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Research article

A general conservative eighth-order compact finite difference scheme for the coupled Schrödinger-KdV equations

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Abstract: In this paper, we present a general conservative eighth-order compact finite difference scheme for solving the coupled Schrödinger-KdV equations numerically. The proposed scheme is decoupled and preserves several physical invariants in discrete sense. The matrices obtained in the eighth-order compact scheme are all circulant symmetric positive definite so that it can be used to solve other similar equations. Numerical experiments on model problems show the better performance of the scheme compared with other numerical schemes.

Keywords: Schrödinger-KdV equations; compact finite difference scheme; periodic boundary condition; conservation; error estimate
Mathematics Subject Classification: 65M06, 65M12, 65M15

1. Introduction

Nonlinear partial differential equations play an important part in various branches of science such as fluid mechanics, solid state physics, plasma physics and quantum mechanics. The coupled Schrödinger-KdV equations are put forward to model nonlinear dynamics of one-dimensional Langmuir and ion-acoustic waves in a system of coordinates moving at the ion-acoustic speed [18, 19]. In detail, we consider the system [9]

$$i\epsilon u_t + pu_{xx} - qvu - s|u|^2 u = 0, \quad (x,t) \in R \times (0,T], \tag{1.1}$$

$$v_t + \alpha v_{xxx} + (\beta v^m + \rho|u|^2)_x = 0, \quad (x,t) \in R \times (0,T], \tag{1.2}$$

$$u(x,t) = u(x+l,t), \quad v(x,t) = v(x+l,t), \quad (x,t) \in R \times (0,T], \tag{1.3}$$

$$u(x,0) = \varphi(x), \quad v(x,0) = \phi(x), \quad x \in R, \tag{1.4}$$

where $i = \sqrt{-1}$, m is a positive integer, $p, q, s, \epsilon, \alpha, \beta, \rho$ are real constants with $p \neq 0$ and $\epsilon, \alpha \geq 0$. The complex-valued function u and the real-valued function v describe electric field of Langmuir oscillations and low-frequency density perturbation, respectively. The initial functions φ and ϕ are given l-periodic functions. Hence, it suffices to take a single period $[0, l]$. For Eqs (1.1) and (1.2), there are four physical invariants to be considered:

The number of plasmons

$$I_1 = \int_0^l |u(x,t)|^2 dx. \tag{1.5}$$

The number of particles

$$I_2 = \int_0^l v(x,t) dx. \tag{1.6}$$

The energy of oscillations

$$I_3 = \int_0^l \left[\frac{q\beta}{m+1} v^{m+1} + p\rho|u_x|^2 + q\rho v|u|^2 + \frac{s\rho}{2}|u|^4 - \frac{q\alpha}{2}(v_x)^2\right] dx, \tag{1.7}$$

and the momentum

$$I_4 = \int_0^l \left[qv^2 - 2\rho\epsilon Im(u\bar{u}_x)\right] dx. \tag{1.8}$$

According to [2], these invariants may connect closely to accurate behaviors in time. Extensive numerical studies have been presented for the coupled Schödinger-KdV equations in the last decade, such as the finite element method [3], radial basis function (RBF) collocation method [4], decomposition [5], variational iteration [6], exponential time differencing three-layer implicit scheme(ETDT-P) [7], homotopy perturbation [8], and fourth-order conservative compact finite difference scheme [9] and so on.

In the aspect of compact difference scheme, which is well known for the narrower stencils, i.e., fewer neighboring nodes it uses, and have less truncation error comparing with typical finite difference schemes. A variety of fourth-order compact methods have been employed solving partial differential equations [9, 12–14, 21–25, 28, 29]. Furthermore, Wang [26] proposed a conservative eighth-order compact difference scheme for the nonlinear Schrödinger equation. In [27], Chen and Chen presented a conservative eighth-order compact difference scheme for the Klein-Gordon-Schrödinger equations. Motivated by ideas in [26, 27], this article aims to construct a new general difference scheme which can deal with the conservativeness of the invariants and convergence theorem easily. In detail, there are following three advantages:

(i) The proposed scheme is compact, linearized, decoupled.

(ii) The proposed scheme preserves several invariants in discrete sense.

(iii) The operator form of scheme is novel and can be easily generalized from the fourth-order compact method to the eight-order method for solving other equations.

The rest of the paper is organized as follows. In Section 2, we introduce an eighth-order conservative compact finite difference scheme and apply it to solve the coupled Schödinger-KdV equations numerically. The discrete conservation properties of the proposed nonlinear scheme is analyzed and the convergence theorem of the linearized scheme is established in Section 3. Numerical experiments are presented in Section 4. Finally, a brief conclusion is given in Section 5.

2. The eighth-order compact finite difference scheme

The domain $\Omega = \{(x,t)\,|\,0 \le x \le l,\ 0 \le t \le T\}$ is discretized into grids described by the set $\{x_j, t_n\}$ of nodes, in which $x_j = jh,\ j = 0, 1, \dots, J = l/h$ and $t_n = n\tau,\ n = 0, 1, \dots, N = T/\tau$, where h and τ are discretization parameters. Briefly, let $u_j^n = u(x_j, t_n)$, $v_j^n = v(x_j, t_n)$ and $\Omega_h = \{x_0, x_1, \dots, x_J\}$. For more convenient discussion, define the following difference operators and notations:

$$\delta_t u_j^n = \frac{u_j^{n+1} - u_j^n}{\tau}, \quad \delta_x^2 u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2},$$
$$\delta_x u_j^n = \frac{u_{j+1}^n - u_j^n}{h}, \quad \delta_{\bar{x}} u_j^n = \frac{u_j^n - u_{j-1}^n}{h}, \quad \delta_{\hat{x}} u_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2h},$$
$$u_j^{n+\frac{1}{2}} = \frac{u_j^{n+1} + u_j^n}{2}, \quad (|u|^2)_j^{n+\frac{1}{2}} = \frac{|u_j^n|^2 + |u_j^{n+1}|^2}{2},$$
$$\mathcal{A}_1 u_j^n = \left(1 + \frac{5h^2}{42}\delta_x^2\right) u_j^n = \frac{1}{42}\left(5u_{j-1}^n + 32u_j^n + 5u_{j+1}^n\right),$$
$$\mathcal{A}_2 u_j^n = \left(1 + \frac{31h^2}{252}\delta_x^2\right) u_j^n = \frac{1}{252}\left(31u_{j-1}^n + 190u_j^n + 31u_{j+1}^n\right),$$
$$\mathcal{B}_1 u_j^n = \left(1 + \frac{20h^2}{70}\delta_x^2 + \frac{h^4}{70}\delta_x^2\delta_x^2\right) u_j^n = \frac{1}{70}\left(u_{j-2}^n + 16u_{j-1}^n + 36u_j^n + 16u_{j+1}^n + u_{j+2}^n\right),$$
$$\mathcal{B}_2 u_j^n = \left(1 + \frac{780h^2}{3780}\delta_x^2 + \frac{23h^4}{3780}\delta_x^2\delta_x^2\right) u_j^n = \frac{1}{3780}\left(23u_{j-2}^n + 688u_{j-1}^n + 2358u_j^n + 688u_{j+1}^n + 23u_{j+2}^n\right),$$
$$\mathcal{J} u_j^n = \left(1 + \frac{h^2}{4}\delta_x^2\right) u_j^n = \frac{1}{4}\left(u_{j-1}^n + 2u_j^n + u_{j+1}^n\right).$$

About the approximate formulas of the first and second-order spatial derivatives at all nodes (with periodic boundary conditions) with the eighth-order accuracy, we have the following lemma. Note that we denote $u_j' = \frac{\partial u(x_j,t)}{\partial x}$ or simply denote $u_j' = (u_x)_j$ in the following lemma. Similarly, the notations u_j'' and u_j''' are the same meaning.

Lemma 1. *[1] For u' and u'', we have the following approximate formulas*

$$\begin{aligned} &u_{j-2}' + 16u_{j-1}' + 36u_j' + 16u_{j+1}' + u_{j+2}' \\ =&\frac{5}{6h}(-5u_{j-2} - 32u_{j-1} + 32u_{j+1} + 5u_{j+2}) + O(h^8), \end{aligned} \tag{2.1}$$

$$\begin{aligned} &23u_{j-2}'' + 688u_{j-1}'' + 2358u_j'' + 688u_{j+1}'' + 23u_{j+2}'' \\ =&\frac{15}{h^2}(31u_{j-2} + 128u_{j-1} - 318u_j + 128u_{j+1} + 31u_{j+2}) + O(h^8). \end{aligned} \tag{2.2}$$

For the convenience to discrete and analyse the equations, we need to rewrite the relations (2.1) and (2.2) to the operators forms.

Lemma 2. *By the definition of the operators above, we have*

$$\mathcal{B}_1 u_j' = \mathcal{A}_1 \delta_{\hat{x}} u_j + O(h^8), \tag{2.3}$$

$$\mathcal{B}_2 u_j'' = \mathcal{A}_2 \delta_x^2 u_j + O(h^8), \tag{2.4}$$

$$\mathcal{B}_1 \mathcal{B}_2 u_j''' = \mathcal{A}_1 \mathcal{A}_2 \delta_{\hat{x}} \delta_x^2 u_j + O(h^8). \tag{2.5}$$

Proof. Assume that there is an operator $\mathcal{A}_1^* u'_j = \lambda_1 u_{j-1} + \lambda_2 u_j + \lambda_1 u_{j+1}$ such that

$$\mathcal{B}_1 u'_j = \mathcal{A}_1^* \delta_{\hat{x}} u_j + O(h^8). \tag{2.6}$$

By computation and the definition of operators above, we have $\lambda_1 = 5/42$ and $\lambda_2 = 32/42$. Hence, $\mathcal{A}_1^* = \mathcal{A}_1$ and (2.3) holds. (2.4) can be proved similarly. At last, combining (2.3) and (2.4), (2.5) follows directly. □

We note that Lemma 2 shows the discrete scheme has the eighth-order accuracy if we use the operators $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$ and $\mathcal{B}_2$ or their combinations to discrete the corresponding derivative values at nodes.

In the temporal discretization, we need to evaluate the function values at mid-nodes $\left(\left(n+\frac{1}{2}\right)\text{-nodes}\right)$. The following lemma is necessary to ensure to approximate the function values at mid-nodes by values at n- and $(n+1)$-nodes, which can be obtained by Taylors expansion.

Lemma 3. *For any smooth function $g(t)$ and $m \in \mathbb{N}^*$, we have*

$$\left(g(t_{n+\frac{1}{2}})\right)^m - \psi(g(t_n), g(t_{n+1})) = O(\tau^2), \tag{2.7}$$

where $t_{n+\frac{1}{2}} = \frac{t_n + t_{n+1}}{2}$ and

$$\psi(u, v) = \frac{1}{m+1} \sum_{k=0}^{m} u^k v^{m-k}. \tag{2.8}$$

Proof. By using Taylor's expansion, we have

$$g(t_{n+1}) = g(t_{n+\frac{1}{2}}) + \frac{\tau}{2} g'(t_{n+\frac{1}{2}}) + O(\tau^2), \tag{2.9}$$

$$g(t_n) = g(t_{n+\frac{1}{2}}) - \frac{\tau}{2} g'(t_{n+\frac{1}{2}}) + O(\tau^2), \tag{2.10}$$

$$(g(t_{n+1}))^z = \left(g(t_{n+\frac{1}{2}})\right)^z + \frac{\tau}{2} z \left(g(t_{n+\frac{1}{2}})\right)^{z-1} \left(g'(t_{n+\frac{1}{2}})\right) + O(\tau^2), \tag{2.11}$$

$$(g(t_n))^z = \left(g(t_{n+\frac{1}{2}})\right)^z - \frac{\tau}{2} z \left(g(t_{n+\frac{1}{2}})\right)^{z-1} \left(g'(t_{n+\frac{1}{2}})\right) + O(\tau^2), \tag{2.12}$$

where $z \in \mathbb{N}^*$. Let $k < \frac{m}{2}, k \in \mathbb{N}$, from (2.11) and (2.12), we can obtain

$$\begin{aligned}
&(g(t_n))^k (g(t_{n+1}))^{m-k} + (g(t_n))^{m-k} (g(t_{n+1}))^k \\
=&(g(t_n))^k (g(t_{n+1}))^k \left[(g(t_n))^{m-2k} + (g(t_{n+1}))^{m-2k}\right] \\
=&\left[\left(g(t_{n+\frac{1}{2}})\right)^{2k} + O(\tau^2)\right]\left[2\left(g(t_{n+\frac{1}{2}})\right)^{m-2k} + O(\tau^2)\right] \\
=&2\left(g(t_{n+\frac{1}{2}})\right)^m + O(\tau^2).
\end{aligned} \tag{2.13}$$

Plugging (2.13) into (2.8), (2.7) immediately follows. □

Denote the approximations of u_j^n and v_j^n by U_j^n and V_j^n, respectively. Ignoring the truncation error terms in Eqs (2.3)–(2.5) and (2.7), we obtain the following implicit compact scheme with truncation error $O(\tau^2 + h^8)$ by using the Crank-Nicolson method for temporal discretization and Lemmas 2 and 3:

$$i\epsilon \mathcal{B}_2(\delta_t U_j^n) + p\mathcal{A}_2 \delta_x^2 U_j^{n+\frac{1}{2}} - q\mathcal{B}_2\left(V_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}}\right) - s\mathcal{B}_2\left((|U|^2)_j^{n+\frac{1}{2}} U_j^{n+\frac{1}{2}}\right) = 0, \tag{2.14}$$

$$\mathcal{B}_1\mathcal{B}_2(\delta_t V_j^n) + \alpha\mathcal{A}_1\mathcal{A}_2\delta_{\hat{x}}\delta_x^2 V_j^{n+\frac{1}{2}} + \beta\mathcal{B}_2\mathcal{A}_1\delta_{\hat{x}}\psi(V_j^n, V_j^{n+1}) + \rho\mathcal{B}_2\mathcal{A}_1\delta_{\hat{x}}(|U|^2)_j^{n+\frac{1}{2}} = 0, \tag{2.15}$$

$$U_j^n = U_{j+J}^n, \quad V_j^n = V_{j+J}^n, \quad n = 0, 1, \ldots, N, \quad j = 1, 2, \ldots, J, \tag{2.16}$$

$$U_j^0 = \varphi(x_j), \quad V_j^0 = \phi(x_j). \tag{2.17}$$

The schemes (2.14 and 2.15) are nonlinear and gotten by discretizing the temporal derivative with the Crank-Nicolson method, which has the second-order $O(\tau^2)$ and discretizing the special derivatives with the operators $\mathcal{B}_1$ and $\mathcal{B}_1\mathcal{B}_2$ for (1.1) and (1.2), respectively, which has the eighth-order $O(h^8)$ by Lemma 2.

As to the linearized form of (2.14 and 2.15), we will discuss in the next section.

3. The conservation and convergence analysis

3.1. Notations and preliminaries

Let $H_p(\Omega_h) = \left\{u \,|\, u = \left\{u_j\right\},\ j = 0, 1, \ldots, J \text{ and } u_j = u_{j+J}\right\}$ denote the space of periodic real- or complex-valued grid functions defined on Ω_h with period J. The discrete inner product and the corresponding discrete L^2-norm on the grid function space $H_p(\Omega_h)$ are defined as

$$\langle u, w\rangle = \sum_{j=1}^{J} u_j \overline{w}_j h, \quad \|u\| = \sqrt{\langle u, u\rangle},$$

where $\overline{w}$ denotes the conjugate of w. Norm $\|\delta_x^2 u\|^2 = \left\langle \delta_x^2 u, \delta_x^2 u\right\rangle$ is well-defined with periodic condition ($u_j = u_{j\pm J}$) and the discrete L^∞- and H^1-norm are defined as

$$\|u\|_\infty = \max_{1\le j\le J}|u_j|, \quad \|u\|_1 = \sqrt{\|u\|^2 + \|\delta_{\overline{x}} u\|^2}.$$

The following lemmas can be easily proved:

Lemma 4. *For any grid functions $u, w \in H_p(\Omega_h)$, we have*

$$\begin{aligned}
&\langle \delta_x u, w\rangle = -\langle u, \delta_{\overline{x}} w\rangle, \quad \langle \delta_{\hat{x}} u, w\rangle = -\langle u, \delta_{\hat{x}} w\rangle,\\
&\left\langle \delta_x^2 u, w\right\rangle = -\langle \delta_{\overline{x}} u, \delta_{\overline{x}} w\rangle = -\langle \delta_x u, \delta_x w\rangle = \left\langle u, \delta_x^2 w\right\rangle,\\
&\langle \mathcal{A}_1 u, w\rangle = \langle u, \mathcal{A}_1 w\rangle = \langle u, w\rangle - \frac{5h^2}{42}\langle \delta_{\overline{x}} u, \delta_{\overline{x}} w\rangle,\\
&\langle \mathcal{A}_2 u, w\rangle = \langle u, \mathcal{A}_2 w\rangle = \langle u, w\rangle - \frac{31h^2}{252}\langle \delta_{\overline{x}} u, \delta_{\overline{x}} w\rangle,\\
&\langle \mathcal{B}_1 u, w\rangle = \langle u, \mathcal{B}_1 w\rangle = \langle u, w\rangle - \frac{20h^2}{70}\langle \delta_{\overline{x}} u, \delta_{\overline{x}} w\rangle + \frac{h^4}{70}\left\langle \delta_x^2 u, \delta_x^2 w\right\rangle,\\
&\langle \mathcal{B}_2 u, w\rangle = \langle u, \mathcal{B}_2 w\rangle = \langle u, w\rangle - \frac{780h^2}{3780}\langle \delta_{\overline{x}} u, \delta_{\overline{x}} w\rangle + \frac{23h^4}{3780}\left\langle \delta_x^2 u, \delta_x^2 w\right\rangle,\\
&\langle \mathcal{J} u, w\rangle = \langle u, \mathcal{J} w\rangle = \langle u, w\rangle - \frac{h^2}{4}\langle \delta_{\overline{x}} u, \delta_{\overline{x}} w\rangle.
\end{aligned}$$

Lemma 5. *For any grid functions $u \in H_p(\Omega_h)$, we have*

$$Re(\langle \delta_{\hat{x}} u, u\rangle) = Re(\langle \delta_{\hat{x}}\mathcal{A}_1 u, u\rangle) = Re(\left\langle \delta_{\hat{x}}\mathcal{B}_1^{-1}\mathcal{A}_1 u, u\right\rangle) = Re(\left\langle \delta_{\hat{x}}\mathcal{B}_2^{-1}\mathcal{A}_2 u, u\right\rangle) = 0.$$

Lemma 6. *[20] For any grid functions $u \in H_p(\Omega_h)$, we have*

$$\|\delta_{\bar{x}}u\| \le \frac{2}{h}\|u\|, \quad \|u\|_\infty \le h^{-\frac{1}{2}}\|u\|,$$
$$\|u\|_\infty^2 \le \varepsilon\|\delta_{\bar{x}}u\|^2 + \left(\frac{1}{\varepsilon} + \frac{1}{l}\right)\|u\|^2 \quad \forall\varepsilon > 0.$$

Lemma 7. *[12] For a real circulant matrix $A = C(b_0, b_1, \dots, b_{n-1})$, all eigenvalues of A are given by*

$$f(\mu_k), k = 0, 1, 2, \dots, n-1,$$

where $f(x) = b_0 + b_1x + b_2x^2 + \dots + b_{n-1}x^{n-1}$, and $\mu_k = \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})$.

3.2. Conservation and error analyses

For the compact schemes (2.14) and (2.15), we have the following conservative properties in the discrete sense. The process of proof is similar to [9]. Since it still has some difference, so for the convenience to read, we give the detail of proof as following:

Theorem 1. *The compact schemes* (2.14) and (2.15) *preserve the discrete conservation laws of the numbers of plasmons and particles, i.e.,*

$$\|U^n\|^2 = \|U^0\|^2 \tag{3.1}$$

and

$$\sum_{j=1}^{J} V_j^n h = \sum_{j=1}^{J} V_j^0 h, \tag{3.2}$$

where $U^n = (U_1^n, U_2^n, \dots, U_J^n)^T$.

Proof. Setting G^n is a vector with the component

$$G_j^n = qV_j^{n+\frac{1}{2}}U_j^{n+\frac{1}{2}} + s(|U|^2)_j^{n+\frac{1}{2}}U_j^{n+\frac{1}{2}}, \tag{3.3}$$

then (2.14) can be written as

$$i\epsilon\delta_t(\mathcal{B}_2U_j^n) + p\mathcal{A}_2\delta_x^2U_j^{n+\frac{1}{2}} = \mathcal{B}_2G_j^n. \tag{3.4}$$

Computing the inner product $\langle\cdot,\cdot\rangle$ on both sides of Eq (3.4) with $U^{n+\frac{1}{2}}$, $\mathcal{A}_2^{-1}G^n$, $\delta_t(\mathcal{A}_2^{-1}U^n)$, $\mathcal{A}_2^{-1}\delta_x^2G^n$ and $\delta_t(\mathcal{A}_2^{-1}\delta_x^2U^n)$, respectively, and applying Lemma 4, we obtain

$$\begin{aligned} &i\epsilon\left\langle\delta_t(\mathcal{B}_2U^n), U^{n+\frac{1}{2}}\right\rangle - p\left\langle\mathcal{A}_2\delta_{\bar{x}}U^{n+\frac{1}{2}}, \delta_{\bar{x}}U^{n+\frac{1}{2}}\right\rangle = \left\langle\mathcal{B}_2G^n, U^{n+\frac{1}{2}}\right\rangle \\ =& \left\langle G^n, U^{n+\frac{1}{2}}\right\rangle - \frac{20h^2}{70}\left\langle\delta_{\bar{x}}G^n, \delta_{\bar{x}}U^{n+\frac{1}{2}}\right\rangle + \frac{h^4}{70}\left\langle\delta_x^2G^n, \delta_x^2U^{n+\frac{1}{2}}\right\rangle, \end{aligned} \tag{3.5}$$

$$i\epsilon\left\langle\delta_t(\mathcal{B}_2U^n), \mathcal{A}_2^{-1}G^n\right\rangle - p\left\langle\delta_{\bar{x}}U^{n+\frac{1}{2}}, \delta_{\bar{x}}G^n\right\rangle = \left\langle\mathcal{B}_2G^n, \mathcal{A}_2^{-1}G^n\right\rangle, \tag{3.6}$$

$$i\epsilon\left\langle\delta_t(\mathcal{B}_2U^n), \delta_t(\mathcal{A}_2^{-1}U^n)\right\rangle - p\left\langle\delta_{\bar{x}}U^{n+\frac{1}{2}}, \delta_t(\delta_{\bar{x}}U^n)\right\rangle = \left\langle\mathcal{B}_2G^n, \delta_t(\mathcal{A}_2^{-1}U^n)\right\rangle. \tag{3.7}$$

$$i\epsilon\left\langle\delta_t(\mathcal{B}_2U^n), \mathcal{A}_2^{-1}\delta_x^2G^n\right\rangle + p\left\langle\delta_x^2U^{n+\frac{1}{2}}, \delta_x^2G^n\right\rangle = \left\langle\mathcal{B}_2G^n, \mathcal{A}_2^{-1}\delta_x^2G^n\right\rangle, \tag{3.8}$$

$$i\epsilon \left\langle \delta_t(\mathcal{B}_2 U^n), \delta_t(\mathcal{A}_2^{-1}\delta_x^2 U^n) \right\rangle + p \left\langle \delta_x^2 U^{n+\frac{1}{2}}, \delta_t(\delta_x^2 U^n) \right\rangle = \left\langle \mathcal{B}_2 G^n, \delta_t(\mathcal{A}_2^{-1}\delta_x^2 U^n) \right\rangle. \tag{3.9}$$

By the Hermitian property of inner product and multiplying Eqs (3.7) and (3.9) by $i\epsilon$, we can obtain

$$i\epsilon \left\langle \delta_t(\mathcal{A}_2^{-1} U^n), \mathcal{B}_2 G^n \right\rangle = \epsilon^2 \left\langle \delta_t(\mathcal{A}_2^{-1} U^n), \delta_t(\mathcal{B}_2 U^n) \right\rangle - ip\epsilon \left\langle \delta_t(\delta_{\bar{x}} U^n), \delta_{\bar{x}} U^{n+\frac{1}{2}} \right\rangle. \tag{3.10}$$

$$i\epsilon \left\langle \delta_t(\mathcal{A}_2^{-1}\delta_x^2 U^n), \mathcal{B}_2 G^n \right\rangle = \epsilon^2 \left\langle \delta_t(\mathcal{A}_2^{-1}\delta_x^2 U^n), \delta_t(\mathcal{B}_2 U^n) \right\rangle + ip\epsilon \left\langle \delta_t(\delta_x^2 U^n), \delta_x^2 U^{n+\frac{1}{2}} \right\rangle. \tag{3.11}$$

By Lemma 4 and Eqs (3.6), (3.8), (3.10) and (3.11), it follows that

$$\begin{aligned} &\epsilon^2 \left\langle \delta_t(\mathcal{A}_2^{-1} U^n), \delta_t(\mathcal{B}_2 U^n) \right\rangle - ip\epsilon \left\langle \delta_t(\delta_{\bar{x}} U^n), \delta_{\bar{x}} U^{n+\frac{1}{2}} \right\rangle \\ =&p \left\langle \delta_{\bar{x}} U^{n+\frac{1}{2}}, \delta_{\bar{x}} G^n \right\rangle + \left\langle \mathcal{B}_2 G^n, \mathcal{A}_2^{-1} G^n \right\rangle. \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\epsilon^2 \left\langle \delta_t(\mathcal{A}_2^{-1}\delta_x^2 U^n), \delta_t(\mathcal{B}_2 U^n) \right\rangle + ip\epsilon \left\langle \delta_t(\delta_x^2 U^n), \delta_x^2 U^{n+\frac{1}{2}} \right\rangle \\ =& - p \left\langle \delta_x^2 U^{n+\frac{1}{2}}, \delta_x^2 G^n \right\rangle + \left\langle \mathcal{B}_2 G^n, \mathcal{A}_2^{-1}\delta_x^2 G^n \right\rangle. \end{aligned} \tag{3.13}$$

Multiplying by p, $\frac{20h^2}{70}$ and $\frac{h^4}{70}$ in Eqs (3.5), (3.12) and (3.13), respectively, and eliminating the term $\left\langle \delta_{\bar{x}} U^{n+\frac{1}{2}}, \delta_{\bar{x}} G^n \right\rangle$, we obtain

$$\begin{aligned} &ip\epsilon \left\langle \delta_t(\mathcal{B}_2 U^n), U^{n+\frac{1}{2}} \right\rangle + i\frac{20h^2 p\epsilon}{70} \left\langle \delta_{\bar{x}} U^{n+\frac{1}{2}}, \delta_t(\delta_{\bar{x}} U^n) \right\rangle - i\frac{h^4 p\epsilon}{70} \left\langle \delta_x^2 U^{n+\frac{1}{2}}, \delta_t(\delta_x^2 U^n) \right\rangle \\ &+ \frac{20h^2\epsilon^2}{70} \left\langle \delta_t(\mathcal{B}_2 U^n), \delta_t(\mathcal{A}_2^{-1} U^n) \right\rangle + \frac{h^4\epsilon^2}{70} \left\langle \delta_t(\mathcal{B}_2 U^n), \delta_t(\mathcal{A}_2^{-1}\delta_x^2 U^n) \right\rangle - p \left\langle G^n, U^{n+\frac{1}{2}} \right\rangle \\ &- \frac{20h^2}{70} \left\langle \mathcal{A}_2^{-1} G^n, \mathcal{B}_2 G^n \right\rangle - \frac{h^4}{70} \left\langle \mathcal{A}_2^{-1}\delta_x^2 G^n, \mathcal{B}_2 G^n \right\rangle - p^2 \left\langle \mathcal{A}_2 \delta_{\bar{x}} U^{n+\frac{1}{2}}, \delta_{\bar{x}} U^{n+\frac{1}{2}} \right\rangle = 0. \end{aligned} \tag{3.14}$$

From the definition of G^n we can see that $\left\langle G^n, U^{n+\frac{1}{2}} \right\rangle$ is a real number. Hence, the imaginary part of (3.14) is zero, i.e.,

$$\mathrm{Re}\left(\left\langle \delta_t(\mathcal{B}_2 U^n), U^{n+\frac{1}{2}} \right\rangle + \frac{20h^2}{70} \left\langle \delta_{\bar{x}} U^{n+\frac{1}{2}}, \delta_t(\delta_{\bar{x}} U^n) \right\rangle - \frac{h^4}{70} \left\langle \delta_x^2 U^{n+\frac{1}{2}}, \delta_t(\delta_x^2 U^n) \right\rangle \right) = 0. \tag{3.15}$$

Applying Lemma 4 in (3.15), we can obtain immediately that

$$\|U^{n+1}\|^2 = \|U^n\|^2.$$

Computing the inner product $\langle \cdot, \cdot \rangle$ on both sides of Eq (2.15) with $\mathbf{1} := (1, 1, \ldots, 1)^T \in H_p(\Omega_h)$, we can obtain

$$\begin{aligned} \langle \delta_t(\mathcal{B}_1\mathcal{B}_2 V^n), \mathbf{1} \rangle + \alpha \left\langle \mathcal{A}_1\mathcal{A}_2 \delta_{\hat{x}} \delta_x^2 V^{n+\frac{1}{2}}, \mathbf{1} \right\rangle + \beta \left\langle \mathcal{B}_2\mathcal{A}_1 \delta_{\hat{x}} \psi(V^n, V^{n+1}), \mathbf{1} \right\rangle \\ + \rho \left\langle \mathcal{B}_2\mathcal{A}_1 \delta_{\hat{x}} (|U|^2)^{n+\frac{1}{2}}, \mathbf{1} \right\rangle = 0, \end{aligned} \tag{3.16}$$

where

$$(|U|^2)^{n+\frac{1}{2}} := ((|U|^2)_1^{n+\frac{1}{2}}, (|U|^2)_2^{n+\frac{1}{2}}, \ldots, (|U|^2)_J^{n+\frac{1}{2}}),$$

using the periodic conditions, one can have the equation

$$\langle \delta_t(\mathcal{B}_1\mathcal{B}_2 V^n), \mathbf{1} \rangle = 0,$$

i.e.,

$$\left\langle \mathcal{B}_1\mathcal{B}_2 V^{n+1}, \mathbf{1} \right\rangle = \langle \mathcal{B}_1\mathcal{B}_2 V^{n}, \mathbf{1} \rangle . \tag{3.17}$$

With the periodic conditions, (3.2) immediately satisfies. The proof is finished. □

Hereinafter we define

$$U^n := (U_1^n, U_2^n, \dots, U_J^n)^T, \quad (U^n)^{\cdot 2} := ((U_1^n)^2, (U_2^n)^2, \dots, (U_J^n)^2)^T,$$

and

$$U^n.V^n := (U_1^n V_1^n, U_2^n V_2^n, \dots, U_J^n V_J^n)^T, \quad \psi(U^n, V^n) := \frac{1}{m+1}\sum_{k=0}^{m} (U^n)^{\cdot k}.(V^n)^{\cdot(m-k)}.$$

The compact schemes (2.14) and (2.15) are equivalent to the following matrix equations:

$$i\epsilon \mathbf{B}_2(\delta_t U^n) + p\mathbf{A}_2\delta_x^2 U^{n+\frac{1}{2}} - q\mathbf{B}_2\left(V^{n+\frac{1}{2}}.U^{n+\frac{1}{2}}\right) - s\mathbf{B}_2\left((|U|^2)^{n+\frac{1}{2}}.U^{n+\frac{1}{2}}\right) = 0, \tag{3.18}$$

$$\mathbf{B}_1\mathbf{B}_2(\delta_t V^n) + \alpha\mathbf{A}_1\mathbf{A}_2\delta_{\hat{x}}\delta_x^2 V^{n+\frac{1}{2}} + \beta\mathbf{B}_2\mathbf{A}_1\delta_{\hat{x}}\psi(V^n, V^{n+1}) + \rho\mathbf{B}_2\mathbf{A}_1\delta_{\hat{x}}(|U|^2)^{n+\frac{1}{2}} = 0, \tag{3.19}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$ and $\mathbf{B}_2$ are $J \times J$ matrices corresponding to the operators $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively,

$$\mathbf{A}_1 = \frac{1}{42}\begin{pmatrix} 32 & 5 & 0 & \cdots & 5 \\ 5 & 32 & 5 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 5 & 32 & 5 \\ 5 & \cdots & 0 & 5 & 32 \end{pmatrix}, \quad \mathbf{A}_2 = \frac{1}{252}\begin{pmatrix} 190 & 31 & 0 & \cdots & 31 \\ 31 & 190 & 31 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 31 & 190 & 31 \\ 31 & \cdots & 0 & 31 & 190 \end{pmatrix},$$

$$\mathbf{B}_1 = \frac{1}{70}\begin{pmatrix} 36 & 16 & 1 & 0 & \cdots & 1 & 16 \\ 16 & 36 & 16 & 1 & \ddots & 0 & 1 \\ 1 & 16 & 36 & 16 & \ddots & 0 & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \ddots & 16 & 36 & 16 & 1 \\ 1 & 0 & \ddots & 1 & 16 & 36 & 16 \\ 16 & 1 & \cdots & 0 & 1 & 16 & 36 \end{pmatrix},$$

$$\mathbf{B}_2 = \frac{1}{3780}\begin{pmatrix} 2358 & 688 & 23 & 0 & \cdots & 23 & 688 \\ 688 & 2358 & 688 & 23 & \ddots & 0 & 23 \\ 23 & 688 & 2358 & 688 & \ddots & 0 & 0 \\ 0 & 23 & \ddots & \ddots & \ddots & 23 & 0 \\ 0 & 0 & \ddots & 688 & 2358 & 688 & 23 \\ 23 & 0 & \ddots & 23 & 688 & 2358 & 688 \\ 688 & 23 & \cdots & 0 & 23 & 688 & 2358 \end{pmatrix}.$$

By the properties of circulant matrices, we can see that matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$ and $\mathbf{B}_2$ are circulant symmetric positive definite [10]. Let $\mathcal{A}_1^{-1}$, $\mathcal{A}_2^{-1}$, $\mathcal{B}_1^{-1}$ and $\mathcal{B}_2^{-1}$ denote inverse operators of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. The matrices corresponding to the operators δ_x^2, $\delta_{\hat{x}}$, $\mathcal{A}_1^{-1}$, $\mathcal{A}_2^{-1}$, $\mathcal{B}_1^{-1}$ and $\mathcal{B}_2^{-1}$ are also circulant, therefore, they commute under multiplication.

The compact schemes (2.14) and (2.15) can be equivalently written as

$$i\epsilon(\delta_t U_j^n) + p\mathcal{B}_2^{-1}\mathcal{A}_2\delta_x^2 U_j^{n+\frac{1}{2}} = qV_j^{n+\frac{1}{2}}U_j^{n+\frac{1}{2}} + s(|U|^2)_j^{n+\frac{1}{2}}U_j^{n+\frac{1}{2}}, \tag{3.20}$$

$$\delta_t V_j^n + \delta_{\hat{x}}\left(\alpha\mathcal{B}_1^{-1}\mathcal{B}_2^{-1}\mathcal{A}_1\mathcal{A}_2\delta_x^2 V_j^{n+\frac{1}{2}} + \beta\mathcal{B}_1^{-1}\mathcal{A}_1\psi(V_j^n, V_j^{n+1}) + \rho\mathcal{B}_1^{-1}\mathcal{A}_1(|U|^2)_j^{n+\frac{1}{2}}\right) = 0. \tag{3.21}$$

which can be obtained by multiplying $\mathcal{B}_2^{-1}$ and $\mathcal{B}_1^{-1}\mathcal{B}_2^{-1}$ in both side of (2.14) and (2.15), respectively.

By applying Lemma 7, we can obtain the following result:

Lemma 8. *[14] For any grid function $u \in H_p(\Omega_h)$, we have the inequalities*

$$\frac{32}{63}\|u\|^2 \le \left\langle \mathcal{B}_2^{-1}\mathcal{A}_2 u, u\right\rangle \le \frac{105}{26}\|u\|^2,$$
$$\frac{1}{35}\|u\|^2 \le \left\langle \mathcal{A}_1^{-1}\mathcal{B}_1 u, u\right\rangle \le \frac{21}{11}\|u\|^2.$$

Define

$$|||u|||_Q^2 = \left\langle \mathcal{B}_2^{-1}\mathcal{A}_2 u, u\right\rangle, \quad |||u|||_P^2 = \left\langle \mathcal{A}_1^{-1}\mathcal{B}_1 u, u\right\rangle.$$

Lemma 8 shows that $|||\cdot|||_Q$ and $|||\cdot|||_P$ are norms on $H_p(\Omega_h)$ equivalent to the discrete L^2-norm $\|\cdot\|$. For the proof of the following theorem, we want the relations (3.20) and (3.21).

Theorem 2. *The compact schemes* (2.14) and (2.15) *preserve the energy of oscillations in discrete sense, i.e.,*

$$\begin{aligned}
&\frac{q\beta}{m+1}\sum_{j=1}^{J}(V_j^{n+1})^{m+1}h + p\rho|||\delta_{\bar{x}}U^{n+1}|||_Q^2 + q\rho\sum_{j=1}^{J}V_j^{n+1}|U_j^{n+1}|^2h + \frac{s\rho}{2}\|U^{n+1}\|_{L^4}^4 - \frac{q\alpha}{2}|||\delta_{\bar{x}}V^{n+1}|||_Q^2\\
=&\frac{q\beta}{m+1}\sum_{j=1}^{J}(V_j^0)^{m+1}h + p\rho|||\delta_{\bar{x}}U^0|||_Q^2 + q\rho\sum_{j=1}^{J}V_j^0|U_j^0|^2h + \frac{s\rho}{2}\|U^0\|_{L^4}^4 - \frac{q\alpha}{2}|||\delta_{\bar{x}}V^0|||_Q^2
\end{aligned} \tag{3.22}$$

where

$$\|U\|_{L^4}^4 = \sum_{j=1}^{J}|U_j|^4h.$$

Proof. Computing the inner product $\langle\cdot,\cdot\rangle$ on both sides of Eq (3.20) with $\delta_t U^n$, we can obtain the following equation with the commutativity under multiplication of circulant matrices:

$$\begin{aligned}
&i\epsilon\langle\delta_t U^n, \delta_t U^n\rangle - p\left\langle \mathcal{B}_2^{-1}\mathcal{A}_2\delta_{\bar{x}}U^{n+\frac{1}{2}}, \delta_t\delta_{\bar{x}}U^n\right\rangle\\
=&q\left\langle V^{n+\frac{1}{2}}.U^{n+\frac{1}{2}}, \delta_t U^n\right\rangle + s\left\langle (|U|^2)^{n+\frac{1}{2}}.U^{n+\frac{1}{2}}, \delta_t U^n\right\rangle.
\end{aligned} \tag{3.23}$$

Then taking the real part of Eq (3.24), we obtain

$$
\begin{aligned}
&-\frac{p}{2\tau}\left(|||\delta_{\bar{x}}U^{n+1}|||_Q^2-|||\delta_{\bar{x}}U^{n}|||_Q^2\right)\\
&=\frac{q}{2\tau}\left(\left\langle V^{n+\frac{1}{2}}.U^{n+1},U^{n+1}\right\rangle-\left\langle V^{n+\frac{1}{2}}.U^{n},U^{n}\right\rangle\right)\\
&\quad+\frac{s}{2\tau}\left(\left\langle (|U|^2)^{n+\frac{1}{2}}.U^{n+1},U^{n+1}\right\rangle-\left\langle (|U|^2)^{n+\frac{1}{2}}.U^{n},U^{n}\right\rangle\right).
\end{aligned}
\tag{3.24}
$$

Multiplying Eq (3.24) with 2τ and summing from 0 to n, we obtain

$$
\begin{aligned}
&p|||\delta_{\bar{x}}U^{n+1}|||_Q^2+\frac{s}{2}\|U^{n+1}\|_{L^4}^4+q\left\langle V^{n+\frac{1}{2}},|U^{n+1}|^{.2}\right\rangle-\frac{q}{2}\sum_{k=1}^{n}\left\langle V^{k+1}-V^{k-1},|U^k|^{.2}\right\rangle\\
&=p|||\delta_{\bar{x}}U^{0}|||_Q^2+\frac{s}{2}\|U^{0}\|_{L^4}^4+q\left\langle V^{\frac{1}{2}},|U^{0}|^{.2}\right\rangle.
\end{aligned}
\tag{3.25}
$$

Setting W^n is a vector with the component

$$
W_j^n=\alpha\mathcal{B}_2^{-1}\mathcal{A}_2\delta_x^2V_j^{n+\frac{1}{2}}+\beta\psi(V_j^n,V_j^{n+1})+\rho(|U|^2)_j^{n+\frac{1}{2}},
$$

then Eq (3.21) can be written as

$$
\delta_tV_j^n+\delta_{\hat{x}}\mathcal{B}_1^{-1}\mathcal{A}_1W_j^n=0. \tag{3.26}
$$

Computing the inner product $\langle\cdot,\cdot\rangle$ on both sides of Eq (3.26) with W^n and applying Lemma 5, we can obtain

$$
\alpha\left\langle\delta_tV^n,\mathcal{B}_2^{-1}\mathcal{A}_2\delta_x^2V^{n+\frac{1}{2}}\right\rangle+\beta\left\langle\delta_tV^n,\psi(V^n,V^{n+1})\right\rangle+\rho\left\langle\delta_tV^n,(|U|^2)^{n+\frac{1}{2}}\right\rangle=0. \tag{3.27}
$$

It follows from definition (2.8) that

$$
\left\langle\delta_tV^n,\psi(V^n,V^{n+1})\right\rangle=\frac{1}{(m+1)\tau}\left(\sum_{j=1}^{J}(V_j^{n+1})^{m+1}h-\sum_{j=1}^{J}(V_j^{n})^{m+1}h\right).
$$

It is easy to see that

$$
\left\langle\delta_tV^n,\mathcal{B}_2^{-1}\mathcal{A}_2\delta_x^2V^{n+\frac{1}{2}}\right\rangle=-\frac{1}{2\tau}(|||\delta_{\bar{x}}V^{n+1}|||_Q^2-|||\delta_{\bar{x}}V^{n}|||_Q^2).
$$

Multiplying Eq (3.27) with 2τ and summing from 0 to n, we have

$$
\begin{aligned}
&\alpha|||\delta_{\bar{x}}V^{n+1}|||_Q^2-\frac{2\beta}{m+1}\sum_{j=1}^{J}(V_j^{n+1})^{m+1}h\\
&-\rho\sum_{k=0}^{n}\left\langle V^{k+1}-V^{k},|U^{k+1}|^{.2}+|U^k|^{.2}\right\rangle\\
&=\alpha|||\delta_{\bar{x}}V^{0}|||_Q^2-\frac{2\beta}{m+1}\sum_{j=1}^{J}(V_j^{0})^{m+1}h.
\end{aligned}
\tag{3.28}
$$

Since

$$\begin{aligned}&\sum_{k=0}^{n}\left\langle V^{k+1}-V^{k},|U^{k+1}|^{\cdot 2}+|U^{k}|^{\cdot 2}\right\rangle\\ =&\sum_{k=1}^{n}\left\langle V^{k+1}-V^{k-1},|U^{k}|^{\cdot 2}\right\rangle+\left\langle V^{n+1}-V^{n},|U^{n+1}|^{\cdot 2}\right\rangle+\left\langle V^{1}-V^{0},|U^{0}|^{\cdot 2}\right\rangle,\end{aligned}$$

the Eq (3.28) becomes

$$\begin{aligned}&\alpha|||\delta_{\bar{x}}V^{n+1}|||_Q^2-\frac{2\beta}{m+1}\sum_{j=1}^{J}(V_j^{n+1})^{m+1}h-\rho\sum_{k=1}^{n}\left\langle V^{k+1}-V^{k-1},|U^{k}|^{2}\right\rangle-\rho\left\langle V^{n+1}-V^{n},|U^{n+1}|^{2}\right\rangle\\ =&\,\alpha|||\delta_{\bar{x}}V^{0}|||_Q^2-\frac{2\beta}{m+1}\sum_{j=1}^{J}(V_j^{0})^{m+1}h+\rho\left\langle V^{1}-V^{0},|U^{0}|^{2}\right\rangle.\end{aligned}\tag{3.29}$$

Multiplying Eqs (3.25) and (3.29) with ρ and $\frac{q}{2}$, respectively, and subtracting the results, (3.22) follows immediately. □

In the following convergence analysis, we will take the symbol C as a general positive constant independent of h and τ, not necessarily the same at different occurrences. We assume that there is a positive constant Y^* such that the exact solutions u and v of the coupled system satisfy

$$\max\{\|u^n\|_\infty,\|u_t^n\|_\infty,\|v^n\|_\infty,\|v_t^n\|_\infty\}\le Y^*,\quad 0\le n\le N.\tag{3.30}$$

Let $Y_0=2(Y^*+1)^2$ and define a smooth function $\Psi(r,s)\in C^\infty(R^2)$ as

$$\Psi(r,s)=\begin{cases}\psi(r,s), & r^2+s^2\le Y_0,\\ 0, & r^2+s^2\ge Y_0+1.\end{cases}\tag{3.31}$$

Since schemes (2.14) and (2.15) are nonlinear, we change it into the following linearized compact scheme to reduce computational cost:

$$i\epsilon\mathcal{B}_2\left(\frac{U_j^{0*}-U_j^0}{\tau}\right)+\frac{p}{2}\mathcal{A}_2\delta_x^2(U_j^{0*}+U_j^0)-q\mathcal{B}_2\left(V_j^0U_j^0\right)=s\mathcal{B}_2\left(|U_j^0|^2U_j^0\right),\tag{3.32}$$

$$\mathcal{B}_1\mathcal{B}_2\left(\frac{V_j^{0*}-V_j^0}{\tau}\right)+\frac{\alpha}{2}\mathcal{A}_1\mathcal{A}_2\delta_{\hat{x}}\delta_x^2(V_j^{0*}+V_j^0)+\beta\mathcal{B}_2\mathcal{A}_1\delta_{\hat{x}}\psi(V_j^0,V_j^0)=-\rho\mathcal{B}_2\mathcal{A}_1\delta_{\hat{x}}|U_j^0|^2,\tag{3.33}$$

$$i\epsilon\mathcal{B}_2(\delta_tU_j^n)+p\mathcal{A}_2\delta_x^2U_j^{n+\frac12}-q\mathcal{B}_2\left(\hat{V}_j^n\hat{U}_j^n\right)=s\mathcal{B}_2\left(\widehat{(|U|^2)}_j^n\hat{U}_j^n\right),\tag{3.34}$$

$$\mathcal{B}_1\mathcal{B}_2(\delta_tV_j^n)+\alpha\mathcal{A}_1\mathcal{A}_2\delta_{\hat{x}}\delta_x^2V_j^{n+\frac12}+\beta\mathcal{B}_2\mathcal{A}_1\delta_{\hat{x}}\Psi(V_j^n,V_j^{n*})=-\rho\mathcal{B}_2\mathcal{A}_1\delta_{\hat{x}}\widehat{(|U|^2)}_j^n,\tag{3.35}$$

where $\hat{U}^0=(U^{0*}+U^0)/2$, $\hat{U}^n=3U^n/2-U^{n-1}/2$, and $U^{n*}=2U^n-U^{n-1}$ for $n\ge 1$.

We can prove that the temporal and spatial convergence rates of the linearized compact schemes (3.34) and (3.35) are second- and eighth-order, respectively.

Lemma 9. *Let $\{y_n\}$ be a nonnegative real sequence, c a nonnegative constant, d and τ are positive constants. If*

$$y_{n+1} \le c + d\tau \sum_{i=0}^{n} y_i \quad for \quad n \ge 0,$$

then

$$y_{n+1} \le (c + d\tau y_0) e^{d\tau(n+1)} \quad for \quad n \ge 0.$$

Theorem 3. *Suppose that $u, v \in C^4(0, T; C^{11}(R))$ are the exact solutions to Eqs (1.1) and (1.2), $h^8\tau^{-1} = o(1)$, and that assumption (3.30) holds. Let U and V be the solutions of (3.34) and (3.35). Then there exists a constant $C = C(Y^*, T)$ such that*

$$\max_{0<n\le N} \{\|u^n - U^n\|_1 + \|v^n - V^n\|_1\} \le C(\tau^2 + h^8),$$

for h and τ sufficiently small.

Proof. Let $E_u^n = u^n - U^n$ and $E_v^n = v^n - V^n$. By Eqs (1.1), (1.2), (3.34) and (3.35), and ignoring the subindex j, we obtain

$$i\epsilon\mathcal{B}_2(\delta_t E_u^n) + p\mathcal{A}_2\delta_x^2 E_u^{n+\frac{1}{2}} = q\mathcal{B}_2 T_1^n + s\mathcal{B}_2 T_2^n + r_u^n, \tag{3.36}$$

$$\mathcal{B}_1\mathcal{B}_2(\delta_t E_v^n) + \alpha\mathcal{A}_1\mathcal{A}_2\delta_{\hat{x}}\delta_x^2 E_v^{n+\frac{1}{2}} = -\beta\mathcal{B}_2\mathcal{A}_1\delta_{\hat{x}} T_3^n - \rho\mathcal{B}_2\mathcal{A}_1\delta_{\hat{x}} T_4^n + r_v^n, \tag{3.37}$$

where

$$T_1^n = \hat{v}^n.\hat{u}^n - \hat{V}^n.\hat{U}^n, \quad T_2^n = \widehat{(|u|^2)}^n.\hat{u}^n - \widehat{(|U|^2)}^n.\hat{U}^n,$$
$$T_3^n = \Psi(v^n, v^{n*}) - \Psi(V^n, V^{n*}), \quad T_4^n = \widehat{(|u|^2)}^n - \widehat{(|U|^2)}^n.$$

By the assumption (3.30) and definition (3.31), one can see that $\Psi((v^n, v^{n+1})) = \psi((v^n, v^{n+1}))$, and hence, the truncation errors r_u^n and r_v^n are such that $r_u^n = O(\tau^2 + h^8)$ and $r_v^n = O(\tau^2 + h^8)$. Under the smoothness assumption of u and v, we have

$$\delta_t r_u^n = O(\tau^2 + h^8) \quad \text{and} \quad \delta_t r_v^n = O(\tau^2 + h^8).$$

From (3.36) and (3.37), we can obtain the following equations:

$$i\epsilon\delta_t E_u^n + p\mathcal{B}_2^{-1}\mathcal{A}_2\delta_x^2 E_u^{n+\frac{1}{2}} = qT_1^n + sT_2^n + R_u^n, \tag{3.38}$$

$$\mathcal{A}_1^{-1}\mathcal{B}_1(\delta_t E_v^n) + \alpha\mathcal{B}_2^{-1}\mathcal{A}_2\delta_{\hat{x}}\delta_x^2 E_v^{n+\frac{1}{2}} = -\beta\delta_{\hat{x}} T_3^n - \rho\delta_{\hat{x}} T_4^n + R_v^n, \tag{3.39}$$

where $R_u^n = \mathcal{B}_2^{-1} r_u^n$ and $R_v^n = \mathcal{B}_2^{-1}\mathcal{A}_1^{-1} r_u^n$.

We use the induction argument as in [15–17] to estimate the error bounds. To obtain our error estimate, we assume that there exists a constant $h_0 > 0$ such that, for $0 < h \le h_0$,

$$\max\{\|E_u^n\|_\infty, \|E_v^n\|_\infty, \|\delta_t E_u^{n-1}\|_\infty, \|\delta_t E_v^{n-1}\|_\infty\} \le 1, \quad 1 \le n \le k. \tag{3.40}$$

Since $E_u^0 = E_v^0 = 0$, it is easy to see that

$$\|E_u^1\|_1 \le C(\tau^2 + h^8) \quad \text{and} \quad \|E_v^1\|_1 \le C(\tau^2 + h^8).$$

For $n \ge 1$, by computing the inner product $\langle\cdot,\cdot\rangle$ on both sides of (3.38) with $E_u^{n+\frac{1}{2}}$. we can obtain following equation by Lemma 4:

$$i\epsilon\left\langle \delta_t E_u^n, E_u^{n+\frac{1}{2}}\right\rangle - p\left\langle \mathcal{B}_2^{-1}\mathcal{A}_2\delta_{\bar{x}}E_u^{n+\frac{1}{2}}, \delta_{\bar{x}}E_u^{n+\frac{1}{2}}\right\rangle = q\left\langle T_1^n, E_u^{n+\frac{1}{2}}\right\rangle + s\left\langle T_2^n, E_u^{n+\frac{1}{2}}\right\rangle + \left\langle R_u^n, E_u^{n+\frac{1}{2}}\right\rangle. \tag{3.41}$$

Taking the imaginary part of (3.41), we can obtain the inequality

$$\frac{\epsilon}{2\tau}\{\|E_u^{n+1}\|^2 - \|E_u^n\|^2\} \le \frac{q^2}{2}\|T_1^n\|^2 + \frac{s^2}{2}\|T_2^n\|^2 + \frac{1}{2}\|R_u^n\|^2 + \frac{3}{2}\|E_u^{n+\frac{1}{2}}\|^2. \tag{3.42}$$

By computing the inner product $\langle\cdot,\cdot\rangle$ on both sides of (3.39) with $E_v^{n+\frac{1}{2}}$. we can obtain following equation by Lemma 4:

$$\begin{aligned}&\left\langle \mathcal{A}_1^{-1}\mathcal{B}_1(\delta_t E_v^n), E_v^{n+\frac{1}{2}}\right\rangle - \alpha\left\langle \mathcal{B}_2^{-1}\mathcal{A}_2\delta_{\hat{x}}\delta_{\bar{x}}E_v^{n+\frac{1}{2}}, \delta_{\bar{x}}E_v^{n+\frac{1}{2}}\right\rangle \\ &= \beta\left\langle T_3^n, \delta_{\hat{x}}E_v^{n+\frac{1}{2}}\right\rangle + \rho\left\langle T_4^n, \delta_{\hat{x}}E_v^{n+\frac{1}{2}}\right\rangle + \left\langle R_v^n, E_v^{n+\frac{1}{2}}\right\rangle.\end{aligned} \tag{3.43}$$

Since

$$\begin{gathered}T_1^n = \hat{E}_v^n.\hat{u}^n + \hat{E}_u^n.\hat{v}^n - \hat{E}_u^n.\hat{E}_v^n, \\ T_2^n = (\widehat{|u|^{.2}})^n.\hat{E}_u^n + \left[2\mathrm{Re}(\widehat{\overline{u}.E_u})^n - (\widehat{|E_u|^{.2}})^n\right].\hat{u}^n - \left[2\mathrm{Re}(\widehat{\overline{u}.E_u})^n - (\widehat{|E_u|^{.2}})^n\right].\hat{E}_u^n, \\ T_4^n = (\widehat{\overline{u}.E_u})^n + (\widehat{u.\overline{E_u}})^n - (\widehat{|E_u|^{.2}})^n,\end{gathered}$$

we can have the inequality

$$\|T_1^n\|^2 + \|T_2^n\|^2 + \|T_3^n\|^2 + \|T_4^n\|^2 \le C(\|E_u^{n-1}\|^2 + \|E_u^n\|^2 + \|E_v^{n-1}\|^2 + \|E_v^n\|^2). \tag{3.44}$$

By Lemma 5, Eq (3.43) and inequality (3.44), we have

$$\begin{aligned}&\frac{1}{2\tau}\{|||E_v^{n+1}|||_P^2 - |||E_v^n|||_P^2\} \le C\{\|E_u^{n-1}\|^2 + \|E_u^n\|^2 + \|E_v^{n-1}\|^2\} \\ &+ C\{\|E_v^n\|^2 + \|E_v^{n+1}\|^2 + \|\delta_{\bar{x}}E_v^n\|^2 + \|\delta_{\bar{x}}E_v^{n+1}\|^2 + \|R_v^n\|^2\}.\end{aligned} \tag{3.45}$$

Summing inequalities (3.42) and (3.45) side by side, and using inequality (3.44), we can have following inequality with $E_u^0 = E_v^0 = 0$:

$$\epsilon\|E_u^{k+1}\|^2 + |||E_v^{k+1}|||_P^2 \le C\tau\sum_{n=1}^{k+1}\{\|E_u^n\|^2 + \|E_v^n\|^2 + \|\delta_{\bar{x}}E_v^n\|^2 + \|R_u^{n-1}\|^2 + \|R_v^{n-1}\|^2\}. \tag{3.46}$$

By Computing the inner product $\langle\cdot,\cdot\rangle$ on both sides of (3.38) with $\delta_t E_u^n$. we can obtain following equation by Lemma 4:

$$i\epsilon\left\langle \delta_t E_u^n, \delta_t E_u^n\right\rangle - p\left\langle \mathcal{B}_2^{-1}\mathcal{A}_2\delta_{\bar{x}}E_u^{n+\frac{1}{2}}, \delta_{\bar{x}}\delta_t E_u^n\right\rangle = q\left\langle T_1^n, \delta_t E_u^n\right\rangle + s\left\langle T_2^n, \delta_t E_u^n\right\rangle + \left\langle R_u^n, \delta_t E_u^n\right\rangle. \tag{3.47}$$

Taking the real part of (3.47) and summing from 0 to k, we can obtain

$$\begin{aligned}\frac{p}{2\tau}|||\delta_{\bar{x}}E_u^{k+1}|||_Q^2 =& -q\mathrm{Re}\left(\sum_{n=0}^{k}\langle T_1^n,\delta_t E_u^n\rangle\right) - s\mathrm{Re}\left(\sum_{n=0}^{k}\langle T_2^n,\delta_t E_u^n\rangle\right) - \mathrm{Re}\left(\sum_{n=0}^{k}\langle R_u^n,\delta_t E_u^n\rangle\right)\\ &:= M_1^k + M_2^k + M_3^k.\end{aligned} \tag{3.48}$$

By using a method of summation by parts together with assumptions (3.30) and (3.40), we have the inequalities

$$|M_1^k| + |M_2^k| \le C\sum_{n=1}^{k}\{\|E_u^n\|^2 + \|E_v^n\|^2\} + \frac{C}{\tau}\|E_u^{k+1}\|^2,$$
$$|M_3^k| \le C\sum_{n=1}^{k}\{\|E_u^n\|^2 + \|\delta_t R_u^{n-1}\|^2\} + \frac{C}{\tau}\|E_u^{k+1}\|^2 + \frac{C}{\tau}\|R_u^k\|^2.$$

By (3.48) and the above estimates, we can obtain

$$|||\delta_{\bar{x}}E_u^{k+1}|||_Q^2 \le C\{\|E_u^{k+1}\|^2 + \|R_u^k\|^2\} + C\tau\sum_{n=1}^{k}\{\|E_u^n\|^2 + \|E_v^n\|^2 + \|\delta_t R_u^{n-1}\|^2\}. \tag{3.49}$$

For any real-valued grid function f, an operator Θ is defined by

$$\Theta f_j = \sum_{k=1}^{j-1} f_k h + \frac{h}{2}f_j,\quad j = 1,2,\dots,J,\quad \Theta f_0 = \sum_{k=1}^{J-1} f_k h + \frac{h}{2}f_J, \tag{3.50}$$

with $\Theta f_j = \Theta f_{j+J}$. The following results can be easily proved:

$$\delta_x^2\Theta f_j = \delta_{\hat{x}} f_j,\quad \delta_{\hat{x}}\Theta f_j = \frac{1}{4}(f_{j-1} + 2f_j + f_{j+1}) = \mathcal{J} f_j, \tag{3.51}$$

$$\langle f,\Theta f\rangle = \sum_{j=1}^{J} f_j\cdot\Theta f_j h = \frac{1}{2}\left(\sum_{j=1}^{J} f_j h\right)^2 \ge 0, \tag{3.52}$$

$$\|\Theta f\|^2 \le \frac{l^2}{2}\|f\|^2. \tag{3.53}$$

Then define a matrix $\mathbf{J}$ corresponding to the operator $\mathcal{J}$, i.e.,

$$\mathbf{J} = \frac{1}{4}\begin{pmatrix} 2 & 1 & 0 & \cdots & 1\\ 1 & 2 & 1 & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & 1 & 2 & 1\\ 1 & \cdots & 0 & 1 & 2\end{pmatrix}_{J\times J}.$$

It's obvious that $\mathbf{J}$ is invertible and $\mathbf{J}^{-1}$ is circulant symmetric positive definite as the scale J of matrix is an odd integer. By computing the inner product $\langle\cdot,\cdot\rangle$ on both sides of (3.39) with $\delta_t(\mathcal{J}^{-1}\Theta E_v)^n$ and applying Lemma 4, (3.51) and (3.52), we can obtain

$$\begin{aligned}&\left\langle \mathcal{J}^{-1}\mathcal{A}_1^{-1}\mathcal{B}_1(\delta_t E_v^n), \delta_t(\Theta E_v)^n\right\rangle + \alpha\left\langle \mathcal{B}_2^{-1}\mathcal{A}_2\delta_{\bar{x}}E_v^{n+\frac{1}{2}}, \delta_t\delta_{\bar{x}}E_v^n\right\rangle\\ &= \beta\left\langle T_3^n, \delta_t E_v^n\right\rangle + \rho\left\langle T_4^n, \delta_t E_v^n\right\rangle + \left\langle R_v^n, \delta_t(\mathcal{J}^{-1}\Theta E_v)^n\right\rangle.\end{aligned} \tag{3.54}$$

Since $\mathbf{J}$, $\mathbf{A}_1$, $\mathbf{B}_1$ are circulant symmetric positive definite, so there exists $\mathbf{G}$ such that $\mathbf{J}^{-1}\mathbf{A}_1^{-1}\mathbf{B}_1 = \mathbf{G}\mathbf{G}^T$. By (3.51) and (3.52), we can have

$$\begin{aligned}&\left\langle \mathcal{J}^{-1}\mathcal{A}_1^{-1}\mathcal{B}_1(\delta_t E_v^n), \delta_t(\Theta E_v)^n\right\rangle = \langle \delta_t(GE_v)^n, \delta_t(\Theta(GE_v))^n\rangle\\ &= \frac{1}{2}\left(h\sum_{j=1}^{J}\delta_t(GE_v)_j^n\right)^2 := C^n \geq 0.\end{aligned} \tag{3.55}$$

Summing Eq (3.54) from 0 to k together with (3.55), we can obtain

$$\begin{aligned}\sum_{n=0}^{k} C^n + \frac{\alpha}{2\tau}|||\delta_{\bar{x}}E_v^{k+1}|||_Q^2 &= \beta\sum_{n=0}^{k}\left\langle T_3^n, \delta_t E_v^n\right\rangle + \rho\sum_{n=0}^{k}\left\langle T_4^n, \delta_t E_v^n\right\rangle + \sum_{n=0}^{k}\left\langle R_v^n, \delta_t(\mathcal{J}^{-1}\Theta E_v)^n\right\rangle\\ &:= M_4^k + M_5^k + M_6^k.\end{aligned} \tag{3.56}$$

By using a method of summation by parts together with assumptions (3.30) and (3.40), we have the inequalities

$$|M_4^k| + |M_5^k| \leq C\sum_{n=1}^{k}\{\|E_v^n\|^2 + \|E_u^n\|^2\} + \frac{C}{\tau}\|E_v^{k+1}\|^2,$$

$$|M_6^k| \leq C\sum_{n=1}^{k}\{\|\mathcal{J}^{-1}\Theta E_v^n\|^2 + \|\delta_t R_v^{n-1}\|^2\} + \frac{C}{\tau}\|\mathcal{J}^{-1}\Theta E_v^{k+1}\|^2 + \frac{C}{\tau}\|R_v^k\|^2.$$

Noticing that $\mathcal{J}(\mathbf{I} - \frac{h^2}{4}\delta_x^2 + \frac{h^4}{16}\delta_x^2\delta_x^2 - \frac{h^6}{64}\delta_x^2\delta_x^2\delta_x^2) = \mathbf{I} - \frac{h^8}{256}\delta_x^2\delta_x^2\delta_x^2\delta_x^2$, we have

$$\mathcal{J}^{-1} = \mathbf{I} - \frac{h^2}{4}\delta_x^2 + \frac{h^4}{16}\delta_x^2\delta_x^2 - \frac{h^6}{64}\delta_x^2\delta_x^2\delta_x^2 + O(h^8).$$

By using Lemma 6 and (3.53), the above inequality can be written as

$$|M_6^k| \leq C\sum_{n=1}^{k}\{\|E_v^n\|^2 + \|\delta_t R_v^{n-1}\|^2\} + \frac{C}{\tau}\|E_v^{k+1}\|^2 + \frac{C}{\tau}\|R_v^k\|^2,$$

Multiplying Eq (3.56) with 2τ, we can obtain

$$|||\delta_{\bar{x}}E_v^{k+1}|||_Q^2 \leq C\tau\sum_{n=1}^{k}\{\|E_u^n\|^2 + \|E_v^n\|^2 + \|\delta_t R_v^{n-1}\|^2\} + C\{\|E_v^{k+1}\|^2 + \|R_v^k\|^2\}. \tag{3.57}$$

Since the norms $\|\cdot\|$, $|||\cdot|||_P$, and $|||\cdot|||_Q$ are equivalent, we can have the following inequality by summing (3.46), (3.49) and (3.57):

$$\begin{aligned}\|E_u^{k+1}\|_1^2 + \|E_v^{k+1}\|_1^2 &\leq C\{\|R_u^k\|^2 + \|R_v^k\|^2\}\\ &+ C\tau\sum_{n=1}^{k+1}\{\|E_v^n\|_1^2 + \|E_u^n\|^2 + \|\delta_t R_u^{n-1}\|^2 + \|\delta_t R_v^{n-1}\|^2 + \|R_u^{n-1}\|^2 + \|R_v^{n-1}\|^2\}.\end{aligned} \tag{3.58}$$

Taking τ sufficiently small and applying Lemmas 8 and 9, we can obtain

$$\|E_u^{k+1}\|_1^2 + \|E_v^{k+1}\|_1^2 \le C(\tau^4 + h^{16}). \tag{3.59}$$

Moreover, we need to confirm the inequality in (3.40) holds for $n = k + 1$ to complete our proof. We can get the following inequalities by Lemma 6:

$$\|E_u^{k+1}\|_\infty \le C\|E_u^{k+1}\|_1 \le C(Y^*, h_0, T)(\tau^2 + h^8),$$
$$\|\delta_t E_u^k\|_\infty \le \tau^{-1}\|E_u^{k+1} - E_u^k\|_\infty \le C(Y^*, h_0, T)(\tau + h^8\tau^{-1}),$$

and similar inequalities hold for $\|E_v^{k+1}\|_\infty$ and $\|\delta_t E_u^k\|_\infty$. Then it's easy to see that the inequalities above hold for $n = k + 1$ when $h^8\tau^{-1} = o(1)$, i.e., $h^8\tau^{-1} \to 0$ as $h \to 0$, and taking h sufficiently small, which implies that assumption (3.40) is valid for $n = k + 1$. The proof is finished. □

Corollary 1. *By applying Lemma 6, we can obtain the following optimal order convergence rate under the same conditions in Theorem 3:*

$$\max_{0<n\le N}\{\|u^n - U^n\|_\infty + \|v^n - V^n\|_\infty\} \le C(\tau^2 + h^8).$$

4. Numerical experiments

In this section, some numerical examples are presented to illustrate the conservative properties and eighth-order accuracy of the proposed compact scheme. The ultimate compact schemes (3.32)–(3.35) can be written as the following linear matrix equations:

$$C_1U^{0*} = D_1U^0 + E_1(U^0, V^0),$$
$$C_2V^{0*} = D_2V^0 + E_2(U^0, V^0),$$
$$C_1U^{n+1} = D_1U^n + F_1\big((\widehat{|U|^2})^n, \hat{U}^n, \hat{V}^n\big),$$
$$C_2V^{n+1} = D_2V^n + F_2\big(V^n, V^{n*}, (\widehat{|U|^2})^n\big),$$

where E_1, E_2, F_1 and F_2 are nonlinear terms. Our numerical experiments are conducted using Matlab (R2019b). The invariants I_1, I_2, I_3 and I_4 are tested by the discrete formulations:

$$I_{1h}^n = h\sum_{j=1}^{J}|U_j^n|^2, \quad I_{2h}^n = h\sum_{j=1}^{J}V_j^n,$$

$$I_{3h}^n = h\sum_{j=1}^{J}\Big(\frac{q\beta}{m+1}(V_j^n)^{m+1} + p\rho|\mathcal{B}_1^{-1}\mathcal{A}_1\delta_{\hat{x}}U_j^n|^2 + q\rho V_j^n|U_j^n|^2 + \frac{s\rho}{2}|U_j^n|^4 - \frac{q\alpha}{2}(\mathcal{B}_1^{-1}\mathcal{A}_1\delta_{\hat{x}}V_j^n)^2\Big),$$

$$I_{4h}^n = h\sum_{j=1}^{J}\Big(q(V_j^n)^2 - 2\rho\epsilon Im(U_j^n\mathcal{B}_1^{-1}\mathcal{A}_1\delta_{\hat{x}}\overline{U}_j^n)\Big),$$

and the errors of invariants are defined as

$$E_1 = |I_{1h}^n - I_{1h}^0|, \quad E_2 = |I_{2h}^n - I_{2h}^0|, \quad E_3 = |I_{3h}^n - I_{3h}^0|, \quad E_4 = |I_{4h}^n - I_{4h}^0|.$$

Moreover, the accuracy of the proposed scheme is tested by the discrete L^2- norm ($\|u - U\| + \|v - V\|$) and L^∞- norm ($\|u - U\|_\infty + \|v - V\|_\infty$).

Example 1. *[8] We consider the following Cauchy problem:*

$$\begin{aligned} &iu_t + u_{xx} - vu = 0, \quad (x,t) \in R \times (0,T], \\ &v_t + v_{xxx} + (3v^2 + |u|^2)_x = 0, \quad (x,t) \in R \times (0,T], \\ &u(x,0) = \varphi(x), \quad v(x,0) = \phi(x), \quad x \in R, \end{aligned}$$

whose exact solutions are given by $u(x,t) = \exp(i(x + t/4))$ and $v(x,t) = 3/4$. we then compute the equations with $h = \pi/20$ and $\tau = 0.001$ in the spatial interval $[0, 2\pi]$. The errors of the numerical invariants at different time are listed in Table 1, which indicates that the proposed compact scheme preserves the conservation properties. Table 2 shows that the convergence rate of the proposed compact scheme is eighth-order in space.

Table 1. Errors of invariants at different time: $h = \pi/20$, $\tau = 0.001$.

t	E_1	E_2	E_3	E_4
1	3.73346E-11	1.11910E-13	9.03455E-12	7.48273E-11
5	1.86450E-10	5.69322E-13	4.50811E-11	3.73753E-10
10	3.72820E-10	1.32072E-12	8.96581E-11	7.47615E-10

Table 2. Convergence rates at different time: $h = \pi/10$, $\tau = 0.1$.

t	h	τ	$L^2 - error$	Rate	$L^\infty - error$	Rate
2	h	τ	1.09158E-03		4.35476E-04	
	$h/2$	$\tau/16$	4.80890E-06	7.82649	1.91847E-06	7.82649
	$h/4$	$\tau/256$	1.89240E-08	7.98935	7.54999E-09	7.98927
5	h	τ	2.94823E-03		1.17617E-03	
	$h/2$	$\tau/16$	1.20760E-05	7.93156	4.81764E-06	7.93156
	$h/4$	$\tau/256$	4.73303E-08	7.99517	1.88923E-08	7.99439
10	h	τ	6.04262E-03		2.41066E-03	
	$h/2$	$\tau/16$	2.41903E-05	7.96460	9.65142E-06	7.96447
	$h/4$	$\tau/256$	1.02970E-07	7.87605	4.42506E-08	7.76890

Example 2. *[3] We consider the following coupled equations:*

$$\begin{aligned} &i\epsilon u_t + \frac{3}{2}u_{xx} - \frac{1}{2}vu = 0, \quad (x,t) \in R \times (0,T], \\ &v_t + \frac{1}{2}v_{xxx} + \frac{1}{2}(v^2 + |u|^2)_x = 0, \quad (x,t) \in R \times (0,T], \end{aligned}$$

with exact solutions

$$\begin{aligned} u(x,t) &= -\frac{6\sqrt{3}c}{5}\frac{\tanh(\xi)}{\cosh(\xi)}\exp\left(ic\left(\left(\frac{3}{20\epsilon} - \frac{\epsilon c}{6}\right)t - \frac{\epsilon}{3}x\right)\right), \\ v(x,t) &= -\frac{9c}{5}\frac{1}{\cosh^2(\xi)}, \quad \xi = \sqrt{c/10}(x + ct), \end{aligned}$$

where c is an arbitrary positive constant. In addition, we set the artificial boundary conditions $u(a,t) = u(b,t) = 0$ and $v(a,t) = v(b,t) = 0$ to satisfy the physical condition that $|u|$ and v tend to zero as $|x| \to \infty$. Our simulations are conducted by taking $\epsilon = 1$, the traveling wave speed $c = 0.45$ and initial conditions

$$u(x,0) = -\frac{6\sqrt{3}c}{5}\frac{\tanh(\xi)}{\cosh(\xi)}\exp(ic(-\frac{\epsilon}{3}x)),$$
$$v(x,0) = -\frac{9c}{5}\frac{1}{\cosh^2(\xi)}, \quad \xi = \sqrt{c/10}(x+ct).$$

Table 3 lists the numerical solutions at $t = 0.001$, with $h = 0.25$, $\tau = 0.00001$ and $[a,b] = [-30,30]$, where the scheme MECS expands $[a,b]$ to $[-150,150]$ for reducing boundary truncation error. Compared with the numerical results obtained by the fourth-order compact scheme (FCS) in [9] and exponential time differencing three-layer implicit scheme with Padé approximation (ETDT-P) in [7]. We can see that the eighth-order compact scheme (ECS) and modified eighth-order compact scheme (MECS) give better approximations. In addition, MECS gives much more accurate error estimate than ECS, which is caused by boundary truncation error. The numerical solution profiles of $|U|$ and V, as well as the contours in Figure 1 show that the waves traveling with a speed $c = 0.45$ keep the shape and hight, which are in good agreement with the exact solutions.

Table 3. Comparison of numerical solutions with exact solutions and other methods: $t = 0.001$, $\tau = 0.00001$, $h = 0.25$.

	x	MECS	ECS	FCS	ETDT-P	Exact solution
ImU	-20	3.7904E-03	3.7904E-03	3.7904E-03	3.7904E-03	3.7904E-03
	-10	2.1428E-01	2.1428E-01	2.1428E-01	2.1428E-01	2.1428E-01
	0	-3.013332E-09	-3.013332E-09	-3.0140E-09	-2.4973E-09	-3.013332E-09
	10	2.1424E-01	2.1424E-01	2.1424E-01	2.1424E-01	2.1424E-01
	20	3.7915E-03	3.7915E-03	3.7915E-03	3.7915E-03	3.7915E-03
$\|\|\mathrm{Im}E_u\|\|$		5.1605E-14	1.5738E-05	1.4412E-05	3.8279E-05	
ReU	-20	-2.6597E-02	-2.6597E-02	-2.6597E-02	-2.6597E-02	-2.6597E-02
	-10	1.5188E-02	1.5188E-02	1.5188E-02	1.5188E-02	1.5188E-02
	0	-8.928390E-05	-8.928390E-05	-8.928328E-05	-8.9282E-05	-8.928390E-05
	10	-1.5200E-02	-1.5200E-02	-1.5200E-02	-1.5200E-02	-1.5200E-02
	20	2.6592E-02	2.6592E-02	2.6592E-02	2.6592E-02	2.6592E-02
$\|\|\mathrm{Re}E_u\|\|$		3.9746E-14	9.7531E-05	8.0273E-05	7.5941E-06	
V	-20	-6.6886E-04	-6.6886E-04	-6.6886E-04	-6.6886E-04	-6.6886E-04
	-10	-4.5256E-02	-4.5256E-02	-4.5256E-02	-4.5256E-02	-4.5256E-02
	0	-8.1000E-01	-8.1000E-01	-8.1000E-01	-8.1000E-01	-8.1000E-01
	10	-4.5239E-02	-4.5239E-02	-4.5239E-02	-4.5239E-02	-4.5239E-02
	20	-6.6861E-04	-6.6861E-04	-6.6861E-04	-6.6861E-04	-6.6861E-04
$\|\|E_v\|\|$		7.6034E-14	1.1311E-06	7.2736E-07	1.0331E-07	

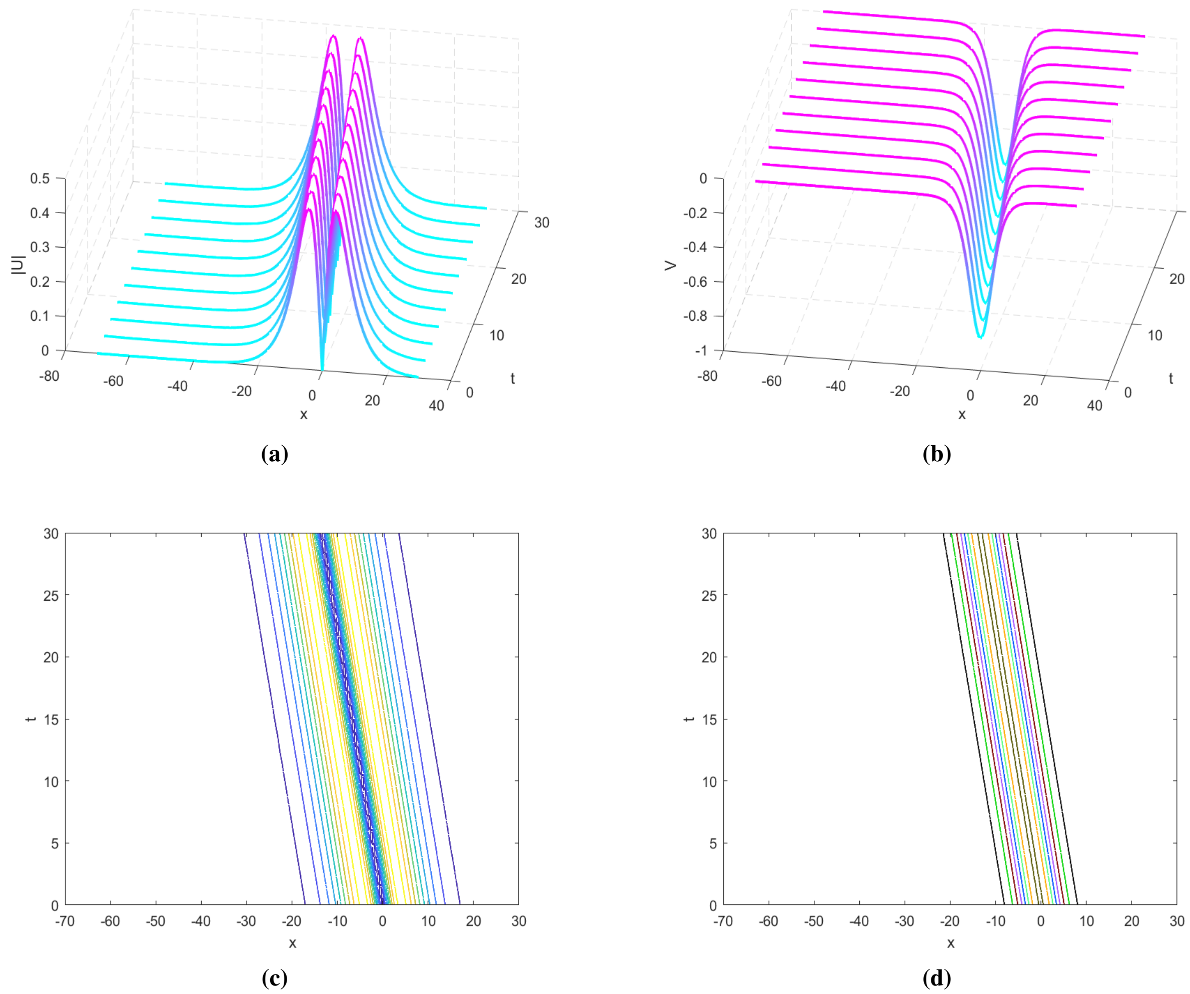

Figure 1. Numerical solution profiles of $|U|$ and V(a and b) and the contours(c and d): $t \in [0, 30]$, $[a, b] = [-70, 30]$, $h = 0.5$, $\tau = 0.001$.

Example 3. *[11] We consider the following coupled equations:*

$$iu_t + u_{xx} - \sigma vu + |u|^2 u = 0, \quad (x, t) \in R \times (0, T],$$
$$v_t + v_{xxx} + \frac{1}{2}(v^2 - \sigma|u|^2)_x = 0, \quad (x, t) \in R \times (0, T],$$

with exact solutions

$$u(x, t) = \exp(i(\omega t + cx/2))\frac{\sqrt{2C^*(1 + 6\sigma)}}{\cosh(\sqrt{C^*}(x - ct))}, \quad C^* = c^2/4 + \omega^2,$$
$$v(x, t) = \frac{12C^*}{\cosh^2(\sqrt{C^*}(x - ct))}, \quad 2c = 1 + \sqrt{1 + \frac{\sigma}{3}(1 + 6\sigma)},$$

where $\sigma \in (-1/6, 0)$ and $\omega \in R$. Set the artificial boundary conditions $u(a, t) = u(b, t) = 0$ and $v(a, t) = v(b, t) = 0$. Our simulations are conducted by taking $\sigma = -1/12$, $\omega = 0$, $[a, b] = [-40, 70]$, the traveling wave speed $c = (1 + \sqrt{71/72})/2$ and initial conditions

$$u(x,0) = \exp(icx/2)\frac{\sqrt{2C^*(1+6\sigma)}}{\cosh\left(\sqrt{C^*}x\right)}, \quad C^* = c^2/4 + \omega^2,$$
$$v(x,0) = \frac{12C^*}{\cosh^2\left(\sqrt{C^*}x\right)}, \quad 2c = 1 + \sqrt{1 + \frac{\sigma}{3}(1+6\sigma)}.$$

The errors of the numerical invariants at different times are listed in Table 4, which indicates that the proposed compact scheme preserves the conservation properties. Table 5 shows that the convergence rate of the proposed compact scheme is eighth-order in space. The numerical solution profiles of $|U|$ and V, as well as the contours in Figure 2 show that the waves traveling with a speed $c = 0.99652$ keep the shape and hight, which are in good agreement with the exact solutions.

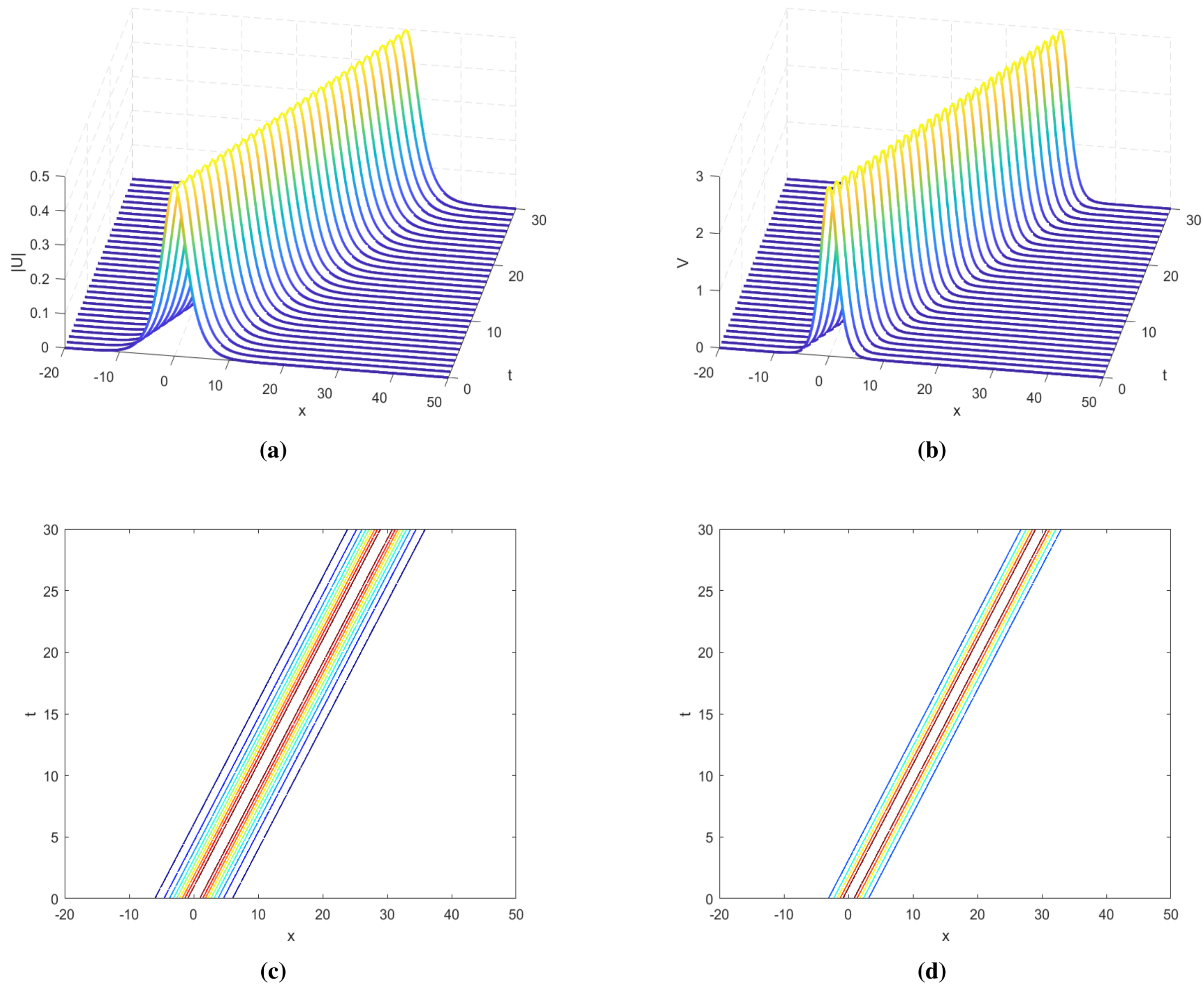

Figure 2. Numerical solution profiles of $|U|$ and V(a and b) and the contours(c and d): $t \in [0, 30]$, $h = 0.25$, $\tau = 0.001$.

Table 4. Errors of invariants at different time: $h = 0.1$, $\tau = 0.001$.

t	E_1	E_2	E_3	E_4
1	1.35891E-13	8.96330E-10	1.65457E-10	3.32290E-10
5	7.79488E-13	9.67230E-08	8.24029E-10	1.65427E-09
10	1.53033E-12	2.48881E-07	1.64719E-09	3.30639E-09

Table 5. Convergence rates at different time: $h = 1$, $\tau = 0.1$.

t	h	τ	$L^2 - error$	Rate	$L^\infty - error$	Rate
1	h	τ	2.83547E-02		1.60049E-02	
	$h/2$	$\tau/16$	8.47660E-05	8.38589	5.63783E-05	8.14915
	$h/4$	$\tau/256$	3.28192E-07	8.01280	2.20134E-07	8.00062
5	h	τ	7.81002E-02		3.78102E-02	
	$h/2$	$\tau/16$	2.60905E-04	8.22566	1.49546E-04	7.98205
	$h/4$	$\tau/256$	1.01440E-06	8.00675	5.81734E-07	8.00601
10	h	τ	1.44349E-01		7.50822E-02	
	$h/2$	$\tau/16$	4.75731E-04	8.24520	2.60189E-04	8.17277
	$h/4$	$\tau/256$	1.84463E-06	8.01067	1.00971E-06	8.00947

5. Conclusions

In this paper, we propose an eighth-order compact finite difference scheme by constructing several circulant symmetric positive definite matrices to obtain the numerical solution of coupled Schrödinger-KdV equations. The performance of proposed compact scheme is evaluated by conservation properties and error estimate. Numerical examples demonstrate the better performance of the proposed compact scheme in accuracy compared with FCS and ETDT-P given in [7, 9]. Since the matrices have good properties, we can discuss the possibility that the proposed compact scheme can be applied to other equations such as nonlinear Dirac equation [21], generalized Rosenau-RLW equation [22], Klein-Gordon-Schrödinger equation [23], coupled Gross-Pitaevskii equations [24] and regularized long wave equation [25].

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Conflict of interest

The authors declare no conflicts of interest.

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