AIMS Mathematics, 8(5): 10513–10528.
DOI:10.3934/math.2023533
Received: 7 December 2022
Revised: 20 February 2023
Accepted: 22 February 2023
Published: 02 March 2023

AIMS Mathematics

http://www.aimspress.com/journal/Math

*Research article*

# Existence and energy decay rate of the solutions for the wave equation with a nonlinear distributed delay

**Tae Gab Ha*** **and Seyun Kim**

Department of Mathematics, and Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju 54896, Republic of Korea

* **Correspondence:** Email: tgha@jbnu.ac.kr.

**Abstract:** This paper is concerned with the wave equation having a nonlinear distributed delay. First, we prove the local existence of the solutions by using the semigroup theory, where the source term is globally Lipschitz. Next, we establish the global existence of solutions and the energy decay result under the local Lipschitz source and suitable conditions on the initial data.

**Keywords:** wave equations; nonlinear distributed delay; existence of solutions; energy decay
**Mathematics Subject Classification:** 35A01, 35B35, 35B40, 35L05

## 1. Introduction

In this paper, we consider the following wave equation with nonlinear distributed delay:

$$\begin{cases} u_{tt} - \Delta u + \mu_0 f(u_t) + \int_{\tau_1}^{\tau_2} \mu(s) g(u_t(x, t-s)) ds = h(u) & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{in} \quad \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) & \text{in} \quad \Omega, \\ u_t(x, -t) = f_0(x, -t) & \text{in} \quad \Omega \times (0, \tau_2), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is a bounded domain with smooth boundary $\partial\Omega$, $\mu(s) \in L^\infty([\tau_1, \tau_2]; \mathbb{R}_+)$ with $\tau_2 > \tau_1 > 0$, and $\mu_0$ is for some positive constant that will be specified later.

Time delay effect occurs in many various phenomena depending on past states as well as on present situations, so time delay problem is widely applied in many engineering and biology fields [1, 21]. Hence the partial differential equations with such circumstance have been studied by many researchers (see [3, 4, 10–16, 22, 23] and a list on references therein). For example, Nicaise and Pignotti [13] considered the wave equation with a delay concentrated at a time

$$u'' - \Delta u + \mu_0 u' + \mu_1 u'(x, t - s) = 0.$$

They proved that this equation is either exponentially stable under condition $\mu_0 > \mu_1$ or unstable under condition $\mu_0 \le \mu_1$. Benaissa et al. [4] studied the global existence and energy decay of solutions to a viscoelastic wave equation with delay term in the nonlinear internal feedback. They proved the global existence result using Galerkin's method and the asymptotic behavior of solutions using a perturbed energy method. While there are many results dealing with a delay concentrated at a time, there are relatively few researches dealing with a distributed delay. The distributed delay is important and has been studied in many problems (see [6–8, 14, 19]). For instance, Nicaise and Pignotti [14] has studied the wave equation with linear distributed delay

$$u'' - \Delta u + \mu_0 u' + \int_{\tau_1}^{\tau_2} a(x)\mu(s)u'(x, t - s)ds = 0.$$

They proved the global well-posedness by the semigroup theory and the exponential stability under the assumption

$$\mu_0 > \|a\|_\infty \int_{\tau_1}^{\tau_2} \mu(s)ds.$$

Raposo et al. [19] proved the well-posedness using the semigroup theory and the exponential stability exploiting the dissipative properties of the linear operator associated to damped model using the Gearhart-Huang-Pruss theorem for the wave equation with frictional damping and nonlocal time-delayed condition. Recently, Choucha et al. [7] studied a coupled Lame system with distributed delay, viscoelastic, and logarithmic source terms. They proved an exponential decay of solutions by using Lyapunov functional method. But, the above mentioned references were considered a linear distributed delay. There is none, as far as we know, well-posedness result dealing with a nonlinear distributed delay.

Motivated by previous works, the goal of the paper is to study the existence and energy decay of the solutions for the wave equation with a nonlinear distributed delay. We prove the local existence of the solutions by using the semigroup theory, where the source term is globally Lipschitz and then establish the global existence of solutions and the energy decay result under the local Lipschitz source and suitable conditions on the initial data.

Throughout this paper, we use standard functional spaces and $L^p(\Omega)$-norm is denoted by $\| \cdot \|_p$, and $(u, v) = \int_\Omega u(x)v(x)dx$. The following assumptions are made on the nonlinear functions $f$ and $g$.

(A1) $f : \mathbb{R} \to \mathbb{R}$ is a continuous monotone increasing function with $f(0) = 0$.

(A2) There exist positive constants $\nu, M_1, M_2$ and a convex increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $H \in C^1(\mathbb{R}_+) \cap C^2((0, \infty))$ satisfying $H(0) = 0$, and $H$ is linear in $[0, \nu]$ or $H'(0) = 0$ and $H'' > 0$ on $(0, \nu]$ such that

$$M_1 s^2 \le s f(s) \le M_2 s^2 \quad \text{for} \quad |s| > \nu,$$
$$s^2 + f^2(s) \le H^{-1}(s f(s)) \quad \text{for} \quad |s| \le \nu.$$

(A3) $g$ is an odd nondecreasing Lipschitz function.

(A4) $\alpha_1 s g(s) \le G(s) \le \alpha_2 s f(s)$, where $\alpha_1, \alpha_2$ are some positive constants, and $G(s) = \int_0^s g(r)dr$.

## 2. Well-posedness

### 2.1. Globally Lipschitz source

We first deal with the case where the source $h$ is globally Lipschitz from $H_0^1(\Omega)$ into $L^2(\Omega)$. We will prove the problem (1.1) are well-posed using the semigroup theory.

Let us set $u_t(x, t - \rho s) := y(x, \rho, t, s)$, $\rho \in (0, 1)$, $s \in (\tau_1, \tau_2)$. Then the problem (1.1) is transformed into

$$\begin{cases} u_{tt} - \Delta u + \mu_0 f(u_t) + \int_{\tau_1}^{\tau_2} \mu(s) g(y(x, 1, t, s)) ds = h(u) & \text{in} \quad \Omega \times (0, \infty), \\ s y_t(x, \rho, t, s) + y_\rho(x, \rho, t, s) = 0 & \text{in} \quad \Omega \times (0, 1) \times (0, \infty) \times (\tau_1, \tau_2), \\ u = 0 & \text{in} \quad \partial\Omega \times (0, \infty), \\ y(x, 0, t, s) = u_t(x, t) & \text{in} \quad \Omega \times (0, \infty) \times (\tau_1, \tau_2), \\ u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) & \text{in} \quad \Omega, \\ y(x, \rho, 0, s) = f_0(x, -\rho s) & \text{in} \quad \Omega \times (0, 1) \times (\tau_1, \tau_2). \end{cases} \tag{2.1}$$

If we set $u_t := v$. and $U := (u, v, y)^T$, then (2.1) can be rewritten as

$$\begin{cases} U' + \mathcal{A} U = 0, \\ U(0) = (u_0, v_0, f_0)^T, \end{cases} \tag{2.2}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} \begin{bmatrix} u \\ v \\ y \end{bmatrix} = \begin{bmatrix} -v \\ -\Delta u + \mu_0 f(v) + \int_{\tau_1}^{\tau_2} \mu(s) g(y(., 1, s)) ds - h(u) \\ s^{-1} y_\rho \end{bmatrix}$$

with $D(\mathcal{A}) := \{(u, v, y)^T \in (H^2(\Omega) \cap H_0^1) \times H_0^1(\Omega) \times L^2(\Omega \times (\tau_1, \tau_2); H^1(0, 1)) : v(x) = y(x, 0, s) \quad \text{in} \quad \Omega\}$. So we see that in order to obtain the existence of the solutions to the problem (1.1), it is sufficient to show that the problem (2.2) admits a solution.

We define the Hilbert space $\mathcal{H}$

$$\mathcal{H} := H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))$$

with inner product

$$\left\langle \begin{pmatrix} u \\ v \\ y \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{y} \end{pmatrix} \right\rangle_{\mathcal{H}} := \int_\Omega \nabla u \nabla \tilde{u} + v \tilde{v} dx + \int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 s \mu(s) y(x, \rho, s) \tilde{y}(x, \rho, s) d\rho ds dx.$$

**Theorem 2.1.** *Assume that (A1) and (A3) hold. In addition, h is globally Lipschitz from $H_0^1(\Omega)$ into $L^2(\Omega)$. Then* (2.2) *has a unique solution*

$$u \in C(0,\infty; H_0^1(\Omega)) \cap C^1(0,\infty; L^2(\Omega))$$

*for* $(u_0, v_0, f_0)^T \in \mathcal{H}$.

*Proof.* First, we will show that the operator $\mathcal{A}$ is $w$-accretive on $\mathcal{H}$. Let $U = (\xi,\eta,y)^T, V = (\tilde{\xi},\tilde{\eta},\tilde{y})^T \in D(\mathcal{A})$. Then we get

$$\begin{aligned}\langle(\mathcal{A}+wI)U-(\mathcal{A}+wI)V, U-V\rangle_{\mathcal{H}} =& -\int_\Omega \nabla(\eta-\tilde{\eta})\nabla(\xi-\tilde{\xi})dx + w\int_\Omega \nabla(\xi-\tilde{\xi})^2dx\\ &-\int_\Omega \Delta(\xi-\tilde{\xi})(\eta-\tilde{\eta})dx + \mu_0\int_\Omega (f(\eta)-f(\tilde{\eta}))(\eta-\tilde{\eta})dx\\ &-\int_\Omega (h(\xi)-h(\tilde{\xi}))(\eta-\tilde{\eta})dx + w\|\eta-\tilde{\eta}\|^2\\ &+\int_\Omega\int_{\tau_1}^{\tau_2}\mu(s)(g(y(x,1,s))-g(\tilde{y}(x,1,s))ds(\eta-\tilde{\eta})dx\\ &+\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 \mu(s)(y_\rho-\tilde{y}_\rho)(y-\tilde{y})d\rho dsdx\\ &+w\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 s\mu(s)(y-\tilde{y})^2 d\rho dsdx.\end{aligned}$$

Using Hölder's and Young's inequalities, we obtain for sufficiently large $w$ and small $\varepsilon$,

$$\begin{aligned}\langle(\mathcal{A}+wI)U-(\mathcal{A}+wI)V, U-V\rangle_{\mathcal{H}} \geq& wc\|\xi-\tilde{\xi}\|_{H^1}^2 - \frac{L_h^2}{2}\|\xi-\tilde{\xi}\|_{H^1}^2 - \frac{1}{2}\|\eta-\tilde{\eta}\|_2^2\\ &+w\|\eta-\tilde{\eta}\|_2^2 - \frac{C}{2}\|\mu(s)\|_\infty\|\eta-\tilde{\eta}\|_2^2 - C(\varepsilon)\|\eta-\tilde{\eta}\|_2^2\\ &-L_g^2\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}\mu(s)(y(x,1,s)-\tilde{y}(x,1,s))^2dsdx\\ &+\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\mu(s)(y(x,1,s)-\tilde{y}(x,1,s))^2dsdx\\ \geq& \left(wc-\frac{L_h^2}{2}\right)\|\xi-\tilde{\xi}\|_{H_1}^2\\ &+\left(\frac{1}{2}-L_g^2\varepsilon\right)\int_\Omega\int_{\tau_1}^{\tau_2}\mu(s)(y(x,1,s)-\tilde{y}(x,1,s))^2dsdx\\ &+\left(w-\frac{C}{2}\|\mu(s)\|_\infty-\frac{1}{2}-C(\varepsilon)\right)\|\eta-\tilde{\eta}\|_2^2\\ \geq& 0,\end{aligned}$$

where $L_h$ and $L_g$ are Lipschitz constants for $h$ and $g$, respectively. Thus $\mathcal{A}$ is $w$-accretive.

Next, we will show that $\mathcal{A}+wI$ is a maximal monotone operator. To this end, it is sufficient to show that $R(\lambda I+\mathcal{A}) = \mathcal{H}$ for sufficiently large constant $\lambda$.

Given $(k,l,m)\in\mathcal{H}$, We seek $U=(\xi,\eta,y)^T\in D(\mathcal{A})$ satisfying

$$\lambda\xi-\eta=k, \tag{2.3}$$

$$\lambda\eta-\Delta\xi+\mu_0 f(\eta)+\int_{\tau_1}^{\tau_2}\mu(s)g(y(x,1,s))ds-h(\xi)=l, \tag{2.4}$$

$$\lambda y+s^{-1}y_\rho=m. \tag{2.5}$$

From (2.3) and (2.5) with $y(x,0,s)=\eta(x)$, we have $y(x,1,s)=\eta(x)e^{-\lambda s}+X_0$ , where

$$X_0:=se^{-\lambda s}\int_0^1 m(x,\tau)e^{\lambda\tau s}d\tau.$$

Hence, by (2.4) we obtain

$$T\eta:=\lambda\eta-\frac{1}{\lambda}\Delta\eta+\mu_0 f(\eta)+\int_{\tau_1}^{\tau_2}\mu(s)g(\eta(x)e^{-\lambda s}+X_0)ds-h(\frac{\eta+k}{\lambda})=l+\frac{1}{\lambda}\Delta k.$$

We will show that $T:H_0^1(\Omega)\to H^{-1}(\Omega)$ is surjective. Let $\hat{\eta}:=\eta e^{-\lambda s}+X_0$, and let

$$\begin{aligned}\mathcal{B}\eta=&\mu_0 f(\eta),\\ C\eta=&-\frac{1}{\lambda}\Delta\eta+\int_{\tau_1}^{\tau_2}\mu(s)g(\hat{\eta})ds+\lambda\eta-h(\frac{\eta+k}{\lambda}).\end{aligned}$$

It easy to see that $\mathcal{B}$ is maximal monotone. From the fact that $g$ and $h$ are global Lipschitz, and $\lambda I-\frac{1}{\lambda}\Delta$ is continuous and coercive, we infer that, for large constant $\lambda>0$,

$$\begin{aligned}(C\eta_1-C\eta_2,\eta_1-\eta_2)_2=&\lambda\|\eta_1-\eta_2\|_2^2+\frac{1}{\lambda}(-\Delta\eta_1+\Delta\eta_2,\eta_1-\eta_2)_{L^2}-\left(h(\frac{\eta_1+k}{\lambda})-h(\frac{\eta_2+k}{\lambda}),\eta_1-\eta_2\right)_{L^2}\\ &+\left(\int_{\tau_1}^{\tau_2}\mu(s)(g(\hat{\eta}_1)-g(\hat{\eta}_2))ds,\eta_1-\eta_2\right)_{L^2}\\ \ge&\lambda\|\eta_1-\eta_2\|_2^2-L_g\|\mu(s)\|_\infty\int_{\tau_1}^{\tau_2}e^{-\lambda s}ds\|\eta_1-\eta_2\|_2\\ &-\frac{L_h^2}{2\lambda^2}\|\eta_1-\eta_2\|_{H^1}^2-\frac{1}{2}\|\eta_1-\eta_2\|_2^2+\frac{c}{\lambda}\|\eta_1-\eta_2\|_{H^1}^2\\ \ge&0,\end{aligned}$$

where $c$ is positive constant. Therefore $C$ is maximal monotone operator and coercive. Thus $\mathcal{B}+C$ is maximal monotone, which implies that $T$ is surjective. So we obtain

$$\xi=\frac{\eta+k}{\lambda}\in H_0^1,$$

$$y(x,\rho,s)=\eta(x)e^{-\lambda\rho s}+se^{-\lambda\rho s}\int_0^\rho me^{\lambda\tau s}d\tau\in L^2\left(\Omega\times(\tau_1,\tau_2);H^1(0,1)\right),$$

$$y(x,0,s)=\eta(x),$$

$$-\Delta\xi=l-\lambda\eta-\mu_0 f(\eta)-\int_{\tau_1}^{\tau_2}\mu(s)g\left(g\left(y(x,1,t,s)\right)\right)ds+h(\xi)\in L^2(\Omega).$$

Thus we have $(\xi,\eta,y)^T\in D(\mathcal{A})$, consequently, the operator $\mathcal{A}+wI$ is maximal monotone.

From the fact the density of $D(\mathcal{A})$ in $\mathcal{H}$ and the nonlinear semigroup theory [2, 20], the proof of Theorem 2.1 is completed. □

*2.2. Locally Lipschitz source*

In this subsection, we loosen the globally Lipschitz condition on the source by allowing $h$ to be locally Lipschitz continuous. We first introduce the Legendre transformation. Let $G^*$ be conjugate of convex function $G$ . It is defined by $G^*(s) = \sup_{t\in\mathbb{R}_+}(st - G(t))$. $G^*$ is called Legendre transform of $G$ . By definition, we have

$$G^*(s) = s\left(G'\right)^{-1}(s) - G\left[\left(G'\right)^{-1}(s)\right], \quad \forall s \geq 0, \tag{2.6}$$

and

$$st \leq G^*(s) + G(t), \quad \forall s, t \geq 0. \tag{2.7}$$

By the assumption (A4) and (2.6), we obtain

$$\begin{aligned} G^*\left(g\left(y(x,1,t,s)\right)\right) &= y(x,1,t,s)g\left(y(x,1,t,s)\right) - G\left(y(x,1,t,s)\right) \\ &\leq \left(1-\alpha_1\right)y(x,1,t,s)g\left(y(x,1,t,s)\right). \end{aligned} \tag{2.8}$$

Let us define functional

$$\xi(t) = \frac{1}{2}\int_\Omega |u_t|^2 + |\nabla u|^2 dx + \xi_0 \int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 sG(y(x,\rho,t,s))d\rho ds dx, \tag{2.9}$$

where

$$\xi_0 > \frac{1-\alpha_1}{\alpha_1}\|\mu(s)\|_\infty. \tag{2.10}$$

To estimate this subsection, we need the following additional assumption:

$$\mu_0 - \xi_0\alpha_2(\tau_2-\tau_1) - \alpha_2\int_{\tau_1}^{\tau_2}\mu(s)ds > 0. \tag{2.11}$$

**Theorem 2.2.** *Assume that (A1), (A3), (A4),* (2.10) *and* (2.11) *hold, and let* $h : H_0^1(\Omega) \to L^2(\Omega)$ *be local Lipschitz continuous function. Then* (1.1) *has unique local solution for* $(u_0, v_0, f_0)^T \in \mathcal{H}$ *such that*

$$u \in C(0, T_{max}; H_0^1(\Omega)) \cap C^1(0, T_{max}; L^2(\Omega)).$$

*Proof.* Define

$$h_k(u) := \begin{cases} h(u), & \text{if} \quad \|u\|_{H^1} \leq k, \\ h\left(\frac{ku}{\|u\|_{H^1}}\right), & \text{if} \quad \|u\|_{H^1} \geq k, \end{cases}$$

where $k$ is a positive constant. With this truncated function $h_k$, we consider the following problem:

$$\begin{cases} u_{tt} - \Delta u + \mu_0 f(u_t) + \int_{\tau_1}^{\tau_2}\mu(s)g\left(y(x,1,t,s)\right)ds = h_k(u) & \text{in} \quad \Omega\times(0,\infty), \\ u = 0 & \text{in} \quad \partial\Omega\times(0,\infty), \\ u(0) = u_0 \in H_0^1(\Omega) \quad \text{and} \quad u_t(0) = u_1 \in L^2(\Omega) & \text{in} \quad \Omega. \end{cases} \tag{2.12}$$

Since $h_k : H_0^1(\Omega) \to L^2(\Omega)$ is global Lipschitz with Lipschitz constant $L_{h_k}$ for each $k$ (see [9]), by Theorem 2.1, the problem (2.12) has a unique solution $u_k \in C(0,\infty; H_0^1(\Omega)) \cap C^1(0,\infty; L^2(\Omega))$. To simplify the notation in the rest of the proof, we shall express $u_k$ as $u$. We will use the following notation: $\int_Q = \int_0^T \int_\Omega$, $dQ = d\Omega dt$.

Multiplying (2.12) by $u_t$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_2^2 + \|u_t\|_2^2\right) + \mu_0 \int_\Omega f(u_t)u_t dx + \int_\Omega \int_{\tau_1}^{\tau_2} \mu(s)u_t(x,t)g\left(y(x,1,t,s)\right) dsdx = \int_\Omega h_k(u)u_t dx.$$

Since

$$\begin{aligned}\xi_0 \frac{d}{dt}\int_\Omega \int_{\tau_1}^{\tau_2}\int_0^1 sG\left(y(x,\rho,t,s)\right) d\rho dsdx &= -\xi_0 \int_\Omega \int_{\tau_1}^{\tau_2}\int_0^1 \frac{d}{d\rho}G\left(y(x,\rho,t,s)\right) d\rho dsdx \\ &= -\xi_0 \int_\Omega \int_{\tau_1}^{\tau_2}[G\left(y(x,1,t,s)\right) - G\left(y(x,0,t,s)\right)]dsdx,\end{aligned}$$

we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_2^2 + \|u_t\|_2^2\right) + \xi_0 \frac{d}{dt}\int_\Omega \int_{\tau_1}^{\tau_2}\int_0^1 sG\left(y(x,\rho,t,s)\right) d\rho dsdx \\ &\quad = -\mu_0 \int_\Omega f(u_t)u_t dx - \int_\Omega \int_{\tau_1}^{\tau_2} \mu(s)u_t(x,t)g\left(y(x,1,t,s)\right) dsdx + \int_\Omega h_k(u)u_t dx \\ &\quad - \xi_0 \int_\Omega \int_{\tau_1}^{\tau_2}[G\left(y(x,1,t,s)\right) - G\left(y(x,0,t,s)\right)]dsdx.\end{aligned} \tag{2.13}$$

Integrating (2.13) over (0,T) and using (2.9), we get

$$\begin{aligned}\xi(T) - \xi(0) = -\mu_0 \int_Q f(u_t)u_t dQ - \int_Q \int_{\tau_1}^{\tau_2} \mu(s)u_t(x,t)g\left(y(x,1,t,s)\right) dsdQ + \int_Q h_k(u)u_t dQ \\ - \xi_0 \int_Q \int_{\tau_1}^{\tau_2}[G\left(y(x,1,s)\right) - G\left(y(x,0,s)\right)]dsdQ.\end{aligned}$$

By using (2.7) and (2.8), we deduce that

$$\begin{aligned}
\xi(T)-\xi(0) \le& -\mu_0\int_Q f(u_t)u_t d\mathrm{Q} - \int_Q\int_{\tau_1}^{\tau_2}\mu(s)u_t(x,t)g\left(y(x,1,t,s)\right)dsd\mathrm{Q} + \int_Q h_k(u)u_t d\mathrm{Q}\\
&+\xi_0\alpha_2\int_Q\int_{\tau_1}^{\tau_2} u_t f(u_t)dsd\mathrm{Q} - \xi_0\alpha_1\int_Q\int_{\tau_1}^{\tau_2} y(x,1,t,s)g\left(y(x,1,t,s)\right)dsd\mathrm{Q}\\
\le& -\mu_0\int_Q f(u_t)u_t d\mathrm{Q} + \xi_0\alpha_2\int_Q\int_{\tau_1}^{\tau_2} u_t f(u_t)dsd\mathrm{Q}\\
&+\int_Q\int_{\tau_1}^{\tau_2}\mu(s)[G(|u_t|)+G^*\left(|g\left(y(x,1,t,s)\right)|\right)]dsd\mathrm{Q} + \int_Q h_k(u)u_t d\mathrm{Q}\\
&-\xi_0\alpha_1\int_Q\int_{\tau_1}^{\tau_2} y(x,1,t,s)g\left(y(x,1,t,s)\right)dsd\mathrm{Q}\\
\le& -\mu_0\int_Q f(u_t)u_t d\mathrm{Q} + \xi_0\alpha_2\int_Q\int_{\tau_1}^{\tau_2} u_t f(u_t)dsd\mathrm{Q} + \alpha_2\int_Q\int_{\tau_1}^{\tau_2}\mu(s)u_t f(u_t)dsd\mathrm{Q} \qquad (2.14)\\
&+\int_Q\int_{\tau_1}^{\tau_2}\mu(s)(1-\alpha_1)y(x,1,t,s)g\left(y(x,1,t,s)\right)dsd\mathrm{Q}\\
&-\xi_0\alpha_1\int_Q\int_{\tau_1}^{\tau_2} y(x,1,t,s)g\left(y(x,1,t,s)\right)dsd\mathrm{Q} + \int_Q h_k(u)u_t d\mathrm{Q}\\
=& -\left[\mu_0 - \xi_0\alpha_2(\tau_2-\tau_1) - \alpha_2\int_{\tau_1}^{\tau_2}\mu(s)ds\right]\int_Q f(u_t)u_t d\mathrm{Q}\\
&+\int_Q\int_{\tau_1}^{\tau_2}\left(\mu(s)(1-\alpha_1)-\xi_0\alpha_1\right)y(x,1,t,s)g\left(y(x,1,t,s)\right)dsd\mathrm{Q}\\
&+\int_Q h_k(u)u_t d\mathrm{Q}.
\end{aligned}$$

From the assumptions (2.10), (2.11), and using the Young inequality, (2.14) is rewritten as

$$\begin{aligned}
\frac{1}{2}\left(\|u_t(T)\|_2^2+\|u(T)\|_{H^1}^2\right) \le& \int_Q h_k(u)u_t d\mathrm{Q} + \xi(0)\\
\le& \int_0^T \|h_k(u)\|_2\|u_t\|_2 dt + \xi(0)\\
\le& L_h(k)\int_0^T \|u\|_{H^1}\|u_t\|_2 dt + \int_0^T \|h_k(0)\|_2\|u_t\|_2 dt + \xi(0)\\
\le& \left(L_h^2(k)+1\right)\int_0^T \|u\|_{H^1}^2 + \|u_t\|_2^2 dt + C_{h_k}T + \xi(0).
\end{aligned}$$

Hence, by Gronwall's inequality, we obtain

$$\left(\|u_t(t)\|_2^2+\|u(t)\|_{H^1}^2\right) \le \left(2\xi(0)+2C_{h_k}T\right)e^{2T\left(L_h^2(k)+1\right)}.$$

If we choose $T$ such that $2\xi(0)+2C_{h_k}T < k^2$, there exists

$$T_k = \min\left\{T, \frac{1}{2\left(L_h^2(k)+1\right)}\ln\left(\frac{k^2}{2\xi(0)+2C_{h_k}T}\right)\right\}$$

such that $\|u(t)\|_{H^1} < k$ for all $t < T_k$. Hence the original problem same as problem (2.12) for $t < T_k$. By repeating the same process with initial data $u(T_k)$ and for large $k$, we have maximal time $T_{max}$. Thus the proof of Theorem 2.2 is completed. □

## 3. Asymptotic behavior

In this section, we prove the global existence and energy decay of the solutions to the problem (1.1) when

$$h(u) = u|u|^{p-2} \quad 2 < p \leq \frac{2n-2}{n-2} \quad \text{if} \quad n \geq 3, \quad 2 < p < \infty \quad \text{if} \quad n = 1, 2.$$

Since $h : H_0^1(\Omega) \to L^2(\Omega)$ is locally Lipschitz, Theorem 2.2 allows of this polynomial growth source. In the following section, the symbol $C$ is a generic positive constant, which may be different in various occurrences. The energy associated to the problem (1.1) is defined by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \xi_0 \int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 sG(y(x, \rho, t, s))d\rho ds dx - \frac{1}{p}\|u\|_p^p.$$

Then from (2.13) and (2.14), we have

$$\begin{aligned} E'(t) &= \mu_0 \int_\Omega f(u_t)u_t dx - \int_\Omega \int_{\tau_1}^{\tau_2} \mu(s)u_t(x,t)g\left(y(x,1,t,s)\right) ds dx \\ &\quad - \xi_0 \int_\Omega \int_{\tau_1}^{\tau_2} [G\left(y(x,1,t,s)\right) - G\left(y(x,0,t,s)\right)] ds dx \\ &\leq -\left[\mu_0 - \xi_0\alpha_2(\tau_2 - \tau_1) - \alpha_2 \int_{\tau_1}^{\tau_2} \mu(s) ds\right] \int_\Omega f(u_t)u_t dx \\ &\quad + \int_\Omega \int_{\tau_1}^{\tau_2} \left(\mu(s)(1-\alpha_1) - \xi_0\alpha_1\right) y(x,1,t,s) g\left(y(x,1,t,s)\right) ds dx \\ &\leq 0, \end{aligned} \tag{3.1}$$

which implies that $E(t)$ is a nonincreasing function.

We now set

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p,$$

$$I(u) = \|\nabla u\|_2^2 - \|u\|_p^p$$

and

$$N = \{u \in H_0^1(\Omega) : I(u) = 0, \|\nabla u\|_2 \neq 0\}.$$

Then we know that [18]

$$d = \inf_{u \in H_0^1(\Omega)/\{0\}} \sup_{\lambda \geq 0} J(\lambda u) = \inf_{u \in N} J(u) = \frac{p-2}{2p}\left(\frac{1}{C_*^p}\right)^{\frac{2}{p-2}},$$

where

$$C_* = \sup \frac{\|u\|_p}{\|\nabla u\|_2}.$$

From the relationship $J(u) = \frac{1}{2}I(u) + \frac{p-2}{2p}\|u\|_p^p$, the energy $E(t)$ is rewritten as

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + J(u) + \xi_0 \int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 sG(y)d\rho ds dx \\ &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}I(u) + \frac{p-2}{2p}\|u\|_p^p + \xi_0 \int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 sG(y)d\rho ds dx. \end{aligned} \tag{3.2}$$

**Lemma 3.1.** *If $E(0) < d$ and $I(u_0) > 0$, then $I(u(t)) > 0$ for all $t \in [0, T_{max})$.*

*Proof.* The proof is same as that of [17, Lemma 4.1], so we omit it here. □

**Theorem 3.1.** *Under the assumptions on Theorem 2.2 and Lemma 3.1,* (1.1) *has a unique global solution for* $(u_0, v_0, f_0)^T \in \mathcal{H}$.

*Proof.* It suffices to show that $\|u_t\|_2^2 + \|\nabla u\|_2^2$ is bounded independent of $t$. By Lemma 3.1 and (3.2), we get

$$\|u_t\|_2^2 \leq 2E(t) \leq 2E(0) \leq 2d$$

and

$$\frac{p-2}{2p}\|\nabla u\|_2^2 \leq \frac{p-2}{2p}I(u) + \frac{p-2}{2p}\|u\|_p^p \leq 2E(t) \leq 2E(0) \leq 2d.$$

□

Under the assumption Lemma 3.1 and by definition of energy, we easily obtain

$$\|u\|_p^p \leq \left(\frac{p-2}{2p} + \epsilon\right)^{-1} E(t), \text{ for } t \geqq 0, \tag{3.3}$$

where $\epsilon$ is a for some sufficiently small positive constant. Now we recall the technical lemma which will play an essential role when establishing the energy decay.

**Lemma 3.2.** *(see [5]) Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing differentiable function and $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex and increasing function s.t $\Psi(0) = 0$. Assume that*

$$\int_S^T \Psi(E(t))dt \leq CE(S), \quad \forall 0 \leq S \leq T, \text{ for some positive constant } C.$$

*Then $E$ satisfies the following estimate:*

$$E(t) \leq \psi^{-1}\left(h(t) + \psi(E(0))\right), \quad \forall t \geq 0,$$

*where $\psi(t) = \int_t^1 \frac{1}{\Psi(s)}ds$ for $t > 0$, $h(t) = 0$ for $0 \leq t \leq \frac{E(0)}{\Psi(E(0))}$, and*

$$h^{-1}(t) = t + \frac{\psi^{-1}\left(t + \psi(E(0))\right)}{\Psi\left(\psi^{-1}(t + \psi(E(0)))\right)}, \quad \forall t > 0.$$

Thanks to [3] idea, we obtain following theorem.

**Theorem 3.2.** *Under the assumptions on Theorem 3.1, we obtain the following energy decay property:*

$$E(t) \leq \psi^{-1}\left(h(t) + \psi(E(0))\right), \quad \forall t > 0,$$

*where* $\psi(t) = \int_t^1 \frac{1}{\omega\varphi(s)} ds$ *for* $t > 0, h(t) = 0$ *for* $0 \leq t \leq \frac{E(0)}{\omega\varphi(E(0))}$, *and*

$$h^{-1}(t) = t + \frac{\psi^{-1}\left(t + \psi(E(0))\right)}{\omega\varphi\left(\psi^{-1}(t + \psi(E(0)))\right)}, \quad \forall t > 0$$

*for some positive constant* $\omega$, *and*

$$\varphi(s) = \begin{cases} s & \text{if} \quad H \quad \text{is linear on} \quad [0, \nu], \\ sH'(\varepsilon_0 s) & \text{if} \quad H'(0) = 0 \quad \text{and} \quad H'' > 0 \quad \text{on} \quad (0, \nu] \end{cases}$$

*for some positive constant* $\epsilon_0$.

*Proof.* We multiply the first equation of (2.1) by $A\frac{\varphi(E)}{E}u$ where $A = e^{-2\tau_2}$, and then integrate the obtained result over $(S, T) \times \Omega$. Then we have

$$\begin{aligned} 0 = &A\left[\frac{\varphi(E)}{E}(u, u_t)\right]_S^T - A\int_S^T \frac{\varphi(E)}{E}\|u_t\|_2^2 dt - A\int_S^T \left(\frac{\varphi(E)}{E}\right)'(u, u_t)dt + A\int_S^T \frac{\varphi(E)}{E}\|\nabla u\|_2^2 dt \\ &+ \mu_0 A\int_S^T \frac{\varphi(E)}{E}\int_\Omega u f(u_t)dxdt + A\int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2} \mu(s)ug(y(x, 1, t, s))dsdxdt \\ &- A\int_S^T \frac{\varphi(E)}{E}\|u\|_p^p dt. \end{aligned}$$

Similarly, multiplying the second equation of (2.1) by $\xi_0\frac{\varphi(E)}{E}e^{-2s\rho}g(y(x, \rho, t, s))$, and integrating over $(S, T) \times \Omega \times (\tau_1, \tau_2) \times (0, 1)$, we obtain

$$\begin{aligned} 0 &= \xi_0\int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2s\rho}g(y)y_t + e^{-2s\rho}g(y)y_\rho d\rho dsdxdt \\ &= \xi_0\left[\frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2s\rho}G(y)d\rho dsdx\right]_S^T - \xi_0\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2s\rho}G(y)d\rho dsdxdt \\ &\quad + \xi_0\left[\int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2} e^{-2s\rho}G(y)dsdxdt\right]_0^1 + 2\xi_0\int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2s\rho}G(y)d\rho dsdxdt. \end{aligned}$$

Combining the above two equations, we get the following equation:

$$
\begin{aligned}
2A\int_S^T \varphi(E)dt \le\ & 2A\int_S^T \frac{\varphi(E)}{E}\|u_t\|_2^2dt - A\left[\frac{\varphi(E)}{E}(u,u_t)\right]_S^T + A\int_S^T\left(\frac{\varphi(E)}{E}\right)'(u,u_t)dt\\
&-\mu_0A\int_S^T\frac{\varphi(E)}{E}\int_\Omega uf(u_t)dxdt - A\int_S^T\frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2}\mu(s)ug(y(x,1,t,s))dsdxdt\\
&+\frac{(p-2)A}{p}\int_S^T\frac{\varphi(E)}{E}\|u\|_p^pdt - \xi_0\left[\frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2s\rho}G(y)d\rho dsdx\right]_S^T\\
&+\xi_0\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2s\rho}G(y)d\rho dsdxdt\\
&-\xi_0\left(\int_S^T\frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2}\left[e^{-2s}G(y(1))-G(y(0))\right]dsdxdt\right)\\
:=\ & I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9.
\end{aligned}
\tag{3.4}
$$

Now we are going to estimate terms on the right hand side of (3.4).

From (3.2) and (3.3), we have

$$
I_6 := \frac{(p-2)A}{p}\int_S^T\frac{\varphi(E)}{E}\|u\|_p^pdt \le \frac{(p-2)A}{p}\left(\frac{p-2}{2p}+\epsilon\right)^{-1}\int_S^T\varphi(E)dt. \tag{3.5}
$$

We know by assumption of $\varphi(s)$, $\frac{\varphi(s)}{s}$ is nondecreasing and $E$ is nonnegative and decreasing. Also by using Young's inequality and Poincaré inequality, we obtain

$$
\begin{aligned}
I_2 :\ &= -A\left[\frac{\varphi(E)}{E}(u,u_t)\right]_S^T\\
&= -A\frac{\varphi(E(T))}{E(T)}(u(T),u_t(T)) + \frac{\varphi(E(S))}{E(S)}(u(S),u_t(S))\\
&\leqslant C_1\frac{\varphi(E(T))}{E(T)}E(T) + C_2\frac{\varphi(E(S))}{E(S)}E(S)\\
&\leqslant C\varphi(E(S)).
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
I_3 := A\int_S^T\left(\frac{\varphi(E)}{E}\right)'(u,u_t)dt &\le C\int_S^T\left|\frac{\varphi'(E)E'E-\varphi(E)E'}{E^2}\right|Edt\\
&\le C\int_S^T -\varphi'(E)E'dt + C\int_S^T -\frac{\varphi(E)}{E}E'dt \le C\varphi(E(S)).
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
|I_4| := A&\left|-\mu_0\int_S^T\frac{\varphi(E)}{E}\int_\Omega uf(u_t)dxdt\right|\\
&\le C\int_S^T\frac{\varphi(E)}{E}\left(\varepsilon\|\nabla u\|_2^2 + C(\varepsilon)\|f(u_t)\|_2^2\right)dt\\
&\le C\varepsilon\int_S^T\varphi(E)dt + C\int_S^T\frac{\varphi(E)}{E}\|f(u_t)\|_2^2dt.
\end{aligned}
\tag{3.8}
$$

Since $g$ is a nondecreasing odd Lipschitz function and $g^2(y(x,1,t,s) \leqslant L_g g(y(x,1,t,s))y(x,1,t,s)$, we have

$$\begin{aligned}
|I_5| &:= A\left|-\int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2}\mu(s)ug(y(x,1,t,s))dsdxdt\right| \\
&\le C\varepsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2}|\nabla u|^2\,dsdxdt + C(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2} g^2(y(x,1,t,s))dsdxdt \\
&\le C\varepsilon\int_S^T \frac{\varphi(E)}{E}\int_{\tau_1}^{\tau_2}E(t)dsdt + C\int_S^T \frac{\varphi(E)}{E}\int_{\tau_1}^{\tau_2}-E'(t)dsdt \\
&\le C\varepsilon\int_S^T \varphi(E)dt + C\varphi(E(S)).
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
I_7 &:= -\xi_0\left[\frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2s\rho}G(y)d\rho dsdx\right]_S^T \\
&= \xi_0\frac{\varphi(E(S))}{E(S)}\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2s\rho}G(y(S))d\rho dsdx - \xi_0\frac{\varphi(E(T))}{E(T)}\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2s\rho}G(y(T))d\rho dsdx \\
&\le C\varphi(E(S)).
\end{aligned} \tag{3.10}$$

$$I_8 := \xi_0\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2s\rho}G(y)d\rho dsdxdt \le \xi_0\int_S^T\left(\frac{\varphi(E)}{E}\right)'E(t)dt \le C\varphi(E(S)). \tag{3.11}$$

By assumption ($A4$) and (3.1), we obtain

$$\begin{aligned}
I_9 &:= -\xi_0\left(\int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2}\left[e^{-2s}G(y(1)) - G(y(0))\right]dsdxdt\right) \\
&\le C\xi_0\int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_{\tau_1}^{\tau_2} f(u_t)u_t dsdxdt \le C\xi_0\int_S^T\frac{\varphi(E)}{E}(-E'(t))dt \le C\varphi(E(S)).
\end{aligned} \tag{3.12}$$

Replacing (3.5)–(3.12) in (3.4) and taking $\epsilon$ sufficiently small, we arrive at

$$\int_S^T \varphi(E)dt \le C\varphi(E(S)) + C\int_S^T \frac{\varphi(E)}{E}\|f(u_t)\|_2^2dt + C\int_S^T\frac{\varphi(E)}{E}\|u_t\|_2^2dt. \tag{3.13}$$

We are now going to estimate the last two terms of right-hand side of (3.13). We consider the two cases with respect to the conditions on $H$.

**Case 1.** $H$ is linear on $[0,\nu]$.

From the assumption (A2) and the definition of the energy $E(t)$, we have

$$\int_S^T \frac{\varphi(E)}{E}\|u_t\|_2^2dt \le C\int_S^T\frac{\varphi(E)}{E}\int_\Omega u_t f(u_t)dxdt \le C\int_S^T\frac{\varphi(E)}{E}(-E')dt \le C\varphi(E(S)).$$

Put

$$\Omega_1 = \{x\in\Omega : |u_t| > \nu\},\ \Omega_2 = \{x\in\Omega : |u_t|\le\nu\}.$$

By assumption $(A1),(A2)$ and using (3.1), we have

$$\begin{aligned}
&\int_S^T \frac{\varphi(E)}{E}\|f(u')\|_2^2 dt\\
&\quad= \int_S^T \frac{\varphi(E)}{E}\int_{\Omega_1}|f(u')|^2\,dxdt + \int_S^T \frac{\varphi(E)}{E}\int_{\Omega_2}|f(u')|^2\,dxdt\\
&\quad\leqslant \max\{M_1, M_2\}\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_1} f(u')u'dxdt + C_1\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_2} f(u')u'dxdt\\
&\quad\leqslant C_2\int_S^T \frac{\varphi(E)}{E}\int_{\Omega} f(u')u'dxdt\\
&\quad\leqslant C_2\int_S^T \frac{\varphi(E)}{E}(-E')dt\\
&\quad\leqslant C\varphi(E(S)).
\end{aligned}$$

**Case 2.** $H'(0) = 0, H'' > 0 \ on \ (0, \nu]$. By the assumption (A2), we obtain

$$\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_1}|u_t|^2 + |f(u_t)|^2\,dxdt \le C\int_S^T \frac{\varphi(E)}{E}\int_{\Omega} u_t f(u_t)dxdt \le C\int_S^T \frac{\varphi(E)}{E}(-E')dt \le C\varphi(E(S)).$$

On the other hand, by applying Jensen's inequality for concave function we deduce that

$$\begin{aligned}
\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_2}|u_t|^2 + |f(u_t)|^2\,dxdt &\le \int_S^T \frac{\varphi(E)}{E}\int_{\Omega_2} H^{-1}(u_t f(u_t))dxdt\\
&\le C\int_S^T \frac{\varphi(E)}{E}H^{-1}\left(\frac{1}{meas(\Omega_2)}\int_{\Omega_2} u_t f(u_t)dx\right)dt.
\end{aligned}$$

By using Legendre transform (2.6) and (2.7) for $H$ and $\varphi(s) := sH'(\varepsilon_0 s)$, we obtain

$$\begin{aligned}
\int_S^T \frac{\varphi(E)}{E}H^{-1}\left(\frac{1}{meas(\Omega_2)}\int_{\Omega_2} u_t f(u_t)dx\right)dt &\le C\int_S^T H^*\left(\frac{\varphi(E)}{E}\right)dt + C\int_S^T\int_{\Omega} u_t f(u_t)dxdt\\
&\le \varepsilon_0 C\int_S^T \varphi(E)dt + CE(S).
\end{aligned}$$

Therefore, choosing $\varepsilon_0$ small enough we get in both cases

$$\int_S^T \varphi(E)dt \le C\left(E(S) + \varphi(E(S))\right) \le C\left(1 + \frac{\varphi(E(S))}{E(S)}\right)E(S) \le CE(S).$$

Thus, applying Lemma 3.2 with $\Psi(t) = \omega\varphi(t)$, the proof of Theorem 3.2 is completed. □


## Acknowledgments

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2022R1I1A3055309).

## Conflict of interest

The authors declare no conflict of interest.

## References


1. W. Aiello, H. Freedman, A time-delay model of single-species growth with stage structure, *Math. Biosci.*, **101** (1990), 139–153. https://doi.org/10.1016/0025-5564(90)90019-U
2. V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Dordrecht: Springer, 1976.
3. A. Benaissa, A. Benaissa, S. Messaoudi, Global existence and energy decay of solutions for the wave equation with a time varying delay term in the weakly nonlinear internal feedbacks, *J. Math. Phys.*, **53** (2012), 123514. https://doi.org/10.1063/1.4765046
4. A. Benaissa, A. Benguessoum, S. Messaoudi, Global existence and energy decay of solutions to a viscoelastic wave equation with a delay term in the non–linear internal feedback, *Int. J. Dyn. Syst. Differ.*, **5** (2014), 1–26. https://doi.org/10.1504/IJDSDE.2014.067080
5. A. Benaissa, A. Guesmia, Energy decay for wave equations of $\varphi$-Laplacian type with weakly nonlinear dissipation, *Electron. J. Differ. Eq.*, **2008** (2008), 1–22.
6. L. Bouzettouta, F. Hebhoub, K. Ghennam, S. Benferdi, Exponential stability for a nonlinear Timoshenko system with distributed delay, *Int. J. Anal. Appl.*, **19** (2021), 77–90. https://doi.org/10.28924/2291-8639-19-2021-77
7. A. Choucha, S. Boulaaras, D. Ouchenane, Exponential decay of solutions for a viscoelastic coupled lame system with logarithmic source and distributed delay terms, *Math. Methods Appl. Sci.*, **44** (2021), 4858–4880. https://doi.org/10.1002/mma.7073
8. A. Choucha, D. Ouchenane, S. Boulaaras, Well posedness and stability result for a thermoelastic laminated Timoshenko beam with distributed delay term, *Math. Methods Appl. Sci.*, **43** (2020), 9983–10004. https://doi.org/10.1002/mma.6673
9. I. Chueshov, M. Eller, I. Lasiecka, On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation, *Commun. Part. Diff. Eq.*, **27** (2002), 1901–1951. https://doi.org/10.1081/PDE-120016132
10. R. Datko, J. Lagnese, M. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, *SIAM J. Control Optim.*, **24** (1986), 152–156. https://doi.org/10.1137/0324007
11. B. Feng, Global well-posedness and stability for a viscoelastic plate equation with a time delay, *Math. Probl. Eng.*, **2015** (2015), 585021. https://doi.org/10.1155/2015/585021
12. N. Mezouar, S. Boulaaras, A. Allahem, Global existence of solutions for the viscoelastic Kirchhoff equation with logarithmic source terms, *Complexity*, **2020** (2020), 7105387. https://doi.org/10.1155/2020/7105387
13. S. Nicaise, C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.*, **45** (2006), 1561–1585. https://doi.org/10.1137/060648891

14. S. Nicaise, C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, *Differ. Integral Equ.*, **21** (2008), 935–958. https://doi.org/10.57262/die/1356038593

15. S. Nicaise, C. Pignotti, J. Valein, Exponential stability of the wave equation with boundary time-varying delay, *Discrete Cont. Dyn.-S*, **4** (2011), 693–722. https://doi.org/10.3934/dcdss.2011.4.693

16. S. Nicaise, J. Valein, E. Fridaman, Stability of the heat and of the wave equations with boundary time-varying delays, *Discrete Cont. Dyn.-S*, **2** (2009), 559–581. https://doi.org/10.3934/dcdss.2009.2.559

17. S. Park, Global existence, energy decay and blow-up of solutions for wave equations with time delay and logarithmic source, *Adv. Differ. Equ.*, **2020** (2020), 631. https://doi.org/10.1186/s13662-020-03037-6

18. L. Payne, D. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, *Israel J. Math.*, **22** (1975), 273–303. https://doi.org/10.1007/BF02761595

19. C. Raposo, H. Nguyen, J. Ribeiro, V. Barros, Well-posedness and exponential stability for a wave equation with nonlocal time-delay condition, *Electron. J. Differ. Eq.*, **2017** (2017), 279.

20. R. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Providence: American Mathematical Society, 1997.

21. H. Suh, Z. Bien, Use of time-delay actions in the controller design, *IEEE T. Automat. Contr.*, **25** (1980), 600–603. https://doi.org/10.1109/TAC.1980.1102347

22. Y. Xie, G. Xu, Exponential stability of 1-d wave equation with the boundary time delay based on the interior control, *Discrete Cont. Dyn.-S*, **10** (2017), 557–579. https://doi.org/10.3934/dcdss.2017028

23. X. Yang, J. Zhang, Y. Lu, Dynamics of the nonlinear Timoshenko system with variable delay, *Appl. Math. Optim.*, **83** (2021), 297–326. https://doi.org/10.1007/s00245-018-9539-0


AIMS Press