_1 > P_4 > P_2$

From Table 3, we can see that the aggregation results are slightly different with parameter q increasing in the T-SLDFWA operator and the ranking of the alternatives is still the same. So the method based on the T-SLDFWA operator is stable. Further, we can find that for alternative P_3 (The optimal alternative) with the increase of parameter q, the score function becomes smaller in general, while for the other alternatives the score function becomes larger with the increase of parameter q. In general, different decision makers can set different values to parameter q on the basis of their preferences.

6.1.2. Performances of the proposed operators

We tested the performances of the proposed operators by using the information from Example 6.1. The ranking results by the proposed operators are shown in Table 4.

From Table 4, we can see that there are slight differences between the ranking results derived from the proposed operators, but the best and the first choice were the same for all methods.

Table 4. Comparative analysis of the proposed operators.

Operators	Ranking	Optimal Alternative
T-SLDFWA	$P_3 > P_1 > P_2 > P_4$	P_3
T-SLDFOWA	$P_3 > P_1 > P_2 > P_4$	P_3
T-SLDFHWA	$P_3 > P_1 > P_4 > P_2$	P_3
T-SLDFWG	$P_3 > P_1 > P_2 > P_4$	P_3
T-SLDFOWG	$P_3 > P_2 > P_1 > P_4$	P_3
T-SLDFHWG	$P_3 > P_1 > P_4 > P_2$	P_3

6.1.3. Influences of the score functions

Three types of SFs were proposed in Section 3.1, namely, the SF, QSF, and ESF. For each SF, an AF has been provided to compare the T-SLDFNs. To depict the influence of the proposed SFs, we used them to rank the alternative products in Example 6.1. The ranking orders using the T-SLDFWA operator and T-SLDFWG operator are shown in Tables 5 and 6.

In Table 5, it is clear that the ranking results generated by using all the SFs are almost the same, and their optimal selections are the same.

In Table 6, we can observe that the ranking results are the same for all SFs. Additionally, it should be noted that for all SFs, the results from both operators are almost equivalent, which proves the validity of the proposed SFs very well. The column charts of the SF, QSF, and ESF values based on T-SLDFWA and T-SLDFWG operators are given in Figures 2 and 3.

Table 5. Ranking order using the T-SLDFWA operator.

Score Function	Score Values	Ranking Results
κ	$\kappa(P_1) = -0.0395, \kappa(P_2) = -0.1905, \kappa(P_3) = 0.5662, \kappa(P_4) = -0.3147$	$P_3 > P_1 > P_2 > P_4$
π	$\pi(P_1) = 0.1229, \pi(P_2) = -0.0547, \pi(P_3) = 0.4196, \pi(P_4) = 0.0803$	$P_3 > P_1 > P_4 > P_2$
Γ	$\Gamma(P_1) = 0.6532, \Gamma(P_2) = 0.6034, \Gamma(P_3) = 0.8551, \Gamma(P_4) = 0.613$	$P_3 > P_1 > P_4 > P_2$

Table 6. Ranking order using the T-SLDFWG operator.

Score Function	Score Values	Ranking Results
κ	$\kappa(P_1) = -0.3071, \kappa(P_2) = -0.6078, \kappa(P_3) = -0.1887, \kappa(P_4) = -0.7207$	$P_3 > P_1 > P_2 > P_4$
π	$\pi(P_1) = -0.058, \pi(P_2) = -0.42, \pi(P_3) = 0.0028, \pi(P_4) = -0.4558$	$P_3 > P_1 > P_2 > P_4$
Γ	$\Gamma(P_1) = 0.5639, \Gamma(P_2) = 0.4649, \Gamma(P_3) = 0.6038, \Gamma(P_4) = 0.4284$	$P_3 > P_1 > P_2 > P_4$

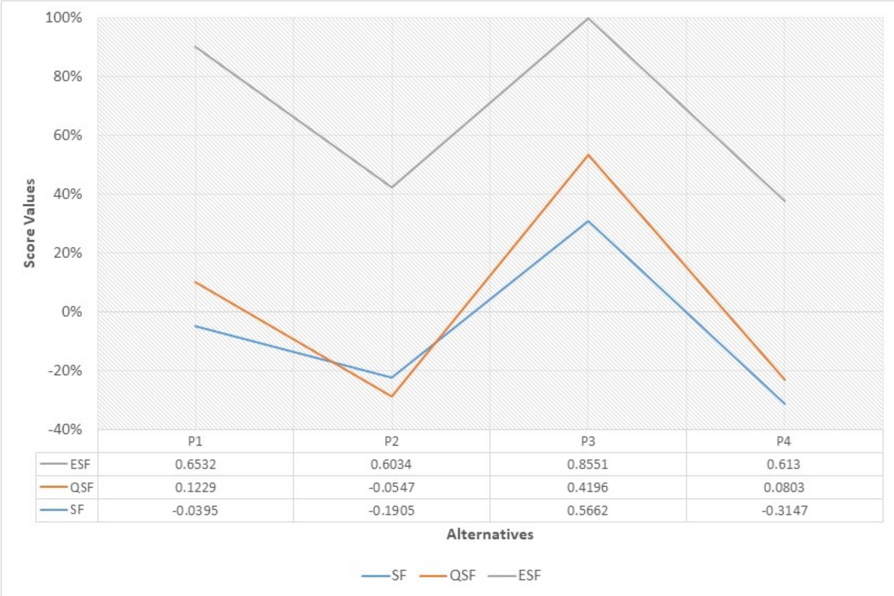


Figure 2. The comparison of SF, QSF, and ESF under the T-SLDFWA operator.

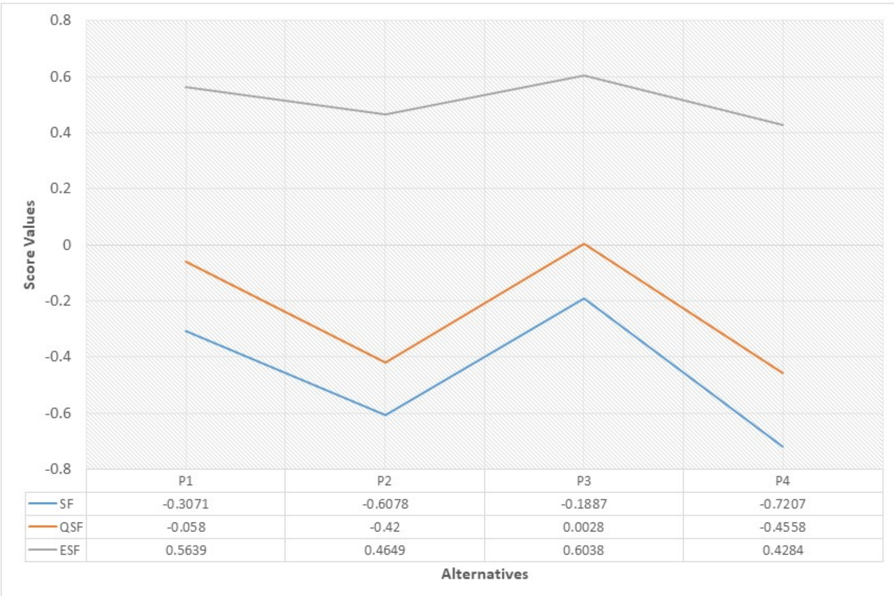


Figure 3. The comparison of the SF, QSF, and ESF under the T-SLDFWG operator.

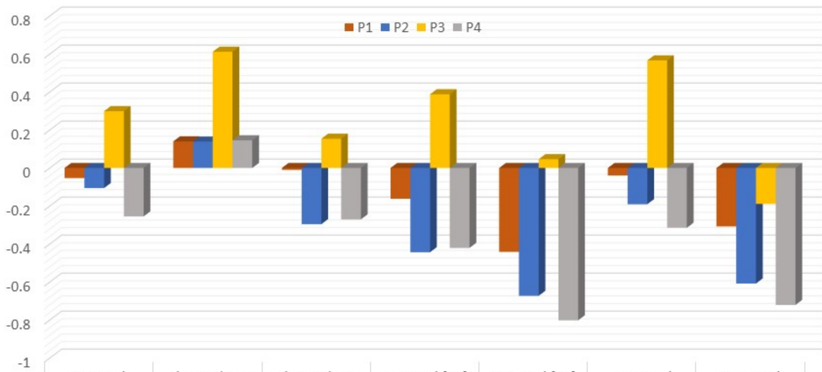
7. Comparative study

The above section displays the method and detailed calculation processes of the proposed operators. Besides the proposed operators, there are many aggregation operators to solve quantitative MADM problems. Among them, we underline, for their relevance in this comparison, the LDFWG operator [11], q-LDF weighted averaging (q-LDFWA) operator [12], q-LDF weighted geometric (q-LDFWG) operator [12], SLDF weighted averaging (SLDFWA) operator [45] and SLDF weighted geometric (SLDFWG) operator [45]. To accomplish the comparison, we used the above operators to aggregate the same data presented in Example 6.1.

It is noteworthy that the q-LDFS only has two membership grades, T-grade and F-grade, accompanied by two RPs. The T-SLDFS is characterized by three membership grades, which are T-grade, I-grade, and F-grade, and three RPs. Thus, the q-LDFS is a special case of the T-SLDFS and can be easily written in the form of a T-SLDFS. Therefore, to compare the proposed model with those in [12], we set I-grade to 0 in the proposed operator and assigned two RPs instead of three. The LDFS and SLDFS are also considered special cases of the T-SLDFS. In particular, we put $q = 1$ and I-grade= 0 in the suggested operators when we compare the LDFS with the suggested operators and $q = 1$ while comparing SLDFS operators with the proposed operators. The computed results are summarized in Table 7, and the geometrical interpretation of the results is shown in Figure 4.

Table 7. Comparative analysis of the proposed operators with existing operators.

Methods	Operators	Score Values	Ranking Results
Riaz and Hashmi [11] $I_{\Lambda=0,\nu=0}$ ($q = 1$)	WG	$\kappa(P_1) = -0.0528, \kappa(P_2) = -0.1052, \kappa(P_3) = 0.2992, \kappa(P_4) = -0.2547$	$P_3 > P_1 > P_2 > P_4$
Almagrabi et al. [12] $I_{\Lambda=0,\nu=0}$ ($q = 3$)	WA	$\kappa(P_1) = 0.1402, \kappa(P_2) = 0.1391, \kappa(P_3) = 0.6117, \kappa(P_4) = 0.1461$	$P_3 > P_4 > P_1 > P_2$
Almagrabi et al. [12] $I_{\Lambda=0,\nu=0}$ ($q = 3$)	WG	$\kappa(P_1) = -0.01, \kappa(P_2) = -0.2957, \kappa(P_3) = 0.154, \kappa(P_4) = -0.2707$	$P_3 > P_1 > P_4 > P_2$
Riaz et al. [45] ($q = 1$)	WA	$\kappa(P_1) = -0.1619, \kappa(P_2) = -0.4433, \kappa(P_3) = 0.3882, \kappa(P_4) = -0.4207$	$P_3 > P_1 > P_4 > P_2$
Riaz et al. [45] ($q = 1$)	WG	$\kappa(P_1) = -0.4416, \kappa(P_2) = -0.6728, \kappa(P_3) = 0.0463, \kappa(P_4) = -0.8015$	$P_3 > P_1 > P_2 > P_4$
Proposed operators ($q = 3$)	WA	$\kappa(P_1) = -0.0395, \kappa(P_2) = -0.1905, \kappa(P_3) = 0.5662, \kappa(P_4) = -0.3147$	$P_3 > P_1 > P_2 > P_4$
Proposed operators ($q = 3$)	WG	$\kappa(P_1) = -0.3071, \kappa(P_2) = -0.6078, \kappa(P_3) = -0.1887, \kappa(P_4) = -0.7207$	$P_3 > P_1 > P_2 > P_4$



	Riaz and Hashmi [11]	Almagrabi et al.[12] (WA)	Almagrabi et al.[12](WG)	Riaz et al.[45] (WA)	Riaz et al.[45] (WG)	Proposed operator(WA)	Proposed operator(WG)
P1	-0.0528	0.1402	-0.01	-0.1619	-0.4416	-0.0395	-0.3071
P2	-0.1052	0.1391	-0.2957	-0.4433	-0.6728	-0.1905	-0.6078
P3	0.2992	0.6117	0.154	0.3882	0.0463	0.5662	-0.1887
P4	-0.2547	0.1461	-0.2707	-0.4207	-0.8015	-0.3147	-0.7207

Figure 4. Geometrical representation of the information given in Table 7.

It is clear from Table 7 that the best alternatives obtained by using the methods of Riaz and Hashmi [11], Almagrabi et al. [12] and Riaz et al. [45] remained our proposed operators. This implies that the suggested methods are authentic and applicable. As mentioned above, LDFNs and q-LDFNs do not have the I-grade, which represents neutrality, which will lead to the lack of some information. The proposed T-SLDFNs include T-grade, I-grade, and F-grade, and give decision makers a more flexible environment to avoid information loss in the decision-making process. The method of Riaz et al. [45] is based on SLDFNs, and the rung q in SLDFNs equals 1. Therefore, under this circumstance, some decision evaluation information cannot be effectively expressed.

It should be noted that as the parameter q increases, the allowable area of the evaluated information escalates and we can continue to increase the value of parameter q to satisfy the required information range. This is what happened while applying the T-SLDFNs, as the parameter q in T-SLDFNs is not restricted by a certain value. As a result, the T-SLDFNs are more flexible and can express a wider range of fuzzy information than the SLDFNs.

8. Conclusions

This manuscript briefly described how the proposed theory of T-SLDFS generalizes all of the existing methods. T-SLDFS can express fuzzy information and simulate realistic DM problem scenarios more accurately through the assignment of variable parameter q to the construction of the SLDFS. The formal definition of the T-SLDFS was stated. The operations laws were developed, and some aggregation operators were defined under the T-SLDF environment. Some roperties of these operators were verified. Furthermore, a MADM method was designed on the basis of the proposed operators and SFs. A case study was provided to rank some alternative products. A T-SLDFN is formulated to portray the performance of each alternative product concerning each feature. Then, T-SLDFWA, T-SLDFOWA, T-SLDFHWA, T-SLDFWG, T-SLDFOWG and T-SLDFHWG operators are utilized to aggregate the attribute values. Several types of score functions are used to obtain the ranking results. We see slight differences between the ranking results derived from the proposed operators, but the best and the first choice were the same for all proposed operators. Further, the results demonstrate a great similarity and compatibility while using other evaluation methods such as LDFWG, q-LDFWA, q-LDFWG, SLDFWA and SLDFWG operators. In this study we attempt to handle more complicated MADM problems, however, there are still some limitations in the proposed work. We have only taken into consideration the evaluation information given by T-SLDFS, whereas in factual MADM problems, decision makers can use hybrid evaluation methods by employing the features of soft sets, complex numbers, bipolarity, hesitancy and interval-based membership to better capture the vagueness and uncertainties in some complicated data. In addition, this study addressed only two aggregation operators with their variations, namely, T-SLDFWA, T-SLDFOWA, T-SLDFHWA, T-SLDFWG, T-SLDFOWG and T-SLDFHWG operators. In the future, our targets are to study other generalizations of T-SLDFS such as T-spherical linear Diophantine fuzzy soft set, T-spherical linear Diophantine hesitant fuzzy set, T-spherical linear Diophantine bipolar fuzzy set and interval-valued T-spherical linear Diophantine fuzzy sets. Also, the proposed operators could be extended to Heronian mean, power mean, Hamacher, Bonferroni mean and Dombi's aggregation operators.

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Conflict of interest

The author declares no conflict of interest.

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