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*Research article*

# Blow-up solution and analyticity to a generalized Camassa-Holm equation

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**Abstract:** A generalized Camassa-Holm equation with higher order nonlinear term was studied. First, we give a new blow-up criterion and a new blow-up phenomenon to the Cauchy problem for the equation under some conditions. Then, we focus on the analytical solutions for the equation. Finally, we prove the analyticity of solution for the equation.

**Keywords:** blow-up criterion; blow-up phenomenon; analytical solutions; analyticity; a generalized Camassa-Holm equation
**Mathematics Subject Classification:** 35D05, 35G25, 35L05, 35Q35

## 1. Introduction

In this paper, we are concerned with the Cauchy problem for the generalized Camassa-Holm equation

$$(1-\partial_x^2)u_t + \frac{(k+1)(k+2)}{2}u^k u_x - \frac{k(k-1)}{2}u^{k-2}u_x^3 - 2ku^{k-1}u_x u_{xx} - u^k u_{xxx} = 0, \quad k \in \mathbb{N}^+, \tag{1.1}$$

$$u(0,x) = u_0(x), \tag{1.2}$$

which is introduced by S. Hakkaev and K. Kirchev [14, 15]. In Eq (1.1), $u = u\,(t, x)$ stands for the fluid velocity at time $t > 0$ in the spatial direction. Equation (1.1) admits following conservative laws

$$E = \int_{\mathbb{R}} \frac{1}{2}(u^{k+2} + u^k u_x^2)dx, \tag{1.3}$$

$$F = \int_{\mathbb{R}} \frac{1}{2}(u^2 + u_x^2)dx, \tag{1.4}$$

and

$$H = \int_{\mathbb{R}} u dx. \tag{1.5}$$

An important feature of Eq (1.1) is the existence of traveling solitary waves, interacting like solitons, also called peakons

$$u(t,x) = c^{\frac{1}{k}} e^{-|x-ct|}. \tag{1.6}$$

Since then, Eq (1.1) attracted lots of the attentions in the last few years. The well-posedness of the solutions for Eq (1.1) is obtained by parabolic regularization method [14], Katos semigroup approach [20] and by classical Friedrichss regularization method [21], respectively. In [17], Lai and Wu obtained a sufficient condition for the existence of weak solutions of Eq (1.1) in lower order Sobolev space $H^s(\mathbb{R})$ with $1 < s \leq 3/2$. Yan [23] proved that Eq (1.1) does not depend uniformly continuously on the initial data in $H^s(\mathbb{R})$ with $s < 3/2$ and that the Cauchy problem for the generalized Camassa-Holm equation is locally well-posedness in $B^{3/2}_{2,1}$. In [24], Zhou focused on the persistence property in weighted $L^p$ spaces.

When $k = 1$, Eq (1.1) reduced to the well-known Camassa-Holm equation

$$(1 - \partial_x^2)u_t = -3uu_x + 2u_x u_{xx} + uu_{xxx}, \tag{1.7}$$

which was derived by Camassa and Holm [1] and by Fokas and Fuchssteiner [11]. It describes the motion of shallow water waves and possesses soliton solutions, a Lax pair, a bi-Hamiltonian structures and infinitely many conserved integrals [1, 32], and it can be solved by the Inverse Scattering Method. The dynamic properties related the equation can be found in [3–5, 10, 12–16, 18–20, 22, 25] and the references therein. It is well-known that a major interest in water waves is the existence of breaking waves (solutions that remain bounded but whose slope becomes unbounded in finite time [7]). Comparing with KdV equation, another important feature of Camassa-Holm equation is that it possesses breaking wave [6–9].

To our best knowledge, blow-up, analyticity and analytical solutions have not been investigated yet for the problems (1.1) and (1.2). Inspired by the ideas from [7], the objective of this paper is to investigate the blow-up phenomenon, analyticity and analytical solutions for the problems (1.1) and (1.2). In our blow-up phenomenon analysis, the quantity $\int_{\mathbb{R}}((u^k)_x)^3dx$ plays a key role. Taking advantage of complicated calculation, we obtain the Riccati inequality of quantity $\int_{\mathbb{R}}((u^k)_x)^3dx$ to arrive at a new blow-up result. In addition, we present some analytical solutions for the problems (1.1) and (1.2). Finally, we prove the analyticity. The results we obtained complements earlier results in this direction.

**Notations.** The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty)\times\mathbb{R}$ is denoted by $C_0^\infty$. Let $L^p = L^p(\mathbb{R})(1 \leq p < +\infty)$ be the space of all measurable functions $h$ such that $\| h \|^P_{L^P} = \int_{\mathbb{R}} |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(\mathbb{R})$ with the standard norm $\| h \|_{L^\infty} = \inf_{m(e)=0}\sup_{x\in R\setminus e}|h(t,x)|$. For any real number $s$, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space with the norm defined by

$$\| h \|_{H^s} = \left( \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{h}(t,\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty,$$

where $\hat{h}(t,\xi) = \int_{\mathbb{R}} e^{-ix\xi} h(t,x)dx$.

We denote by $*$ the convolution, using the green function $G(x) = \frac{1}{2}e^{-|x|}$, we have $(1 - \partial_x^2)^{-1} f = G(x) * f$ for all $f \in L^2$,and $p * (u - u_{xx}) = u$. For $T > 0$ and nonnegative number $s$, $C([0,T); H^s(\mathbb{R}))$

denotes the Frechet space of all continuous $H^s$-valued functions on $[0, T)$. For simplicity, throughout this article, we let $c$ denote any positive constant

The Cauchy problems (1.1) and (1.2) is equivalent to

$$u_t + u^k u_x = -\partial_x(1-\partial_x^2)^{-1}\Big(\frac{k(k+3)}{2(k+1)}u^{k+1} + \frac{k}{2}u^{k-1}u_x^2\Big), \tag{1.8}$$

$$u(0,x) = u_0(x), \tag{1.9}$$

which is also equivalent to

$$y_t + u^k y_x + 2ku^{k-1}u_x y + \frac{k(k-1)}{2}(u^k u_x - u^{k-2}u_x^3) = 0, \tag{1.10}$$

$$y = u - u_{xx}, \quad u(0,x) = u_0(x). \tag{1.11}$$

The rest sections are organized as follows. In the second section, we give a blow up criterion and a new blow up phenomenon. Existence of weak solution (CH-type peakon) and analytical solutions are studied in third section. In the forth section, we proved analyticity of strong solutions..

## 2. Blow-up criterion and blow-up phenomenon

We firstly give some useful Lemmas.

**Lemma 2.1.** *Given $u(x,0) = u_0 \in H^s(\mathbb{R})$, $s > 3/2$, then there exist a maximal $T = T(u_0)$ and a unique solution $u$ to the problems (1.1) and (1.2) such that*

$$u = u(\cdot, u_0) \in C([0,T); H^s(\mathbb{R})) \bigcap C^1([0,T); H^{s-1}(\mathbb{R})).$$

*Moreover, the solution depends continuously on the initial data, i.e., the mapping $u_0 \to u(\cdot, u_0) : H^s \to C([0,T); H^s(\mathbb{R})) \bigcap C^1([0,T); H^{s-1}(\mathbb{R}))$ is continuous.*

*Proof.* Using the Kato's theorem [27], we can prove the above theorem. Because there exist some similarities, here we omit the proof of Lemma 2.1, a detailed proof can be found in [26].

**Lemma 2.2.** ( *[2]*) Let $f \in C^1(\mathbb{R})$, $a > 0$, $b > 0$ and $f(0) > \sqrt{\frac{b}{a}}$. If $f'(t) \geq af^2(t) - b$, then

$$f(t) \to +\infty \quad as \quad t \to T = \frac{1}{2\sqrt{ab}} \log\Bigg(\frac{f(0) + \sqrt{\frac{b}{a}}}{f(0) - \sqrt{\frac{b}{a}}}\Bigg). \tag{2.1}$$

**Lemma 2.3.** (*see [23]*) *Let $u_0 \in B^{3/2}_{2,1}$ and $u$ be the corresponding solution to (1.1). Assume that $T$ is the maximal time of existence of the solution to the problems (1.1) and (1.2). If $T < \infty$, then*

$$\int_0^T \| u_x \|_{L^\infty} \, d\tau = +\infty. \tag{2.2}$$

**Remark 1.** For $s > \frac{3}{2}$, it is well known that $H^s \hookrightarrow B^{3/2}_{2,1}$, so we have the following result:

*Let $u_0 \in H^s$ with $s > \frac{3}{2}$ and $u$ be the corresponding solution to (1.1). Assume that $T$ is the maximal time of existence of the solution to the problems (1.1) and (1.2). If $T < \infty$, then*

$$\int_0^T \| u_x \|_{L^\infty} d\tau = +\infty. \tag{2.3}$$

**Lemma 2.4.** *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Let $T > 0$ be the maximum existence time of the solution $u$ to the problems (1.1) and (1.2) with the initial data $u_0$. Then the corresponding solution $u$ blows up in finite time if and only if*

$$\lim_{t\to T^-} \| u^{k-1}u_x \|_{L^\infty} = +\infty.$$

*Proof*. Applying Lemma 2.1 and a simple density argument, it suffices to consider the case $s = 3$. Let $T > 0$ be the maximal time of existence of solution $u$ to the problems (1.1) and (1.2) with initial data $u_0 \in H^3(\mathbb{R})$. From Lemma 2.1 we know that $u \in C([0,T);H^3(\mathbb{R})) \bigcap C^1([0,T);H^2(\mathbb{R}))$. Due to $y = u - u_{xx}$, by direct computation, one has

$$\| y \|_{L^2}^2 = \int_{\mathbb{R}} (u - u_{xx})^2 dx = \int_{\mathbb{R}} (u^2 + 2u_x^2 + u_{xx}^2) dx. \tag{2.4}$$

So,

$$\| u \|_{H^2}^2 \leq \| y \|_{L^2}^2 \leq 2 \| u \|_{H^2}^2 . \tag{2.5}$$

Multiplying equation (1.10) by $2y$ and integrating by parts and using the interpolation $\| u_x \|_{L^\infty}^2 \leq C \| u \|_{H^1} \| y \|_{L^2}$, we obtain

$$\begin{aligned} \frac{d}{dt} \| y \|_{L^2}^2 = 2\int_{\mathbb{R}} yy_t dx &= -3k \int_{\mathbb{R}} u^{k-1}u_x y^2 dx - k(k-1)\int_{\mathbb{R}} u^k u_x y dx + k(k-1)\int_{\mathbb{R}} u^{k-2}u_x^3 y dx \\ &\leq c(\| u^{k-1}u_x \|_{L^\infty} \| y \|_{L^2}^2 + \| u^k \|_{L^\infty} \| y \|_{L^2} \| u_x \|_{L^2} + \| u^{k-2} \|_{L^\infty} \| u_x^2 \|_{L^\infty} \| y \|_{L^2} \| u_x \|_{L^2}) \\ &\leq c(\| u^{k-1}u_x \|_{L^\infty} + C_1) \| y \|_{L^2}^2 . \end{aligned} \tag{2.6}$$

where $C_1 = C_1(\| u_0 \|_{H^1})$.

If there exists a constant $M > 0$ such that $\| u^{k-1}u_x \|_{L^\infty} < M$, from (2.6) we deduce that

$$\frac{d}{dt} \| y \|_{L^2}^2 \leq c(M + C_1) \| y \|_{L^2}^2 . \tag{2.7}$$

By virtue of Gronwall's inequality, one has

$$\| y \|_{L^2}^2 \leq \| y_0 \|_{L^2}^2 e^{c(M+C_1)t}. \tag{2.8}$$

On the other hand, due to $u = p * y$ and $u_x = p_x * y$, then

$$\| u^{k-1}u_x \|_{L^\infty} \leq \| u \|_{L^\infty}^{k-1} \| u_x \|_{L^\infty} \leq \| p \|_{L^2}^{k-1} \| p_x \|_{L^2} \| y \|_{L^2}^k$$

This completes the proof of Lemma 2.4.

Now, we present the blow-up phenomenon.

**Theorem 2.5.** *Let $u_0 \in H^s(\mathbb{R})$ for $s > \frac{3}{2}$. Suppose that $u(t,x)$ be corresponding solution of problems (1.1) and (1.2) with the initial datum $u_0$. If the slope of $u_0^k$ satisfies*

$$\int_{\mathbb{R}} (u_0^k)_x^3 < -\frac{K}{K_1}, \tag{2.9}$$

*where $K = \frac{3k(6k^2+7k-2)}{4(k+1)} \parallel u_0 \parallel_{H^1}^{2k}$ and $K_1 = \frac{1}{2ck\|u_0\|_{H^1}^k}$. Then there exists the lifespan $T < \infty$ such that the corresponding solution $u(t,x)$ blows up in finite time $T$ with*

$$T = \frac{1}{2KK_1} \log\Big(\frac{K_1 h(0) - K}{K_1 h(0) + K}\Big). \tag{2.10}$$

*Proof.* Defining $g(t) = u^k(t,x)$, $h(t) = \int_{\mathbb{R}} g_x^3 dx$, it follows that

$$g_t + g g_x = k u^{k-1} Q, \tag{2.11}$$

where $Q = -\partial_x (1-\partial_x^2)^{-1} (\frac{k(k+2)}{2(k+1)} u^{k+1} + \frac{k}{2} u^{k-1} u_x^2)$.

Differentiating the above Eq (2.11) with respect $x$ yields

$$\begin{aligned} g_{tx} + g g_{xx} = & -\frac{1}{2} g_x^2 + k(k-1) u^{k-2} u_x Q + \frac{k^2(k+2)}{2(k+1)} g^2 - \frac{k^2(k+2)}{2(k+1)} u^{k-1} (1-\partial_x^2)^{-1} (u^{k+1}) \\ & -\frac{k^2}{2} u^{k-1} (1-\partial_x^2)^{-1} (u^{k-1} u_x^2). \end{aligned} \tag{2.12}$$

Multiplying $3g_x^2$ both sides of (2.12) and integrating with respect $x$ over $\mathbb{R}$, one has

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} g_x^3 dx = & -\frac{1}{2} \int_{\mathbb{R}} g_x^4 dx + 3k(k-1) \int_{\mathbb{R}} u^{k-2} u_x g_x^2 Q dx + \frac{3k^2(k+2)}{2(k+1)} \int_{\mathbb{R}} g^2 g_x^2 dx \\ & -\frac{3k^2(k+2)}{2(k+1)} \int_{\mathbb{R}} g_x^2 u^{k-1} (1-\partial_x^2)^{-1} (u^{k+1}) dx - \frac{3k^2}{2} \int_{\mathbb{R}} g_x^2 u^{k-1} (1-\partial_x^2)^{-1} (u^{k-1} u_x^2) dx \\ = & \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5. \end{aligned} \tag{2.13}$$

Using the Hölder's inequality, Yong's inequality and (1.4), from (2.13) we get

$$\begin{aligned} \Gamma_2 = & 3k(k-1) \int_{\mathbb{R}} u^{k-2} u_x g_x^2 Q dx \\ \le & 3k(k-1) \parallel Q \parallel_{L^\infty} (\int_{\mathbb{R}} (u^{k-2} u_x)^2 dx)^{\frac{1}{2}} (\int_{\mathbb{R}} (g_x)^2 dx)^{\frac{1}{2}} \\ \le & \frac{3k(k-1)}{2} \parallel u_0 \parallel_{H^1}^{2k} (\int_{\mathbb{R}} (g_x)^4 dx)^{\frac{1}{2}} \\ \le & \frac{3k(k-1)}{2} (\frac{\parallel u_0 \parallel_{H^1}^{4k}}{2\epsilon} + \frac{\epsilon \int_R (g_x)^4 dx}{2}), \end{aligned} \tag{2.14}$$

$$\Gamma_3 = \frac{3k^2(k+2)}{2(k+1)} \int_{\mathbb{R}} g^2 g_x^2 dx$$

$$
\begin{aligned}
&\leq \frac{3k^2(k+2)}{2(k+1)}(\int_{\mathbb{R}} g^4dx)^{\frac{1}{2}}(\int_{\mathbb{R}}(g_x)^4dx)^{\frac{1}{2}}\\
&\leq \frac{3k^2(k+2)}{2(k+1)} \parallel u_0 \parallel_{H^1}^{2k} (\int_{\mathbb{R}}(g_x)^4dx)^{\frac{1}{2}}\\
&\leq \frac{3k^2(k+2)}{2(k+1)}(\frac{\parallel u_0 \parallel_{H^1}^{4k}}{2\epsilon}+\frac{\epsilon\int_{\mathbb{R}}(g_x)^4dx}{2}),
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
\Gamma_4 &= \frac{3k^2(k+2)}{2(k+1)}\int_{\mathbb{R}} g_x^2u^{k-1}(1-\partial_x^2)^{-1}(u^{k+1})dx\\
&\leq \frac{3k^2(k+2)}{2(k+1)} \parallel (1-\partial_x^2)^{-1}(u^{k+1}) \parallel_{L^\infty} (\int_{\mathbb{R}} u^{2k-2}dx)^{\frac{1}{2}}(\int_{\mathbb{R}}(g_x)^4dx)^{\frac{1}{2}}\\
&\leq \frac{3k^2(k+2)}{2(k+1)} \parallel u_0 \parallel_{H^1}^{2k} (\int_{\mathbb{R}}(g_x)^4dx)^{\frac{1}{2}}\\
&\leq \frac{3k^2(k+2)}{2(k+1)}(\frac{\parallel u_0 \parallel_{H^1}^{4k}}{2\epsilon}+\frac{\epsilon\int_{\mathbb{R}}(g_x)^4dx}{2}),
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
\Gamma_5 &= \frac{3k^2}{2}\int_{\mathbb{R}} g_x^2u^{k-1}(1-\partial_x^2)^{-1}(u^{k-1}u_x^2)dx\\
&\leq \frac{3k^2}{2} \parallel (1-\partial_x^2)^{-1}(u^{k-1}u_x^2) \parallel_{L^\infty} (\int_{\mathbb{R}} u^{2k-2}dx)^{\frac{1}{2}}(\int_{\mathbb{R}}(g_x)^4dx)^{\frac{1}{2}}\\
&\leq \frac{3k^2}{2} \parallel u_0 \parallel_{H^1}^{2k} (\int_{\mathbb{R}}(g_x)^4dx)^{\frac{1}{2}}\\
&\leq \frac{3k^2}{2}(\frac{\parallel u_0 \parallel_{H^1}^{4k}}{2\epsilon}+\frac{\epsilon\int_{\mathbb{R}}(g_x)^4dx}{2}).
\end{aligned}
\tag{2.17}
$$

Combining the above inequalities (2.14)–(2.17), we obtain

$$
\mid \Gamma_2 \mid + \mid \Gamma_3 \mid + \mid \Gamma_4 \mid + \mid \Gamma_5 \mid \leq \frac{3k(6k^2+7k-2)}{8\epsilon(k+1)} \parallel u_0 \parallel_{H^1}^{4k} + \frac{\epsilon}{2}\frac{3k(6k^2+7k-2)}{4(k+1)}\int_{\mathbb{R}}(g_x)^4dx. \tag{2.18}
$$

Choosing $\epsilon = \frac{2(k+1)}{3k(6k^2+7k-2)}$, which results in

$$
\frac{d}{dt}\int_{\mathbb{R}} g_x^3dx \leq -\frac{1}{4}\int_{\mathbb{R}} g_x^4dx + K^2. \tag{2.19}
$$

in which $K^2 = \frac{9k^2(6k^2+7k-2)^2}{16(k+1)^2} \parallel u_0 \parallel_{H^1}^{4k}$. Using the Hölder's inequality, we get

$$
(\int_{\mathbb{R}} g_x^3dx)^2 \leq c\int_{\mathbb{R}} g_x^2dx\int_R g_x^4dx \leq ck^2 \parallel u_0 \parallel_{H^1}^{2k} \int_{\mathbb{R}} g_x^4dx. \tag{2.20}
$$

Combining (2.19) and (2.20), we have

$$
\frac{d}{dt}h(t) \leq -K_1^2h^2(t) + K^2, \tag{2.21}
$$

where $K_1^2 = \frac{1}{4ck^2\|u_0\|_{H^1}^{2k}}$.

It is observed from assumption of Theorem that $h(0) < -\frac{K}{K_1}$, the continuity argument ensures that $h(t) < h(0)$. Lemma 2.1 ($a = K_1^2$ and $b = K^2$) implies that $h(t) \to -\infty$ as $t \to T = \frac{1}{2K_1K} \log \frac{K_1h(0)-K}{K_1h(0)+K}$.

On the other hand, by using the fact that

$$| \int_{\mathbb{R}} g_x^3 dx | \leq \int_{\mathbb{R}} | g_x^3 | dx \leq k^3 \| u^{k-1}u_x(t,x) \|_{L^\infty} \int_{\mathbb{R}} g_x^2 dx = k^3 \| u^{k-1}u_x(t,x) \|_{L^\infty} \| u_0 \|_{H^1}^{2k} . \tag{2.22}$$

Lemma 2.4 implies that the Theorem 2.5 is true. This completes the proof of Theorem 2.5.

## 3. Analytical solutions

The solitons do not belong to the spaces $H^s(\mathbb{R})$ with $s > \frac{3}{2}$ [28, 29], so it motivates us to carry out the study of analytical solutions to problems (1.1) and (1.2).

**Definition 3.1.** *Given initial data $u_0 \in H^s$, $s > \frac{3}{2}$, the function $u$ is said to be a weak solution to the initial-value problems (1.8) and (1.9) if it satisfies the following identity*

$$\int_0^T \int_{\mathbb{R}} u\varphi_t + \frac{1}{k+1}u^{k+1}\varphi_x + G * (\frac{k(k+3)}{2(k+1)}u^{k+1} + \frac{k}{2}u^{k-1}u_x^2)\varphi_x dxdt$$
$$+ \int_R u_0(x)\varphi(0,x)dx = 0 \tag{3.1}$$

*for any smooth test function $\varphi(t,x) \in C_c^\infty([0,T) \times \mathbb{R})$. If $u$ is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.*

**Theorem 3.2** *The peakon function of the form*

$$u(t,x) = p(t)e^{-|x-q(t)|} \tag{3.2}$$

*is a global weak solution to problems (1.1) and (1.2) in the sense of Definition 3.1. Assumed that the functions $p(t)$ and $q(t)$ satisfy*

$$p^{k+1}(t) - p(t)q'(t) - p'(t) = 0,$$

*and*

$$p^{k+1}(t) - p(t)q'(t) + p'(t) = 0.$$

*where $'$ denotes differentiation.*

*Proof.* We firstly claim that

$$u = p(t)e^{-|x-q(t)|} \tag{3.3}$$

is a peakon solution of (1.1) and

$$u_t = p'(t)e^{-|x-q(t)|} + p(t)sign(x-q(t))q'(t)e^{-|x-q(t)|}, \quad u_x = -p(t)sign(x-q(t))e^{-|x-q(t)|}. \tag{3.4}$$

Hence, using (3.1), (3.4) and integration by parts, we derive that

$$\int_0^T \int_{\mathbb{R}} u\varphi_t + \frac{1}{k+1}u^{k+1}\varphi_x dxdt + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx$$

$$
\begin{aligned}
&= -\int_0^T \int_{\mathbb{R}} \varphi(u_t + u^k u_x) dxdt \\
&= -\int_0^T \int_{\mathbb{R}} \varphi[p'(t)e^{-|x-q(t)|} + sign(x - q(t))(p(t)q'(t)e^{-|x-q(t)|} - p^{k+1}e^{-(k+1)|x-q(t)|})]dxdt. \quad (3.5)
\end{aligned}
$$

On the other hand, using (3.4), we obtain

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} G * (\frac{k(k+3)}{2(k+1)}u^{k+1} + \frac{k}{2}u^{k-1}u_x^2)\varphi_x dxdt \\
&= \int_0^T \int_{\mathbb{R}} -\varphi G_x * [\frac{k}{2}u^{k-1}u_x^2 + \frac{k(k+3)}{2(k+1)}u^{k+1}]dxdt \\
&= \int_0^T \int_{\mathbb{R}} -\varphi G_x * [\frac{k(k+2)}{k+1}u^{k+1}]dxdt. \quad (3.6)
\end{aligned}
$$

Note that $G_x = -\frac{1}{2}sign(x)e^{-|x|}$. For $x > q(t)$, directly calculate

$$
\begin{aligned}
&G_x * [\frac{k(k+2)}{k+1}u^{k+1}] \\
&= -\frac{1}{2}\int_{\mathbb{R}} sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}p^{k+1}(t)e^{-(k+1)|y-q(t)|}dy \\
&= -\frac{1}{2}(\int_{-\infty}^{q(t)} + \int_{q(t)}^{x} + \int_{x}^{\infty})sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}p^{k+1}(t)e^{-(k+1)|y-q(t)|}dy \\
&= I_1 + I_2 + I_3. \quad (3.7)
\end{aligned}
$$

We directly compute $I_1$ as follows

$$
\begin{aligned}
I_1 &= -\frac{1}{2}\int_{-\infty}^{q(t)} sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}p^{k+1}(t)e^{-(k+1)|y-q(t)|}dy \\
&= -\frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}(t)\int_{-\infty}^{q(t)} e^{-x-(k+1)q(t)}e^{(k+2)y}dy \\
&= -\frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}(t)e^{-x-(k+1)q(t)}\int_{-\infty}^{q(t)} e^{(k+2)y}dy \\
&= -\frac{1}{2(k+2)}\frac{k(k+2)}{k+1}p^{k+1}(t)e^{-x+q(t)}. \quad (3.8)
\end{aligned}
$$

In a similar procedure,

$$
\begin{aligned}
I_2 &= -\frac{1}{2}\int_{q(t)}^{x} sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}a^{k+1}e^{-(k+1)|y-q(t)|}dy \\
&= -\frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}\int_{q(t)}^{x} e^{-x+(k+1)q(t)}e^{-ky}dy \\
&= -\frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}e^{-x+(k+1)q(t)}\int_{q(t)}^{x} e^{-ky}dy \\
&= \frac{1}{2k}\frac{k(k+2)}{k+1}p^{k+1}(e^{-(k+1)(x-q(t))} - e^{-x+q(t)}), \quad (3.9)
\end{aligned}
$$

and

$$\begin{aligned}
I_3 &= -\frac{1}{2}\int_x^{\infty} sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}p^{k+1}e^{-(k+1)|y-q(t)|}dy \\
&= \frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}\int_x^{\infty} e^{x+(k+1)q(t)}e^{-(k+2)y}dy \\
&= \frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}e^{x+(k+1)q(t)}\int_x^{\infty} e^{-(k+2)y}dy \\
&= \frac{1}{2(k+2)}\frac{k(k+2)}{k+1}p^{k+1}e^{-(k+1)(x-q(t))}. \qquad (3.10)
\end{aligned}$$

Substituting (3.8)–(3.10) into (3.7), we deduce that for $x > q(t)$

$$\begin{aligned}
G_x * [\frac{k(k+2)}{k+1}u^{k+1}] &= \frac{2(k+1)}{k(k+2)}\Omega e^{-x+q(t)} - \frac{2(k+1)}{k(k+2)}\Omega e^{-(k+1)(x-q(t))} \\
&= -p^{k+1}(t)e^{-x+q(t)} + p^{k+1}(t)e^{-(k+1)(x-q(t))}, \qquad (3.11)
\end{aligned}$$

where $\Omega = -\frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}(t)$.

For $x < q(t)$,

$$\begin{aligned}
&G_x * [(\frac{k(k+2)}{k+1}u^{k+1}] \\
&= -\frac{1}{2}\int_{\mathbb{R}} sign(x-y)e^{-|x-y|}(\frac{k(k+2)}{k+1}p^{k+1}(t)e^{-(k+1)|y-q(t)|}dy \\
&= -\frac{1}{2}(\int_{-\infty}^{x} + \int_x^{q(t)} + \int_{q(t)}^{\infty})sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}p^{k+1}(t)e^{-(k+1)|y-q(t)|}dy \\
&= \Delta_1 + \Delta_2 + \Delta_3. \qquad (3.12)
\end{aligned}$$

We directly compute $\Delta_1$ as follows

$$\begin{aligned}
\Delta_1 &= -\frac{1}{2}\int_{-\infty}^{x} sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}p^{k+1}e^{-(k+1)|y-q(t)|}dy \\
&= -\frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}(t)\int_{-\infty}^{x} e^{-x-(k+1)q(t)}e^{(k+2)y}dy \\
&= -\frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}(t)e^{-x-(k+1)q(t)}\int_{-\infty}^{x} e^{(k+2)y}dy \\
&= -\frac{1}{2(k+2)}\frac{k(k+2)}{k+1}p^{k+1}(t)e^{(k+1)(x-q(t))}. \qquad (3.13)
\end{aligned}$$

In a similar procedure,

$$\begin{aligned}
\Delta_2 &= -\frac{1}{2}\int_x^{ct} sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}p^{k+1}e^{-(k+1)|y-q(t)|}dy \\
&= \frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}(t)\int_x^{q(t)} e^{x-(k+1)q(t)}e^{ky}dy \\
&= \frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}(t)e^{x-(k+1)q(t)}\int_x^{q(t)} e^{ky}dy
\end{aligned}$$

$$= \frac{1}{2k}\frac{k(k+2)}{k+1}p^{k+1}(t)(-e^{(k+1)(x-q(t))} + e^{x-q(t)}), \tag{3.14}$$

and

$$\begin{aligned}\Delta_3 &= -\frac{1}{2}\int_{q(t)}^{\infty} sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}p^{k+1}(t)e^{-(k+1)|y-q(t)|}dy\\ &= \frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}(t)\int_{q(t)}^{\infty} e^{x+(k+1)q(t)}e^{-(k+2)y}dy\\ &= \frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}(t)e^{x+(k+1)q(t)}\int_{q(t)}^{\infty} e^{-(k+2)y}dy\\ &= \frac{1}{2(k+2)}\frac{k(k+2)}{k+1}p^{k+1}(t)e^{x-q(t)}.\end{aligned} \tag{3.15}$$

Therefore, from (3.8)–(3.10), we deduce that for $x < q(t)$

$$\begin{aligned}&G_x * [\frac{k(k+2)}{k+1}u^{k+1}]\\ &= \frac{2(k+1)}{k(k+2)}\Theta e^{x-q(t)} - \frac{2(k+1)}{k(k+2)}\Theta e^{(k+1)(x-q(t))}\\ &= p^{k+1}(t)e^{x-q(t)} - p^{k+1}(t)e^{(k+1)(x-q(t))},\end{aligned} \tag{3.16}$$

where $\Theta = \frac{1}{2}\frac{k(k+2)}{k+1}p^{k+1}(t)$.

Recalling $u = p(t)e^{-|x-q(t)|}$, we have

$$\begin{aligned}&p'(t)e^{-|x-q(t)|} + sign(x-q(t))(p(t)q'(t)e^{-|x-q(t)|} - p^{k+1}e^{-(k+1)|x-q(t)|})\\ &= \begin{cases} p'(t)e^{-x+q(t)} + p(t)q'(t)e^{-x+q(t)} - p^{k+1}e^{-(k+1)(x-q(t)}, & \text{for} \quad x > q(t),\\ p'(t)e^{x-q(t)} - p(t)q'(t)e^{x-q(t)} + p^{k+1}e^{(k+1)(x-q(t))}, & \text{for} \quad x \le q(t).\end{cases}\end{aligned}$$

To ensure that $u = p(t)e^{-|x-q(t)|}$ is a global weak solution of (1.1) in the sense of Definition 3.1, we infer that

$$p^{k+1}(t) - p(t)q'(t) - p'(t) = 0 \tag{3.17}$$

and

$$p^{k+1}(t) - p(t)q'(t) + p'(t) = 0 \tag{3.18}$$

hold.

It completes the proof of Theorem 3.2.

**Remark 2.** Solving Eqs (3.17) and (3.18), we get

$$p(t) = c^{\frac{1}{k}}, and \quad q(t) = ct + x_0 \quad c > 0. \tag{3.19}$$

Therefore, we conclude that peakon solution for problems (1.1) and (1.2)

$$u = c^{\frac{1}{k}}e^{-|x-ct-x_0|}, \quad c > 0. \tag{3.20}$$

**Remark 3.** For $x > q(t)$, the solution of problems (1.1) and (1.2) is of following form

$$u = p(t)e^{-x+q(t)}, \tag{3.21}$$

where $p(t)$ and $q(t)$ satisfy

$$p^{k+1}(t) - p(t)q'(t) - p'(t) = 0. \tag{3.22}$$

For $x < q(t)$, the solution of problems (1.1) and (1.2) is of following form

$$u = p(t)e^{x-q(t)}, \tag{3.23}$$

where $p(t)$ and $q(t)$ satisfy

$$p^{k+1}(t) - p(t)q'(t) + p'(t) = 0. \tag{3.24}$$

**Example.** For $x > q(t)$, letting $q(t) = \sqrt{t} + c, c > 0$, from (3.17) we derive that

$$p' + \frac{1}{2\sqrt{t}}p - p^{k+1} = 0. \tag{3.25}$$

(3.25) implies that

$$p = (2\sqrt{t} + \frac{2}{k})^{-\frac{1}{k}}. \tag{3.26}$$

Hence, we obtain from (3.3) the solution of (1.1) for $x > q(t)$.

$$u = (2\sqrt{t} + \frac{2}{k})^{-\frac{1}{k}} e^{-x+\sqrt{t}+c}. \tag{3.27}$$

For $x < q(t)$, letting $q(t) = \sqrt{t} + c, c > 0$, from (3.18) we derive that

$$p' - \frac{1}{2\sqrt{t}}p + p^{k+1} = 0. \tag{3.28}$$

(3.28) implies that

$$p = (2\sqrt{t} - \frac{2}{k})^{-\frac{1}{k}}. \tag{3.29}$$

Therefore, we obtain from (3.3) the solution of (1.1) for $x < q(t)$.

$$u = (2\sqrt{t} - \frac{2}{k})^{-\frac{1}{k}} e^{x-\sqrt{t}-c}. \tag{3.30}$$

## 4. Analyticity of solution

In this section, we focus on the analyticity of the Cauchy problems (1.1) and (1.2) based on a contraction type argument in a suitably chosen scale of the Banach spaces. In order to state the main result, we will need a suitable scale of the Banach spaces as follows. For any $s > 0$, we set

$$E_s = \{u \in C^\infty(\mathbb{R}) : |||u|||_s = \sup_{k \in N_0} \frac{s^k \|\partial^k u\|_{H^2}}{k!/(k+1)^2} < \infty\},$$

where $H^2(\mathbb{R})$ is the Sobolev space of order two on the real line and $N_0$ is the set nonnegative integers. One can easily verify that $E_s$ equipped with the norm $||| \cdot |||_s$ is a Banach space and that, for any $0 < s' < s$, $E_s$ is continuously embedded in $E_{s'}$ with

$$|||u|||_{s'} \leq |||u|||_s.$$

Another simple consequence of the definition is that any $u$ in $E_s$ is a real analytic function on $\mathbb{R}$. Our main theorem is stated as follows.

**Theorem 4.1.** *If the initial data $u_0$ is analytic and belongs to a space $E_{s_0}$, for some $0 < s_0 \leq 1$, then there exist an $\varepsilon > 0$ and a unique solution $u(t, x)$ to the Cauchy problems (1.1) and (1.2) that is analytic on $(-\varepsilon, \varepsilon) \times \mathbb{R}$.*

For the proof of Theorem 4.1, we need the following lemmas

**Lemma 4.2.** ( *[30]*) *Let $0 < s < 1$. There is a constant $C > 0$, independent of $s$, such that for any $u$ and $v$ in $E_s$ we have*

$$|||uv|||_s \leq C|||u|||_s|||v|||_s.$$

**Lemma 4.3.** ( *[30]*) *There is a constant $C > 0$ such that for any $0 < s' < s < 1$, we have $|||\partial_x u|||_{s'} \leq \frac{C}{s-s'}$,and $|||(1-\partial_x^2)^{-1}u|||_s \leq |||u|||_s$, $|||\partial_x(1-\partial_x^2)^{-1}u|||_s \leq |||u|||_s$.*

**Lemma 4.4.** ( *[31]*) Let $\{X_s\}_{0<s<1}$ be a scale of decreasing Banach spaces, namely for any $s' < s$ we have $X_s \subset X_{s'}$ and $|||\cdot|||_{s'} \leq |||\cdot|||_s$. Consider the Cauchy problem

$$\frac{du}{dt} = F(t, u(t)), \tag{4.1}$$

$$u(0, x) = 0. \tag{4.2}$$

Let $T$, $R$ and $C$ be positive constants and assume that $F$ satisfies the following conditions:

(1) If for $0 < s' < s < 1$ the function $t \mapsto u(t)$ is holomorphic in $|t| < T$ and continuous on $|t| \leq T$ with values in $X_s$ and

$$\sup_{|t|\leq T} |||u(t)|||_s < R,$$

then $t \mapsto F(t, u(t))$ is a holomorphic function on $|t| < T$ with values in $X_{s'}$.

(2) For any $0 < s' < s < 1$ and only $u, v \in X_s$ with $|||u|||_s < R$, $|||v|||_s < R$,

$$\sup_{|t|\leq T} |||F(t, u) - F(t, v)|||_{s'} \leq \frac{C}{s - s'}|||u - v|||_s.$$

(3) There exists $M > 0$ such that for any $0 < s < 1$,

$$\sup_{|t|\leq T} |||F(t, 0)|||_s \leq \frac{M}{1 - s},$$

then there exist a $T_0 \in (0, T)$ and a unique function $u(t)$, which for every $s \in (0, 1)$ is holomorphic in $|t| < (1 - s)T_0$ with values in $X_s$, and is a solution to the Cauchy problems (1.1) and (1.2).

Let $u_1 = u$ and $u_2 = u_x$, then the problems (1.1) and (1.2) can be written as a system for $u_1$ and $u_2$.

$$u_{1t} = -u_1^k u_2 - \partial_x(1 - \partial_x^2)^{-1}(\frac{k(k-3)}{2(k-1)}u_1^{k+1} + \frac{k}{2}u_1^{k-1}u_2^2) = F_1(u_1, u_2), \tag{4.3}$$

$$u_{2t} = -ku_1^{k-1}u_2^2 - u_1^k u_{2x} - \partial_x^2(1 - \partial_x^2)^{-1}(\frac{k(k-3)}{2(k-1)}u_1^{k+1} + \frac{k}{2}u_1^{k-1}u_2^2) = F_2(u_1, u_2), \tag{4.4}$$

$$u_1(x, 0) = u(x, 0) = u_0(x), u_2(x, 0) = u_x(x, 0) = u_{0x}(x). \tag{4.5}$$

To apply Lemma 4.4 to prove Theorem 4.1, we rewrite the system (4.3)-(4.5) as

$$\frac{dU}{dt} = F(u_1, u_2), \tag{4.6}$$

$$U(0) = (u_0, u_0'), \tag{4.7}$$

where $U = (u_1, u_2)$ and $F(t, U) = F(u_1, u_2) = (F_1(u_1, u_2), F_2(u_1, u_2))$.

*Proof of Theorem 4.1.* Theorem 4.1 is a straightforward consequence of the Cauchy-Kowalevski theorem [31]. We only need verify the conditions (1)–(3) in the statement of the abstract Cauchy-Kowalevski theorem (see Lemma 4.4) for both $F_1(u_1, u_2)$ and $F_2(u_1, u_2)$ in the systems (4.3)–(4.5), since neither $F_1$ nor $F_2$ depends on $t$ explicitly. For $0 < s' < s < 1$, we derive from Lemmas 4.2 and 4.3 that

$$|||F_1(u_1, u_2)|||_{s'} \leq |||u_1|||_s^k |||u_2|||_s + C|||u_1|||_s^{k+1} + C|||u_1|||_s^{k-1} |||u_2|||_s^2$$

$$\begin{aligned} |||F_2(u_1, u_2)|||_{s'} \leq\ & C|||u_1|||_s^{k-1}|||u_2|||_s^2 + \frac{C}{s-s'}|||u_1|||_s^k|||u_2|||_s + \frac{C}{s-s'}|||u_1|||_s^{k+1} \\ & + \frac{C}{s-s'}|||u_1|||_s^{k-1}|||u_2|||_s^2, \end{aligned}$$

where the constant $C$ depends only on $R$, so condition (1) holds.

Notice that to verify the second condition it is sufficient to estimate

$$\begin{aligned} & |||F(u_1, u_2) - F(v_1, v_2)|||_{s'} \leq |||F_1(u_1, u_2) - F_1(v_1, v_2)|||_{s'} + |||F_2(u_1, u_2) - F_2(v_1, v_2)|||_{s'} \\ & \leq C|||u_1^k u_2 - v_1^k v_2|||_{s'} + C|||\partial_x(1-\partial_x^2)^{-1}(u_1^{k+1} - v_1^{k+1})|||_{s'} \\ & + C|||\partial_x(1-\partial_x^2)^{-1}(u_1^{k-1}u_2^2 - v_1^{k-1}v_2^2)|||_{s'} + C|||u_1^{k-1}u_2^2 - v_1^{k-1}v_2^2|||_{s'} \\ & + C|||u_1^k u_{2x} - v_1^k v_{2x}|||_{s'} + C|||\partial_x^2(1-\partial_x^2)^{-1}(u_1^{k+1} - v_1^{k+1})|||_{s'} \\ & \quad + C|||\partial_x^2(1-\partial_x^2)^{-1}(u_1^{k-1}u_2^2 - v_1^{k-1}v_2^2)|||_{s'}, \end{aligned}$$

Using Lemmas 4.2 and 4.3, we get the following estimates

$$\begin{aligned} |||u_1^k u_2 - v_1^k v_2|||_{s'} & \leq |||u_1^k u_2 - u_1^k v_2|||_{s'} + |||u_1^k v_2 - v_1^k v_2|||_{s'} \\ & \leq C|||u_2 - v_2|||_s|||u_1|||_s^k + |||u_1 - v_1|||_s|||v_2|||_s(\sum_0^{k-1}(|||u_1|||_s^{k-1-i}|||v_1|||_s^i)), \end{aligned}$$

$$\begin{aligned} |||\partial_x(1-\partial_x^2)^{-1}(u_1^{k+1} - v_1^{k+1})|||_{s'} & \leq |||u_1^{k+1} - v_1^{k+1}|||_{s'} \\ & \leq |||u_1 - v_1|||_s(\sum_0^{k}(|||u_1|||_s^{k-i}|||v_1|||_s^i)), \end{aligned}$$

$$\begin{aligned} |||u_1^{k-1}u_2^2 - v_1^{k-1}v_2^2|||_{s'} & \leq |||u_1^{k-1}u_2^2 - u_1^{k-1}v_2^2|||_{s'} + |||u_1^{k-1}v_2^2 - v_1^{k-1}v_2^2|||_{s'} \\ & \leq C|||u_2 - v_2|||_s(|||u_2 + v_2|||_s)|||u_1|||_s^{k-1} + |||u_1 - v_1|||_s|||v_2|||_s^2(\sum_0^{k-2}(|||u_1|||_s^{k-2-i}|||v_1|||_s^i)), \end{aligned}$$

and

$$
\begin{aligned}
&|||\partial_x(1-\partial_x^2)^{-1}(u_1^{k-1}u_2^2-v_1^{k-1}v_2^2)|||_{s'}\\
&\leq C|||u_2-v_2|||_s(|||u_2+v_2|||_s)|||u_1|||_s^{k-1}+|||u_1-v_1|||_s|||v_2|||_s^2(\sum_0^{k-2}(|||u_1|||_s^{k-2-i}|||v_1|||_s^i)),
\end{aligned}
$$

$$
\begin{aligned}
|||u_1^k u_{2x}-v_1^k v_{2x}|||_{s'} &\leq |||u_1^k u_2-u_1^k v_{2x}|||_{s'}+|||u_1^k v_{2x}-v_1^k v_2|||_{s'}\\
&\leq \frac{C}{s-s'}|||u_2-v_2|||_s|||u_1|||_s^k+\frac{C}{s-s'}|||u_1-v_1|||_s|||v_2|||_s(\sum_0^{k-1}(|||u_1|||_s^{k-1-i}|||v_1|||_s^i)),
\end{aligned}
$$

$$
\begin{aligned}
|||\partial_x^2(1-\partial_x^2)^{-1}(u_1^{k+1}-v_1^{k+1})|||_{s'} &\leq \frac{C}{s-s'}|||u_1^{k+1}-v_1^{k+1}|||_{s'}\\
&\leq \frac{C}{s-s'}|||u_1-v_1|||_s(\sum_0^{k}(|||u_1|||_s^{k-i}|||v_1|||_s^i)),
\end{aligned}
$$

$$
\begin{aligned}
&|||\partial_x^2(1-\partial_x^2)^{-1}(u_1^{k-1}u_2^2-v_1^{k-1}v_2^2)|||_{s'}\\
&\leq \frac{C}{s-s'}|||u_2-v_2|||_s(|||u_2+v_2|||_s)|||u_1|||_s^{k-1}+\frac{C}{s-s'}|||u_1-v_1|||_s|||v_2|||_s^2(\sum_0^{k-2}(|||u_1|||_s^{k-2-i}|||v_1|||_s^i)).
\end{aligned}
$$

Therefore, we arrive at

$$
\begin{aligned}
&|||F(u_1,u_2)-F(v_1,v_2)|||_{s'} \leq C|||u_2-v_2|||_s|||u_1|||_s^k+|||u_1-v_1|||_s|||v_2|||_s(\sum_0^{k-1}(|||u_1|||_s^{k-1-i}|||v_1|||_s^i))\\
&+C|||u_1-v_1|||_s(\sum_0^{k}(|||u_1|||_s^{k-i}|||v_1|||_s^i))+\frac{C}{s-s'}|||u_1-v_1|||_s(\sum_0^{k}(|||u_1|||_s^{k-i}|||v_1|||_s^i))\\
&+2C|||u_2-v_2|||_s|||u_2+v_2|||_s|||u_1|||_s^{k-1}+|||u_1-v_1|||_s|||v_2|||_s^2(\sum_0^{k-2}(|||u_1|||_s^{k-2-i}|||v_1|||_s^i))\\
&+\frac{C}{s-s'}|||u_2-v_2|||_s|||u_1|||_s^k+\frac{C}{s-s'}|||u_1-v_1|||_s|||v_2|||_s(\sum_0^{k-1}(|||u_1|||_s^{k-1-i}|||v_1|||_s^i))\\
&+\frac{C}{s-s'}|||u_2-v_2|||_s|||u_2+v_2|||_s|||u_1|||_s^{k-1}+\frac{C}{s-s'}|||u_1-v_1|||_s|||v_2|||_s^2(\sum_0^{k-2}(|||u_1|||_s^{k-2-i}|||v_1|||_s^i)),
\end{aligned}
$$

where the constant $C$ depends only on $R$ and $k$. The conditions (1)–(3) above are easily verified once our system is transformed into a new system with zero initial data as (4.1) and (4.2). So, we have completed the proof of Theorem 4.1.

## 5. Conclusions

In this paper, we focus on several dynamic properties of the Cauchy problems (1.1) and (1.2). We first establish a new blow-up criterion and a blow-up phenomenon for the problem, then we study analytical solutions for the equation by using a new method, here, we present two analytical solutions for the problems (1.1) and (1.2) for the first time. Finally, we study the analyticity in a suitable scale of the Banach spaces . The properties of the problems (1.1) and (1.2) not only present fundamental importance from mathematical point of view but also are of great physical interest.

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## Conflict of interest

There is no conflict of interest.

## References

1. R. Camassa, D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71** (1993), 1661–1664. http://dx.doi.org/10.1103/PhysRevLett.71.1661
2. R. M. Chen, F. Guo, Y. Liu, C. Z. Qu, Analysis on the blow-up of solutions to a class of integrable peakon equations, *J. Funct. Anal.*, **270** (2016), 2343–2374. http://dx.doi.org/10.1016/j.jfa.2016.01.017
3. G. M. Coclite, H. Holden, K. H. Karlsen, Well-posedness of a parabolic-elliptic system, *Discrete Cont. Dyn. syst.*, **13** (2005), 659–682. http://dx.doi.org/10.3934/dcds.2005.13.659
4. G. M. Coclite, H. Holden, Stability of solutions of quasilinear parabolic equations, *J. Math. Anal. Appl.*, **308** (2005), 221–239. http://dx.doi.org/10.1016/j.jmaa.2005.01.026
5. G. M. Coclite, H. Holden, K. H. Karlsen, Global weak solutions to a generalized hyperelastic-rod wave equation, *SIAM J. Math. Anal.*, **37** (2005), 1044–1069. http://dx.doi.org/10.1137/040616711
6. A. Constantin, Existence of permanent and breaking waves for a shallow water equation: A geometric approach, *Ann. I. Fourier*, **50** (2000), 321–362. http://dx.doi.org/10.5802/aif.1757
7. A. Constantin, J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.*, **181** (1998), 229–243. http://dx.doi.org/10.1007/BF02392586
8. A. Constantin, J. Escher, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, *Math. Z.*, **233** (2000), 75–91. http://dx.doi.org/10.1007/PL00004793
9. A. Constantin, J. Escher, Global existence and blow-up for a shallow water equation, *Ann. Scuola Norm. Sci.*, **26** (1998), 303–328.
10. R. Danchin, A note on well-posedness for Camassa-Holm equation, *J. Differ. Equations*, **192** (2003), 429–444. http://dx.doi.org/10.1016/S0022-0396(03)00096-2

11. B. Fuchssteiner, A. S. Fokas, Symplectic structures, their B*ä*klund transformations and hereditary symmetries, *Physica D*, **4** (1981), 47–66. http://dx.doi.org/10.1016/0167-2789(81)90004-X

12. Y. P. Fu, B. L. Guo, Time periodic solution of the viscous Camassa-Holm equation, *J. Math. Anal. Appl.*, **313** (2006), 311–321. http://dx.doi.org/10.1016/j.jmaa.2005.08.073

13. Z. Guo, M. Jiang, Z. Wang, G. Zheng, Global weak solutions to the Camassa-Holm equation, *Discrete Cont. Dyn. Syst.*, **21** (2008), 883–906. http://dx.doi.org/10.3934/dcds.2008.21.883

14. S. Hakkaev, K. Kirchev, Local well-posedness and orbital stability of solitary wave solutions for the generalized Camassa-Holm equation, *Commun. Part. Diff. Eq.*, **30** (2005), 761–781. http://dx.doi.org/10.1081/PDE-200059284

15. S. Hakkaev, K. Kirchev, On the well-posedness and stability of peakons for a generalized Camassa-Holm equation, *Int. J. Nonlinear Sci.*, **1** (2006), 139–148.

16. A. A. Himonas, C. Kenig, Non-uniform dependence on initial data for the CH equation on the line, *Differ. Integral Equ.*, **22** (2009), 201–224. http://dx.doi.org/10.57262/die/1356019770

17. S. Lai, Y. Wu, The local well-posedness and existence of weak solutions for a generalized Camassa-Holm equation, *J. Differ. Equations*, **248** (2010), 2038–2063. http://dx.doi.org/10.1016/j.jde.2010.01.008

18. J. Lenells, Stability of periodic peakons, *Int. Math. Res. Notices*, **2004** (2004), 485–499. http://dx.doi.org/10.1155/S1073792804132431

19. Y. A. Li, P. J. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, *J. Differ. Equations*, **162** (2000), 27–63. http://dx.doi.org/10.1006/jdeq.1999.3683

20. X. X. Liu, Z. Y. Yin, Local well-posedness and stability of peakons for a generalized Dullin-Gottwald-Holm equation, *Nonlinear Anal. Theor.*, **74** (2011), 2497–2507. http://dx.doi.org/10.1016/j.na.2010.12.005

21. Y. S. Mi, C. L. Mu, Well-posedness and analyticity for the Cauchy problem for the generalized Camassa-Holm equation, *J. Math. Anal. Appl.*, **405** (2013), 173–182. http://dx.doi.org/10.1016/j.jmaa.2013.03.020

22. Z. Xin, P. Zhang, On the weak solutions to a shallow water equation, *Commun. Pure Appl. Math.*, **53** (2000), 1411–1433. http://dx.doi.org/10.1002/1097-0312(200011)53:11¡1411::AID-CPA4¿3.0.CO;2-5

23. W. Yan, Y. Li, Y. Zhang, The Cauchy problem for the generalized Camassa-Holm equation in Besov space, *J. Differ. Equations*, **256** (2014), 2876–2901. http://dx.doi.org/10.1016/j.jde.2014.01.023

24. S. Zhou, Persistence properties for a generalized Camassa-Holm equation in weighted $L^p$ spaces, *J. Math. Anal. Appl.*, **410** (2014), 932–938. http://dx.doi.org/10.1016/j.jmaa.2013.09.022

25. Y. Zhou, J. Fan, Regularity criteria for the viscous Camassa-Holm equation, *Int. Math. Res. Notices*, **2009** (2009), 2508-2518. http://dx.doi.org/10.1093/imrn/rnp023

26. L. Ni, Y. Zhou, Well-posedness and persistence properties for the Novikov equation, *J. Differ. Equations*, **250** (2011), 3002–3021. http://dx.doi.org/10.1016/j.jde.2011.01.030

27. T. Kato, *Quasi-linear equations of evolution with application to partial differential equations*, In: Spectral theory and differential equations, Berlin: Springer, 1975. http://dx.doi.org/10.1007/BFb0067080

28. A. Himonas, K. Grayshan, C. Holliman, Ill-posedness for the b-family of equations, *J. Nonlinear Sci.*, **26** (2016), 1175–1190. http://dx.doi.org/10.1007/s00332-016-9302-0

29. E. Novruzov, Construction of peakon-antipeakon solutions and ill-posedness for the b-family of equations, *J. Differ. Equations*, **272** (2021), 544–559. http://dx.doi.org/10.1016/j.jde.2020.10.013

30. A. Himonas, G. Misiolek, Analyticity of the Cauchy problem for an integrable evolution equation, *Math. Ann.*, **327** (2003), 575–584. https://doi.org/10.1007/s00208-003-0466-1

31. M. Baouendi, C. Goulaouic, Sharp estimates for analytic pseudodifferential operators and application to the Cauchy problems, *J. Differ. Equations*, **48** (1983), 241–268. https://doi.org/10.1016/0022-0396(83)90051-7

32. E. G. Reyes, Geometric integrability of the Camassa-Holm equation, *Lett. Math. Phys.*, **59** (2002), 117–131. https://doi.org/10.1023/A:1014933316169

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