AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 11037–11056.
DOI: 10.3934/math.2023559
Received: 15 August 2022
Revised: 18 January 2023
Accepted: 02 February 2023
Published: 08 March 2023

Research article

A generalized Halpern-type forward-backward splitting algorithm for solving variational inclusion problems

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Abstract: In this paper, we investigate the problem of finding a zero of sum of two accretive operators in the setting of uniformly convex and q-uniformly smooth real Banach spaces ($q > 1$). We incorporate the inertial and relaxation parameters in a Halpern-type forward-backward splitting algorithm to accelerate the convergence of its sequence to a zero of sum of two accretive operators. Furthermore, we prove strong convergence of the sequence generated by our proposed iterative algorithm. Finally, we provide a numerical example in the setting of the classical Banach space $l_4(\mathbb{R})$ to study the effect of the relaxation and inertial parameters in our proposed algorithm.

Keywords: accretive operator; convergence; generalized duality mapping; relaxation parameter; splitting method
Mathematics Subject Classification: 46N10, 90C25

1. Introduction

Let E be a real Banach space and let A and B be single-valued and multi-valued operators, respectively, on E. Consider the variational inclusion problem:

$$\text{find } u \in E \text{ such that } 0 \in (A+B)u. \tag{1.1}$$

Problem (1.1) has been of interest to many authors largely due to its several applications in convex minimization, variational inequalities and split feasibility problems (see, e.g., [20, 25, 28]). Interestingly, the convex minimization problems arising from image restoration, signal processing and machine learning can be transformed to an inclusion of the form (1.1) (see, e.g., [11, 18, 19, 31] and the referees therein). This interesting connection between problem (1.1) and concrete problems arising from applications have made the problem of approximating zeros of sum of two (monotone or accretive) operators a contemporary problem of interest (see, e.g., [1, 17, 29, 36]).

Many iterative algorithms have been proposed for approximating solutions of problem (1.1) in the setting of Hilbert spaces and Banach spaces (see, e.g., [2,3,18,20,26]). Of interest to us is the forward-backward splitting algorithm (FBSA) which was studied by Passty [32] defined by:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (I + \lambda_n B)^{-1}(x_n - \lambda_n A x_n), \quad n \geq 1, \end{cases} \tag{1.2}$$

in the setting of a real Hilbert space, H. Under the assumption that A is α-inverse strongly monotone, B is maximal monotone and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying some appropriate conditions, Passty [32] proved that the sequence generated by (1.2) converges weakly to a solution of problem (1.1). He also remarked that for the special case when B is the indicator function of a nonempty closed and convex set, Lions [27] also proved weak convergence of the sequence generated by (1.2) to a solution of problem (1.1).

Since strong convergence results are more desirable, in the literature, modifications of the FBSA (1.2) by introducing a projection or Halpern or viscosity approximation techniques have been proposed by many authors which guarantee strong convergence of the modified version of the FBSA (1.2) to a solution of problem (1.1) in the setting of Hilbert spaces and Banach spaces more general than Hilbert spaces see e.g., [3, 21, 39–41] and the references therein.

Due to its simplicity, the Halpern-type modification of the FBSA (1.2) proposed by Takahashi et al. [39] captured our interest. Their algorithm is defined in the following manner: given x_1 and u in H, the next iterate is generated by

$$\begin{cases} y_n = \alpha_n u + (1-\alpha_n)(I + \lambda_n B)^{-1}(I - \lambda_n A)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) y_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $A : H \to H$ is α-inverse strongly monotone, $B : H \to 2^H$ is set-valued maximal monotone and, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ are sequences satisfying some appropriate conditions. Later, in 2016, Pholasa et al. [33] extended the theorem of Takahashi et al. [39] to Banach spaces. They proved the following theorem:

Theorem 1.1. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-inverse strongly accretive of order q and $B : X \to 2^X$ be an m-accretive operator. Assume that $\Omega = (A + B)^{-1}0 \neq \emptyset$. We define a sequence $\{x_n\}$ by the iterative scheme: for any $x_1 \in X$,*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)J^B_{\lambda_n}(x_n - \lambda_n A x_n), \tag{1.4}$$

for each $n \geq 1$, where $u \in X$, $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ are sequences satisfying some appropriate conditions. Then the sequence $\{x_n\}$ converges strongly to a solution of (1.1).

Now, let us recall the inertial acceleration method which is based on a discrete version of a second order dissipative dynamical system (see, e.g., [6, 7, 30] for more about the discretization of the system). The inertial procedure can be regarded as a method of speeding up the convergence properties of existing iterative algorithms (see, e.g., [5, 14–16, 22, 34, 37]). Recently, the inertial procedure is of interest to many researchers with motivations varying from the fact that the method accelerates convergence or for the purpose of academic exercise. For example, it is known that inertial acceleration strategy published by Nesterov improves the theoretical rate of convergence of the forward-backward Algorithm (1.2) (see [11]).

In 2018, Cholamjiak et al. [20] incorporated the inertial acceleration strategy in a Halpern-type FBSA to accelerate the convergence of the sequence to a solution of problem (1.1). They proved the following theorem:

Theorem 1.2. *Let H be a real Hilbert space. Let $A : H \to H$ be an α-inverse strongly monotone operator and $B : H \to 2^H$ be a maximal monotone operator such that $\Omega == (A + B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_0, x_1 \in H$ and*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \alpha_n u + \beta_n y_n + \gamma_n J^B_{\lambda_n}(y_n - A y_n), \quad n \geq 1, \end{cases} \tag{1.5}$$

where $\{\theta_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are real sequences satisfying some appropriate conditions. Then the sequence $\{x_n\}$ converges strongly to a solution of (1.1).

Recently, Adamu et al. [4] extended the result of Cholamjiak et al. [20] to 2-uniformly convex and uniformly Banach spaces and proved a strong convergence theorem. Furthermore, some applications to convex minimization and image restoration problems were presented in their paper to support the results with numerical experiments.

Another acceleration strategy which is currently of interest is the relaxation technique. This method is based on a convex combination of the "x_{n+1}" term in the existing algorithm and "x_n". The influence of this convex combination is what is called the relaxation technique. Interested readers may see, for example, [8, 24] for motivation about this technique. This strategy has been used to accelerate the convergence of the FBSA and some of its modified versions which guarantee strong convergence (see, e.g., [2, 8, 9, 18] and the references therein). Recently, in 2021, Cholamjiak [18] combined the inertial and relaxation acceleration strategies in a modified FBSA. They proved the following theorems:

Theorem 1.3. *Let H be a real Hilbert space and let $A : H \to H$ be monotone and Lipschitz continuous and $B : H \to 2^H$ be maximal monotone. Suppose the solution set of the VIP (1.1) $(A + B)^{-1}0$ is* *nonempty. Let* $x_1 \in H$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} y_n = (I + \lambda_n B)^{-1}(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \theta_n)x_n + \theta_n y_n + \theta_n \lambda_n (A x_n - A y_n), \\ \lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu_n \|x_n - y_n\|}{\|A x_n - B y_n\|}\right\}, \end{cases} \tag{1.6}$$

where $\lambda_1 > 0$, $\{\theta_n\} \subset [a, b] \subset (0, 1)$, $\{\mu_n\} \subset [c, d] \subset (0, 1)$. *Then the sequence generated by (1.6) converges weakly to a solution of problem (1.1).*

Theorem 1.4. *Under the same hypothesis as in Theorem 1.3 above, given* x_0, $x_1 \in H$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} w_n = x_n + \alpha(x_n - x_{n-1}), \\ y_n = (I + \lambda_n B)^{-1}(w_n - \lambda_n A w_n), \\ x_{n+1} = (1 - \theta_n)w_n + \theta_n y_n + \theta_n \lambda_n (A w_n - A y_n), \\ \lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu_n \|w_n - y_n\|}{\|A w_n - B y_n\|}\right\}, \end{cases} \tag{1.7}$$

where $\lambda_0 > 0$, $\theta \in (0, 1]$, $\mu \in (0, 1)$, $\alpha \in [0, 1)$ *such that*

$$\frac{\theta(1 - \mu^2)}{(2 - \theta + \mu\theta)^2} + \frac{1 - \theta}{\theta} > \frac{\alpha(1 + \alpha)}{(1 - \alpha)^2}.$$

Then the sequence $\{x_n\}$ *converges weakly to a solution of problem (1.1).*

Recently, Adamu et al. [2] used the idea of Halpern approximation technique and also combined the inertial and relaxation acceleration strategies in a modified FBSA to obtain strong convergence. They proved the following theorems:

Theorem 1.5. *Let E be a 2-uniformly convex and uniformly smooth real Banach space with dual space,* E^**. Let* $A : E \to E^*$ *be a monotone and L-Lipschitz continuous mapping,* $B : E \to 2^{E^*}$ *be a maximal monotone mapping and* $T : E \to E$ *be a relatively nonexpansive mapping. Assume the solution set* $\Omega = (A + B)^{-1}0 \cap F(T) \neq \emptyset$*, given* $x_1 \in E$*, let* $\{x_n\}$ *be a sequence defined by:*

$$\begin{cases} y_n = J^B_{\lambda_n} J^{-1}(J x_n - \lambda_n A x_n), \\ z_n = J^{-1}(J y_n - \lambda_n (A y_n - A x_n)), \\ u_n = J^{-1}(\beta_n J z_n + (1 - \beta_n) J T z_n), \\ x_{n+1} = J^{-1}((1 - \theta_n) J x_n + \theta_n(\gamma_n J u + (1 - \gamma_n) J u_n)), \end{cases} \tag{1.8}$$

where $J^B_{\lambda_n} = (J + \lambda_n B)^{-1} J$*, J is the normalized duality map,* $\{\theta_n\}, \{\beta_n\} \subset (0, 1]$, $\{\gamma_n\} \subset (0, 1)$ *are sequences that satisfy some appropriate conditions. Then,* $\{x_n\}$ *converges strongly to a solution of problem (1.1).*

Theorem 1.6. *Under the same setting as in Theorem 1.5, given* $x_0, x_1 \in E$*, let* $\{x_n\}$ *be a sequence defined by:*

$$\begin{cases} w_n = J^{-1}(J x_n + \alpha_n (J x_n - J x_{n-1})), \\ y_n = J^B_{\lambda_n} J^{-1}(J w_n - \lambda_n A w_n), \\ z_n = J^{-1}(J y_n - \lambda_n (A y_n - A w_n)), \\ u_n = J^{-1}(\beta_n J z_n + (1 - \beta_n) J T z_n), \\ x_{n+1} = J^{-1}((1 - \theta_n) J w_n + \theta_n(\gamma_n J u + (1 - \gamma_n) J u_n)), \end{cases} \tag{1.9}$$

where $\{\alpha_n\}$ is a real sequence that satisfies some appropriate conditions. The remaining parameters are the same as in Theorem 1.5. Then, $\{x_n\}$ converges strongly to a solution of problem (1.1).

Remark 1.1. *Looking at the results of Cholamjiak et al. [18] and Adamu et al. [2] with their competitive numerical illustrations, it is natural to ask the following question:*

Question 1. *Can the idea of combining inertial and relaxation acceleration strategies be incorporated in an existing algorithm involving accretive operators?*

Motivated by Question 1, our contribution in this paper are the following:

(1) We incorporate the inertial and relaxation acceleration strategies in Algorithm (1.4) of Pholasa et al. [33] to get a relaxed inertial Halpern-type forward-backward splitting algorithm involving accretive operators in Banach spaces for solving problem (1.1).
(2) Unlike in the results of Cholamjiak et al. [18] and Adamu et al. [2], in this paper, we study the effect of the inertia and relaxation parameters and provided the best choice for these parameters in the examples we considered.

The paper is structured as follows. In the next section, we recall some basic concepts on operators in Banach spaces. In Section 3, we prove strong convergence results for relaxed inertial Halpern-type forward-backward splitting algorithm. In the last section, a numerical example is given to illustrate the performance of the proposed algorithms.

2. Preliminaries

The following definitions and lemmas are needed in the proof of our main theorem. Assume that E is a real normed space with dual space E^* and C is a nonempty closed and convex subset of E. For any $a \in E$ and $r > 0$, the notation $B(a, r)$ means the set $\{x \in E : \|x - a\| \leq r\}$. The notation J_q is the generalized duality mapping defined, for any $x \in E$, by

$$J_q(x) := \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\right\}.$$

Observe that when $q = 2$, J_2 is the duality mapping denoted by J. Analytic representations of the generalized duality mapping on some classical Banach spaces can be found in [10].

Let $T : E \to 2^E$ be a set-valued operator. Recall that the operator T is said to be

- a contraction if there exists $k \in (0, 1)$ such that for all $x, y \in E$,

$$\|\eta - \zeta\| \leq k \|x - y\|,$$

  where $\eta \in Tx, \zeta \in Ty$. If $0 < k \leq 1$, then T is called nonexpansive.
- accretive if for all $x, y \in E$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\left\langle \eta - \zeta, j_q(x - y)\right\rangle \geq 0,$$

  where $\eta \in Tx, \zeta \in Ty$.
- strongly accretive if there exists $\gamma > 0$ and for all $x, y \in E$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\left\langle \eta - \zeta, j_q(x - y)\right\rangle \geq \gamma \|x - y\|,$$

  where $\eta \in Tx, \zeta \in Ty$.

- α-inverse strongly accretive (α-isa) of order q, if there exist $\alpha > 0, q > 1$ and for all $x, y \in E$, there exists $j_q(x-y) \in J_q(x-y)$ such that
$$\left\langle \eta - \zeta, j_q(x-y) \right\rangle \geq \alpha \left\| \eta - \zeta \right\|^q ,$$
where $\eta \in Tx, \zeta \in Ty$.
- m-accretive if T is accretive and $R(I + \lambda T) = E$, for all $\lambda > 0$.

Lemma 2.1. *[12]* For $q > 1$, let J_q be the generalized duality mapping. Then, for all $x, y \in E$, there exists $j_q(x+y) \in J_q(x+y)$ such that
$$\|x+y\|^q \leq \|x\|^q + q \left\langle y, j_q(x+y) \right\rangle .$$

Lemma 2.2. *[42]* Let E be a uniformly convex real Banach space and let $q > 1$ and $r > 0$. Then there exist strictly increasing continuous and convex functions $\phi, \psi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $\psi(0) = 0$ such that, for all $x, y \in B(0, r)$,

(i) $\|\lambda x + (1-\lambda) y\|^q \leq \lambda \|x\|^q + (1-\lambda) \|y\|^q - \lambda(1-\lambda)\phi(\|x-y\|)$, for any $\lambda \in [0, 1]$,
(ii) $\psi(\|x-y\|) \leq \|x\|^q - q \left\langle x, j_q(y) \right\rangle + (q-1) \|y\|^q$,

where $j_q(x+y) \in J_q(x+y)$.

Lemma 2.3. *[28]* Let E be a q-uniformly smooth real Banach space and let $A : C \to E$ be an α-isa of order q. Then the following inequality holds for all $x, y \in C$
$$\|(I - \lambda A)\, x - (I - \lambda A)\, y\|^q \leq \|x-y\|^q - \lambda(\alpha q - \kappa_q \lambda^{q-1}) \|Ax - Ay\|^q ,$$
where $\kappa_q > 0$ is the q-uniform smoothness coefficient of E (see, e.g., [42] for an explicit definition of κ_q). In particular, if $0 < \lambda < \alpha q - \kappa_q \lambda^{q-1}$ then $(I - \lambda A)$ is nonexpansive.

Remark 2.1. *Let $A : E \to 2^E$ be an m-accretive map. The resolvent $J_\lambda^A : E \to 2^E$ of A is defined by*
$$J_\lambda^A x := \{u \in E : x \in u + \lambda A u\} .$$
It is well-known that J_λ^A is single valued with $F(J_\lambda^A) := A^{-1}0$ and J_λ^A is firmly nonexpansive. In the sequel we adopt the following notation:
$$W_\lambda^{A,B} := J_\lambda^B (I - \lambda A) = (I - \lambda B)^{-1} (I - \lambda A), \quad \lambda > 0.$$
The following statements are true (see [28]),

(i) *for* $\lambda > 0$, $F(W_\lambda^{A,B}) = (A+B)^{-1}0$,
(ii) *for* $0 < \lambda \leq \varepsilon$ *and* $x \in E$, $\left\| x - W_\lambda^{A,B} x \right\| \leq 2 \left\| x - W_\varepsilon^{A,B} x \right\|$.

Lemma 2.4. *[28]* Let E be a uniformly convex and q-uniformly smooth real Banach space and let $A : E \to E$ be an α-isa mapping of order q and $B : E \to E$ be an m-accretive mapping. Then given $r > 0$, there exists a continuous, strictly increasing and convex function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that for all $x, y \in B(0, r)$,
$$\begin{aligned} \left\| W_\lambda^{A,B} x - W_\lambda^{A,B} y \right\|^q \leq\ & \|x-y\|^q - \lambda \left(\alpha q - \lambda^{q-1}\kappa_q\right) \|Ax - Ay\|^q \\ & - \varphi \left(\|(I - J_\lambda)(I - \lambda A)\, x - (I - J_\lambda)(I - \lambda A)\, y\| \right) . \end{aligned}$$

Lemma 2.5. *[23]* Let $\{d_n\}$ be a sequence of a nonnegative real numbers such that

$$d_{n+1} \leq (1-\vartheta_n)\, d_n + \vartheta_n \tau_n \text{ and } d_{n+1} \leq d_n - \eta_n + \rho_n,$$

where $\{\vartheta_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers, $\{\rho_n\}$ and $\{\tau_n\}$ are real sequences. Then $\lim_{n\to\infty} d_n = 0$ if

(i) $\sum_{n=1}^{\infty} \vartheta_n = \infty$,
(ii) $\lim_{n\to\infty} \rho_n = 0$,
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \tau_{n_k} \leq 0$, for any subsequence $\{n_k\} \subseteq \{n\}$.

Lemma 2.6. *[38]* Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that

$$a_{n+1} \leq a_n + b_n \text{ for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

3. Relaxed inertial Halpern-type forward-backward splitting algorithm

The following assumption is central in the proof of our results.

Assumption 3.1.

(i) *For $q > 1$, let E be a real Banach space that is uniformly convex and q-uniformly smooth and let $A : E \to E$ be an α-isa of order q, and $B : E \to 2^E$ be a set-valued m-accretive operator such that $\Omega := (A+B)^{-1}0 = \{x \in E : 0 \in (Ax + Bx)\}$ is nonempty.*
(ii) *Let $\{\beta_n\} \subseteq (0,1)$ be a sequence such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$.*
(iii) *Let $\{\lambda_n\} \subseteq (0,\infty)$ be a sequence such that $0 < \lambda \leq \kappa_q \lambda_n^{q-1} < \alpha q$.*
(iv) *Let $\{\gamma_n\} \subseteq (0,1)$ be a sequence with $\lim_{n\to\infty} \gamma_n = 0$.*
(v) *Let $\{\theta_n\} \subseteq (0,1]$ be an increasing sequence.*
(vi) *Let $\{\varepsilon_n\} \subseteq (0,\infty)$ be a sequence such that with $\sum_{n=1}^{\infty} \varepsilon_n < \infty$.*

Algorithm 3.1. *Relaxed inertial Halpern-type forward-backward splitting algorithm.*

Step 0. *Choose arbitrary points x_0, $x_1 \in E$, $\alpha \in (0,1)$ and set $n = 1$.*

Step 1. *Choose α_n such that $0 \leq \alpha_n \leq \overline{\alpha}_n$, where*

$$\overline{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{where } x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. *Compute*

$$\begin{cases} y_n = x_n + \alpha_n (x_n - x_{n-1}), \\ v_n = \beta_n u + (1-\beta_n) J_{\lambda_n}^B (y_n - \lambda_n A y_n), \\ x_{n+1} = (1-\theta_n) x_n + \theta_n (\gamma_n y_n + (1-\gamma_n) v_n). \end{cases}$$

Step 2. *Update $n = n + 1$.*

Remark 3.1. *By Assumption 3.1, (vi) and Step 1, we deduce that*

$$\lim_{n\to\infty} \alpha_n \|x_n - x_{n-1}\| = 0.$$

Lemma 3.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1, then $\{x_n\}$ is bounded.*

Proof. Let $W_n := J^B_{\lambda_n}(I - \lambda_n A)$ and $z \in \Omega$. Then, W_n is nonexpansive (see [13], page 8). Using the nonexpansivity of W_n and Remark 2.1, we get

$$\begin{aligned} \|v_n - z\| &= \|\beta_n u + (1-\beta_n) W_n y_n - z\| \\ &= \|\beta_n (u - z) + (1-\beta_n)(W_n y_n - z)\| \\ &\le \beta_n \|u - z\| + (1-\beta_n)\|W_n y_n - z\| \\ &= \beta_n \|u - z\| + (1-\beta_n)\|W_n y_n - W_n z\| \\ &\le \beta_n \|u - z\| + (1-\beta_n)\|y_n - z\|. \end{aligned} \tag{3.2}$$

Using the Inequality (3.2), and the fact that $\theta_n \in (0,1]$, we get

$$\begin{aligned} \|x_{n+1} - z\| &= \|(1-\theta_n) x_n + \theta_n (\gamma_n y_n + (1-\gamma_n) v_n) - z\| \\ &\le (1-\theta_n)\|x_n - z\| + \theta_n \|(\gamma_n y_n + (1-\gamma_n) v_n) - z\| \\ &\le (1-\theta_n)\|x_n - z\| + \theta_n (\gamma_n \|y_n - z\| + (1-\gamma_n)\|v_n - z\|) \\ &= (1-\theta_n)\|x_n - z\| + \theta_n\gamma_n \|y_n - z\| + \theta_n (1-\gamma_n)\|v_n - z\| \\ &\le (1-\theta_n)\|x_n - z\| + \theta_n\gamma_n \|y_n - z\| \\ &\quad + \theta_n (1-\gamma_n)(\beta_n \|u - z\| + (1-\beta_n)\|y_n - z\|) \\ &= (1-\theta_n)\|x_n - z\| + \theta_n\gamma_n \|y_n - z\| \\ &\quad + \beta_n\theta_n (1-\gamma_n)\|u - z\| + \theta_n (1-\gamma_n)(1-\beta_n)\|y_n - z\| \\ &= (1-\theta_n)\|x_n - z\| + (\theta_n\gamma_n + \theta_n(1-\gamma_n)(1-\beta_n))\|y_n - z\| \\ &\quad + \beta_n\theta_n (1-\gamma_n)\|u - z\| \\ &= (1-\theta_n)\|x_n - z\| + (\theta_n - \beta_n\theta_n(1-\gamma_n))\|y_n - z\| \\ &\quad + \beta_n\theta_n (1-\gamma_n)\|u - z\| \\ &= (1-\theta_n)\|x_n - z\| + (\theta_n - \beta_n\theta_n(1-\gamma_n))\|x_n + \alpha_n (x_n - x_{n-1}) - z\| \\ &\quad + \beta_n\theta_n (1-\gamma_n)\|u - z\| \\ &\le (1-\theta_n)\|x_n - z\| + (\theta_n - \beta_n\theta_n(1-\gamma_n))(\|x_n - z\| + \alpha_n \|x_n - x_{n-1}\|) \\ &\quad + \beta_n\theta_n (1-\gamma_n)\|u - z\| \\ &= (1-\beta_n\theta_n(1-\gamma_n))\|x_n - z\| + (\theta_n - \beta_n\theta_n(1-\gamma_n))\alpha_n \|x_n - x_{n-1}\| \\ &\quad + \beta_n\theta_n (1-\gamma_n)\|u - z\| \\ &\le (1-\beta_n\theta_n(1-\gamma_n))\|x_n - z\| + (1 - \beta_n\theta_n(1-\gamma_n))\alpha_n \|x_n - x_{n-1}\| \\ &\quad + \beta_n\theta_n (1-\gamma_n)\|u - z\| \\ &= (1-\beta_n\theta_n(1-\gamma_n))(\|x_n - z\| + \alpha_n \|x_n - x_{n-1}\|) + \beta_n\theta_n (1-\gamma_n)\|u - z\| \\ &\le \max\{\|x_n - z\| + \alpha_n \|x_n - x_{n-1}\|, \|u - z\|\}. \end{aligned}$$

If $\max\{\|x_n - z\| + \alpha_n \|x_n - x_{n-1}\|, \|u - z\|\} = \|u - z\|$, we have that $\{x_n\}$ is bounded. Otherwise, there exists $n_0 \geq 1$ such that

$$\|x_{n+1} - z\| \leq \|x_n - z\| + \alpha_n \|x_n - x_{n-1}\|, \ \forall\, n \geq n_0.$$

From the Eq (3.1), we note that $\alpha_n \leq \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}$, for all $n \geq 1$. Thus,

$$\sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\| \leq \sum_{n=1}^{\infty} \varepsilon_n < \infty.$$

By Lemma 2.6, $\{\|x_n - z\|\}$ has a limit. Therefore, $\{x_n\}$ is bounded.

Next, we prove strong convergence theorem for the sequence generated by our proposed Algorithm 3.1.

Theorem 3.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then $\{x_n\}$ converges strongly to $z \in \Omega$.*

Proof. Let $z \in \Omega$. Using Lemmas 2.1 and 2.4, we have

$$\begin{aligned}
\|v_n - z\|^q &= \left\|\beta_n u + (1-\beta_n) J_{\lambda_n}^B (y_n - \lambda_n A y_n) - z\right\|^q \\
&= \|\beta_n u + (1-\beta_n) W_n y_n - z\|^q \\
&\leq (1-\beta_n)^q \|W_n y_n - W_n z\|^q + q\beta_n \left\langle u - z, j_q(v_n - z)\right\rangle \\
&\leq (1-\beta_n)^q \left(\|y_n - z\|^q - \lambda_n \left(\alpha q - \lambda_n^{q-1}\kappa_q\right) \|Ay_n - Az\|^q \right. \\
&\quad \left. - \varphi\left(\|y_n - \lambda_n (Ay_n - Az) - W_n y_n\|\right)\right) + q\beta_n \left\langle u - z, j_q(v_n - z)\right\rangle \\
&= (1-\beta_n)^q \|y_n - z\|^q - \lambda_n (1-\beta_n)^q \left(\alpha q - \lambda_n^{q-1}\kappa_q\right) \|Ay_n - Az\|^q \\
&\quad - (1-\beta_n)^q \varphi\left(\|y_n - \lambda_n (Ay_n - Az) - W_n y_n\|\right) + q\beta_n \left\langle u - z, j_q(v_n - z)\right\rangle.
\end{aligned} \tag{3.3}$$

Next, using Inequality (3.3), Lemma 2.2 and the fact that $q > 1$, we get that

$$\begin{aligned}
\|x_{n+1} - z\|^q &= \|(1-\theta_n) x_n + \theta_n (\gamma_n y_n + (1-\gamma_n) v_n) - z\|^q \\
&= \|(1-\theta_n)(x_n - z) + \theta_n ((\gamma_n y_n + (1-\gamma_n) v_n) - z)\|^q \\
&\leq (1-\theta_n)\|x_n - z\|^q + \theta_n \|(\gamma_n y_n + (1-\gamma_n) v_n) - z\|^q \\
&\leq (1-\theta_n)\|x_n - z\|^q + \theta_n (\gamma_n \|y_n - z\|^q + (1-\gamma_n)\|v_n - z\|^q) \\
&= (1-\theta_n)\|x_n - z\|^q + \theta_n \gamma_n \|y_n - z\|^q + \theta_n (1-\gamma_n)\|v_n - z\|^q \\
&\leq (1-\theta_n)\|x_n - z\|^q + \theta_n \gamma_n \|y_n - z\|^q + \theta_n (1-\gamma_n)((1-\beta_n)^q \|y_n - z\|^q \\
&\quad -\lambda_n (1-\beta_n)^q \left(\alpha q - \lambda_n^{q-1}\kappa_q\right) \|Ay_n - Az\|^q \\
&\quad - (1-\beta_n)^q \varphi\left(\|y_n - \lambda_n (Ay_n - Az) - W_n y_n\|\right) \\
&\quad + q\beta_n \left\langle u - z, j_q(v_n - z)\right\rangle\Big) \\
&= (1-\theta_n)\|x_n - z\|^q + \theta_n \gamma_n \|y_n - z\|^q + \theta_n (1-\gamma_n)(1-\beta_n)^q \|y_n - z\|^q \\
&\quad -\theta_n (1-\gamma_n) \lambda_n (1-\beta_n)^q \left(\alpha q - \lambda_n^{q-1}\kappa_q\right) \|Ay_n - Az\|^q \\
&\quad -\theta_n (1-\gamma_n)(1-\beta_n)^q \varphi\left(\|y_n - \lambda_n (Ay_n - Az) - W_n y_n\|\right)
\end{aligned}$$

$$
\begin{aligned}
&+\theta_n(1-\gamma_n)\,q\beta_n\left\langle u-z, j_q(v_n-z)\right\rangle\\
=\;&(1-\theta_n)\|x_n-z\|^q+(\theta_n\gamma_n+\theta_n(1-\gamma_n)(1-\beta_n)^q)\|y_n-z\|^q\\
&-\theta_n(1-\gamma_n)\lambda_n(1-\beta_n)^q\left(\alpha q-\lambda_n^{q-1}\kappa_q\right)\|Ay_n-Az\|^q\\
&-\theta_n(1-\gamma_n)(1-\beta_n)^q\varphi(\|y_n-\lambda_n(Ay_n-Az)-W_ny_n\|)\\
&+\theta_n(1-\gamma_n)\,q\beta_n\left\langle u-z, j_q(v_n-z)\right\rangle\\
\le\;&(1-\theta_n)\|x_n-z\|^q+(\theta_n-\theta_n\beta_n(1-\gamma_n))\|y_n-z\|^q\\
&-\theta_n(1-\gamma_n)\lambda_n(1-\beta_n)^q\left(\alpha q-\lambda_n^{q-1}\kappa_q\right)\|Ay_n-Az\|^q\\
&-\theta_n(1-\gamma_n)(1-\beta_n)^q\varphi(\|y_n-\lambda_n(Ay_n-Az)-W_ny_n\|)\\
&+\theta_n(1-\gamma_n)\,q\beta_n\left\langle u-z, j_q(v_n-z)\right\rangle\\
=\;&(1-\theta_n)\|x_n-z\|^q+(\theta_n-\theta_n\beta_n(1-\gamma_n))\|x_n+\alpha_n(x_n-x_{n-1})-z\|^q\\
&-\theta_n(1-\gamma_n)\lambda_n(1-\beta_n)^q\left(\alpha q-\lambda_n^{q-1}\kappa_q\right)\|Ay_n-Az\|^q\\
&-\theta_n(1-\gamma_n)(1-\beta_n)^q\varphi(\|y_n-\lambda_n(Ay_n-Az)-W_ny_n\|)\\
&+\theta_n(1-\gamma_n)\,q\beta_n\left\langle u-z, j_q(v_n-z)\right\rangle\\
=\;&(1-\theta_n)\|x_n-z\|^q+(\theta_n-\theta_n\beta_n(1-\gamma_n))\|(x_n-z)+\alpha_n(x_n-x_{n-1})\|^q\\
&-\theta_n(1-\gamma_n)\lambda_n(1-\beta_n)^q\left(\alpha q-\lambda_n^{q-1}\kappa_q\right)\|Ay_n-Az\|^q\\
&-\theta_n(1-\gamma_n)(1-\beta_n)^q\varphi(\|y_n-\lambda_n(Ay_n-Az)-W_ny_n\|)\\
&+\theta_n(1-\gamma_n)\,q\beta_n\left\langle u-z, j_q(v_n-z)\right\rangle\\
\le\;&(1-\theta_n)\|x_n-z\|^q+(\theta_n-\theta_n\beta_n(1-\gamma_n))\big(\|x_n-z\|^q\\
&+q\alpha_n\left\langle x_n-x_{n-1}, j_q(y_n-z)\right\rangle\big)\\
&-\theta_n(1-\gamma_n)\lambda_n(1-\beta_n)^q\left(\alpha q-\lambda_n^{q-1}\kappa_q\right)\|Ay_n-Az\|^q\\
&-\theta_n(1-\gamma_n)(1-\beta_n)^q\varphi(\|y_n-\lambda_n(Ay_n-Az)-W_ny_n\|)\\
&+\theta_n(1-\gamma_n)\,q\beta_n\left\langle u-z, j_q(v_n-z)\right\rangle\\
=\;&(1-\theta_n)\|x_n-z\|^q+(\theta_n-\theta_n\beta_n(1-\gamma_n))\|x_n-z\|^q\\
&+q\alpha_n(\theta_n-\theta_n\beta_n(1-\gamma_n))\left\langle x_n-x_{n-1}, j_q(y_n-z)\right\rangle\\
&-\theta_n(1-\gamma_n)\lambda_n(1-\beta_n)^q\left(\alpha q-\lambda_n^{q-1}\kappa_q\right)\|Ay_n-Az\|^q\\
&-\theta_n(1-\gamma_n)(1-\beta_n)^q\varphi(\|y_n-\lambda_n(Ay_n-Az)-W_ny_n\|)\\
&+\theta_n(1-\gamma_n)\,q\beta_n\left\langle u-z, j_q(v_n-z)\right\rangle\\
=\;&(1-\theta_n+(\theta_n-\theta_n\beta_n(1-\gamma_n)))\|x_n-z\|^q\\
&+q\alpha_n(\theta_n-\theta_n\beta_n(1-\gamma_n))\left\langle x_n-x_{n-1}, j_q(y_n-z)\right\rangle\\
&-\theta_n(1-\gamma_n)\lambda_n(1-\beta_n)^q\left(\alpha q-\lambda_n^{q-1}\kappa_q\right)\|Ay_n-Az\|^q\\
&-\theta_n(1-\gamma_n)(1-\beta_n)^q\varphi(\|y_n-\lambda_n(Ay_n-Az)-W_ny_n\|)\\
&+\theta_n(1-\gamma_n)\,q\beta_n\left\langle u-z, j_q(v_n-z)\right\rangle\\
=\;&(1-\theta_n\beta_n(1-\gamma_n))\|x_n-z\|^q\\
&+q\alpha_n(\theta_n-\theta_n\beta_n(1-\gamma_n))\left\langle x_n-x_{n-1}, j_q(y_n-z)\right\rangle\\
&-\theta_n(1-\gamma_n)\lambda_n(1-\beta_n)^q\left(\alpha q-\lambda_n^{q-1}\kappa_q\right)\|Ay_n-Az\|^q
\end{aligned}
$$

$$-\theta_n (1-\gamma_n)(1-\beta_n)^q \varphi(\|y_n - \lambda_n (Ay_n - Az) - W_n y_n\|)$$
$$+\theta_n (1-\gamma_n) q\beta_n \left\langle u - z, j_q(v_n - z)\right\rangle.$$

Now, we have

$$\begin{aligned} \|x_{n+1} - z\|^q &\leq (1 - \theta_n\beta_n (1-\gamma_n)) \|x_n - z\|^q \\ &\quad + q\alpha_n (\theta_n - \theta_n\beta_n (1-\gamma_n)) \left\langle x_n - x_{n-1}, j_q(y_n - z)\right\rangle \\ &\quad + \theta_n (1-\gamma_n) q\beta_n \left\langle u - z, j_q(v_n - z)\right\rangle \end{aligned} \tag{3.4}$$

for each $n \geq n_0$, and

$$\begin{aligned} \|x_{n+1} - z\|^q &\leq \|x_n - z\|^q - \theta_n (1-\gamma_n) \lambda_n (1-\beta_n)^q \left(\alpha q - \lambda_n^{q-1}\kappa_q\right) \|Ay_n - Az\|^q \\ &\quad - \theta_n (1-\gamma_n)(1-\beta_n)^q \varphi(\|y_n - \lambda_n (Ay_n - Az) - W_n y_n\|) \\ &\quad + q\alpha_n (\theta_n - \theta_n\beta_n (1-\gamma_n)) \left\langle x_n - x_{n-1}, j_q(y_n - z)\right\rangle \\ &\quad + \theta_n (1-\gamma_n) q\beta_n \left\langle u - z, j_q(v_n - z)\right\rangle. \end{aligned} \tag{3.5}$$

for each $n \geq n_0$. We define the following sequences

$$\begin{aligned} d_n &:= \|x_{n+1} - z\|^q, \\ \vartheta_n &:= (1-\gamma_n)\theta_n\beta_n, \\ \tau_n &:= \frac{q\alpha_n (1 - \beta_n (1-\gamma_n))}{\beta_n (1-\gamma_n)} \left\langle x_n - x_{n-1}, j_q(y_n - z)\right\rangle + q\left\langle u - z, j_q(v_n - z)\right\rangle, \\ \eta_n &:= \theta_n (1-\gamma_n) \lambda_n (1-\beta_n)^q \left(\alpha q - \lambda_n^{q-1}\kappa_q\right) \|Ay_n - Az\|^q \\ &\quad + \theta_n (1-\gamma_n)(1-\beta_n)^q \varphi(\|y_n - \lambda_n (Ay_n - Az) - W_n y_n\|), \\ \rho_n &:= q\alpha_n (\theta_n - \theta_n\beta_n (1-\gamma_n)) \left\langle x_n - x_{n-1}, j_q(y_n - z)\right\rangle \\ &\quad + \theta_n (1-\gamma_n) q\beta_n \left\langle u - z, j_q(v_n - z)\right\rangle. \end{aligned}$$

From the Inequalities (3.4) and (3.5), it means that we have

$$d_{n+1} \leq (1-\vartheta_n) d_n + \vartheta_n \tau_n \quad \text{and} \quad d_{n+1} \leq d_n - \eta_n + \rho_n.$$

By Assumptions 3.1 (ii), (iv) and (v), it follows that

$$\sum_{n=1}^{\infty} \vartheta_n = \sum_{n=1}^{\infty} (1-\gamma_n)\theta_n\beta_n = \infty.$$

Using boundedness of $\{y_n\}$, $\{v_n\}$ and $\{\theta_n\}$, Assumption 3.1 (ii) and Remark 3.1, we obtain

$$\begin{aligned} \lim_{n\to\infty} \rho_n &= \lim_{n\to\infty} \Big(q\alpha_n (\theta_n - \theta_n\beta_n (1-\gamma_n)) \left\langle x_n - x_{n-1}, j_q(y_n - z)\right\rangle \\ &\quad + \theta_n (1-\gamma_n) q\beta_n \left\langle u - z, j_q(v_n - z)\right\rangle \Big) \\ &= 0. \end{aligned}$$

Lastly, by Lemma 2.5, we assume that $\lim_{k\to\infty} \eta_{n_k} = 0$ for any subsequence $\{n_k\} \subseteq \{n\}$. That is,

$$\lim_{k\to\infty} \left(\theta_{n_k} (1-\gamma_{n_k}) \lambda_{n_k} (1-\beta_{n_k})^q \left(\alpha q - \lambda_{n_k}^{q-1}\kappa_q\right) \left\|Ay_{n_k} - Az\right\|^q\right.$$

$$+\theta_{n_k}(1-\gamma_{n_k})(1-\beta_{n_k})^q\varphi\left(\left\|y_{n_k}-\lambda_{n_k}(Ay_{n_k}-Az)-W_{n_k}y_{n_k}\right\|\right)=0.$$

By property of φ, it can be seen that

$$\lim_{k\to\infty}\left\|Ay_{n_k}-Az\right\|=\lim_{k\to\infty}\left\|y_{n_k}-\lambda_{n_k}(Ay_{n_k}-Az)-W_{n_k}y_{n_k}\right\|=0.$$

That is

$$\begin{aligned}0 &= \lim_{k\to\infty}\left\|y_{n_k}-\lambda_{n_k}(Ay_{n_k}-Az)-W_{n_k}y_{n_k}\right\|\\ &= \lim_{k\to\infty}\left\|(W_{n_k}y_{n_k}-y_{n_k})+\lambda_{n_k}(Ay_{n_k}-Az)\right\|\\ &\le \lim_{k\to\infty}\left(\left\|W_{n_k}y_{n_k}-y_{n_k}\right\|+\lambda_{n_k}\left\|Ay_{n_k}-Az\right\|\right)\\ &= \lim_{k\to\infty}\left\|W_{n_k}y_{n_k}-y_{n_k}\right\|+\lim_{k\to\infty}\lambda_{n_k}\left\|Ay_{n_k}-Az\right\|.\end{aligned}$$

By Assumption 3.1 (iv), and $\lim_{k\to\infty}\left\|Ay_{n_k}-Az\right\|=0$, we can write

$$\lim_{k\to\infty}\left\|W_{n_k}y_{n_k}-y_{n_k}\right\|=0. \tag{3.6}$$

In addition, we notice that

$$\begin{aligned}\left\|W_{n_k}y_{n_k}-x_{n_k}\right\| &= \left\|W_{n_k}y_{n_k}-y_{n_k}\right\|+\left\|y_{n_k}-x_{n_k}\right\|\\ &= \left\|W_{n_k}y_{n_k}-y_{n_k}\right\|+\left\|x_{n_k}+\alpha_{n_k}(x_{n_k}-x_{n_k-1})-x_{n_k}\right\|\\ &= \left\|W_{n_k}y_{n_k}-y_{n_k}\right\|+\alpha_{n_k}\left\|x_{n_k}-x_{n_k-1}\right\|.\end{aligned}$$

From Remark 3.1, it implies that

$$\left\|W_{n_k}y_{n_k}-x_{n_k}\right\|\to 0,\quad k\to\infty. \tag{3.7}$$

Also, it should be noted that

$$\begin{aligned}\left\|y_{n_k}-v_{n_k}\right\| &= \left\|x_{n_k}-v_{n_k}\right\|+\alpha_{n_k}\left\|x_{n_k}-x_{n_k-1}\right\|\\ &= \left\|x_{n_k}-\left(\beta_{n_k}u+(1-\beta_{n_k})J^B_{\lambda_{n_k}}(y_{n_k}-\lambda_{n_k}Ay_{n_k})\right)\right\|+\alpha_{n_k}\left\|x_{n_k}-x_{n_k-1}\right\|\\ &= \left\|x_{n_k}-\left(\beta_{n_k}u+(1-\beta_{n_k})J^B_{\lambda_{n_k}}(I-\lambda_{n_k}A)y_{n_k}\right)\right\|+\alpha_{n_k}\left\|x_{n_k}-x_{n_k-1}\right\|\\ &= \left\|x_{n_k}-(\beta_{n_k}u+(1-\beta_{n_k})W_{n_k}y_{n_k})\right\|+\alpha_{n_k}\left\|x_{n_k}-x_{n_k-1}\right\|\\ &= \left\|x_{n_k}-(\beta_{n_k}u+(1-\beta_{n_k})W_{n_k}y_{n_k})+\beta_{n_k}x_{n_k}-\beta_{n_k}x_{n_k}\right\|+\alpha_{n_k}\left\|x_{n_k}-x_{n_k-1}\right\|\\ &= \left\|\beta_{n_k}(x_{n_k}-u)+(1-\beta_{n_k})(x_{n_k}-W_{n_k}y_{n_k})\right\|+\alpha_{n_k}\left\|x_{n_k}-x_{n_k-1}\right\|\\ &\le \beta_{n_k}\left\|x_{n_k}-u\right\|+(1-\beta_{n_k})\left\|x_{n_k}-W_{n_k}y_{n_k}\right\|+\alpha_{n_k}\left\|x_{n_k}-x_{n_k-1}\right\|.\end{aligned}$$

By boundedness of $\{x_{n_k}\}$, Assumption 3.1 (ii), Remarks 3.1 and 3.7, we have

$$\lim_{k\to\infty}\left\|y_{n_k}-v_{n_k}\right\|=0. \tag{3.8}$$

By Assumption 3.1 (iii), there exists $\lambda > 0$ such that $\lambda \leq \lambda_n$, for all $n \geq 1$. Using Remark 2.1, we obtain that

$$\left\|W_\lambda^{A,B} y_{n_k} - y_{n_k}\right\| \leq 2\left\|W_{n_k} y_{n_k} - y_{n_k}\right\|.$$

This implies that

$$\limsup_{k\to\infty} \left\|W_\lambda^{A,B} y_{n_k} - y_{n_k}\right\| \leq \limsup_{k\to\infty} 2\left\|W_{n_k} y_{n_k} - y_{n_k}\right\|.$$

By (3.6), we get $\limsup_{k\to\infty} 2\left\|W_{n_k} y_{n_k} - y_{n_k}\right\| = 0$ which it follows that

$$0 \leq \limsup_{k\to\infty} \left\|W_\lambda^{A,B} y_{n_k} - y_{n_k}\right\| \leq 0,$$

i.e., $\limsup_{k\to\infty} \left\|W_\lambda^{A,B} y_{n_k} - y_{n_k}\right\| = 0$. By the fact that

$$0 \leq \liminf_{k\to\infty} \left\|W_\lambda^{A,B} y_{n_k} - y_{n_k}\right\| \leq \limsup_{k\to\infty} \left\|W_\lambda^{A,B} y_{n_k} - y_{n_k}\right\| = 0$$

then we have $\liminf_{k\to\infty} \left\|W_\lambda^{A,B} y_{n_k} - y_{n_k}\right\| = 0$. Observe that

$$\left\|W_\lambda^{A,B} y_{n_k} - v_{n_k}\right\| \leq \left\|W_\lambda^{A,B} y_{n_k} - y_{n_k}\right\| + \left\|y_{n_k} - v_{n_k}\right\|$$

which implies, by (3.8), that

$$\lim_{k\to\infty} \left\|W_\lambda^{A,B} y_{n_k} - v_{n_k}\right\| = 0. \tag{3.9}$$

Moreover, we have that

$$\begin{aligned}
\left\|W_\lambda^{A,B} v_{n_k} - v_{n_k}\right\| &\leq \left\|W_\lambda^{A,B} v_{n_k} - W_\lambda^{A,B} y_{n_k}\right\| + \left\|W_\lambda^{A,B} y_{n_k} - v_{n_k}\right\| \\
&\leq \left\|v_{n_k} - y_{n_k}\right\| + \left\|W_\lambda^{A,B} y_{n_k} - v_{n_k}\right\|
\end{aligned}$$

by using the fact that W_n is nonexpansive. From (3.8) and (3.9), we get

$$\lim_{k\to\infty} \left\|W_\lambda^{A,B} v_{n_k} - v_{n_k}\right\| = 0. \tag{3.10}$$

We now construct $z_t = tu + (1-t)W_\lambda^{A,B} z_t$ where $t \in (0,1)$. Using theorem of Reich (see [35]), z_t converges strongly to the unique fixed point $z \in F(W_\lambda^{A,B}) = (A+B)^{-1}0$. By the fact that $W_\lambda^{A,B}$ is nonexpansive, using Lemma 2.1, it follows that

$$\begin{aligned}
\left\|z_t - v_{n_k}\right\|^q &= \left\|tu + (1-t)W_\lambda^{A,B} z_t - v_{n_k}\right\|^q \\
&= \left\|tu + (1-t)W_\lambda^{A,B} z_t - v_{n_k} + tv_{n_k} - tv_{n_k}\right\|^q \\
&= \left\|t\left(u - v_{n_k}\right) + (1-t)\left(W_\lambda^{A,B} z_t - v_{n_k}\right)\right\|^q \\
&\leq (1-t)^q \left\|W_\lambda^{A,B} z_t - v_{n_k}\right\|^q + qt\left\langle u - v_{n_k}, j_q\left(z_t - v_{n_k}\right)\right\rangle \\
&\leq (1-t)^q \left(\left\|W_\lambda^{A,B} z_t - W_\lambda^{A,B} v_{n_k}\right\| + \left\|W_\lambda^{A,B} v_{n_k} - v_{n_k}\right\|\right)^q \\
&\quad + qt\left\langle u - v_{n_k}, j_q\left(z_t - v_{n_k}\right)\right\rangle \\
&\leq (1-t)^q \left(\left\|z_t - v_{n_k}\right\| + \left\|W_\lambda^{A,B} v_{n_k} - v_{n_k}\right\|\right)^q + qt\left\langle u - v_{n_k}, j_q\left(z_t - v_{n_k}\right)\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= (1-t)^q \left(\left\|z_t - v_{n_k}\right\| + \left\|W_\lambda^{A,B} v_{n_k} - v_{n_k}\right\|\right)^q \\
&\quad + qt \left\langle u - v_{n_k} + z_t - z_t, j_q\left(z_t - v_{n_k}\right)\right\rangle \\
&= (1-t)^q \left(\left\|z_t - v_{n_k}\right\| + \left\|W_\lambda^{A,B} v_{n_k} - v_{n_k}\right\|\right)^q + qt\left\langle u - z_t, j_q\left(z_t - v_{n_k}\right)\right\rangle \\
&\quad + qt \left\langle z_t - v_{n_k}, j_q\left(z_t - v_{n_k}\right)\right\rangle \\
&= (1-t)^q \left(\left\|z_t - v_{n_k}\right\| + \left\|W_\lambda^{A,B} v_{n_k} - v_{n_k}\right\|\right)^q - qt\left\langle z_t - u, j_q\left(z_t - v_{n_k}\right)\right\rangle \\
&\quad + qt \left\langle z_t - v_{n_k}, j_q\left(z_t - v_{n_k}\right)\right\rangle \\
&\leq (1-t)^q \left(\left\|z_t - v_{n_k}\right\| + \left\|W_\lambda^{A,B} v_{n_k} - v_{n_k}\right\|\right)^q - qt\left\langle z_t - u, j_q\left(z_t - v_{n_k}\right)\right\rangle \\
&\quad + qt \left\|z_t - v_{n_k}\right\|^q.
\end{aligned}
$$

It implies that

$$
\left\langle z_t - u, j_q\left(z_t - v_{n_k}\right)\right\rangle \leq \frac{(1-t)^q}{qt}\left(\left\|z_t - v_{n_k}\right\| + \left\|W_\lambda^{A,B} v_{n_k} - v_{n_k}\right\|\right)^q + \frac{(qt-1)}{qt}\left\|z_t - v_{n_k}\right\|^q.
$$

Hence, we have

$$
\limsup_{k\to\infty}\left\langle z_t - u, j_q\left(z_t - v_{n_k}\right)\right\rangle \leq \frac{(1-t)^q}{qt}C^q + \frac{(qt-1)}{qt}C^q = \left(\frac{(1-t)^q + qt - 1}{qt}\right)C^q \tag{3.11}
$$

where $C = \limsup_{k\to\infty}\left\|z_t - v_{n_k}\right\|$. Observe that $\lim_{t\to 0}\frac{(1-t)^q+qt-1}{qt} = 0$. By the uniform continuity of j_q on bounded sets and the fact that $z_t \to z$ as $t \to 0$, i.e.,

$$
\lim_{t\to 0}\|z_t - z\| = 0, \tag{3.12}
$$

we get

$$
\lim_{t\to 0}\left\|j_q(z_t - v_{n_k}) - j_q(z - v_{n_k})\right\| = 0. \tag{3.13}
$$

Thus, by (3.12) and (3.13), we have

$$
\begin{aligned}
&\left|\left\langle z_t - u, j_q\left(z_t - v_{n_k}\right)\right\rangle - \left\langle z - u, j_q\left(z - v_{n_k}\right)\right\rangle\right| \\
&= \left|\left\langle (z_t - z) + (z - u), j_q\left(z_t - v_{n_k}\right)\right\rangle - \left\langle z - u, j_q\left(z - v_{n_k}\right)\right\rangle\right| \\
&= \left|\left\langle z_t - z, j_q\left(z_t - v_{n_k}\right)\right\rangle + \left\langle z - u, j_q\left(z_t - v_{n_k}\right)\right\rangle - \left\langle z - u, j_q\left(z - v_{n_k}\right)\right\rangle\right| \\
&= \left|\left\langle z_t - z, j_q\left(z_t - v_{n_k}\right)\right\rangle + \left\langle z - u, j_q\left(z_t - v_{n_k}\right) - j_q\left(z - v_{n_k}\right)\right\rangle\right| \\
&\leq \left|\left\langle z_t - z, j_q\left(z_t - v_{n_k}\right)\right\rangle\right| + \left|\left\langle z - u, j_q\left(z_t - v_{n_k}\right) - j_q\left(z - v_{n_k}\right)\right\rangle\right| \\
&\leq \|z_t - z\|\left\|z_t - v_{n_k}\right\|^{q-1} + \|z - u\|\left\|j_q\left(z_t - v_{n_k}\right) - j_q\left(z - v_{n_k}\right)\right\|.
\end{aligned}
$$

Hence, $\lim_{t\to 0}\left\langle z_t - u, j_q\left(z_t - v_{n_k}\right)\right\rangle = \left\langle z - u, j_q\left(z - v_{n_k}\right)\right\rangle$. From (3.11), it can be seen that

$$
\limsup_{k\to\infty}\left\langle z - u, j_q\left(z - v_{n_k}\right)\right\rangle \leq 0. \tag{3.14}
$$

Furthermore, note that, by boundedness of $\{y_n\}$, and Remark 3.1, it can be seen that

$$\limsup_{k\to\infty} \frac{q\alpha_{n_k}(1-\beta_{n_k}(1-\gamma_{n_k}))}{\beta_{n_k}(1-\gamma_{n_k})}\left\|x_{n_k}-x_{n_k-1}\right\|\left\|y_{n_k}-z\right\|^{q-1}\leq 0. \tag{3.15}$$

Now, we obtain, by (3.14) and (3.15), that

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\frac{q\alpha_{n_k}(1-\beta_{n_k}(1-\gamma_{n_k}))}{\beta_{n_k}(1-\gamma_{n_k})}\left\langle x_{n_k}-x_{n_k-1}, j_q(y_{n_k}-z)\right\rangle + q\left\langle u-z, j_q(v_{n_k}-z)\right\rangle\right)\\
&= \limsup_{k\to\infty}\frac{q\alpha_{n_k}(1-\beta_{n_k}(1-\gamma_{n_k}))}{\beta_{n_k}(1-\gamma_{n_k})}\left\langle x_{n_k}-x_{n_k-1}, j_q(y_{n_k}-z)\right\rangle\\
&\quad + q\limsup_{k\to\infty}\left\langle u-z, j_q(v_{n_k}-z)\right\rangle\\
&\leq \limsup_{k\to\infty}\frac{q\alpha_{n_k}(1-\beta_{n_k}(1-\gamma_{n_k}))}{\beta_{n_k}(1-\gamma_{n_k})}\left\|x_{n_k}-x_{n_k-1}\right\|\left\|y_{n_k}-z\right\|^{q-1}\\
&\quad + q\limsup_{k\to\infty}\left\langle u-z, j_q(v_{n_k}-z)\right\rangle\\
&\leq 0.
\end{aligned}$$

That is, $\limsup_{k\to\infty}\tau_{n_k}\leq 0$. By Lemma 2.5, we can conclude that $\lim_{n\to\infty} d_n = 0$. Therefore,

$$\lim_{n\to\infty} x_n = z \in (A+B)^{-1}0,$$

which completes the proof.

Corollary 3.1. *Setting $\alpha_n = 0$ in Algorithm 3.1, we get a relaxed Halpern-type forward-backward splitting algorithm.*

4. Numerical illustration

In this section, we give a numerical example to illustrate the performance of our proposed Algorithm 3.1 in the setting of $l_4(\mathbb{R})$. Furthermore, we shall study the effect of the relaxation parameter and inertial parameter in the performance of our proposed algorithm. We consider the classical Banach space:

$$l_4(\mathbb{R}) = \left\{\{x_n\}\subset\mathbb{R} : \sum_{n=1}^{\infty}|x_n|^4<\infty\right\} \text{ with norm } \|x\| = \Big(\sum_{n=1}^{\infty}|x_n|^4\Big)^{\frac{1}{4}}.$$

For the purpose of numerical illustration, we considered the subspace of $l_4(\mathbb{R})$ consisting of finite nonzero terms

$$D_k := \Big\{\{x_n\}\subset\mathbb{R} : \{x_n\} = \{x_1, x_2, \cdots, x_k, 0, 0, 0, \cdots\}\Big\}, \text{ for some } k\geq 1.$$

This is to enable us compute the norm of a vector $x\in l_4(\mathbb{R})$.

Consider the space D_4. Let $A, B : D_4 \to D_4$ be defined by

$$Ax := 5x + \Big(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 0, 0, \cdots\Big), \quad \text{and} \qquad Bx := \frac{3}{2}x.$$

Then it is easy to show that A is 5-inverse strongly accretive and B is m-accretive. Furthermore, the set $\Omega = -\frac{1}{6.5}\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 0, 0, \cdots\right)$. In Algorithm 3.1, we choose $\beta_n = \frac{1}{1000n+1}$, $\gamma_n = \frac{1}{(n+1)^3}$, $\lambda_n = 0.5$ and we study the performance of the algorithm as we vary θ_n as presented in Table 1. In addition, we initialized the vector u to be zeros and choose $x_0 = (2, 1, 3, 0, 0, 0, 0, \cdots)$ and $x_1 = (2, 0, 1, 1, 0, 0, \cdots)$. We set maximum number of iterations $n = 200$ using a tolerance of 10^{-5}. The results of the experiment are presented in Tables 1 and 2.

Table 1. Numerical results for different values of the θ_n.

θ_n	n	$\|\|x_{n+1} - s\|\|$
$\frac{n}{n+1}$	26	7.12E-06
$\frac{2n}{3n+1}$	9	9.00E-06
$\frac{n}{2n+1}$	9	9.68E-06
$\frac{n}{4n+1}$	22	8.47E-06
$\frac{n}{8n+1}$	48	9.64E-06

Remark 4.1. *The results from the experiment presented in Table 1 above suggests that as the relaxation parameter θ_n increases to one but NOT one, we have a better approximation with less number of iterations.*

From Table 1, we saw that choosing the relaxation parameter $\theta_n = \frac{2n}{3n+1}$ gave the best approximation. So, in the next table, with this choice of $\theta_n = \frac{2n}{3n+1}$, we shall investigate the performance of Algorithm 3.1 as we vary the inertial parameter, α_n. From Step 1, first choose $\alpha_n = \overline{\alpha}_n$. Then, we choose $\overline{\alpha}_n = \alpha$ and vary α_n be a constant less than α. Choosing $\epsilon_n = \frac{1}{(n+1)^6}$ and $\alpha = 0.999$ we obtain the following results:

Table 2. Numerical results for the varied inertial parameter α_n.

α_n	θ_n	n	$\|\|x_{n+1} - s\|\|$
$\frac{n}{n+1}$	$\frac{2n}{3n+1}$	26	7.12E-06
$\overline{\alpha}_n$	$\frac{2n}{3n+1}$	6	8.95E-06
0.9	$\frac{2n}{3n+1}$	199	2.27E14
0.5	$\frac{2n}{3n+1}$	55	9.01E-06
0.1	$\frac{2n}{3n+1}$	10	8.80E-06
0.001	$\frac{2n}{3n+1}$	9	9.01E-06

Remark 4.2. *The results presented in Table 2 above suggest that the choice of the inertial parameter*

defined in Step 1 of Algorithm 3.1 as $\alpha_n = \overline{\alpha}_n$ *gives better approximation and satisfies the tolerance in fewer number of iterations. Also, we observed that in this example as that as we choose the inertial parameter close to one the algorithm diverges.*

5. Conclusions

This paper presents a relaxed inertial Halpern-type forward-backward splitting algorithm involving accretive operators in the setting of uniformly convex and q-uniformly smooth real Banach spaces. Strong convergence of the sequence generated by the proposed method is proved to a solution of the variational inclusion problem (1.1). Furthermore, a relaxed Halpern-type forward-backward splitting algorithm for solving the variational inclusion problem (1.1) is obtained as a corollary. Finally, a numerical example that demonstrates the effect of the inertial and relaxation parameters is presented.

Acknowledgments

The authors acknowledge the financial support provided by Mid-Career Research Grant (N41A640089). Finally, the authors would like to express their gratitude to the anonymous referees for their valuable comments and suggestions which improve this paper.

Conflict of interest

The authors declare that they have no conflict of interest.

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