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Research article

Existence of periodic solutions for a class of (ϕ_1, ϕ_2)-Laplacian discrete Hamiltonian systems

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Abstract: In this paper, we consider the existence of periodic solutions for a class of nonlinear difference systems involving classical (ϕ_1, ϕ_2)-Laplacian. By using the least action principle, we obtain that the system with classical (ϕ_1, ϕ_2)-Laplacian has at least one periodic solution when potential function is (p, q)-sublinear growth condition, subconvex condition. The results obtained generalize and extend some known works.

Keywords: (ϕ_1, ϕ_2)-Laplacian; discrete systems; periodic solutions; the least action principle
Mathematics Subject Classification: 34C25, 37J45, 58E50

1. Introduction and main results

In this paper, we investigate the existence of periodic solutions for the following system involving classical (ϕ_1, ϕ_2)-Laplacian:

$$\begin{cases} \Delta\phi_1(\Delta u_1(n-1)) = \nabla_{u_1} F(n, u_1(n), u_2(n)) \\ \Delta\phi_2(\Delta u_2(n-1)) = \nabla_{u_2} F(n, u_1(n), u_2(n)), \end{cases} \tag{1.1}$$

where Δ is the forward difference operator, $n \in \mathbb{Z}$, $F : \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and ϕ_m, $m = 1, 2$ satisfy the following condition:
$(\mathcal{A}0)$ *ϕ_m is a homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$ such that $\phi_m(0) = 0$, $\phi_m = \nabla\Phi_m$, where $\Phi_m \in C^1(\mathbb{R}^N, [0, +\infty))$ strictly convex and $\Phi_m(0) = 0$, $m = 1, 2$.*
$(\mathcal{F})$ $F(n, x_1, x_2)$ is continuously differentiable in (x_1, x_2), there exist $a_1, a_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b : \mathbb{Z}[1, T] \to \mathbb{R}^+$ such that

$$|F(n, x_1, x_2)| \leq [a_1(|x_1|) + a_2(|x_2|]b(n),$$

$$|\nabla_{x_1} F(n, x_1, x_2)| \le [a_1(|x_1|) + a_2(|x_2|]b(n),$$

$$|\nabla_{x_2} F(n, x_1, x_2)| \le [a_1(|x_1|) + a_2(|x_2|]b(n).$$

Remark 1.1. Assumption ($\mathscr{A}0$) given in [1] is used to characterize the classical homeomorphisms. Moreover, if $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ is coercive (*i.e.*, $\Phi_m(x) \to +\infty$ as $|x| \to \infty$), then there exists $\delta_m = \min_{|x|=1} \Phi_m(x) > 0, m = 1, 2$ such that

$$\Phi_m(x) \ge \delta_m(|x| - 1), \quad x \in \mathbb{R}^N. \tag{1.2}$$

In recent years, critical point theory (see [2–6]) plays an important role in studying the Hamiltonian systems, nonlinear differential equations, nonlinear difference system, etc (see [7–32]). In [1] and [7], by virtue of critical point theorem, Mawhin investigated the existence and multiplicity of periodic solutions for the following nonlinear difference systems with ϕ-Laplacian:

$$\Delta\phi[\Delta u(n-1)] = \nabla_u F[n, u(n)] + h(n) \quad (n \in \mathbb{Z}), \tag{1.3}$$

where ϕ is a homeomorphism from $X \subset \mathbb{R}^N$ onto $Y \subset \mathbb{R}^N$ and the following three different types of homeomorphisms were discussed:
(1) classical homeomorphism: if $\phi : \mathbb{R}^N \to \mathbb{R}^N$;
(2) bounded homeomorphism: if $\phi : \mathbb{R}^N \to B_a$ $(a < +\infty)$;
(3) singular homeomorphism: if $\phi : B_a \subset \mathbb{R}^N \to \mathbb{R}^N$,
where B_a is a ball with its center at origin and radius a.

Inspired by [1, 7], in [20], by using some abstract critical point theorems, the authors obtained some multiplicity results of periodic solutions for difference systems involving (ϕ_1, ϕ_2)-Laplacian. In [21], by using Clark theorem, the authors obtained that a class of nonlinear difference systems involving classical (ϕ_1, ϕ_2)-Laplacian has at least m distinct pairs of homoclinic solutions. In [22], by using an important three critial point theorem, the authors obtained that a class of (ϕ_1, ϕ_2)-Laplacian system has at least three T-periodic solutions. In [23], by using the least action principle and saddle point theorem, the authors obtained that system with classical and bounded (ϕ_1, ϕ_2)-Laplacian has at least one periodic solution when F has (p, q)-sublinear growth. In [24], the authors assumed that the nonlinear term F in system satisfies a corresponding sub-linear growth condition in Orlicz-Sobolev space, by using the least action principle, they obtained that nonlinear and non-homogeneous elliptic system involving (ϕ_1, ϕ_2)-Laplacian has at least a nontrivial solution, and by using the genus theory, obtained that system has infinitely many solutions under an additional symmetric condition.

As $\Phi_1(x) = \Phi_2(x) = \frac{1}{2}|x|^2$ and $F(n, x, y) \equiv F(n, x)$, system (1.1) reduces to the following second order discrete Hamiltonian system:

$$\Delta^2 u(n-1) = \nabla F(n, u(n)), \quad \forall\, n \in \mathbb{Z}. \tag{1.4}$$

In [10], Guan and Yang investigated the existence of periodic solutions for system (1.4). By using the variational minimizing method and the saddle point theorem, they obtained that system (1.4) has at least one T-periodic solution.

Theorem A. (see [10]) *Suppose that $F(n, x) = F_1(n, x) + F_2(x)$ satisfies*
(H_1) $F(n, x) \in C^1(\mathbb{R}^N, \mathbb{R})$ *for any* $n \in \mathbb{Z}$, $F(n + T, x) = F(n, x)$ *for all* $(n, x) \in \mathbb{Z} \times \mathbb{R}^N$, T *is a positive*

integer;
(H_2) *there exist* $f, g : \mathbb{Z}[1,T] \to \mathbb{R}^+$ *and* $\alpha \in [0,1)$ *such that*

$$|\nabla F_1(n,x)| \le f(n)|x|^{\alpha} + g(n), \ \textit{for all } (n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N;$$

(H_3) *there exist constants* $r > 0$ *and* $\gamma \in [0,2)$ *such that*

$$(\nabla F_2(x) - \nabla F_2(y), x - y) \ge -r|x-y|^{\gamma}, \ \textit{for all } x, y \in \mathbb{R}^N;$$

(H_4) $\liminf_{|x| \to +\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) > \dfrac{1}{8\sin^2 \frac{\pi}{T}} \sum_{n=1}^{T} f^2(n).$

Then system (1.4) has at least one T-periodic solution.

Theorem B. (see [10]) *Suppose that* $F(n,x) = F_1(n,x) + F_2(x)$ *satisfies* (H_1), (H_2), (H_3) *and the following conditions:*
(H_5) *there exist* $\delta \in [0,2)$ *and* $C > 0$ *such that*

$$(\nabla F_2(x) - \nabla F_2(y), x - y) \le C|x-y|^{\delta}, \ \textit{for all } x, y \in \mathbb{R}^N;$$

(H_6) $\limsup_{|x| \to +\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) < -\dfrac{3}{8\sin^2 \frac{\pi}{T}} \sum_{n=1}^{T} f^2(n).$

Then system (1.4) possesses at least one T-periodic solution.

Motivated by [10, 20–23], in this paper, we consider the existence of periodic solutions for system (1.1) and obtain the following main results.

Theorem 1.1. *Suppose that* $F(n, x_1, x_2) = F_1(n, x_1, x_2) + F_2(x_1, x_2)$ *and the following conditions hold:*
$(\mathcal{A}1)$ *there exist constants* $d_1 > \frac{1}{p}$, $d_2 > \frac{1}{q}$, $p > 1$ *and* $q > 1$ *such that*

$$\Phi_1(x_1) + \Phi_2(x_2) \ge d_1|x_1|^p + d_2|x_2|^q, \qquad \forall\ x_1, x_2 \in \mathbb{R}^N;$$

(F_0) $F(n, x_1, x_2) \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$ *for every* $n \in \mathbb{Z}$, *and* $F(n+T, x_1, x_2) = F(n, x_1, x_2)$ *for all* $(n, x_1, x_2) \in \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N$;
(F_1) *there exist* $f_i, h_i : \mathbb{Z}[1,T] \to \mathbb{R}^+, i = 1,2$ *and* $\alpha_1 \in [0, p-1)$, $\alpha_2 \in [0, q-1)$ *such that*

$$|\nabla_{x_1} F_1(n, x_1, x_2)| \le f_1(n)|x_1|^{\alpha_1} + h_1(n),$$

$$|\nabla_{x_2} F_1(n, x_1, x_2)| \le f_2(n)|x_2|^{\alpha_2} + h_2(n),$$

for all $(n, x_1, x_2) \in \mathbb{Z}[1,T] \times \mathbb{R}^N \times \mathbb{R}^N$;
(F_2) *there exist constants* $r_i \in (0, +\infty), i = 1, 2$, $\gamma_1 \in [0, p)$ *and* $\gamma_2 \in [0, q)$ *such that*

$$(\nabla_{x_1} F_2(x_1, x_2) - \nabla_{y_1} F_2(y_1, y_2), x_1 - y_1) \ge -r_1|x_1 - y_1|^{\gamma_1}$$

and

$$(\nabla_{x_2} F_2(x_1, x_2) - \nabla_{y_2} F_2(y_1, y_2), x_2 - y_2) \ge -r_2|x_2 - y_2|^{\gamma_2}$$

for all (x_1, x_2)*,* $(y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N$*;*
(F_3)

$$\lim_{|x_1|+|x_2|\to\infty} \inf \frac{1}{|x_1|^{q\alpha_1} + |x_2|^{p\alpha_2}} \sum_{n=1}^{T} F(n, x_1, x_2) > M$$

where

$$M = \max\left\{\frac{2^{q\alpha_1}}{q}[C(p, p')]^{\frac{q}{p}} \sum_{n=1}^{T} f_1(n)^q, \quad \frac{2^{p\alpha_2}}{p}[C(q, q')]^{\frac{p}{q}} \sum_{n=1}^{T} f_2(n)^p\right\},$$

$$C(p, p') = \min\left\{\frac{(T-1)^{2p-1}}{T^{p-1}}, \ \frac{T^{p-1}\Theta(p', p)}{(p'+1)^{p/p'}}\right\}, \ \ 1/p + 1/p' = 1,$$

$$\Theta(p', p) = \sum_{n=1}^{T}\left[\left(\frac{n}{T}\right)^{p'+1} + \left(1 - \frac{n}{T} + \frac{1}{T}\right)^{p'+1} - \frac{2}{T^{p'+1}}\right]^{p/p'}.$$

Then system (1.1) possesses at least one T*-periodic solution.*

Theorem 1.2. *Suppose that* $F(n, x_1, x_2) = F_1(n, x_1, x_2) + F_2(x_1, x_2)$*,*$(\mathcal{A}1)$*,*$(F_0)$*,*$(F_1)$*,*$(F_3)$ *and the following conditions hold:*
(F_4) *there exist constants* $r_1 \in [0, \frac{pd_1-1}{C(p,p')})$*,* $r_2 \in [0, +\infty)$*,* $r_3 \in [0, \frac{qd_2-1}{C(q,q')})$*,* $r_4 \in [0, +\infty)$*,* $\alpha_0 \in [0, p)$ *and* $\beta_0 \in [0, q)$ *such that*

$$(\nabla_{x_1} F_2(x_1, x_2) - \nabla_{y_1} F_2(y_1, y_2), x_1 - y_1) \geq -r_1|x_1 - y_1|^p - r_2|x_1 - y_1|^{\alpha_0}$$

and

$$(\nabla_{x_2} F_2(x_1, x_2) - \nabla_{y_2} F_2(y_1, y_2), x_2 - y_2) \geq -r_3|x_2 - y_2|^q - r_4|x_2 - y_2|^{\beta_0}$$

for all (x_1, x_2)*,* $(y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N$*.*
Then system (1.1) possesses at least one T*-periodic solution.*

By Theorem 1.2, it is easy to obtain the following corollary.

Corollary 1.3. *Suppose that* $F(n, x_1, x_2) = F_1(n, x_1, x_2) + F_2(x_1, x_2)$*,*$(\mathcal{A}1)$*,*$(F_0)$*,*$(F_1)$*,*$(F_3)$ *and the following conditions hold:*
(F_5) *there exist constants* $r_1 \in [0, \frac{pd_1-1}{C(p,p')})$*,* $r_2 \in [0, +\infty)$*,* $r_3 \in [0, \frac{qd_2-1}{C(q,q')})$*,* $r_4 \in [0, +\infty)$*,* $\alpha_0 \in [0, p)$ *and* $\beta_0 \in [0, q)$ *such that*

$$|\nabla_{x_1} F_2(x_1, x_2) - \nabla_{y_1} F_2(y_1, y_2)| \geq -r_1|x_1 - y_1|^{p-1} - r_2|x_1 - y_1|^{\alpha_0 - 1}$$

and

$$|\nabla_{x_2} F_2(x_1, x_2) - \nabla_{y_2} F_2(y_1, y_2)| \geq -r_3|x_2 - y_2|^{q-1} - r_4|x_2 - y_2|^{\beta_0 - 1}$$

for all (x_1, x_2)*,* $(y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N$*.*
Then system (1.1) possesses at least one T*-periodic solution.*

Remark 1.2. Theorem 1.1, Theorem 1.2 and Corollary 1.3 generalize Theorem A. In fact, when $\Phi_1(x) = \Phi_2(x) = \frac{1}{2}|x|^2$ and $F(n, x, y) \equiv F(n, x)$, System (1.1) reduces to system (1.4) and Theorem 1.1 become Theorem A. Moreover, Theorem 1.2 and Corollary 1.3 is still a new result, which shows that (F_2) can be weaken to (F_4).

Theorem 1.4. *Suppose that $F(n,x_1,x_2) = F_1(n,x_1,x_2) + F_2(n,x_1,x_2)$, F_1 and F_2 satisfy $(\mathcal{F})$, $(\mathcal{A}1)$, (F_0),(F_1) and the following conditions hold:*
(F_6)

$$F_2(n,x_1,x_2) \geq 0,$$

for all $(n,x_1,x_2) \in \mathbb{Z}\times\mathbb{R}^N\times\mathbb{R}^N$ with $|x_1| > 1$, $|x_2| > 1$;
(F_7) *$F_2(n,.,.)$ is (λ,μ)-subconvex with $\lambda = (\lambda_1,\lambda_2)$, $\mu = (\mu_1,\mu_2)$ and $\lambda_1 > \frac{1}{p}$, $\mu_1 < 2^{p-1}\lambda_1^p$, $\lambda_2 > \frac{1}{q}$, $\mu_2 < 2^{q-1}\lambda_2^q$, that is*

$$F_2(n,\lambda_1(x_1+y_1),\lambda_2(x_2+y_2)) \leq \mu_1 F_2(n,x_1,x_2) + \mu_2 F_2(n,y_1,y_2);$$

(F_8)

$$\lim_{|x_1|+|x_2|\to\infty} \frac{1}{|x_1|^{q\alpha_1}+|x_2|^{p\alpha_2}}\Big(\sum_{n=1}^{T} F_1(n,x_1,x_2) + \frac{1}{\mu_1}\sum_{n=1}^{T} F_2(n,\lambda_1 x_1,\lambda_2 x_2)\Big) > M$$

where $M = \max\Big\{\frac{2^{q\alpha_1}}{q}[C(p,p')]^{\frac{q}{p}}\sum_{n=1}^{T} f_1(n)^q,\quad \frac{2^{p\alpha_2}}{p}[C(q,q')]^{\frac{p}{q}}\sum_{n=1}^{T} f_2(n)^p\Big\}$.
Then system (1.1) possesses at least one T-periodic solution.

2. Preliminaries

First, we present some basic notations. We wse $(\cdot,\cdot)$ and $|\cdot|$ to denote the inner product and the Euclidean norm in $\mathbb{R}^N$. Let

$$H_T = \{v := \{v(n)\} | v(n+T) = v(n), v(n) \in \mathbb{R}^N, n \in \mathbb{Z}\}.$$

For $1 < s < +\infty$ and $v \in H_T$, we define

$$\|v\|_s = \Big(\sum_{n=1}^{T} |\Delta v(n)|^s + \sum_{n=1}^{T} |v(n)|^s\Big)^{1/s}.$$

For $v \in H_T$, set

$$\|v\|_{[r]} = \Big(\sum_{n=1}^{T} |v(n)|^r\Big)^{1/r}, r > 1 \quad \text{and} \quad \|v\|_\infty = \max_{n\in\mathbb{Z}[1,T]} |v(n)|.$$

Let $E = H_T \times H_T$. For $u = (u_1,u_2)^\tau \in E$, define

$$\|u\| = \|u_1\|_p + \|u_2\|_q.$$

For any $u \in H_T$, it can be expressed as $u(n) = \overline{u} + \widetilde{u}(n)$, where $\overline{u} = \frac{1}{T}\sum_{n=1}^{T} u(n)$, and $\sum_{n=1}^{T} \widetilde{u}(n) = 0$.

Lemma 2.1. (see [16])*Let $\widetilde{u}(n) = (\widetilde{u}_1(n),\widetilde{u}_2(n))^\tau$, then*

$$\sum_{n=1}^{T} |\widetilde{u}_m(n)|^s \leq C(s,s')\sum_{n=1}^{T} |\Delta u_m(n)|^s,\ m = 1,2.$$

where $1/s + 1/s' = 1$ *and* $s, s' > 1$.

Lemma 2.2. (see [20])*For any* $u = (u_1, u_2)^\tau,\ v = (v_1, v_2)^\tau \in E$*, the following two equalities hold:*

$$-\sum_{n=1}^{T}(\Delta\phi_1(\Delta u_1(n-1)), v_1(n)) = \sum_{n=1}^{T}(\phi_1(\Delta u_1(n)), \Delta v_1(n)),$$

$$-\sum_{n=1}^{T}(\Delta\phi_2(\Delta u_2(n-1)), v_2(n)) = \sum_{n=1}^{T}(\phi_2(\Delta u_2(n)), \Delta v_2(n)).$$

Define

$$\mathcal{J}(u) = \mathcal{J}(u_1, u_2) = \sum_{n=1}^{T}\left[\Phi_1(\Delta u_1(n)) + \Phi_2(\Delta u_2(n)) + F(n, u_1(t), u_2(n))\right].$$

then

$$\begin{aligned}
\langle \mathcal{J}'(u), v\rangle &= \langle \mathcal{J}'(u_1, u_2), (v_1, v_2)\rangle \\
&= \sum_{n=1}^{T}\left[(\phi_1(\Delta u_1(n)), \Delta v_1(n)) + (\phi_2(\Delta u_2(n)), \Delta v_2(n))\right. \\
&\quad \left. +(\nabla_{u_1}F(n, u_1(n), u_2(n)), v_1(n)) + (\nabla_{u_2}F(n, u_1(n), u_2(n)), v_2(n))\right] \\
&= -\sum_{n=1}^{T}\left[(\Delta\phi_1(\Delta u_1(n-1)), v_1(n)) + (\Delta\phi_2(\Delta u_2(n-1)), v_2(n))\right. \\
&\quad \left. -(\nabla_{u_1}F(n, u_1(n), u_2(n)), v_1(n)) - (\nabla_{u_2}F(n, u_1(n), u_2(n)), v_2(n))\right]
\end{aligned}$$

and then it is easy to obtain that critical point of $\mathcal{J}$ in E is T-periodic solution of system (1.1).

Assume that E is a real Banach space and for $\varphi \in C^1(E, \mathbb{R})$, we say that φ satisfies the Palais-Smale(PS) condition if any sequence $(u_n) \subset E$ for which $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

Lemma 2.3. (see [6])*Assume that* X *is a real Banach space,* $\varphi \in C^1(X, \mathbb{R})$ *is bounded from below and satisfies the (PS) condition, then* $c = \inf_{u\in X}\varphi(u)$ *is a critical value of* φ.

3. proofs

Proof of Theorem 1.1. It follows from (F_1), Hölder inequality, Young inequality and Lemma 2.1 that

$$\begin{aligned}
&\left|\sum_{n=1}^{T}\left[F_1(n, u_1(n), u_2(n)) - F_1(n, u_1(n), \overline{u}_2)\right]\right| \\
=&\left|\sum_{n=1}^{T}\int_0^1(\nabla_{\overline{u}_2+s\widetilde{u}_2(n)}F_1(n, u_1(n), \overline{u}_2 + s\widetilde{u}_2(n)), \widetilde{u}_2(n))ds\right| \\
\leq&\sum_{n=1}^{T}\int_0^1 f_2(n)|\overline{u}_2 + s\widetilde{u}_2(n)|^{\alpha_2}|\widetilde{u}_2(n)|ds + \sum_{n=1}^{T}\int_0^1 h_2(n)|\widetilde{u}_2(n)|ds
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{n=1}^{T} f_2(n)(|\overline{u}_2| + |\widetilde{u}_2(n)|)^{\alpha_2}|\widetilde{u}_2(n)| + \sum_{n=1}^{T} h_2(n)|\widetilde{u}_2(n)| \\
&\le \sum_{n=1}^{T} 2^{\alpha_2} f_2(n)|\overline{u}_2|^{\alpha_2}|\widetilde{u}_2(n)| + \sum_{n=1}^{T} 2^{\alpha_2} f_2(n)|\widetilde{u}_2(n)|^{\alpha_2+1} + \sum_{n=1}^{T} h_2(n)|\widetilde{u}_2(n)| \\
&\le 2^{\alpha_2}|\overline{u}_2|^{\alpha_2}\Big(\sum_{n=1}^{T} f_2(n)^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{T}|\widetilde{u}_2(n)|^q\Big)^{\frac{1}{q}} + 2^{\alpha_2}\Big(\sum_{n=1}^{T} f_2(n)^{\frac{q}{q-\alpha_2-1}}\Big)^{\frac{q-\alpha_2-1}{q}}\Big(\sum_{n=1}^{T}|\widetilde{u}_2(n)|^q\Big)^{\frac{\alpha_2+1}{q}} \\
&\quad +\Big(\sum_{n=1}^{T} h_2(n)^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{T}|\widetilde{u}_2(n)|^q\Big)^{\frac{1}{q}} \\
&\le 2^{\alpha_2}|\overline{u}_2|^{\alpha_2}\Big(\sum_{n=1}^{T} f_2(n)^p\Big)^{\frac{1}{p}}[C(q,q')]^{\frac{1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{1}{q}} \\
&\quad +2^{\alpha_2}\Big(\sum_{n=1}^{T} f_2(n)^{\frac{q}{q-\alpha_2-1}}\Big)^{\frac{q-\alpha_2-1}{q}}[C(q,q')]^{\frac{\alpha_2+1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\alpha_2+1}{q}} \\
&\quad +\Big(\sum_{n=1}^{T} h_2(n)^p\Big)^{\frac{1}{p}}[C(q,q')]^{\frac{1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{1}{q}} \\
&\le \frac{2^{p\alpha_2}}{p}|\overline{u}_2|^{p\alpha_2}[C(q,q')]^{\frac{p}{q}}\sum_{n=1}^{T} f_2(n)^p + \frac{1}{q}\sum_{n=1}^{T}|\Delta u_2(n)|^q \\
&\quad +2^{\alpha_2}[C(q,q')]^{\frac{\alpha_2+1}{q}}\Big(\sum_{n=1}^{T} f_2(n)^{\frac{q}{q-\alpha_2-1}}\Big)^{\frac{q-\alpha_2-1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\alpha_2+1}{q}} \\
&\quad +[C(q,q')]^{\frac{1}{q}}\Big(\sum_{n=1}^{T} h_2(n)^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{1}{q}} \qquad (3.1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\sum_{n=1}^{T}[F_1(n,u_1(n),\overline{u}_2) - F_1(n,\overline{u}_1,\overline{u}_2)]\Big| \\
&= \Big|\sum_{n=1}^{T}\int_0^1(\nabla_{\overline{u}_1(n)+s\widetilde{u}_1(n)}F_1(n,\overline{u}_1(n)+s\widetilde{u}_1(n),\overline{u}_2),\widetilde{u}_1(n))ds\Big| \\
&\le \sum_{n=1}^{T}\int_0^1 f_1(n)|\overline{u}_1 + s\widetilde{u}_1(n)|^{\alpha_1}|\widetilde{u}_1(n)|ds + \sum_{n=1}^{T}\int_0^1 h_1(n)|\widetilde{u}_1(n)|ds \\
&\le \sum_{n=1}^{T} f_1(n)(|\overline{u}_1| + |\widetilde{u}_1(n)|)^{\alpha_1}|\widetilde{u}_1(n)| + \sum_{n=1}^{T} h_1(n)|\widetilde{u}_1(n)| \\
&\le 2^{\alpha_1}|\overline{u}_1|^{\alpha_1}\Big(\sum_{n=1}^{T} f_1(n)^q\Big)^{\frac{1}{q}}\Big(\sum_{n=1}^{T}|\widetilde{u}_1(n)|^p\Big)^{\frac{1}{p}} + 2^{\alpha_1}\Big(\sum_{n=1}^{T} f_1(n)^{\frac{p}{p-\alpha_1-1}}\Big)^{\frac{p-\alpha_1-1}{p}}\Big(\sum_{n=1}^{T}|\widetilde{u}_1(n)|^p\Big)^{\frac{\alpha_1+1}{p}} \\
&\quad +\Big(\sum_{n=1}^{T} h_1(n)^q\Big)^{\frac{1}{q}}\Big(\sum_{n=1}^{T}|\widetilde{u}_1(n)|^p\Big)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2^{\alpha_1}|\overline{u}_1|^{\alpha_1}\Big(\sum_{n=1}^{T} f_1(n)^q\Big)^{\frac{1}{q}}[C(p,p')]^{\frac{1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{1}{p}} \\
&\quad +2^{\alpha_1}\Big(\sum_{n=1}^{T} f_1(n)^{\frac{p}{p-\alpha_1-1}}\Big)^{\frac{p-\alpha_1-1}{p}}[C(p,p')]^{\frac{\alpha_1+1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\alpha_1+1}{p}} \\
&\quad +\Big(\sum_{n=1}^{T} h_1(n)^q\Big)^{\frac{1}{q}}[C(p,p')]^{\frac{1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{1}{p}} \\
&\leq \frac{2^{q\alpha_1}}{q}|\overline{u}_1|^{q\alpha_1}[C(p,p')]^{\frac{q}{p}}\sum_{n=1}^{T} f_1(n)^q+\frac{1}{p}\sum_{n=1}^{T}|\Delta u_1(n)|^p \\
&\quad +2^{\alpha_1}[C(p,p')]^{\frac{\alpha_1+1}{p}}\Big(\sum_{n=1}^{T} f_1(n)^{\frac{p}{p-\alpha_1-1}}\Big)^{\frac{p-\alpha_1-1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\alpha_1+1}{p}} \\
&\quad +[C(p,p')]^{\frac{1}{p}}\Big(\sum_{n=1}^{T} h_1(n)^q\Big)^{\frac{1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{1}{p}}. \qquad (3.2)
\end{aligned}
$$

Then by (F_2), Hölder inequality and Lemma 2.1, we have

$$
\begin{aligned}
&\sum_{n=1}^{T}[F_2(u_1(n),u_2(n))-F_2(u_1(n),\overline{u}_2)] \\
&= \sum_{n=1}^{T}\int_0^1(\nabla_{\overline{u}_2+s\widetilde{u}_2(n))}F_2(u_1(n),\overline{u}_2+s\widetilde{u}_2(n)),\widetilde{u}_2(n))ds \\
&= \sum_{n=1}^{T}\int_0^1\frac{1}{s}(\nabla_{\overline{u}_2+s\widetilde{u}_2(n)}F_2(u_1(n),\overline{u}_2+s\widetilde{u}_2(n))-\nabla_{\overline{u}_2}F(\overline{u}_1,\overline{u}_2),s\widetilde{u}_2(n))ds \\
&\geq -r_2\sum_{n=1}^{T}\int_0^1\frac{1}{s}|s\widetilde{u}_2(n)|^{\gamma_2}ds \\
&\geq -\frac{r_2}{\gamma_2}\sum_{n=1}^{T}|\widetilde{u}_2(n)|^{\gamma_2} \\
&\geq -\frac{r_2}{\gamma_2}T^{\frac{q-\gamma_2}{q}}\Big(\sum_{n=1}^{T}|\widetilde{u}_2(n)|^q\Big)^{\frac{\gamma_2}{q}} \\
&\geq -\frac{r_2}{\gamma_2}T^{\frac{q-\gamma_2}{q}}[C(q,q')]^{\frac{\gamma_2}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\gamma_2}{q}} \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{n=1}^{T}[F_2(u_1(n),\overline{u}_2)-F_2(\overline{u}_1,\overline{u}_2)] \\
&= \sum_{n=1}^{T}\int_0^1(\nabla_{\overline{u}_1+s\widetilde{u}_1(n)}F_2(\overline{u}_1+s\widetilde{u}_1(n),\overline{u}_2),\widetilde{u}_1(n))ds
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{T}\int_0^1 \frac{1}{s}(\nabla_{\overline{u}_1+s\widetilde{u}_1(n)}F_2(\overline{u}_1+s\widetilde{u}_1(n),\overline{u}_2-\nabla_{\overline{u}_1}F(\overline{u}_1,\overline{u}_2),s\widetilde{u}_1(n))ds\\
&\geq -r_1\sum_{n=1}^{T}\int_0^1 \frac{1}{s}|s\widetilde{u}_1(n)|^{\gamma_1}ds\\
&\geq -\frac{r_1}{\gamma_1}\sum_{n=1}^{T}|\widetilde{u}_1(n)|^{\gamma_1}\\
&\geq -\frac{r_1}{\gamma_1}T^{\frac{p-\gamma_1}{p}}\Big(\sum_{n=1}^{T}|\widetilde{u}_1(n)|^{p}\Big)^{\frac{\gamma_1}{p}}\\
&\geq -\frac{r_1}{\gamma_1}T^{\frac{p-\gamma_1}{p}}[C(p,p')]^{\frac{\gamma_1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^{p}\Big)^{\frac{\gamma_1}{p}}. \qquad (3.4)
\end{aligned}
$$

Then by $(\mathcal{A}1)$, (3.1),(3.2),(3.3),(3.4), we have

$$
\begin{aligned}
&\mathcal{J}(u_1,u_2)\\
=\ & \sum_{n=1}^{T}[\Phi_1(\Delta u_1(n))+\Phi_2(\Delta u_2(n))+F(n,u_1(n),u_2(n))]\\
\geq\ & d_1\sum_{n=1}^{T}|\Delta u_1(n)|^p+d_2\sum_{n=1}^{T}|\Delta u_2(n)|^q+\sum_{n=1}^{T}[F_1(n,u_1(n),u_2(n))-F_1(n,u_1(n),\overline{u}_2)]\\
&+\sum_{n=1}^{T}[F_1(n,u_1(n),\overline{u}_2)-F_1(n,\overline{u}_1,\overline{u}_2)]+\sum_{n=1}^{T}[F_2(u_1(n),u_2(n))-F_2(u_1(n),\overline{u}_2)]\\
&+\sum_{n=1}^{T}[F_2(u_1(n),\overline{u}_2)-F_2(\overline{u}_1,\overline{u}_2)]+\sum_{n=1}^{T}F(n,\overline{u}_1,\overline{u}_2)\\
\geq\ & (d_1-\frac{1}{p})\sum_{n=1}^{T}|\Delta u_1(n)|^p+(d_2-\frac{1}{q})\sum_{n=1}^{T}|\Delta u_2(n)|^q\\
&-[C(p,p')]^{\frac{1}{p}}\Big(\sum_{n=1}^{T}h_1(n)^q\Big)^{\frac{1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{1}{p}}\\
&-[C(q,q')]^{\frac{1}{q}}\Big(\sum_{n=1}^{T}h_2(n)^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{1}{q}}\\
&-2^{\alpha_1}[C(p,p')]^{\frac{\alpha_1+1}{p}}\Big(\sum_{n=1}^{T}f_1(n)^{\frac{p}{p-\alpha_1-1}}\Big)^{\frac{p-\alpha_1-1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\alpha_1+1}{p}}\\
&-2^{\alpha_2}[C(q,q')]^{\frac{\alpha_2+1}{q}}\Big(\sum_{n=1}^{T}f_2(n)^{\frac{q}{q-\alpha_2-1}}\Big)^{\frac{q-\alpha_2-1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\alpha_2+1}{q}}\\
&-\frac{r_1}{\gamma_1}T^{\frac{p-\gamma_1}{p}}[C(p,p')]^{\frac{\gamma_1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\gamma_1}{p}}-\frac{r_2}{\gamma_2}T^{\frac{q-\gamma_2}{q}}[C(q,q')]^{\frac{\gamma_2}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\gamma_2}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2^{q\alpha_1}}{q}|\overline{u}_1|^{q\alpha_1}[C(p,p')]^{\frac{q}{p}}\sum_{n=1}^{T}f_1(n)^q-\frac{2^{p\alpha_2}}{p}|\overline{u}_2|^{p\alpha_2}[C(q,q')]^{\frac{p}{q}}\sum_{n=1}^{T}f_2(n)^p+\sum_{n=1}^{T}F(n,\overline{u}_1,\overline{u}_2)\\
\geq\ & (d_1-\frac{1}{p})\sum_{n=1}^{T}|\Delta u_1(n)|^p+(d_2-\frac{1}{q})\sum_{n=1}^{T}|\Delta u_2(n)|^q\\
&-[C(p,p')]^{\frac{1}{p}}\Big(\sum_{n=1}^{T}h_1(n)^q\Big)^{\frac{1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{1}{p}}\\
&-[C(q,q')]^{\frac{1}{q}}\Big(\sum_{n=1}^{T}h_2(n)^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{1}{q}}\\
&-2^{\alpha_1}[C(p,p')]^{\frac{\alpha_1+1}{p}}\Big(\sum_{n=1}^{T}f_1(n)^{\frac{p}{p-\alpha_1-1}}\Big)^{\frac{p-\alpha_1-1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\alpha_1+1}{p}}\\
&-2^{\alpha_2}[C(q,q')]^{\frac{\alpha_2+1}{q}}\Big(\sum_{n=1}^{T}f_2(n)^{\frac{q}{q-\alpha_2-1}}\Big)^{\frac{q-\alpha_2-1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\alpha_2+1}{q}}\\
&-\frac{r_1}{\gamma_1}T^{\frac{p-\gamma_1}{p}}[C(p,p')]^{\frac{\gamma_1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\gamma_1}{p}}-\frac{r_2}{\gamma_2}T^{\frac{q-\gamma_2}{q}}[C(q,q')]^{\frac{\gamma_2}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\gamma_2}{q}}\\
&+(|\overline{u}_1|^{q\alpha_1}+|\overline{u}_2|^{p\alpha_2})\Bigg(\frac{1}{|\overline{u}_1|^{q\alpha_1}+|\overline{u}_2|^{p\alpha_2}}\sum_{n=1}^{T}F(n,\overline{u}_1,\overline{u}_2)-M\Bigg),
\end{aligned}
\tag{3.5}
$$

where $M=\max\left\{\frac{2^{q\alpha_1}}{q}[C(p,p')]^{\frac{q}{p}}\sum_{n=1}^{T}f_1(n)^q,\frac{2^{p\alpha_2}}{p}[C(q,q')]^{\frac{p}{q}}\sum_{n=1}^{T}f_2(n)^p\right\}$. Note that (F_3), $\alpha_1\in[0,p-1)$, $\alpha_2\in[0,q-1)$, $\gamma_1\in[0,p)$ and $\gamma_2\in[0,q)$. Hence, together with (3.5), implies that

$$
\mathcal{J}(u_1,u_2)\to+\infty,\quad \text{as}\|(u_1,u_2)^\tau\|\to\infty. \tag{3.6}
$$

Hence $\mathcal{J}$ is bounded from below and (PS) condition holds. Then by Lemma 2.3, it is easy to know that $\mathcal{J}$ has at least one critical point u_0 such that

$$
\mathcal{J}(u_o)=\inf_{u\in E}\mathcal{J}(u).
$$

Thus the proof is complete. □

Proof of Theorem 1.2. It follows from (F_4), Hölder inequality, Young inequality and Lemma 2.1 that

$$
\begin{aligned}
&\sum_{n=1}^{T}[F_2(u_1(n),u_2(n))-F_2(u_1(n),\overline{u}_2)]\\
=\ &\sum_{n=1}^{T}\int_0^1(\nabla_{\overline{u}_2+s\widetilde{u}_2(n)}F_2(u_1(n),\overline{u}_2+s\widetilde{u}_2(n)),\widetilde{u}_2(n))ds\\
=\ &\sum_{n=1}^{T}\int_0^1\frac{1}{s}(\nabla_{\overline{u}_2+s\widetilde{u}_2(n)}F_2(u_1(n),\overline{u}_2+s\widetilde{u}_2(n))-\nabla_{\overline{u}_2}F(\overline{u}_1,\overline{u}_2),s\widetilde{u}_2(n))ds
\end{aligned}
$$

$$
\begin{aligned}
&\geq \sum_{n=1}^{T}\int_0^1 \frac{1}{s}(-r_3|s\widetilde{u}_2(n)|^q - r_4|s\widetilde{u}_2(n)|^{\beta_0})ds \\
&\geq -\frac{r_3}{q}\sum_{n=1}^{T}|\widetilde{u}_2(n)|^q - \frac{r_4}{\beta_0}\sum_{n=1}^{T}|\widetilde{u}_2(n)|^{\beta_0} \\
&\geq -\frac{r_3}{q}C(q,q')\sum_{n=1}^{T}|\Delta u_2(n)|^q - \frac{r_4}{\beta_0}T^{\frac{q-\beta_0}{q}}[C(q,q')]^{\frac{\beta_0}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\beta_0}{q}} \qquad (3.7)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{n=1}^{T}[F_2(u_1(n),\overline{u}_2) - F_2(\overline{u}_1,\overline{u}_2)] \\
&= \sum_{n=1}^{T}\int_0^1 (\nabla_{\overline{u}_1+s\widetilde{u}_1(n)}F_2(\overline{u}_1 + s\widetilde{u}_1(n),\overline{u}_2),\widetilde{u}_1(n))ds \\
&= \sum_{n=1}^{T}\int_0^1 \frac{1}{s}(\nabla_{\overline{u}_1+s\widetilde{u}_1(n)}F_2(\overline{u}_1 + s\widetilde{u}_1(n),\overline{u}_2 - \nabla_{\overline{u}_1}F(\overline{u}_1,\overline{u}_2), s\widetilde{u}_1(n))ds \\
&\geq \sum_{n=1}^{T}\int_0^1 \frac{1}{s}(-r_1|s\widetilde{u}_1(n)|^p - r_2|s\widetilde{u}_2(n)|^{\alpha_0})ds \\
&\geq -\frac{r_1}{p}\sum_{n=1}^{T}|\widetilde{u}_1(n)|^p - \frac{r_2}{\alpha_0}\sum_{n=1}^{T}|\widetilde{u}_1(n)|^{\alpha_0} \\
&\geq -\frac{r_1}{p}C(p,p')\sum_{n=1}^{T}|\Delta u_1(n)|^p - \frac{r_2}{\alpha_0}T^{\frac{p-\alpha_0}{p}}[C(p,p')]^{\frac{\alpha_0}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\alpha_0}{p}} \qquad (3.8)
\end{aligned}
$$

Then by ($\mathcal{A}$1), (3.1),(3.2),(3.7),(3.8), we have

$$
\begin{aligned}
&\mathcal{J}(u_1,u_2) \\
&= \sum_{n=1}^{T}[\Phi_1(\Delta u_1(n)) + \Phi_2(\Delta u_2(n)) + F(n,u_1(n),u_2(n))] \\
&\geq \Big(d_1 - \frac{1 + r_1C(p,p')}{p}\Big)\sum_{n=1}^{T}|\Delta u_1(n)|^p + \Big(d_2 - \frac{1 + r_3C(q,q')}{q}\Big)\sum_{n=1}^{T}|\Delta u_2(n)|^q \\
&\quad -[C(p,p')]^{\frac{1}{p}}\Big(\sum_{n=1}^{T}h_1(n)^q\Big)^{\frac{1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{1}{p}} \\
&\quad -[C(q,q')]^{\frac{1}{q}}\Big(\sum_{n=1}^{T}h_2(n)^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{1}{q}} \\
&\quad -2^{\alpha_1}[C(p,p')]^{\frac{\alpha_1+1}{p}}\Big(\sum_{n=1}^{T}f_1(n)^{\frac{p}{p-\alpha_1-1}}\Big)^{\frac{p-\alpha_1-1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\alpha_1+1}{p}} \\
&\quad -2^{\alpha_2}[C(q,q')]^{\frac{\alpha_2+1}{q}}\Big(\sum_{n=1}^{T}f_2(n)^{\frac{q}{q-\alpha_2-1}}\Big)^{\frac{q-\alpha_2-1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\alpha_2+1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{r_2}{\alpha_0}T^{\frac{p-\alpha_0}{p}}[C(p,p')]^{\frac{\alpha_0}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\alpha_0}{p}}-\frac{r_4}{\beta_0}T^{\frac{q-\beta_0}{q}}[C(q,q')]^{\frac{\beta_0}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\beta_0}{q}}\\
&-\frac{2^{q\alpha_1}}{q}|\bar{u}_1|^{q\alpha_1}[C(p,p')]^{\frac{q}{p}}\sum_{n=1}^{T}f_1(n)^q-\frac{2^{q\alpha_2}}{p}|\bar{u}_2|^{p\alpha_2}[C(q,q')]^{\frac{p}{q}}\sum_{n=1}^{T}f_2(n)^p+\sum_{n=1}^{T}F(n,\bar{u}_1,\bar{u}_2)\\
\geq\ &\Big(d_1-\frac{1+r_1C(p,p')}{p}\Big)\sum_{n=1}^{T}|\Delta u_1(n)|^p+\Big(d_2-\frac{1+r_3C(q,q')}{q}\Big)\sum_{n=1}^{T}|\Delta u_2(n)|^q\\
&-[C(p,p')]^{\frac{1}{p}}\Big(\sum_{n=1}^{T}h_1(n)^q\Big)^{\frac{1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{1}{p}}\\
&-[C(q,q')]^{\frac{1}{q}}\Big(\sum_{n=1}^{T}h_2(n)^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{1}{q}}\\
&-2\alpha_1[C(p,p')]^{\frac{\alpha_1+1}{p}}\Big(\sum_{n=1}^{T}f_1(n)^{\frac{p}{p-\alpha_1-1}}\Big)^{\frac{p-\alpha_1-1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\alpha_1+1}{p}}\\
&-2\alpha_2[C(q,q')]^{\frac{\alpha_2+1}{q}}\Big(\sum_{n=1}^{T}f_2(n)^{\frac{q}{q-\alpha_2-1}}\Big)^{\frac{q-\alpha_2-1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\alpha_2+1}{q}}\\
&-\frac{r_2}{\alpha_0}T^{\frac{p-\alpha_0}{p}}[C(p,p')]^{\frac{\alpha_0}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\alpha_0}{p}}-\frac{r_4}{\beta_0}T^{\frac{q-\beta_0}{q}}[C(q,q')]^{\frac{\beta_0}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\beta_0}{q}}\\
&+(|\bar{u}_1|^{q\alpha_1}+|\bar{u}_2|^{p\alpha_2})\Big(\frac{1}{|\bar{u}_1|^{q\alpha_1}+|\bar{u}_2|^{p\alpha_2}}\sum_{n=1}^{T}F(n,\bar{u}_1,\bar{u}_2)-M\Big),
\end{aligned}
\tag{3.9}
$$

where $M = \max\Big\{\frac{2^{q\alpha_1}}{q}[C(p,p')]^{\frac{q}{p}}\sum_{n=1}^{T}f_1(n)^q, \frac{2^{p\alpha_2}}{p}[C(q,q')]^{\frac{p}{q}}\sum_{n=1}^{T}f_2(n)^p\Big\}$. Note that (F_3), $r_1 \in [0, \frac{pd_1-1}{C(p,p')})$,$r_3 \in [0, \frac{qd_2-1}{C(q,q')})$,$\alpha_0 \in [0,p)$ and $\beta_0 \in [0,q)$. Hence, together with (3.9), implies that

$$
\mathcal{J}(u_1,u_2)\to+\infty, \quad \text{as } \|(u_1,u_2)^{\tau}\|\to\infty. \tag{3.10}
$$

Hence $\mathcal{J}$ is bounded from below and (PS) condition holds. Then by Lemma 2.3, it is easy to know that $\mathcal{J}$ has at least one critical point u_0 such that

$$
\mathcal{J}(u_o)=\inf_{u\in E}\mathcal{J}(u).
$$

Thus the proof is complete. □

Proof of Theorem 1.3. Let $\beta_i = \log_{2\lambda_i}(2\mu_i)$,$i = 1,2$ then $\beta_1 < p, \beta_2 < q$. For $|x_i| > 1, i = 1,2$ there exist positive integers l, m such that

$$
l-1\leq\log_{2\lambda_1}|x_1|\leq l,\ \ m-1\leq\log_{2\lambda_2}|x_2|\leq m
$$

then

$$
|x_1|^{\beta_1}>(2\lambda_1)^{(l-1)\beta_1}=(2\mu_1)^{l-1},\ \ |x_1|\leq(2\lambda_1)^l,
$$

$$|x_2|^{\beta_2} > (2\lambda_2)^{(m-1)\beta_2} = (2\mu_2)^{m-1}, \ \ |x_2| \le (2\lambda_2)^m$$

by ($\mathcal{F}$), (F_6) and (F_7), we have

$$\begin{aligned}
& F_2(n, x_1, x_2) = F_2(n, \lambda_1(\frac{x_1}{2\lambda_1} + \frac{x_1}{2\lambda_1}), \lambda_2(\frac{x_2}{2\lambda_2} + \frac{x_2}{2\lambda_2})) \\
\le \ & \mu_1 F_2(n, \frac{x_1}{2\lambda_1}, \frac{x_2}{2\lambda_2}) + \mu_2 F_2(n, \frac{x_1}{2\lambda_1}, \frac{x_2}{2\lambda_2}) \\
\le \ & 2\mu_1 F_2(n, \frac{x_1}{2\lambda_1}, \frac{x_2}{2\lambda_2}) + 2\mu_2 F_2(n, \frac{x_1}{2\lambda_1}, \frac{x_2}{2\lambda_2}) \\
\le \ & 2\mu_1^2 F_2(n, \frac{x_1}{(2\lambda_1)^2}, \frac{x_2}{(2\lambda_2)^2}) + 4\mu_1\mu_2 F_2(n, \frac{x_1}{(2\lambda_1)^2}, \frac{x_2}{(2\lambda_2)^2}) + 2\mu_2^2 F_2(n, \frac{x_1}{(2\lambda_1)^2}, \frac{x_2}{(2\lambda_2)^2}) \\
\le \ & (2\mu_1)^2 F_2(n, \frac{x_1}{(2\lambda_1)^2}, \frac{x_2}{(2\lambda_2)^2}) + (2\mu_2)^2 F_2(n, \frac{x_1}{(2\lambda_1)^2}, \frac{x_2}{(2\lambda_2)^2}) \\
\le \ & \cdots \le (2\mu_1)^K F_2(n, \frac{x_1}{(2\lambda_1)^K}, \frac{x_2}{(2\lambda_2)^K}) + (2\mu_2)^K F_2(n, \frac{x_1}{(2\lambda_1)^K}, \frac{x_2}{(2\lambda_2)^K}) \\
\le \ & ((2\mu_1)^{K-l+1}|x_1|^{\beta_1} + (2\mu_2)^{K-m+1}|x_2|^{\beta_2}) a_0 b(n)
\end{aligned} \tag{3.11}$$

for a.e. $n \in [0, T]$ and $|x_1|, |x_2| > 1$, where $a_0 = \max_{0\le s\le 1} a_i(s), i = 1, 2$. Then, for a.e. $n \in [0, T]$ and all $|x_1|, |x_2|$, we have

$$F_2(n, x_1, x_2) \le ((2\mu_1)^{K-l+1}|x_1|^{\beta_1} + (2\mu_2)^{K-m+1}|x_2|^{\beta_2} + 1) a_0 b(n) \tag{3.12}$$

Then by ($\mathcal{A}1$), (3.1),(3.2), (3.12), we have

$$\begin{aligned}
& \mathcal{J}(u_1, u_2) \\
= \ & \sum_{n=1}^{T} [\Phi_1(\Delta u_1(n)) + \Phi_2(\Delta u_2(n)) + F(n, u_1(n), u_2(n))] \\
\ge \ & d_1 \sum_{n=1}^{T} |\Delta u_1(n)|^p + d_2 \sum_{n=1}^{T} |\Delta u_2(n)|^q + \sum_{n=1}^{T} [F_1(n, u_1(n), u_2(n)) - F_1(n, u_1(n), \overline{u}_2)] \\
& + \sum_{n=1}^{T} [F_1(n, u_1(n), \overline{u}_2) - F_1(n, \overline{u}_1, \overline{u}_2)] + \sum_{n=1}^{T} F_1(n, \overline{u}_1, \overline{u}_2) + \sum_{n=1}^{T} F_2(n, u_1(n), u_2(n)) \\
\ge \ & d_1 \sum_{n=1}^{T} |\Delta u_1(n)|^p + d_2 \sum_{n=1}^{T} |\Delta u_2(n)|^q - \frac{2^{q\alpha_1}}{q} |\overline{u}_1|^{q\alpha_1} [C(p, p')]^{\frac{q}{p}} \sum_{n=1}^{T} f_1(n)^q \\
& - \frac{1}{p} \sum_{n=1}^{T} |\Delta u_1(n)|^p - \frac{2^{p\alpha_2}}{p} |\overline{u}_2|^{p\alpha_2} [C(q, q')]^{\frac{p}{q}} \sum_{n=1}^{T} f_2(n)^p - \frac{1}{q} \sum_{n=1}^{T} |\Delta u_2(n)|^q \\
& - [C(p, p')]^{\frac{1}{p}} \Big(\sum_{n=1}^{T} h_1(n)^q \Big)^{\frac{1}{q}} \Big(\sum_{n=1}^{T} |\Delta u_1(n)|^p \Big)^{\frac{1}{p}} \\
& - [C(q, q')]^{\frac{1}{q}} \Big(\sum_{n=1}^{T} h_2(n)^p \Big)^{\frac{1}{p}} \Big(\sum_{n=1}^{T} |\Delta u_2(n)|^q \Big)^{\frac{1}{q}} \\
& - 2\alpha_1 [C(p, p')]^{\frac{\alpha_1+1}{p}} \Big(\sum_{n=1}^{T} f_1(n)^{\frac{p}{p-\alpha_1-1}} \Big)^{\frac{p-\alpha_1-1}{p}} \Big(\sum_{n=1}^{T} |\Delta u_1(n)|^p \Big)^{\frac{\alpha_1+1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&-2\alpha_2[C(q,q')]^{\frac{\alpha_2+1}{q}}\Big(\sum_{n=1}^{T} f_2(n)^{\frac{q}{q-\alpha_2-1}}\Big)^{\frac{q-\alpha_2-1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\alpha_2+1}{q}}\\
&+\sum_{n=1}^{T}F_1(n,\overline{u}_1,\overline{u}_2)+\frac{1}{\mu_1}\sum_{n=1}^{T}F_2(n,\lambda_1\overline{u}_1,\lambda_2\overline{u}_2)-\frac{\mu_2}{\mu_1}\sum_{n=1}^{T}F_2(n,-\widetilde{u}_1(n),-\widetilde{u}_2(n))\\
\geq\ &(d_1-\frac{1}{p})\sum_{n=1}^{T}|\Delta u_1(n)|^p+(d_2-\frac{1}{q})\sum_{n=1}^{T}|\Delta u_2(n)|^q\\
&-[C(p,p')]^{\frac{1}{p}}\Big(\sum_{n=1}^{T}h_1(n)^q\Big)^{\frac{1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{1}{p}}\\
&-[C(q,q')]^{\frac{1}{q}}\Big(\sum_{n=1}^{T}h_2(n)^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{1}{q}}\\
&-2\alpha_1[C(p,p')]^{\frac{\alpha_1+1}{p}}\Big(\sum_{n=1}^{T} f_1(n)^{\frac{p}{p-\alpha_1-1}}\Big)^{\frac{p-\alpha_1-1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\alpha_1+1}{p}}\\
&-2\alpha_2[C(q,q')]^{\frac{\alpha_2+1}{q}}\Big(\sum_{n=1}^{T} f_2(n)^{\frac{q}{q-\alpha_2-1}}\Big)^{\frac{q-\alpha_2-1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\alpha_2+1}{q}}\\
&-\frac{\mu_2}{\mu_1}\sum_{n=1}^{T}((2\mu_1)^{K-l+1}|-\widetilde{u}_1|^{\beta_1}+(2\mu_2)^{K-m+1}|-\widetilde{u}_2|^{\beta_2}+1)a_0b(n)\\
&+(|\overline{u}_1|^{q\alpha_1}+|\overline{u}_2|^{p\alpha_2})\Big(\frac{1}{|\overline{u}_1|^{q\alpha_1}+|\overline{u}_2|^{p\alpha_2}}(\sum_{n=1}^{T}F_1(n,\overline{u}_1,\overline{u}_2)+\frac{1}{\mu_1}\sum_{n=1}^{T}F_2(n,\lambda_1\overline{u}_1,\lambda_2\overline{u}_2))-M\Big)\\
\geq\ &(d_1-\frac{1}{p})\sum_{n=1}^{T}|\Delta u_1(n)|^p+(d_2-\frac{1}{q})\sum_{n=1}^{T}|\Delta u_2(n)|^q\\
&-[C(p,p')]^{\frac{1}{p}}\Big(\sum_{n=1}^{T}h_1(n)^q\Big)^{\frac{1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{1}{p}}\\
&-[C(q,q')]^{\frac{1}{q}}\Big(\sum_{n=1}^{T}h_2(n)^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{1}{q}}\\
&-2\alpha_1[C(p,p')]^{\frac{\alpha_1+1}{p}}\Big(\sum_{n=1}^{T} f_1(n)^{\frac{p}{p-\alpha_1-1}}\Big)^{\frac{p-\alpha_1-1}{p}}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\alpha_1+1}{p}}\\
&-2\alpha_2[C(q,q')]^{\frac{\alpha_2+1}{q}}\Big(\sum_{n=1}^{T} f_2(n)^{\frac{q}{q-\alpha_2-1}}\Big)^{\frac{q-\alpha_2-1}{q}}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\alpha_2+1}{q}}\\
&-a_0\frac{\mu_2}{\mu_1}\sum_{n=1}^{T}b(n)-C_1\frac{\mu_2}{\mu_1}\Big(\sum_{n=1}^{T}|\Delta u_1(n)|^p\Big)^{\frac{\beta_1}{p}}-C_2\frac{\mu_2}{\mu_1}\Big(\sum_{n=1}^{T}|\Delta u_2(n)|^q\Big)^{\frac{\beta_2}{q}}\\
&+(|\overline{u}_1|^{q\alpha_1}+|\overline{u}_2|^{p\alpha_2})\Big(\frac{1}{|\overline{u}_1|^{q\alpha_1}+|\overline{u}_2|^{p\alpha_2}}(\sum_{n=1}^{T}F_1(n,\overline{u}_1,\overline{u}_2)+\frac{1}{\mu_1}\sum_{n=1}^{T}F_2(n,\lambda_1\overline{u}_1,\lambda_2\overline{u}_2))-M\Big)
\end{aligned}
\tag{3.13}
$$

where $C_1 = (2\mu_1)^{K-l+1}[\sum_{n=1}^{T} b(n)]^{\frac{p-\beta_1}{p}}[C(p,p')]^{\frac{p-\beta_1}{p}}$, $C_2 = (2\mu_1)^{K-m+1}[\sum_{n=1}^{T} b(n)]^{\frac{q-\beta_2}{q}}[C(q,q')]^{\frac{q-\beta_2}{q}}$, $M = \max\left\{\frac{2^{q\alpha_1}}{q}[C(p,p')]^{\frac{q}{p}}\sum_{n=1}^{T} f_1(n)^q, \frac{2^{p\alpha_2}}{p}[C(q,q')]^{\frac{p}{q}}\sum_{n=1}^{T} f_2(n)^p\right\}$. Hence, together with (3.13), implies that

$$\mathcal{J}(u_1,u_2) \to +\infty, \quad \text{as} \|(u_1,u_2)^\tau\| \to \infty. \tag{3.14}$$

Hence $\mathcal{J}$ is bounded from below and (PS) condition holds. Then by Lemma 2.3, it is easy to know that $\mathcal{J}$ has at least one critical point u_0 such that

$$\mathcal{J}(u_o) = \inf_{u\in E} \mathcal{J}(u).$$

Thus the proof is complete. □

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Conflict of interest

The authors declare that they have no competing interests.

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