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Research article

Direct quaternion method-based stability criteria for quaternion-valued Takagi-Sugeno fuzzy BAM delayed neural networks using quaternion-valued Wirtinger-based integral inequality

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Abstract: This paper investigates the global asymptotic stability problem for a class of quaternion-valued Takagi-Sugeno fuzzy BAM neural networks with time-varying delays. By applying Takagi-Sugeno fuzzy models, we first consider a general form of quaternion-valued Takagi-Sugeno fuzzy BAM neural networks with time-varying delays. Then, we apply the Cauchy-Schwarz algorithm and homeomorphism principle to obtain sufficient conditions for the existence and uniqueness of the equilibrium point. By utilizing suitable Lyapunov-Krasovskii functionals and newly developed quaternion-valued Wirtinger-based integral inequality, some sufficient criteria are obtained to guarantee the global asymptotic stability of the considered networks. Further, the results of this paper are presented in the form of quaternion-valued linear matrix inequalities, which can be solved using the MATLAB YALMIP toolbox. Two numerical examples are presented with their simulations to demonstrate the validity of the theoretical analysis.

Keywords: global asymptotic stability; quaternion-valued neural networks; Lyapunov-Krasovskii functional; Takagi-Sugeno fuzzy; time-varying delays
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1. Introduction

Over the past few years, neural networks (NNs) and their generalizations have attracted considerable attention for their ability to address the problems involving associative memory, secure communications, signal and image processing, optimization, and others [1–4]. In 1987, Kosko created a significant type of two-layer hetero-associative memory network named bidirectional associative memory (BAM) NNs [5]. Then, BAM NNs have attracted significant attention due to their wide range of applications, such as data mining, pattern recognition, automatic control, and others [6–8]. In recent years, several theoretical studies regarding various dynamics of BAM NNs based on Lyapunov-Krasovskii functionals (LKFs) and linear matrix inequalities (LMIs) have been published [9–17].

There is no doubt that real-, complex-valued NNs have been successfully applied to a variety of engineering applications [18–23]. However, these two NNs have some limitations when it comes to symmetry detection and high-dimensional data problems [24]. In order to address these issues, some scholars have developed quaternion-valued NNs by incorporating quaternions into standard NNs. Moreover, quaternion-valued NNs have shown superior performance than real-, complex-valued NNs because of their general representation and high-efficiency [25–27]. Therefore, the research on quaternion-valued NNs has become a popular topic in modern science [28–31]. Recently, several researchers have investigated various dynamics of quaternion-valued NNs using LKFs and LMI methods [15, 16, 29–31]. Particularly, by utilizing homeomorphism principle and the Lyapunov stability method, the global μ-stability analysis for quaternion-valued NNs with unbounded distributed delays has been investigated in [29], besides the quaternion-valued NNs are decomposed into two complex-valued NNs. By applying the direct quaternion method and LMI algorithm, new stability conditions for neutral-type quaternion-valued NNs with discrete delays were established in [30]. Some exponential stability criteria of quaternion-valued discrete-time NNs with leakage and discrete delays were presented in [31]. There are similar results to be found in [15, 16].

In [32], Takagi-Sugeno (T-S) proposed the fuzzy systems, which have been effectively utilized for modeling and analyzing complex nonlinear systems [32, 33]. The T-S fuzzy system has the advantage of being able to approximate a nonlinear system with a set of linear models. It should be noted that, unlike typical NN structures, T-S fuzzy NNs have fuzzy operations, and they are able to preserve the direct correlation between the cells. Due to their good approximation properties, T-S fuzzy NNs have proved to be an important research topic. Many papers have proposed the idea of incorporating fuzzy logic into the NNs in order to enhance their performance [34–37]. For example, by considering impulsive effects and time delays, the authors of [36] have determined the exponential convergence for T-S fuzzy complex-valued NNs.

On the other hand, time delays inherently occur in NN implementations, and they can cause undesirable system behaviors. Therefore, it is essential to study how delays affect the system's dynamics. Recently, various time delays have been extensively studied in a variety of dynamic models, especially in control theory [38–40]. Furthermore, there are two types of stability criteria that are associated with delayed NNs: delay-dependent stability criteria and delay-independent stability criteria. Recently, delay-dependent stability criteria have received significant attention compared to delay-independent criteria because of their practical importance [41, 42]. On the other hand, the main purpose of delay-dependent stability criteria is to obtain less conservatism of the obtained results. Generally, there are two factors that contribute to less conservative stability conditions, namely the augmented LKFs and the tighter bound integral inequalities. Recently, a number of studies have developed various integral inequalities in order to handle integral terms in the real domain [44–46]. However, only Jensen's inequality has been utilized since the beginning to deal with integral terms in the quaternion domain. To fill such gaps, a new quaternion-valued integral inequality has been developed in this paper, which includes the famous Wirtinger-based integral inequality (WBII) [44] and the reciprocal convex combination (RCC) lemma [43].

Following the above discussions, in this paper, we aim to investigate the global asymptotic stability of quaternion-valued T-S fuzzy BAM NNs by applying the direct quaternion method. There are several results discussed in the literature regarding various stability problems of quaternion-valued BAM NNs, however, quaternion-valued T-S fuzzy BAM NNs have not been fully explored and are not receiving much attention, which motivates us to investigate this topic. This paper has the following main merits: 1) To represent more realistic dynamical behaviors of quaternion-valued NNs, we present a general form of the quaternion-valued T-S fuzzy BAM NNs with time delays for the first time in this paper. 2) The direct quaternion method is employed to examine the global asymptotic stability of quaternion-valued T-S fuzzy BAM NNs for the first time in this paper. 3) By considering suitable LKFs that contain double integral terms and by employing newly developed quaternion-valued WBII, enhanced stability conditions for the concerned NN model are derived in the form of quaternion-valued LMIs, which could be verified directly by MATLAB YALMIP toolbox. 4) The proofs for the quaternion-valued WBII and quaternion-valued RCC lemma are presented for the first time in this paper.

The paper is structured as follows: Section 2 provides the problem model, definitions of global asymptotic stability, assumptions about activation functions and time-varying delays, and some useful lemmas. The results of this study are stated in Section 3; Theorem (3.1) presents sufficient criteria for the existence and uniqueness of the equilibrium point; Theorem (3.2) provides sufficient criteria for the global asymptotic stability of the considered network models. In Corollary (3.4), (3.6), the results of stability criteria are discussed in a special case. Section 4 discusses two numerical case studies that demonstrate the feasibility of the derived results. Section 5 shows the conclusion of this paper.

2. Mathematical formulation and problem definition

2.1. Notations

Let the set of all quaternion, complex, and real numbers are denoted by $\mathbf{H}, \mathbf{C}$ and $\mathbf{R}$, respectively. The symbols $\mathbf{H}^n$, $\mathbf{C}^n$ and $\mathbf{R}^n$ stand for n-dimensional quaternion, complex and real vectors, respectively. The quaternion, complex and real matrices of size $n \times n$ are represented by the symbols $\mathbf{H}^{n\times n}$, $\mathbf{C}^{n\times n}$ and $\mathbf{R}^{n\times n}$, respectively. Let the matrix $\mathcal{P} < 0$ ($\mathcal{P} > 0$) means $\mathcal{P}$ is negative (positive) definite matrix. The block diagonal matrix is shown in $diag\{\cdot\}$. The conjugate transposition and matrix transposition, respectively, are denoted by the superscripts $*$ and T. The symmetric term in a matrix is showed by $\star$.

2.2. Quaternion algebra

The quaternion was first invented by Hamilton in 1843. The skew field of a quaternion is denoted by

$$z = z^R + iz^I + jz^J + kz^K \in \mathbf{H},$$

where $z^R, z^I, z^J, z^K \in \mathbf{R}$, z is the quaternion-valued input and i, j, k are the quaternion basis which subjects to Hamilton's multiplication rules as follows:

$$\begin{aligned} &k^2 = j^2 = i^2 = -1 \\ &jk = -kj = i,\ ki = -ik = j,\ ij = -ji = k. \end{aligned}$$

The conjugate of the quaternion as follows:

$$\bar{z} = z^R - iz^I - jz^J - kz^K \in \mathbf{H}.$$

The modulus of the quaternion as follows:

$$|z| = \sqrt{z\bar{z}} = \sqrt{(z^R)^2 + (z^I)^2 + (z^J)^2 + (z^K)^2}.$$

Let $x = x^R + ix^I + jx^J + kx^K \in \mathbf{H}$ and $y = y^R + iy^I + jy^J + ky^K \in \mathbf{H}$. The addition and multiplication of two quaternions can be accomplished as follows:

$$\begin{aligned} x + y &= (x^R + y^R) + i(x^I + y^I) + j(x^J + y^J) + k(x^K + y^K), \\ xy &= (x^R y^R - x^I y^I - x^J y^J - x^K y^K) + i(x^R y^I + x^I y^R + x^J y^K - x^K y^J) \\ &\quad + j(x^R y^J + x^J y^R - x^I y^K + x^K y^I) + k(x^R y^K + x^K y^I + x^I y^J - x^J y^I). \end{aligned}$$

2.3. Problem formulation

In this paper, we consider the quaternion-valued BAM NNs with time-varying delays as follows:

$$\begin{cases} \dot{p}_r(t) = & -d_{1_r} p_r(t) + \sum_{s=1}^{m} a_{1_{rs}} f_{1_s}(q_s(t - \ell(t))) + J_{1_r},\ r = 1, 2, ..., n, \\ p_r(t) = & \varphi_{1_r}(t),\ t \in [-\ell, 0], \\ \dot{q}_s(t) = & -d_{2_s} q_s(t) + \sum_{r=1}^{n} a_{2_{sr}} f_{2_r}(p_r(t - \ell(t))) + J_{2_s},\ s = 1, 2, ..., m, \\ q_s(t) = & \varphi_{2_s}(t),\ t \in [-\ell, 0], \end{cases} \tag{2.1}$$

or equivalently

$$\begin{cases} \dot{p}(t) = & -\mathcal{D}_1 p(t) + \mathcal{A}_1 f_1(q(t - \ell(t))) + \mathcal{J}_1, \\ p(t) = & \varphi_1(t),\ t \in [-\ell, 0], \\ \dot{q}(t) = & -\mathcal{D}_2 q(t) + \mathcal{A}_2 f_2(p(t - \ell(t))) + \mathcal{J}_2, \\ q(t) = & \varphi_2(t),\ t \in [-\ell, 0], \end{cases} \tag{2.2}$$

where $p(t) = [p_1(t), ..., p_n(t)]^T \in \mathbf{H}^n$, $q(t) = [q_1(t), ..., q_m(t)]^T \in \mathbf{H}^m$ are the state vectors; $f_1(q(\cdot)) = [f_{1_1}(q_1(\cdot)), ..., f_{1_m}(q_m(\cdot))]^T \in \mathbf{H}^m$, $f_2(p(\cdot)) = [f_{2_1}(p_1(\cdot)), ..., f_{2_n}(p_n(\cdot))]^T \in \mathbf{H}^n$ are the vector valued activation functions; $\mathcal{J}_1 = [J_{1_1}, ..., J_{1_n}]^T \in \mathbf{H}^n$, $\mathcal{J}_2 = [J_{2_1}, ..., J_{2_m}]^T \in \mathbf{H}^m$ are the input vectors. $\mathcal{D}_1 = diag\{d_{1_1}, ..., d_{1_n}\} \in \mathbf{R}^{n\times n}$, $\mathcal{D}_2 = diag\{d_{2_1}, ..., d_{2_m}\} \in \mathbf{R}^{m\times m}$ are the self-feedback connection weight matrices with each $d_{1_r} > 0,\ d_{2_s} > 0,\ r = 1, ..., n,\ s = 1, ..., m$. $\mathcal{A}_1 \in \mathbf{H}^{n\times m}, \mathcal{A}_2 \in \mathbf{H}^{m\times n}$ are the delayed interconnection weight matrices. $\varphi_1 \in \mathscr{C}([-\ell, 0], \mathbf{H}^n)$ and $\varphi_2 \in \mathscr{C}([-\ell, 0], \mathbf{H}^m)$ are the initial conditions.

This paper assumes that the activation functions $f_1(q(t-\ell(t)))$, $f_2(p(t-\ell(t)))$ and transmission delays $\ell(t)$ satisfy the following conditions:

Assumption 1: The activation functions f_{1s} and f_{2r} are Lipschitz continuous; that is, there exist positive constants $l_s^{f_1} > 0, l_r^{f_2} > 0$, such that for all $r = 1, 2, ..., n$, $s = 1, 2, ..., m$

$$|f_{1s}(x) - f_{1s}(y)| \leq l_s^{f_1}|x - y|, \ \forall x, y \in \mathbf{H},$$
$$|f_{2r}(x) - f_{2r}(y)| \leq l_r^{f_2}|x - y|, \ \forall x, y \in \mathbf{H}.$$

Furthermore, we define $\mathcal{L}_{f_1} = diag\{l_1^{f_1}, l_2^{f_1}, ..., l_m^{f_1}\}$, $\mathcal{L}_{f_2} = diag\{l_1^{f_2}, l_2^{f_2}, ..., l_n^{f_2}\}$.

Assumption 2: The delay $\ell(t) : \mathbf{R} \to \mathbf{R}$ is differentiable and bounded, which satisfy the conditions $0 \leq \ell(t) \leq \ell$, $\dot{\ell}(t) \leq \mu < 1$, where ℓ and μ are real numbers.

2.4. Preliminaries

In order to derive our main results, we present some definition and lemmas.

Definition 2.1. For the NN model (2.2) with any initial conditions $\varphi_1 \in \mathscr{C}([-\ell, 0], \mathbf{H}^n)$ and $\varphi_2 \in \mathscr{C}([-\ell, 0], \mathbf{H}^m)$, the trivial solution is called globally asymptotically stable (GAS) if $\lim\limits_{t\to+\infty} \{\|p(t, \varphi_1)\|^2 + \|q(t, \varphi_2)\|^2\} = 0$, where $p(t, \varphi_1)$ and $q(t, \varphi_2)$ are the solutions of NN (2.2) at time t under the initial conditions φ_1 and φ_2, respectively.

Lemma 2.2. *[47] For any vectors $p, q \in \mathbf{H}^n$ and a scalar $\epsilon > 0$, then the following inequality holds: $p^*q + q^*p \leq \epsilon p^*p + \epsilon^{-1}q^*q$.*

Lemma 2.3. *[47] Let $\mathcal{H}(p, q) : \mathbf{H}^{n+m} \to \mathbf{H}^{n+m}$ is a continuous map that fulfills the following criteria:*
(i) $\mathcal{H}(p, q)$ is injective on $\mathbf{H}^{n+m}$,
(ii) $\|\mathcal{H}(p, q)\| \to \infty$ as $\|(p, q)\| \to \infty$, then $\mathcal{H}$ is homeomorphism of $\mathbf{H}^{n+m}$ onto itself.

Lemma 2.4. *[48] Let $\alpha, \beta \in \mathbf{H}$; $\mathcal{X}, \mathcal{Y} \in \mathbf{H}^{n\times n}$ and Hermitian matrix $0 < \mathcal{P} \in \mathbf{H}^{n\times n}$, then*

(1) $|\alpha + \beta| \leq |\alpha| + |\beta|$ and $|\alpha\beta| = |\beta\alpha| = |\alpha||\beta|$,
(2) $(\overline{\mathcal{X}})^T = \overline{(\mathcal{X}^T)}$,
(3) $(\mathcal{X}\mathcal{Y})^* = \mathcal{Y}^*\mathcal{X}^*$,
(4) $(\mathcal{X}\mathcal{Y})^{-1} = \mathcal{Y}^{-1}\mathcal{X}^{-1}$, if $\mathcal{X}$ and $\mathcal{Y}$ are invertible,
(5) $(\mathcal{A}^*)^{-1} = (\mathcal{A}^{-1})^*$, if $\mathcal{A}$ is invertible,
(6) Any quaternion α can be formulated uniquely as $\alpha = \gamma_1 + \gamma_2 j$, where $\gamma_1, \gamma_2 \in \mathbf{C}$,
(7) $j\gamma = \overline{\gamma} j$ or $j\gamma j^* = \overline{\gamma}$, for all $\gamma \in \mathbf{C}$,
(8) There exist an invertible matrix $\mathcal{Z} \in \mathbf{H}^{n\times n}$, i.e. $\mathcal{P} = \mathcal{Z}^*\mathcal{Z}$.

Lemma 2.5. *[49] Let $\mathcal{X}, \mathcal{Y} \in \mathbf{H}^{n\times n}$; $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2 \in \mathbf{C}^{n\times n}$ and $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 j$, $\mathcal{Y} = \mathcal{Y}_1 + \mathcal{Y}_2 j$. Then*

(1) $\mathcal{X}^* = \mathcal{X}_1^* - \mathcal{X}_2^T j$,
(2) $\mathcal{X}\mathcal{Y} = (\mathcal{X}_1\mathcal{Y}_1 - \mathcal{X}_2\overline{\mathcal{Y}}_2) + (\mathcal{X}_1\mathcal{Y}_2 + \mathcal{X}_2\overline{\mathcal{Y}}_1)j$.

Lemma 2.6. *[50] A Hermitian matrix $\mathcal{W} = \mathcal{W}^R + i\mathcal{W}^I + j\mathcal{W}^J + k\mathcal{W}^K \in \mathbf{H}^{n\times n}$, then $\mathcal{W} < 0$ is equivalent to*

$$\begin{bmatrix} \mathcal{W}^R & -\mathcal{W}^J & -\mathcal{W}^I & \mathcal{W}^K \\ \mathcal{W}^J & \mathcal{W}^R & \mathcal{W}^K & \mathcal{W}^I \\ \mathcal{W}^I & -\mathcal{W}^K & \mathcal{W}^R & -\mathcal{W}^J \\ -\mathcal{W}^K & -\mathcal{W}^I & \mathcal{W}^J & \mathcal{W}^R \end{bmatrix} < 0.$$

where $\mathcal{W}^R = Re(\mathcal{W})$, $\mathcal{W}^I = Im(\mathcal{W})$, $\mathcal{W}^J = Im(\mathcal{W})$ and $\mathcal{W}^K = Im(\mathcal{W})$.

Remark 2.7. *As indicated in Lemma* (3.3)*, the $n \times n$ Hermitian matrix equals the* $4n \times 4n$ *real matrix. With the aid of this lemma, it is possible to transform the quaternion-valued LMIs into the real-valued LMIs and vice versa.*

Lemma 2.8. *(Quaternion-valued WBII) For every differentiable function $w : [a, b] \to \mathbf{H}^n$ and Hermitian matrix $0 < \mathcal{W} = \mathcal{W}^R + i\mathcal{W}^I + j\mathcal{W}^J + k\mathcal{W}^K \in \mathbf{H}^{n\times n}$, the following criteria holds:*

$$\int_a^b w^*(s)\mathcal{W}w(s)ds \geq \frac{1}{b-a}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}^* \begin{bmatrix}\mathcal{W} & 0\\ 0 & 3\mathcal{W}\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix},$$

where

$$\zeta_1 = \int_a^b w(s)ds,$$
$$\zeta_2 = \int_a^b w(s)ds - \frac{2}{b-a}\int_a^b\int_u^b w(s)dsdu.$$

Proof: Let $w(s) = w^R(s) + iw^I(s) + jw^J(s) + kw^K(s) \in \mathbf{H}$, $\zeta_1 = \zeta_1^R + i\zeta_1^I + j\zeta_1^J + k\zeta_1^K \in \mathbf{H}$, $\zeta_2 = \zeta_2^R + i\zeta_2^I + j\zeta_2^J + k\zeta_2^K \in \mathbf{H}$, $\mathcal{W} = \mathcal{W}^R + i\mathcal{W}^I + j\mathcal{W}^J + k\mathcal{W}^K \in \mathbf{H}^{n\times n}$, where $\mathcal{W}^* = \mathcal{W} \Leftrightarrow (\mathcal{W}^R)^T = \mathcal{W}^R$, $-(\mathcal{W}^I)^T = \mathcal{W}^I$, $-(\mathcal{W}^J)^T = \mathcal{W}^J$, $(\mathcal{W}^K)^T = -\mathcal{W}^K$. Using Lemma (2.6) and WBII [44], we get

$$\int_a^b w^*(s)\mathcal{W}w(s)ds = \int_a^b \begin{bmatrix} w^R(s)\\ w^I(s)\\ w^J(s)\\ w^K(s)\end{bmatrix}^T \begin{bmatrix} \mathcal{W}^R & -\mathcal{W}^J & -\mathcal{W}^I & \mathcal{W}^K \\ \mathcal{W}^J & \mathcal{W}^R & \mathcal{W}^K & \mathcal{W}^I \\ \mathcal{W}^I & -\mathcal{W}^K & \mathcal{W}^R & -\mathcal{W}^J \\ -\mathcal{W}^K & -\mathcal{W}^I & \mathcal{W}^J & \mathcal{W}^R \end{bmatrix}\begin{bmatrix} w^R(s)\\ w^I(s)\\ w^J(s)\\ w^K(s)\end{bmatrix} ds,$$

$$\geq \frac{1}{b-a}\begin{bmatrix}\zeta_1^R\\ \zeta_1^I\\ \zeta_1^J\\ \zeta_1^K\\ \zeta_2^R\\ \zeta_2^I\\ \zeta_2^J\\ \zeta_2^K\end{bmatrix}^T \begin{bmatrix} \mathcal{W}^R & -\mathcal{W}^J & -\mathcal{W}^I & \mathcal{W}^K & 0 & 0 & 0 & 0\\ \mathcal{W}^J & \mathcal{W}^R & \mathcal{W}^K & \mathcal{W}^I & 0 & 0 & 0 & 0\\ \mathcal{W}^I & -\mathcal{W}^K & \mathcal{W}^R & -\mathcal{W}^J & 0 & 0 & 0 & 0\\ -\mathcal{W}^K & -\mathcal{W}^I & \mathcal{W}^J & \mathcal{W}^R & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 3\mathcal{W}^R & -3\mathcal{W}^J & -3\mathcal{W}^I & 3\mathcal{W}^K\\ 0 & 0 & 0 & 0 & 3\mathcal{W}^J & 3\mathcal{W}^R & 3\mathcal{W}^K & 3\mathcal{W}^I\\ 0 & 0 & 0 & 0 & 3\mathcal{W}^I & -3\mathcal{W}^K & 3\mathcal{W}^R & -3\mathcal{W}^J\\ 0 & 0 & 0 & 0 & -3\mathcal{W}^K & -3\mathcal{W}^I & 3\mathcal{W}^J & 3\mathcal{W}^R \end{bmatrix}\begin{bmatrix}\zeta_1^R\\ \zeta_1^I\\ \zeta_1^J\\ \zeta_1^K\\ \zeta_2^R\\ \zeta_2^I\\ \zeta_2^J\\ \zeta_2^K\end{bmatrix},$$

$$= \frac{1}{b-a}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}^* \begin{bmatrix}\mathcal{W} & 0\\ 0 & 3\mathcal{W}\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}.$$

Lemma 2.9. *(Quaternion-valued RCC lemma) For any vectors* $\zeta_1, \zeta_2 \in \mathbf{H}^n$*, Hermitian matrix* $0 < \mathcal{E} \in \mathbf{H}^{n\times n}$*, any matrix* $\mathcal{F} \in \mathbf{H}^{n\times n}$ *and any* $\alpha \in (0,1)$*, such that* $\begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{F}^* & \mathcal{E} \end{bmatrix} \geq 0$*, the following criteria holds*

$$\frac{1}{\alpha}\zeta_1^*\mathcal{E}\zeta_1 + \frac{1}{1-\alpha}\zeta_2^*\mathcal{E}\zeta_2 \geq \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}^* \begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{F}^* & \mathcal{E} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}.$$

Proof: *Let* $\zeta_1 = \zeta_1^R + i\zeta_1^I + j\zeta_1^J + k\zeta_1^K \in \mathbf{H}$*;* $\zeta_2 = \zeta_2^R + i\zeta_2^I + j\zeta_2^J + k\zeta_2^K \in \mathbf{H}$*;* $\mathcal{E} = \mathcal{E}^R + i\mathcal{E}^I + j\mathcal{E}^J + k\mathcal{E}^K \in \mathbf{H}^{n\times n}$*, where* $\mathcal{E}^* = \mathcal{E} \Leftrightarrow (\mathcal{E}^R)^T = \mathcal{E}^R$*,* $-(\mathcal{E}^I)^T = \mathcal{E}^I$*,* $-(\mathcal{E}^J)^T = \mathcal{E}^J$*,* $(\mathcal{E}^K)^T = -\mathcal{E}^K$*;* $\mathcal{F} = \mathcal{F}^R + i\mathcal{F}^I + j\mathcal{F}^J + k\mathcal{F}^K \in \mathbf{H}^{n\times n}$*. Using Lemma* (2.6) *and RCC lemma [43], we get*

$$\frac{1}{\alpha}\zeta_1^*\mathcal{E}\zeta_1 + \frac{1}{1-\alpha}\zeta_2^*\mathcal{E}\zeta_2 = \frac{1}{\alpha}\begin{bmatrix} \zeta_1^R \\ \zeta_1^I \\ \zeta_1^J \\ \zeta_1^K \end{bmatrix}^T \begin{bmatrix} \mathcal{E}^R & -\mathcal{E}^J & -\mathcal{E}^I & \mathcal{E}^K \\ \mathcal{E}^J & \mathcal{E}^R & \mathcal{E}^K & \mathcal{E}^I \\ \mathcal{E}^I & -\mathcal{E}^K & \mathcal{E}^R & -\mathcal{E}^J \\ -\mathcal{E}^K & -\mathcal{E}^I & \mathcal{E}^J & \mathcal{E}^R \end{bmatrix} \begin{bmatrix} \zeta_1^R \\ \zeta_1^I \\ \zeta_1^J \\ \zeta_1^K \end{bmatrix}$$

$$+ \frac{1}{1-\alpha}\begin{bmatrix} \zeta_2^R \\ \zeta_2^I \\ \zeta_2^J \\ \zeta_2^K \end{bmatrix}^T \begin{bmatrix} \mathcal{E}^R & -\mathcal{E}^J & -\mathcal{E}^I & \mathcal{E}^K \\ \mathcal{E}^J & \mathcal{E}^R & \mathcal{E}^K & \mathcal{E}^I \\ \mathcal{E}^I & -\mathcal{E}^K & \mathcal{E}^R & -\mathcal{E}^J \\ -\mathcal{E}^K & -\mathcal{E}^I & \mathcal{E}^J & \mathcal{E}^R \end{bmatrix} \begin{bmatrix} \zeta_2^R \\ \zeta_2^I \\ \zeta_2^J \\ \zeta_2^K \end{bmatrix},$$

$$\geq \begin{bmatrix} \zeta_1^R \\ \zeta_1^I \\ \zeta_1^J \\ \zeta_1^K \\ \zeta_2^R \\ \zeta_2^I \\ \zeta_2^J \\ \zeta_2^K \end{bmatrix}^T \begin{bmatrix} \mathcal{E}^R & -\mathcal{E}^J & -\mathcal{E}^I & \mathcal{E}^K & \mathcal{F}^R & -\mathcal{F}^J & -\mathcal{F}^I & \mathcal{F}^K \\ \mathcal{E}^J & \mathcal{E}^R & \mathcal{E}^K & \mathcal{E}^I & \mathcal{F}^J & \mathcal{F}^R & \mathcal{F}^K & \mathcal{F}^I \\ \mathcal{E}^I & -\mathcal{E}^K & \mathcal{E}^R & -\mathcal{E}^J & \mathcal{F}^I & -\mathcal{F}^K & \mathcal{F}^R & -\mathcal{F}^J \\ -\mathcal{E}^K & -\mathcal{E}^I & \mathcal{E}^J & \mathcal{E}^R & -\mathcal{F}^K & -\mathcal{F}^I & \mathcal{F}^J & \mathcal{F}^R \\ (\mathcal{F}^R)^T & (\mathcal{F}^J)^T & (\mathcal{F}^I)^T & -(\mathcal{F}^K)^T & \mathcal{E}^R & -\mathcal{E}^J & -\mathcal{E}^I & \mathcal{E}^K \\ -(\mathcal{F}^J)^T & (\mathcal{F}^R)^T & -(\mathcal{F}^K)^T & -(\mathcal{F}^I)^T & \mathcal{E}^J & \mathcal{E}^R & \mathcal{E}^K & \mathcal{E}^I \\ -(\mathcal{F}^I)^T & (\mathcal{F}^K)^T & (\mathcal{F}^R)^T & (\mathcal{F}^J)^T & \mathcal{E}^I & -\mathcal{E}^K & \mathcal{E}^R & -\mathcal{E}^J \\ (\mathcal{F}^K)^T & (\mathcal{F}^I)^T & -(\mathcal{F}^J)^T & (\mathcal{F}^R)^T & -\mathcal{E}^K & -\mathcal{E}^I & \mathcal{E}^J & \mathcal{E}^R \end{bmatrix} \begin{bmatrix} \zeta_1^R \\ \zeta_1^I \\ \zeta_1^J \\ \zeta_1^K \\ \zeta_2^R \\ \zeta_2^I \\ \zeta_2^J \\ \zeta_2^K \end{bmatrix},$$

$$= \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}^* \begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{F}^* & \mathcal{E} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}.$$

3. Main results

This section presents new sufficient criteria to ensure the global asymptotic stability of the considered NN model based on the LKFs and LMI method, as well as quaternion-valued WBII.

3.1. Existence and uniqueness of the equilibrium point

In this subsection, we discuss sufficient criteria that guarantee the existence and uniqueness of the equilibrium point for NNs (2.2).

Theorem 3.1. *Under Assumptions 1 and 2, the NN model* (2.2) *has a unique equilibrium point if there exist positive definite Hermitian matrices* $0 < \mathcal{P}_1$*,* $0 < \mathcal{P}_2$*, and diagonal matrix* $0 < \mathcal{G}_1$*,* $0 < \mathcal{G}_2$*, such that the following LMIs hold:*

$$\begin{bmatrix} -\mathcal{P}_1\mathcal{D}_1 - \mathcal{D}_1\mathcal{P}_1 + \mathcal{L}_{f_2}^*\mathcal{G}_2\mathcal{L}_{f_2} & \mathcal{P}_1\mathcal{A}_1 \\ \star & -\mathcal{G}_1 \end{bmatrix} < 0, \tag{3.1}$$

$$\begin{bmatrix} -\mathcal{P}_2\mathcal{D}_2 - \mathcal{D}_2\mathcal{P}_2 + \mathcal{L}_{f_1}^*\mathcal{G}_1\mathcal{L}_{f_1} & \mathcal{P}_2\mathcal{A}_2 \\ \star & -\mathcal{G}_2 \end{bmatrix} < 0, \tag{3.2}$$

Proof: Define the function $\mathcal{H}(p,q) : \mathbf{H}^{n+m} \to \mathbf{H}^{n+m}$ by

$$\mathcal{H}(p,q) = -\begin{bmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{bmatrix}\begin{bmatrix} f_1(q) \\ f_2(p) \end{bmatrix} + \begin{bmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \end{bmatrix}. \tag{3.3}$$

We start by demonstrating that $\mathcal{H}(p,q)$ is injective. Assume by contradiction that there exist $\begin{bmatrix} p \\ q \end{bmatrix}$, $\begin{bmatrix} p' \\ q' \end{bmatrix} \in \mathbf{H}^{n+m}$, $\begin{bmatrix} p \\ q \end{bmatrix} \neq \begin{bmatrix} p' \\ q' \end{bmatrix}$, such that $\mathcal{H}(p,q) = \mathcal{H}(p',q')$, or equivalently

$$\mathcal{H}(p,q) - \mathcal{H}(p',q') = -\begin{bmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{bmatrix}\begin{bmatrix} p-p' \\ q-q' \end{bmatrix} + \begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{bmatrix}\begin{bmatrix} f_1(q)-f_1(q') \\ f_2(p)-f_2(p') \end{bmatrix} = 0. \tag{3.4}$$

Pre multiplication on both sides of (3.4) with $\begin{bmatrix} p-p' \\ q-q' \end{bmatrix}^* \begin{bmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_2 \end{bmatrix}$, we get

$$\begin{bmatrix} p-p' \\ q-q' \end{bmatrix}^* \begin{bmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_2 \end{bmatrix} \times \left(-\begin{bmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{bmatrix}\begin{bmatrix} p-p' \\ q-q' \end{bmatrix} + \begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{bmatrix}\begin{bmatrix} f_1(q)-f_1(q') \\ f_2(p)-f_2(p') \end{bmatrix} \right) = 0, \tag{3.5}$$

that is

$$\begin{aligned} &-(p-p')^*\mathcal{P}_1\mathcal{D}_1(p-p') - (q-q')^*\mathcal{P}_2\mathcal{D}_2(q-q') + (p-p')^*\mathcal{P}_1\mathcal{A}_1(f_1(q)-f_1(q')) \\ &+ (q-q')^*\mathcal{P}_2\mathcal{A}_2(f_2(p)-f_2(p')) = 0. \end{aligned} \tag{3.6}$$

Applying the complex conjugate, we get

$$\begin{aligned} &-(p-p')^*\mathcal{D}_1\mathcal{P}_1(p-p') - (q-q')^*\mathcal{D}_2\mathcal{P}_2(q-q') + (f_1(q)-f_1(q'))^*\mathcal{A}_1^*\mathcal{P}_1(p-p') \\ &+ (f_2(p)-f_2(p'))^*\mathcal{A}_2^*\mathcal{P}_2(q-q') = 0. \end{aligned} \tag{3.7}$$

By combining (3.6) and (3.7), one has

$$\begin{aligned} &-(p-p')^*(\mathcal{P}_1\mathcal{D}_1 + \mathcal{D}_1\mathcal{P}_1)(p-p') - (q-q')^*(\mathcal{P}_2\mathcal{D}_2 + \mathcal{D}_2\mathcal{P}_2)(q-q') \\ &+ 2(p-p')^*\mathcal{P}_1\mathcal{A}_1(f_1(q)-f_1(q')) + 2(q-q')^*\mathcal{P}_2\mathcal{A}_2(f_2(p)-f_2(p')) = 0. \end{aligned} \tag{3.8}$$

From Lemma (2.2), there exist diagonal matrices $0 < \mathcal{G}_1$, $0 < \mathcal{G}_2$, yields

$$\begin{aligned} 2(p-p')^*\mathcal{P}_1\mathcal{A}_1(f_1(q)-f_1(q')) \le\ & (p-p')^*\mathcal{P}_1\mathcal{A}_1\mathcal{G}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1(p-p') \\ & + (f_1(q)-f_1(q'))^*\mathcal{G}_1(f_1(q)-f_1(q')), \end{aligned} \tag{3.9}$$

$$\begin{aligned} 2(q-q')^*\mathcal{P}_2\mathcal{A}_2(f_2(p)-f_2(p')) \le\ & (q-q')^*\mathcal{P}_2\mathcal{A}_2\mathcal{G}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2(q-q') \\ & + (f_2(p)-f_2(p'))^*\mathcal{G}_2(f_2(p)-f_2(p')). \end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) in (3.8), we get

$$-(p-p')^*(\mathcal{P}_1\mathcal{D}_1 + \mathcal{D}_1\mathcal{P}_1)(p-p') - (q-q')^*(\mathcal{P}_2\mathcal{D}_2 + \mathcal{D}_2\mathcal{P}_2)(q-q')$$

$$+ (p-p')^*\mathcal{P}_1\mathcal{A}_1\mathcal{G}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1(p-p') + (f_1(q)-f_1(q'))^*\mathcal{G}_1(f_1(q)-f_1(q'))$$
$$+ (q-q')^*\mathcal{P}_2\mathcal{A}_2\mathcal{G}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2(q-q') + (f_2(p)-f_2(p'))^*\mathcal{G}_2(f_2(p)-f_2(p')) \leq 0. \quad (3.11)$$

Based on Assumption 1, we get

$$(f_1(q)-f_1(q'))^*\mathcal{G}_1(f_1(q)-f_1(q')) \leq (q-q')^*\mathcal{L}_{f_1}^*\mathcal{G}_1\mathcal{L}_{f_1}(q-q'), \quad (3.12)$$
$$(f_2(p)-f_2(p'))^*\mathcal{G}_2(f_2(p)-f_2(p')) \leq (p-p')^*\mathcal{L}_{f_2}^*\mathcal{G}_2\mathcal{L}_{f_2}(p-p'). \quad (3.13)$$

Incorporating (3.11)–(3.13), we get

$$\begin{aligned}
&(p-p')^*(-\mathcal{P}_1\mathcal{D}_1-\mathcal{D}_1\mathcal{P}_1)(p-p') + (q-q')^*(-\mathcal{P}_2\mathcal{D}_2-\mathcal{D}_2\mathcal{P}_2)(q-q')\\
&\quad + (p-p')^*\mathcal{P}_1\mathcal{A}_1\mathcal{G}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1(p-p') + (q-q')^*\mathcal{L}_{f_1}^*\mathcal{G}_1\mathcal{L}_{f_1}(q-q')\\
&\quad + (q-q')^*\mathcal{P}_2\mathcal{A}_2\mathcal{G}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2(q-q') + (p-p')^*\mathcal{L}_{f_2}^*\mathcal{G}_2\mathcal{L}_{f_2}(p-p') \leq 0,\\
&(p-p')^*(-\mathcal{P}_1\mathcal{D}_1-\mathcal{D}_1\mathcal{P}_1+\mathcal{P}_1\mathcal{A}_1\mathcal{G}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1+\mathcal{L}_{f_2}^*\mathcal{G}_2\mathcal{L}_{f_2})(p-p')\\
&\quad + (q-q')^*(-\mathcal{P}_2\mathcal{D}_2-\mathcal{D}_2\mathcal{P}_2+\mathcal{P}_2\mathcal{A}_2\mathcal{G}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2+\mathcal{L}_{f_1}^*\mathcal{G}_1\mathcal{L}_{f_1})(q-q') \leq 0, \quad (3.14)
\end{aligned}$$

which implies that

$$-\mathcal{P}_1\mathcal{D}_1-\mathcal{D}_1\mathcal{P}_1+\mathcal{P}_1\mathcal{A}_1\mathcal{G}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1+\mathcal{L}_{f_2}^*\mathcal{G}_2\mathcal{L}_{f_2} < 0, \quad (3.15)$$
$$-\mathcal{P}_2\mathcal{D}_2-\mathcal{D}_2\mathcal{P}_2+\mathcal{P}_2\mathcal{A}_2\mathcal{G}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2+\mathcal{L}_{f_1}^*\mathcal{G}_1\mathcal{L}_{f_1} < 0. \quad (3.16)$$

Using Schur's complement can be directly inferred from the criteria which can be immediately deduced from (3.1) and (3.2). This contradicts that $\mathcal{H}(p,q)-\mathcal{H}(p',q') < 0$ with initial conditions. Hence the function $\mathcal{H}(p,q)$ is injective.

Now, we shall show that $\|\mathcal{H}(p,q)\| \to \infty$ as $\|(p,q)\| \to \infty$. We infer that from (3.15), (3.16) and small constant, $\epsilon > 0$ exist, such that

$$-\mathcal{P}_1\mathcal{D}_1-\mathcal{D}_1\mathcal{P}_1+\mathcal{P}_1\mathcal{A}_1\mathcal{G}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1+\mathcal{L}_{f_2}^*\mathcal{G}_2\mathcal{L}_{f_2} < -\epsilon I_n,$$
$$-\mathcal{P}_2\mathcal{D}_2-\mathcal{D}_2\mathcal{P}_2+\mathcal{P}_2\mathcal{A}_2\mathcal{G}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2+\mathcal{L}_{f_1}^*\mathcal{G}_1\mathcal{L}_{f_1} < -\epsilon I_m.$$

Taking $(p',q') = (0,0)$, and using (3.14) and the above relations, we have

$$\begin{aligned}
&\begin{bmatrix} p \\ q \end{bmatrix}^* \begin{bmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_2 \end{bmatrix} (\mathcal{H}(p,q)-\mathcal{H}(0,0))\\
&\leq p^*(-\mathcal{P}_1\mathcal{D}_1-\mathcal{D}_1\mathcal{P}_1+\mathcal{P}_1\mathcal{A}_1\mathcal{G}_1^{-1}\mathcal{A}_1^*\mathcal{P}_1+\mathcal{L}_{f_2}^*\mathcal{G}_2\mathcal{L}_{f_2})p\\
&\quad + q^*(-\mathcal{P}_2\mathcal{D}_2-\mathcal{D}_2\mathcal{P}_2\mathcal{P}_2\mathcal{A}_2\mathcal{G}_2^{-1}\mathcal{A}_2^*\mathcal{P}_2+\mathcal{L}_{f_1}^*\mathcal{G}_1\mathcal{L}_{f_1})q\\
&\leq -\epsilon(\|p\|^2+\|q\|^2). \quad (3.17)
\end{aligned}$$

Using the Cauchy-Schwarz inequality, then (3.17) becomes:

$$\epsilon(\|p\|^2+\|q\|^2) \leq 2\|(p,q)\|\,\|\mathcal{P}_1\|\,\|\mathcal{P}_2\|\,(\|\mathcal{H}(p,q)\|+\|\mathcal{H}(0,0)\|), \quad (3.18)$$

which gives result that $\|\mathcal{H}(p,q)\| \to \infty$ as $\|(p,q)\| \to \infty$. Hence, the map $\mathcal{H}(p,q)$ satisfies all conditions in Lemma (2.3) and is homeomorphism of $\mathbf{H}^{n+m}$ onto itself. Then, there exist (p^*,q^*) such that $\mathcal{H}(p^*,q^*) = 0$, that is, NN (2.2) has a unique equilibrium point (p^*,q^*).

Let $u(t) = p(t) - p^*$, $v(t) = q(t) - q^*$, we can get

$$\begin{cases} \dot{u}(t) = & -\mathcal{D}_1 u(t) + \mathcal{A}_1 g_1(v(t-\ell(t))), \\ u(t) = & \phi_1(t),\ t \in [-\ell, 0], \\ \dot{v}(t) = & -\mathcal{D}_2 v(t) + \mathcal{A}_2 g_2(u(t-\ell(t))), \\ v(t) = & \phi_2(t),\ t \in [-\ell, 0], \end{cases} \tag{3.19}$$

where $g_1(v(t-\ell(t))) = f_1(q(t-\ell(t)) + q^* + J_1) - f_1(q^* + J_1)$ and $g_2(u(t-\ell(t))) = f_2(p(t-\ell(t)) + p^* + J_2) - f_2(p^* + J_2)$, $\phi_1 = \varphi_1 - p^*$, $\phi_2 = \varphi_2 - q^*$, $\phi_1 \in \mathscr{C}([-\ell, 0], \mathbf{H}^n)$ and $\phi_2 \in \mathscr{C}([-\ell, 0], \mathbf{H}^m)$ are continuous functions.

Assumption 3: The activation functions g_{1s} and g_{2r} are Lipschitz continuous; that is, there exist positive constants $l_s^{g_1} > 0, l_r^{g_2} > 0$, such that for all $r = 1, 2, ..., n$, $s = 1, 2, ..., m$

$$\begin{aligned} |g_{1s}(x) - g_{1s}(y)| &\leq l_s^{g_1}|x - y|,\ \forall x, y \in \mathbf{H}, \\ |g_{2r}(x) - g_{2r}(y)| &\leq l_r^{g_2}|x - y|,\ \forall x, y \in \mathbf{H}. \end{aligned}$$

Furthermore, we define $\mathcal{L}_{g_1} = diag\{l_1^{g_1}, l_2^{g_1}, ..., l_m^{g_1}\}$, $\mathcal{L}_{g_2} = diag\{l_1^{g_2}, l_2^{g_2}, ..., l_n^{g_2}\}$.

3.2. Global asymptotic stability of quaternion-valued T-S fuzzy BAM neural networks

Based on the works [32, 33, 36], the T-S fuzzy delayed quaternion-valued NNs can be described as bellow.
Plant Rule h:
If $\{\vartheta_1(t)$ is $\eta_{h1}\}$, $\{\vartheta_2(t)$ is $\eta_{h2}\}$, ..., $\{\vartheta_g(t)$ is $\eta_{hg}\}$.
Then

$$\begin{cases} \dot{u}(t) = & -\mathcal{D}_1^h u(t) + \mathcal{A}_1^h g_1(v(t-\ell(t))), \\ u(t) = & \phi_1(t),\ t \in [-\ell, 0], \\ \dot{v}(t) = & -\mathcal{D}_2^h v(t) + \mathcal{A}_2^h g_2(u(t-\ell(t))), \\ v(t) = & \phi_2(t),\ t \in [-\ell, 0], \end{cases} \tag{3.20}$$

where the premise variables are $\vartheta_c(t),\ c = 1, ..., g$, the fuzzy sets are $\eta_{hc},\ h = 1, ..., m$ and m is the total number of If-Then rules.

Hence, the T-S fuzzy quaternion-valued NN can be achieved by inferring from the fuzzy NN model

(3.20), as follows

$$\begin{cases} \dot{u}(t) = & \dfrac{\sum_{h=1}^{m} w_h(\vartheta(t))\Big\{-\mathcal{D}_1^h u(t) + \mathcal{A}_1^h g_1(v(t-\ell(t)))\Big\}}{\sum_{h=1}^{m} w_h(\vartheta(t))}, \\ u(t) = & \phi_1(t),\ t \in [-\ell, 0], \\ \dot{v}(t) = & \dfrac{\sum_{h=1}^{m} w_h(\vartheta(t))\Big\{-\mathcal{D}_2^h v(t) + \mathcal{A}_2^h g_2(u(t-\ell(t)))\Big\}}{\sum_{h=1}^{m} w_h(\vartheta(t))}, \\ v(t) = & \phi_2(t),\ t \in [-\ell, 0], \end{cases} \tag{3.21}$$

or equivalently

$$\begin{cases} \dot{u}(t) = & \sum_{h=1}^{m} \psi_h(\vartheta(t))\Big\{-\mathcal{D}_1^h u(t) + \mathcal{A}_1^h g_1(v(t-\ell(t)))\Big\}, \\ u(t) = & \phi_1(t),\ t \in [-\ell, 0], \\ \dot{v}(t) = & \sum_{h=1}^{m} \psi_h(\vartheta(t))\Big\{-\mathcal{D}_2^h v(t) + \mathcal{A}_2^h g_2(u(t-\ell(t)))\Big\}, \\ v(t) = & \phi_2(t),\ t \in [-\ell, 0], \end{cases} \tag{3.22}$$

where $\vartheta(t) = (\vartheta_1(t), ..., \vartheta_g(t))^T$, $\psi_h(\vartheta(t)) = \frac{w_h(\vartheta(t))}{\sum_{h=1}^{m} w_h(\vartheta(t))}$ and $w_h(\vartheta(t)) = \prod_{c=1}^{g} \eta_{hc}(\vartheta(t))$. The term $\eta_{hc}(\vartheta(t))$ is the grade membership of $\vartheta_c(t)$ in η_{hc}. It is stated that $w_h(\vartheta(t)) \geq 0,\ h = 1, ..., m$ and $\sum_{h=1}^{m} w_h(\vartheta(t)) > 0$ for all $t \geq 0$. By fuzzy set theory, we have $\psi_h(\vartheta(t)) \geq 0,\ h = 1, ..., m$ and $\sum_{h=1}^{m} \psi_h(\vartheta(t)) = 1$ for all $t \geq 0$.

Theorem 3.2. *Let the activation function and time delay satisfy Assumptions 1–3. Suppose that there exist Hermitian matrices* $0 < \mathcal{P}_1$, $0 < \mathcal{P}_2$, $0 < \mathcal{Q}_1$, $0 < \mathcal{Q}_2$, $0 < \mathcal{R}_1$, $0 < \mathcal{R}_2$, $0 < \mathcal{S}_1$, $0 < \mathcal{S}_2$, *and diagonal matrix* $0 < \mathcal{G}_1$, $0 < \mathcal{G}_2$, *such that the following conditions holds for all* $h = 1, 2, ..., m$

$$\Omega^h = (\Omega_{i,j}^h)_{12\times 12} < 0, \tag{3.23}$$

where $\Omega_{1,1}^h = -\mathcal{P}_1\mathcal{D}_1^h - \mathcal{D}_1^h\mathcal{P}_1 + \mathcal{Q}_1 + \mathcal{R}_1 + \ell^2\mathcal{S}_1$, $\Omega_{1,10}^h = \mathcal{P}_1\mathcal{A}_1^h$, $\Omega_{2,2}^h = -(1-\mu)\mathcal{R}_1 + \mathcal{L}_{g_2}^*\mathcal{G}_1\mathcal{L}_{g_2}$, $\Omega_{3,3}^h = -\mathcal{Q}_1$, $\Omega_{4,4}^h = -\mathcal{G}_1$, $\Omega_{4,7}^h = \mathcal{A}_2^{*h}\mathcal{P}_2$, $\Omega_{5,5}^h = -4\mathcal{S}_1$, $\Omega_{5,6}^h = 6\mathcal{S}_1$, $\Omega_{6,6}^h = -12\mathcal{S}_1$, $\Omega_{7,7}^h = -\mathcal{P}_2\mathcal{D}_2^h - \mathcal{D}_2^h\mathcal{P}_2 + \mathcal{Q}_2 + \mathcal{R}_2 + \ell^2\mathcal{S}_2$, $\Omega_{8,8}^h = -(1-\mu)\mathcal{R}_2 + \mathcal{L}_{g_1}^*\mathcal{G}_2\mathcal{L}_{g_1}$, $\Omega_{9,9}^h = -\mathcal{Q}_2$, $\Omega_{10,10}^h = -\mathcal{G}_2$, $\Omega_{11,11}^h = -4\mathcal{S}_2$, $\Omega_{11,12}^h = 6\mathcal{S}_2$, $\Omega_{12,12}^h = -12\mathcal{S}_2$, *then the equilibrium point of NN model* (3.22) *is GAS.*

Proof: Consider the LKF (3.24) for NNs (3.22) described by

$$\mathcal{V}(t, u(t), v(t), h) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}^* \begin{bmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_2 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + \int_{t-\ell}^{t} \begin{bmatrix} u(s) \\ v(s) \end{bmatrix}^* \begin{bmatrix} \mathcal{Q}_1 & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix} \begin{bmatrix} u(s) \\ v(s) \end{bmatrix} ds$$

$$+\int_{t-\ell(t)}^{t}\begin{bmatrix}u(s)\\v(s)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{R}_1 & 0\\0 & \mathcal{R}_2\end{bmatrix}\begin{bmatrix}u(s)\\v(s)\end{bmatrix}ds$$
$$+\ell\int_{-\ell}^{0}\int_{t+r}^{t}\begin{bmatrix}u(s)\\v(s)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{S}_1 & 0\\0 & \mathcal{S}_2\end{bmatrix}\begin{bmatrix}u(s)\\v(s)\end{bmatrix}dsdr. \tag{3.24}$$

The time-derivative of $\mathcal{V}(t,u(t),v(t),h)$ can be obtained as

$$\begin{aligned}
\dot{\mathcal{V}}(t,u(t),v(t),h) = {} & 2\begin{bmatrix}u(t)\\v(t)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{P}_1 & 0\\0 & \mathcal{P}_2\end{bmatrix}\begin{bmatrix}\dot{u}(t)\\\dot{v}(t)\end{bmatrix}+\begin{bmatrix}u(t)\\v(t)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{Q}_1 & 0\\0 & \mathcal{Q}_2\end{bmatrix}\begin{bmatrix}u(t)\\v(t)\end{bmatrix}\\
& -\begin{bmatrix}u(t-\ell)\\v(t-\ell)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{Q}_1 & 0\\0 & \mathcal{Q}_2\end{bmatrix}\begin{bmatrix}u(t-\ell)\\v(t-\ell)\end{bmatrix}+\begin{bmatrix}u(t)\\v(t)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{R}_1 & 0\\0 & \mathcal{R}_2\end{bmatrix}\begin{bmatrix}u(t)\\v(t)\end{bmatrix}\\
& -(1-\dot{\ell}(t))\begin{bmatrix}u(t-\ell(t))\\v(t-\ell(t))\end{bmatrix}^{*}\begin{bmatrix}\mathcal{R}_1 & 0\\0 & \mathcal{R}_2\end{bmatrix}\begin{bmatrix}u(t-\ell(t))\\v(t-\ell(t))\end{bmatrix}\\
& +\ell^2\begin{bmatrix}u(t)\\v(t)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{S}_1 & 0\\0 & \mathcal{S}_2\end{bmatrix}\begin{bmatrix}u(t)\\v(t)\end{bmatrix}-\ell\int_{t-\ell}^{t}\begin{bmatrix}u(s)\\v(s)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{S}_1 & 0\\0 & \mathcal{S}_2\end{bmatrix}\begin{bmatrix}u(s)\\v(s)\end{bmatrix}ds,\\
\leq {} & 2\begin{bmatrix}u(t)\\v(t)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{P}_1 & 0\\0 & \mathcal{P}_2\end{bmatrix}\begin{bmatrix}\sum_{h=1}^{m}\psi_h(\vartheta(t))\Big\{-\mathcal{D}_1^hu(t)+\mathcal{A}_1^hg_1(v(t-\ell(t)))\Big\}\\\sum_{h=1}^{m}\psi_h(\vartheta(t))\Big\{-\mathcal{D}_2^hv(t)+\mathcal{A}_2^hg_2(u(t-\ell(t)))\Big\}\end{bmatrix}\\
& +\begin{bmatrix}u(t)\\v(t)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{Q}_1 & 0\\0 & \mathcal{Q}_2\end{bmatrix}\begin{bmatrix}u(t)\\v(t)\end{bmatrix}-\begin{bmatrix}u(t-\ell)\\v(t-\ell)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{Q}_1 & 0\\0 & \mathcal{Q}_2\end{bmatrix}\begin{bmatrix}u(t-\ell)\\v(t-\ell)\end{bmatrix}\\
& +\begin{bmatrix}u(t)\\v(t)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{R}_1 & 0\\0 & \mathcal{R}_2\end{bmatrix}\begin{bmatrix}u(t)\\v(t)\end{bmatrix}-(1-\mu)\begin{bmatrix}u(t-\ell(t))\\v(t-\ell(t))\end{bmatrix}^{*}\begin{bmatrix}\mathcal{R}_1 & 0\\0 & \mathcal{R}_2\end{bmatrix}\\
& \times\begin{bmatrix}u(t-\ell(t))\\v(t-\ell(t))\end{bmatrix}+\ell^2\begin{bmatrix}u(t)\\v(t)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{S}_1 & 0\\0 & \mathcal{S}_2\end{bmatrix}\begin{bmatrix}u(t)\\v(t)\end{bmatrix}\\
& -\ell\int_{t-\ell}^{t}\begin{bmatrix}u(s)\\v(s)\end{bmatrix}^{*}\begin{bmatrix}\mathcal{S}_1 & 0\\0 & \mathcal{S}_2\end{bmatrix}\begin{bmatrix}u(s)\\v(s)\end{bmatrix}ds,\\
= {} & \sum_{h=1}^{m}\psi_h(\vartheta(t))\Big\{u^*(t)(-\mathcal{P}_1\mathcal{D}_1^h-\mathcal{D}_1^h\mathcal{P}_1+\mathcal{Q}_1+\mathcal{R}_1+\ell^2\mathcal{S}_1)u(t)\\
& +u^*(t)(\mathcal{P}_1\mathcal{A}_1^h)g_1(v(t-\ell(t)))+v^*(t)(-\mathcal{P}_2\mathcal{D}_2^h-\mathcal{D}_2^h\mathcal{P}_2+\mathcal{Q}_2+\mathcal{R}_2\\
& +\ell^2\mathcal{S}_2)v(t)+v^*(t)(\mathcal{P}_2\mathcal{A}_2^h)g_2(u(t-\ell(t))))-u^*(t-\ell)(\mathcal{Q}_1)\\
& \times u(t-\ell)-v^*(t-\ell)(\mathcal{Q}_2)v(t-\ell)-u^*(t-\ell(t))((1-\mu)\mathcal{R}_1)\\
& \times u(t-\ell(t))-v^*(t-\ell(t))((1-\mu)\mathcal{R}_2)v(t-\ell(t))\\
& -\ell\int_{t-\ell}^{t}u^*(s)\mathcal{S}_1u(s)ds-\ell\int_{t-\ell}^{t}v^*(s)\mathcal{S}_2v(s)ds\Big\}.
\end{aligned} \tag{3.25}$$

To obtain tighter bounds for the integral terms $-\ell\int_{t-\ell}^{t}u^*(s)\mathcal{S}_1u(s)ds$, $-\ell\int_{t-\ell}^{t}v^*(s)\mathcal{S}_2v(s)ds$ in (3.25), the proposed Lemma (2.8) were applied as follow:

$$
\begin{aligned}
-\ell\int_{t-\ell}^{t}u^*(s)\mathcal{S}_1u(s)ds \leq & -\begin{bmatrix}\int_{t-\ell}^{t}u(s)ds\\ \int_{t-\ell}^{t}u(s)ds-\frac{2}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}u(s)dsdr\end{bmatrix}^*\\
&\times\begin{bmatrix}\mathcal{S}_1 & 0\\ 0 & 3\mathcal{S}_1\end{bmatrix}\begin{bmatrix}\int_{t-\ell}^{t}u(s)ds\\ \int_{t-\ell}^{t}u(s)ds-\frac{2}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}u(s)dsdr\end{bmatrix}\\
= & -\Big[\int_{t-\ell}^{t}u(s)ds\Big]^*\mathcal{S}_1\Big[\int_{t-\ell}^{t}u(s)ds\Big]-\Big[\int_{t-\ell}^{t}u(s)ds\Big]^*3\mathcal{S}_1\\
&\times\Big[\int_{t-\ell}^{t}u(s)ds\Big]+\Big[\int_{t-\ell}^{t}u(s)ds\Big]^*6\mathcal{S}_1\Big[\frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}u(s)dsdr\Big]\\
&-\Big[\frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}u(s)dsdr\Big]^*12\mathcal{S}_1\Big[\frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}u(s)dsdr\Big],
\end{aligned} \tag{3.26}
$$

$$
\begin{aligned}
-\ell\int_{t-\ell}^{t}v^*(s)\mathcal{S}_2v(s)ds \leq & -\begin{bmatrix}\int_{t-\ell}^{t}v(s)ds\\ \int_{t-\ell}^{t}v(s)ds-\frac{2}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}v(s)dsdr\end{bmatrix}^*\\
&\times\begin{bmatrix}\mathcal{S}_2 & 0\\ 0 & 3\mathcal{S}_2\end{bmatrix}\begin{bmatrix}\int_{t-\ell}^{t}v(s)ds\\ \int_{t-\ell}^{t}v(s)ds-\frac{2}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}v(s)dsdr\end{bmatrix}\\
= & -\Big[\int_{t-\ell}^{t}v(s)ds\Big]^*\mathcal{S}_2\Big[\int_{t-\ell}^{t}v(s)ds\Big]-\Big[\int_{t-\ell}^{t}v(s)ds\Big]^*3\mathcal{S}_2\\
&\times\Big[\int_{t-\ell}^{t}v(s)ds\Big]+\Big[\int_{t-\ell}^{t}v(s)ds\Big]^*6\mathcal{S}_2\Big[\frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}v(s)dsdr\Big]\\
&-\Big[\frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}v(s)dsdr\Big]^*12\mathcal{S}_2\Big[\frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}v(s)dsdr\Big].
\end{aligned} \tag{3.27}
$$

Combining (3.25)–(3.27), we get

$$
\begin{aligned}
&\dot{\mathcal{V}}(t,u(t),v(t),h) \leq \sum_{h=1}^{m}\psi_h(\vartheta(t))\Big\{u^*(t)(-\mathcal{P}_1\mathcal{D}_1^h-\mathcal{D}_1^h\mathcal{P}_1+\mathcal{Q}_1+\mathcal{R}_1+\ell^2\mathcal{S}_1)u(t)+u^*(t)\\
&\times(\mathcal{P}_1\mathcal{A}_1^h)g_1(v(t-\ell(t)))+v^*(t)(-\mathcal{P}_2\mathcal{D}_2^h-\mathcal{D}_2^h\mathcal{P}_2+\mathcal{Q}_2+\mathcal{R}_2+\ell^2\mathcal{S}_2)v(t)+v^*(t)(\mathcal{P}_2\mathcal{A}_2^h)\\
&\times g_2(u(t-\ell(t)))-u^*(t-\ell)(\mathcal{Q}_1)u(t-\ell)-v^*(t-\ell)(\mathcal{Q}_2)v(t-\ell)-u^*(t-\ell(t))\\
&\times((1-\mu)\mathcal{R}_1)u(t-\ell(t))-v^*(t-\ell(t))((1-\mu)\mathcal{R}_2)v(t-\ell(t))\\
&-\Big[\int_{t-\ell}^{t}u(s)ds\Big]^*(4\mathcal{S}_1)\Big[\int_{t-\ell}^{t}u(s)ds\Big]+\Big[\int_{t-\ell}^{t}u(s)ds\Big]^*(6\mathcal{S}_1)\Big[\frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}u(s)dsdr\Big]\\
&-\Big[\frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}u(s)dsdr\Big]^*(12\mathcal{S}_1)\Big[\frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t}u(s)dsdr\Big]
\end{aligned}
$$

$$-\Big[\int_{t-\ell}^{t} v(s)ds\Big]^{*}(4\mathcal{S}_2)\Big[\int_{t-\ell}^{t} v(s)ds\Big]+\Big[\int_{t-\ell}^{t} v(s)ds\Big]^{*}(6\mathcal{S}_2)\Big[\frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t} v(s)dsdr\Big]$$
$$-\Big[\frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t} v(s)dsdr\Big]^{*}(12\mathcal{S}_2)\Big[\frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t} v(s)dsdr\Big]\Big\}. \tag{3.28}$$

From Assumption 3, there exist two diagonal matrices $0 < \mathcal{G}_1$, $0 < \mathcal{G}_2$ such that

$$0 \le u^*(t-\ell(t))\mathcal{L}_{g_2}^*\mathcal{G}_1\mathcal{L}_{g_2}u(t-\ell(t)) - g_2^*(u(t-\ell(t)))\mathcal{G}_1 g_2(u(t-\ell(t))), \tag{3.29}$$
$$0 \le v^*(t-\ell(t))\mathcal{L}_{g_1}^*\mathcal{G}_2\mathcal{L}_{g_1}v(t-\ell(t)) - g_1^*(v(t-\ell(t)))\mathcal{G}_2 g_1(v(t-\ell(t))). \tag{3.30}$$

By combining (3.28)–(3.30), we obtain

$$\dot{\mathcal{V}}(t,u(t),v(t),h) \le \sum_{h=1}^{m}\psi_h(\vartheta(t))\Big\{\xi^*(t)\Omega^h\xi(t)\Big\}. \tag{3.31}$$

where

$$\xi(t) = \Big[u^*(t)\;\; u^*(t-\ell(t))\;\; u^*(t-\ell)\;\; g_2^*(u(t-\ell(t)))\;\; \int_{t-\ell}^{t} u^*(s)ds\;\; \frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t} u^*(s)dsdr$$
$$v^*(t)\;\; v^*(t-\ell(t))\;\; v^*(t-\ell)\;\; g_1^*(v(t-\ell(t)))\;\; \int_{t-\ell}^{t} v^*(s)ds\;\; \frac{1}{\ell}\int_{-\ell}^{0}\int_{t+r}^{t} v^*(s)dsdr\Big]^*.$$

It is obvious that for $\Omega^h < 0, h = 1,2,...,m$, it shows that the NN (3.22) is GAS according to the Lyapunov stability theory. This completes the proof.

The following shows how our results can be specialized to different cases.

Remark 3.3. *When $\ell(t) = \ell$, the NN model* (3.22) *becomes:*

$$\begin{cases} \dot{u}(t) = & \sum_{h=1}^{m}\psi_h(\vartheta(t))\Big\{-\mathcal{D}_1^h u(t) + \mathcal{A}_1^h g_1(v(t-\ell))\Big\}, \\ u(t) = & \phi_1(t),\; t\in[-\ell,0], \\ \dot{v}(t) = & \sum_{h=1}^{m}\psi_h(\vartheta(t))\Big\{-\mathcal{D}_2^h v(t) + \mathcal{A}_2^h g_2(u(t-\ell))\Big\}, \\ v(t) = & \phi_2(t),\; t\in[-\ell,0], \end{cases} \tag{3.32}$$

Corollary 3.4. *Let the activation function satisfy Assumptions 3 and time delay $\ell > 0$. Suppose that there exist Hermitian matrices $0 < \mathcal{P}_1$, $0 < \mathcal{P}_2$, $0 < \mathcal{Q}_1$, $0 < \mathcal{Q}_2$, $0 < \mathcal{S}_1$, $0 < \mathcal{S}_2$, and diagonal matrix $0 < \mathcal{G}_1$, $0 < \mathcal{G}_2$, such that the following conditions holds for all $h = 1,2,...,m$*

$$\bar{\Omega}^h = (\bar{\Omega}_{i,j}^h)_{10\times 10} < 0, \tag{3.33}$$

where $\bar{\Omega}_{1,1}^h = -\mathcal{P}_1\mathcal{D}_1^h - \mathcal{D}_1^h\mathcal{P}_1 + \mathcal{Q}_1 + \ell^2\mathcal{S}_1$, $\bar{\Omega}_{1,8}^h = \mathcal{P}_1\mathcal{A}_1^h$, $\bar{\Omega}_{2,2}^h = -\mathcal{Q}_1 + \mathcal{L}_{g_2}^*\mathcal{G}_1\mathcal{L}_{g_2}$, $\bar{\Omega}_{3,3}^h = -\mathcal{G}_1$, $\bar{\Omega}_{3,6}^h = \mathcal{A}_2^{*h}\mathcal{P}_2$, $\bar{\Omega}_{4,4}^h = -4\mathcal{S}_1$, $\bar{\Omega}_{4,5}^h = 6\mathcal{S}_1$, $\bar{\Omega}_{5,5}^h = -12\mathcal{S}_1$, $\bar{\Omega}_{6,6}^h = -\mathcal{P}_2\mathcal{D}_2^h - \mathcal{D}_2^h\mathcal{P}_2 + \mathcal{Q}_2 + \ell^2\mathcal{S}_2$, $\bar{\Omega}_{7,7}^h = -\mathcal{Q}_2 + \mathcal{L}_{g_1}^*\mathcal{G}_2\mathcal{L}_{g_1}$, $\bar{\Omega}_{8,8}^h = -\mathcal{G}_2$, $\bar{\Omega}_{9,9}^h = -4\mathcal{S}_2$, $\bar{\Omega}_{9,10}^h = 6\mathcal{S}_2$, $\bar{\Omega}_{10,10}^h = -12\mathcal{S}_2$, then the equilibrium point of NN model (3.32) *is GAS.*

Proof: Replacing $\mathcal{R}_1 = \mathcal{R}_2 = 0$ in LKF (3.24). The remaining proof is similar to that in Theorem (3.2), and so it is omitted.

Remark 3.5. *When* $m = 1$ *it is a special form of NN model* (3.22). *For simplicity, we deleted the superscript* "1".

$$\begin{cases} \dot{u}(t) = & -\mathcal{D}_1 u(t) + \mathcal{A}_1 g_1(v(t - \ell(t))), \\ u(t) = & \phi_1(t), \ t \in [-\ell, 0], \\ \dot{v}(t) = & -\mathcal{D}_2 v(t) + \mathcal{A}_2 g_2(u(t - \ell(t))), \\ v(t) = & \phi_2(t), \ t \in [-\ell, 0], \end{cases} \tag{3.34}$$

Corollary 3.6. *Let the activation function and time delay satisfy Assumptions 1-3. Suppose that there exist Hermitian matrices* $0 < \mathcal{P}_1$, $0 < \mathcal{P}_2$, $0 < \mathcal{Q}_1$, $0 < \mathcal{Q}_2$, $0 < \mathcal{R}_1$, $0 < \mathcal{R}_2$, $0 < \mathcal{S}_1$, $0 < \mathcal{S}_2$, *and diagonal matrix* $0 < \mathcal{G}_1$, $0 < \mathcal{G}_2$, *such that the following conditions holds*

$$\tilde{\Omega} = (\tilde{\Omega}_{i,j})_{12\times 12} < 0, \tag{3.35}$$

where $\tilde{\Omega}_{1,1} = -\mathcal{P}_1\mathcal{D}_1 - \mathcal{D}_1\mathcal{P}_1 + \mathcal{Q}_1 + \mathcal{R}_1 + \ell^2\mathcal{S}_1$, $\tilde{\Omega}_{1,10} = \mathcal{P}_1\mathcal{A}_1$, $\tilde{\Omega}_{2,2} = -(1-\mu)\mathcal{R}_1 + \mathcal{L}_{g_2}^*\mathcal{G}_1\mathcal{L}_{g_2}$, $\tilde{\Omega}_{3,3} = -\mathcal{Q}_1$, $\tilde{\Omega}_{4,4} = -\mathcal{G}_1$, $\tilde{\Omega}_{4,7} = \mathcal{A}_2^*\mathcal{P}_2$, $\tilde{\Omega}_{5,5} = -4\mathcal{S}_1$, $\tilde{\Omega}_{5,6} = 6\mathcal{S}_1$, $\tilde{\Omega}_{6,6} = -12\mathcal{S}_1$, $\tilde{\Omega}_{7,7} = -\mathcal{P}_2\mathcal{D}_2 - \mathcal{D}_2\mathcal{P}_2 + \mathcal{Q}_2 + \mathcal{R}_2 + \ell^2\mathcal{S}_2$, $\tilde{\Omega}_{8,8} = -(1-\mu)\mathcal{R}_2 + \mathcal{L}_{g_1}^*\mathcal{G}_2\mathcal{L}_{g_1}$, $\tilde{\Omega}_{9,9} = -\mathcal{Q}_2$, $\tilde{\Omega}_{10,10} = -\mathcal{G}_2$, $\tilde{\Omega}_{11,11} = -4\mathcal{S}_2$, $\tilde{\Omega}_{11,12} = 6\mathcal{S}_2$, $\tilde{\Omega}_{12,12} = -12\mathcal{S}_2$, *then the equilibrium point of NN model* (3.34) *is GAS.*

Proof: Consider the same LKF (3.24). The remaining proof is similar to that in Theorem (3.2), and so it is omitted.

Remark 3.7. *Recently, several methods have been proposed to investigate the stability of delayed quaternion-valued NNs [28]- [31]. As an example, (i) The real-valued decomposition method [28]; (ii) The complex-valued decomposition method [29]; (iii) The direct quaternion method [30]. In general, real-valued and complex-valued decomposition methods have two problems; i.e. they increase the size of the systems, which makes mathematical challenges. In addition, due to the non-commutative nature of quaternion multiplication, some data regarding quaternions may be loss in decomposition. As a result, rather than employing the real or complex decomposition methods, this paper explored the GAS criteria for quaternion-valued T-S fuzzy BAM NNs directly.*

Remark 3.8. *The authors of [41] used WBII to investigate the Mittag-Leffler synchronization criteria of complex-valued memristive NNs by real-valued decomposition method. The authors of [42] used complex-valued WBII to analyze the global* μ*-stability criteria of complex-valued BAM NNs by direct method. In comparison to [41,42], we extended the WBII into the quaternion domain and its proof has been presented for the first time. Furthermore, the GAS criteria for quaternion-valued T-S fuzzy BAM NNs are established by using new quaternion-valued WBII and direct quaternion method.*

4. Numerical evaluations

This section includes two numerical evaluations to emphasize the applicability of this analysis.
Example 1: Determine the two neuron quaternion-valued BAM NNs as given below.

$$\begin{cases} \dot{p}_r(t) = & -d_{1r}p_r(t) + \sum_{s=1}^{2} a_{1rs}f_{1s}(q_s(t-\ell(t))) + J_{1r},\ r = 1,2, \\ p_r(t) = & \varphi_{1r}(t),\ t \in [-\ell, 0], \\ \dot{q}_s(t) = & -d_{2s}q_s(t) + \sum_{r=1}^{2} a_{2sr}f_{2r}(p_r(t-\ell(t))) + J_{2s},\ s = 1,2, \\ q_s(t) = & \varphi_{2s}(t),\ t \in [-\ell, 0], \end{cases} \tag{4.1}$$

where $d_{1_1} = d_{1_2} = 3$, $d_{2_1} = d_{2_2} = 2$, $a_{1_{11}} = 1.2 + i - 1.5j - 0.8k$, $a_{1_{12}} = 1 + 1.2i + 1.3j - 1.5k$, $a_{1_{21}} = 1.4 - 2.7i - 2j - 1.3k$, $a_{1_{22}} = 0.5 + 0.8i - 1.4j - 1.7k$, $a_{2_{11}} = 1.2 + i - 1.5j - 0.8k$, $a_{2_{12}} = 1+1.2i+1.3j-1.5k$, $a_{2_{21}} = 1.4-2.7i-2j-1.3k$, $a_{2_{22}} = 0.5+0.8i-1.4j-1.7k$, $J_{1_1} = 0.2+0.1i-0.2j-0.1k$, $J_{1_2} = 0.1 + 0.2i - 0.1j - 0.1k$, $J_{2_1} = 0.2 + 0.2i - 0.1j - 0.2k$, $J_{2_2} = 0.3 + 0.1i - 0.1j - 0.2k$.

The activation functions are regarded as $f_{1s}(q_s(\cdot)) = 0.5\ tanh(q_s(\cdot)) + 0.5\ tanh(q_s(\cdot))i + 0.5\ tanh(q_s(\cdot))j + 0.5\ tanh(q_s(\cdot))k$, $s = 1,2$, $f_{2r}(p_r(\cdot)) = 0.5\ tanh(p_r(\cdot)) + 0.5\ tanh(p_r(\cdot))i + 0.5\ tanh(p_r(\cdot))j + 0.5\ tanh(p_r(\cdot))k$, $r = 1,2$. Then Assumption 1 holds with $l_1^{f_1} = l_2^{f_1} = l_1^{f_2} = l_2^{f_2} = 0.25$. The delay $\ell(t)$ is regarded as $\ell(t) = 0.4 + 0.3\ sin(t)$ implying that the maximum permissible upper bound is $\ell = 0.7$. It is observable that $0 \le \dot{\ell}(t) \le \mu = 0 \le 0.3\ cos(t) \le 0.3$. By employing MATLAB YALMIP toolbox, the LMI conditions (3.1) and (3.2) in Theorem (3.1) are verified, and the feasibility are

$$\mathcal{P}_1 = \begin{bmatrix} 0.6718 & -0.0034 - 0.0270i + 0.0052j - 0.0277k \\ -0.0034 + 0.0270i - 0.0052 + 0.0277k & 0.6384 \end{bmatrix},$$
$$\mathcal{P}_2 = \begin{bmatrix} 0.4972 & 0.0066 + 0.0196i + 0.0040j + 0.0226k \\ 0.0066 - 0.0196i - 0.0040j - 0.0226k & 0.5029 \end{bmatrix},$$
$$\mathcal{G}_1 = \begin{bmatrix} 8.6913 & 0 \\ 0 & 8.6976 \end{bmatrix},\ \mathcal{G}_2 = \begin{bmatrix} 8.7196 & 0 \\ 0 & 8.6904 \end{bmatrix}.$$

Based on example 1, we conclude that all the conditions associated with Theorem (3.1) are fulfilled; this means that the NN model (2.2) has a unique equilibrium point.
Example 2: Determine the two neuron quaternion-valued T-S fuzzy BAM NNs with $h = 1,2$ as given below:

$$\begin{cases} \dot{u}_r(t) = & \sum_{h=1}^{2} \psi_h(\vartheta(t)) \Big\{ -d_{1r}^{\ h}u_r(t) + \sum_{s=1}^{2} a_{1rs}^{\ h} g_{1s}(v_s(t-\ell(t))) \Big\}, \\ u_r(t) = & \phi_{1r}(t),\ t \in [-\ell, 0],\ r,s = 1,2, \\ \dot{v}_s(t) = & \sum_{h=1}^{2} \psi_h(\vartheta(t)) \Big\{ -d_{2s}^{\ h}v_s(t) + \sum_{r=1}^{2} a_{2sr}^{\ h} g_{2r}(u_r(t-\ell(t))) \Big\}, \\ v_s(t) = & \phi_{2s}(t),\ t \in [-\ell, 0],\ r,s = 1,2. \end{cases} \tag{4.2}$$

Plant Rule 1: IF $\{\vartheta_1(t)$ is $\eta_{11}\}$, THEN

$$\begin{cases} \dot{u}_r(t) = & -d_{1r}^{1}u_r(t) + \sum_{s=1}^{2} a_{1rs}^{1} g_{1s}(v_s(t-\ell(t))), \\ u_r(t) = & \phi_{1r}(t),\ t \in [-\ell, 0],\ r, s = 1, 2, \\ \dot{v}_s(t) = & -d_{2s}^{1}v_s(t) + \sum_{r=1}^{2} a_{2sr}^{1} g_{2r}(u_r(t-\ell(t))), \\ v_s(t) = & \phi_{2s}(t),\ t \in [-\ell, 0],\ r, s = 1, 2. \end{cases}$$

Plant Rule 2: IF $\{\vartheta_2(t)$ is $\eta_{22}\}$, THEN

$$\begin{cases} \dot{u}_r(t) = & -d_{1r}^{2}u_r(t) + \sum_{s=1}^{2} a_{1rs}^{2} g_{1s}(v_s(t-\ell(t))), \\ u_r(t) = & \phi_{1r}(t),\ t \in [-\ell, 0],\ r, s = 1, 2, \\ \dot{v}_s(t) = & -d_{2s}^{2}v_s(t) + \sum_{r=1}^{2} a_{2sr}^{2} g_{2r}(u_r(t-\ell(t))), \\ v_s(t) = & \phi_{2s}(t),\ t \in [-\ell, 0],\ r, s = 1, 2, \end{cases}$$

where $d_{11}^{1} = d_{12}^{1} = 7$, $d_{21}^{1} = d_{22}^{1} = 9$, $d_{11}^{2} = d_{12}^{2} = 8$, $d_{21}^{2} = d_{22}^{2} = 9$, $a_{111}^{1} = 1.2 + i - 1.5j - 0.8k$, $a_{112}^{1} = 1 + 1.2i + 1.3j - 1.5k$, $a_{121}^{1} = 1.4 - 2.7i - 2j - 1.3k$, $a_{122}^{1} = 0.5 + 0.8i - 1.4j - 1.7k$, $a_{211}^{1} = 1.1 - 1.4i - 1.3j - 1.2k$, $a_{212}^{1} = 2.1 + 1.3i - 0.9j - 1.1k$, $a_{221}^{1} = 1.3 + 1.2i - 1.2j + 1.1k$, $a_{222}^{1} = -1.5 + i + 1.2j + 1.4k$, $a_{111}^{2} = 1.2 + i - 1.5j - 0.8k$, $a_{112}^{2} = 1 + 1.2i + 1.3j - 1.5k$, $a_{121}^{2} = 1.4 - 2.7i - 2j - 1.3k$, $a_{122}^{2} = 0.5 + 0.8i - 1.4j - 1.7k$, $a_{211}^{2} = 1.1 - 1.4i - 1.3j - 1.2k$, $a_{212}^{2} = 2.1 + 1.3i - 0.9j - 1.1k$, $a_{221}^{2} = 1.3 + 1.2i - 1.2j + 1.1k$, $a_{222}^{2} = -1.5 + i + 1.2j + 1.4k$.

The activation functions are chosen as $g_{1s}(v_s(\cdot)) = 0.5\,tanh(v_s(\cdot)) + 0.5\,tanh(v_s(\cdot))i + 0.5\,tanh(v_s(\cdot))j + 0.5\,tanh(v_s(\cdot))k$, $s = 1, 2$, $g_{2r}(u_r(\cdot)) = 0.5\,tanh(u_r(\cdot)) + 0.5\,tanh(u_r(\cdot))i + 0.5\,tanh(u_r(\cdot))j + 0.5\,tanh(u_r(\cdot))k$, $s, r = 1, 2$. Then Assumption 3 hold with $l_s^{g_1} = l_r^{g_2} = 0.25$, $s, r = 1, 2$. The delay $\ell(t)$ is regarded as $\ell(t) = 0.4 + 0.3\,sin(t)$, implying that the maximum permissible upper bound is $\ell = 0.7$. It is observable that $0 \le \dot{\ell}(t) \le \mu = 0 \le 0.3\,cos(t) \le 0.3$. Furthermore, the membership functions are considered as $\psi_1(\vartheta(t)) = \frac{1}{1+e^{-2t}}$, $\psi_2(\vartheta(t)) = 1 - \frac{1}{1+e^{-2t}}$. The LMI condition (3.23) in Theorem (3.2) are verified by applying MATLAB YALMIP toolbox, and the feasibility are

$$\mathcal{P}_1 = \begin{bmatrix} 0.6965 & -0.0028 - 0.0287i - 0.0118j - 0.0094k \\ -0.0028 + 0.0287i + 0.0118j + 0.0094k & 0.6708 \end{bmatrix},$$

$$\mathcal{P}_2 = \begin{bmatrix} 0.5718 & -0.0104 + 0.0184i - 0.0024j + 0.0245k \\ -0.0104 - 0.0184i + 0.0024j - 0.0245k & 0.5231 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 3.2612 & -0.0105 - 0.1072i - 0.0439j - 0.0351k \\ -0.0105 + 0.1072i + 0.0439j + 0.0351k & 3.1652 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 3.2225 & -0.0466 + 0.0824i - 0.0108j + 0.1098k \\ -0.0466 - 0.0824i + 0.0108j - 0.1098k & 3.0044 \end{bmatrix},$$

$$\mathcal{R}_1 = \begin{bmatrix} 4.0153 & -0.0160 - 0.1624i - 0.0666j - 0.0531k \\ -0.0160 + 0.1624i + 0.0666j + 0.0531k & 3.8695 \end{bmatrix},$$

$$\mathcal{R}_2 = \begin{bmatrix} 3.9566 & -0.0706 + 0.1248i - 0.0164j + 0.1664k \\ -0.0706 - 0.1248i + 0.0164j - 0.1664k & 3.6254 \end{bmatrix},$$
$$\mathcal{S}_1 = \begin{bmatrix} 0.2526 & -0.0000 - 0.0002i - 0.7085j - 0.5655k \\ -0.0000 + 0.0002i + 0.7085j + 0.5655k & 0.2525 \end{bmatrix},$$
$$\mathcal{S}_2 = \begin{bmatrix} 0.2526 & -0.0001 + 0.0001i - 0.0174j + 0.1772k \\ -0.0001 - 0.0001i + 0.0174j - 0.1772k & 0.2522 \end{bmatrix},$$
$$\mathcal{G}_1 = \begin{bmatrix} 3.5058 & 0 \\ 0 & 3.4995 \end{bmatrix}, \ \mathcal{G}_2 = \begin{bmatrix} 3.5033 & 0 \\ 0 & 3.4888 \end{bmatrix}.$$

Under randomly selected 10 initial values of $\phi_{11}(t)$, $\phi_{12}(t)$, $\phi_{21}(t)$ and $\phi_{22}(t)$, the time responses of states of the NNs (4.2) $u_1^R(t)$, $u_1^I(t)$, $u_1^J(t)$, $u_1^K(t)$, $u_2^R(t)$, $u_2^I(t)$, $u_2^J(t)$, $u_2^K(t)$, $v_1^R(t)$, $v_1^I(t)$, $v_1^J(t)$, $v_1^K(t)$, $v_2^R(t)$, $v_2^I(t)$, $v_2^J(t)$, $v_2^K(t)$ are illustrated in Figures (1)–(8). According to example 2, we can see that the equilibrium point of the NN model (3.22) is GAS since all the conditions in Theorem (3.2) have been fulfilled, which is in accordance with Theorem (3.2).

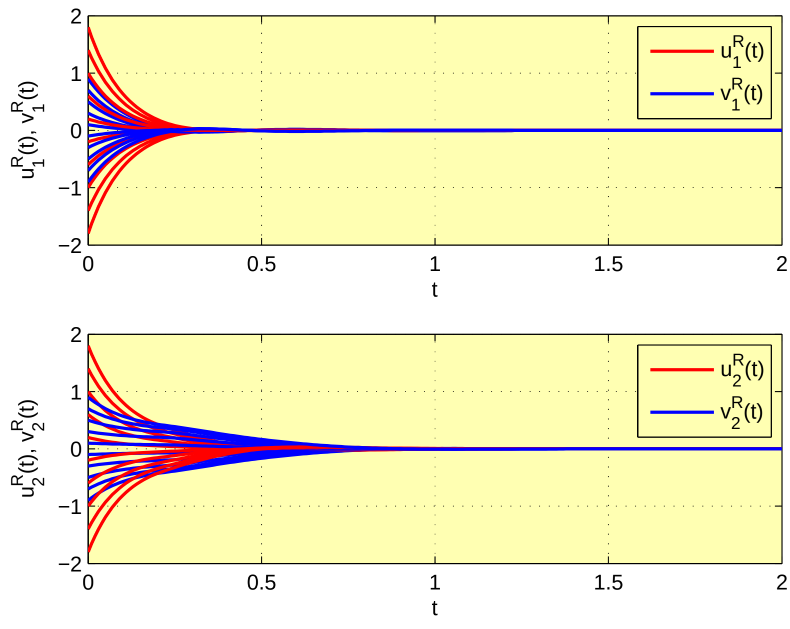

Figure 1. Time representation of the states $u(t)^R$, $v(t)^R$ of NNs (4.2).

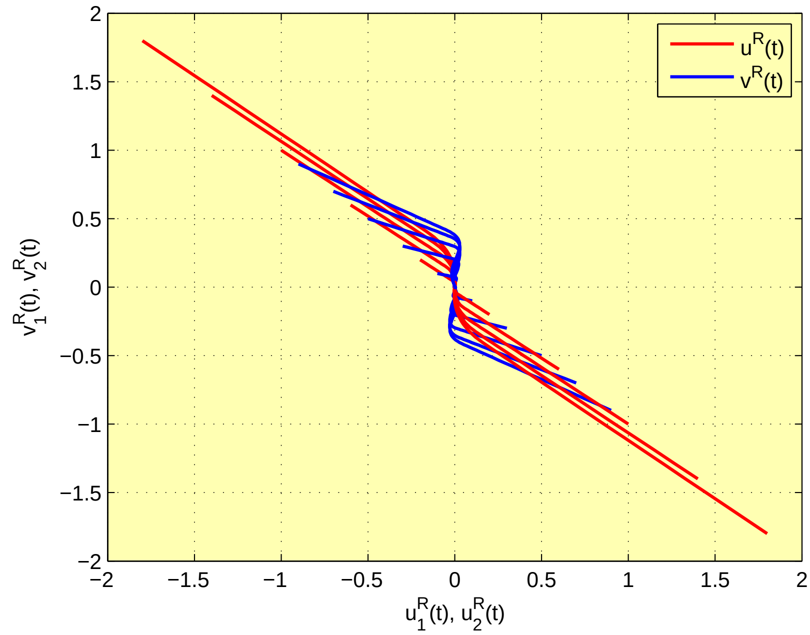

Figure 2. Phase representation of NNs (4.2) between the real subspace $u(t)^R, v(t)^R$.

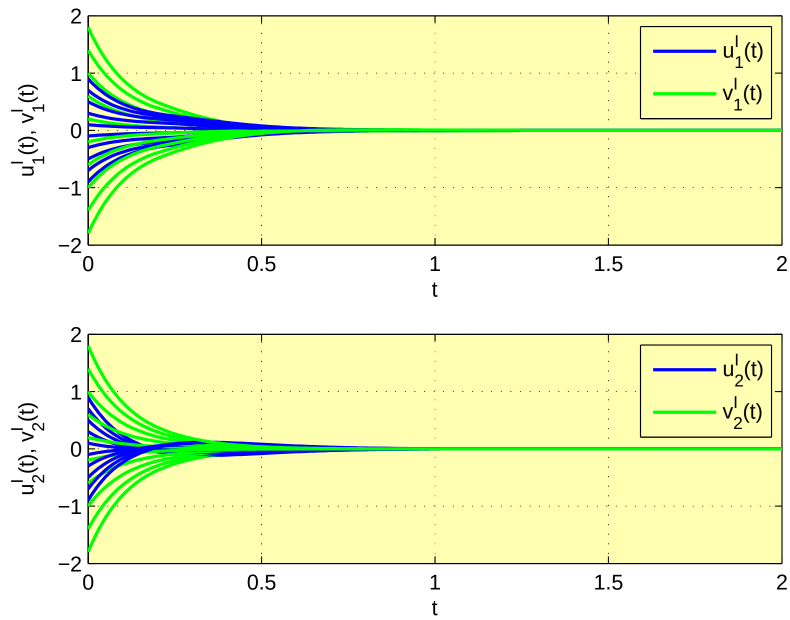

Figure 3. Time representation of the states $u(t)^I, v(t)^I$ of NNs (4.2).

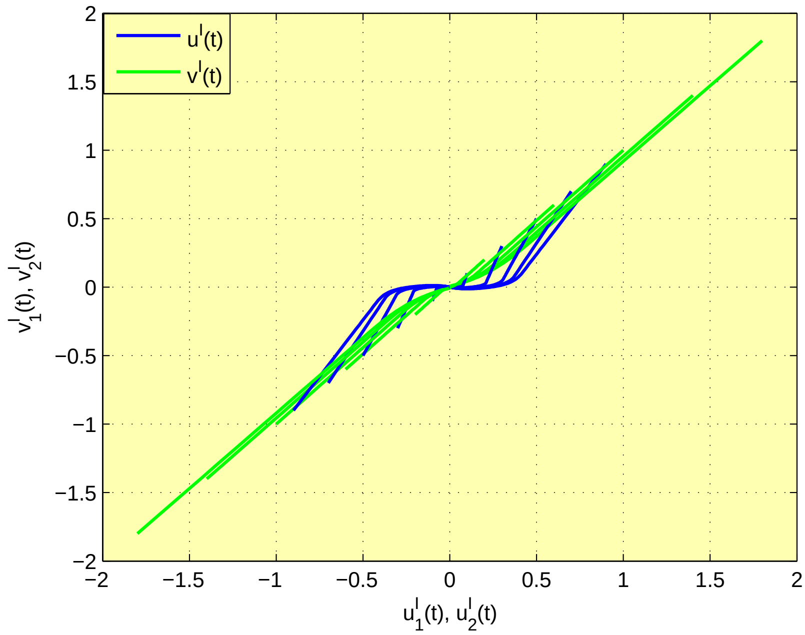

Figure 4. Phase representation of NNs (4.2) between the imaginary subspace $u(t)^I, v(t)^I$.

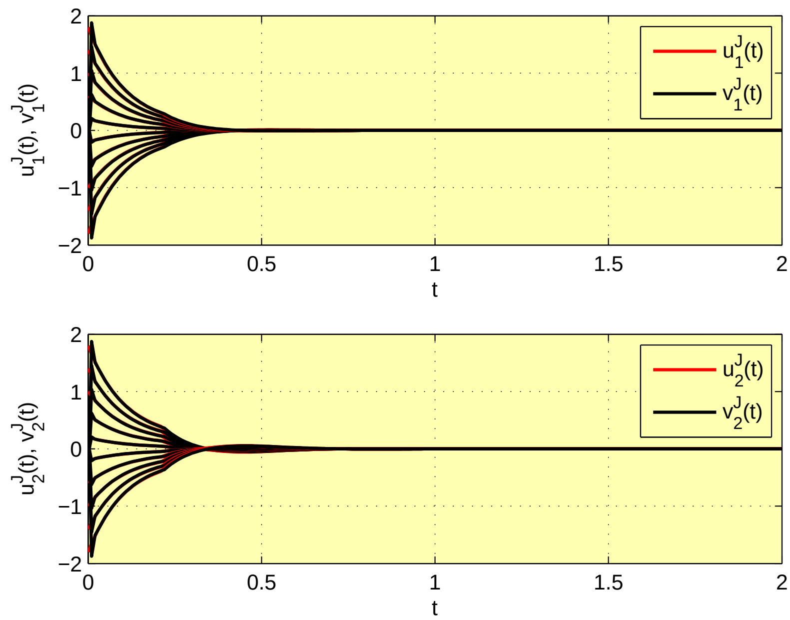

Figure 5. Time representation of the states $u(t)^J, v(t)^J$ of NNs (4.2).

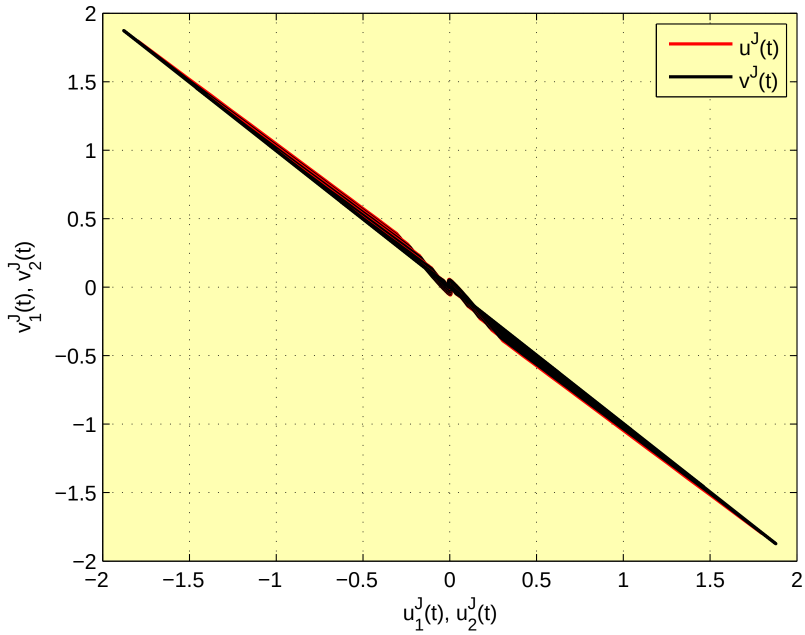

Figure 6. Phase representation of NNs (4.2) between the imaginary subspace $u(t)^J, v(t)^J$.

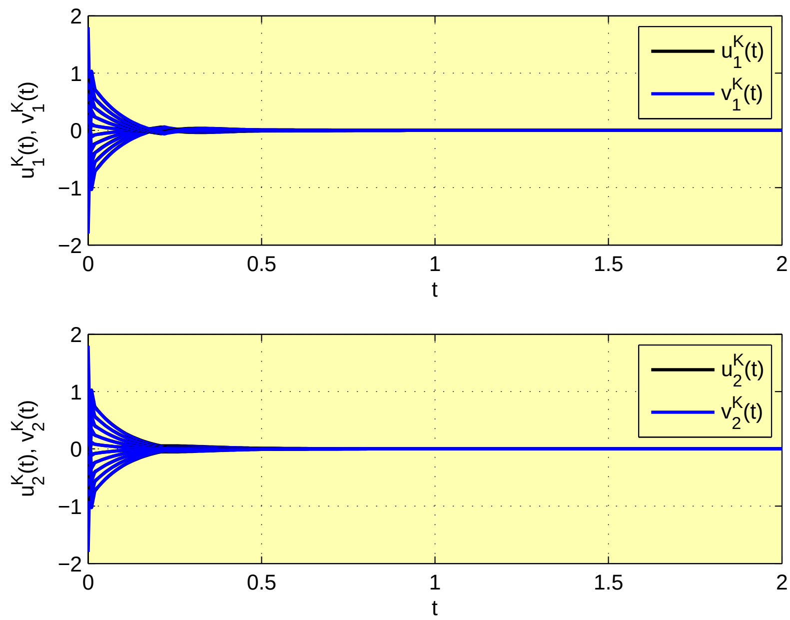

Figure 7. Time representation of the states $u(t)^K, v(t)^K$ of NNs (4.2).

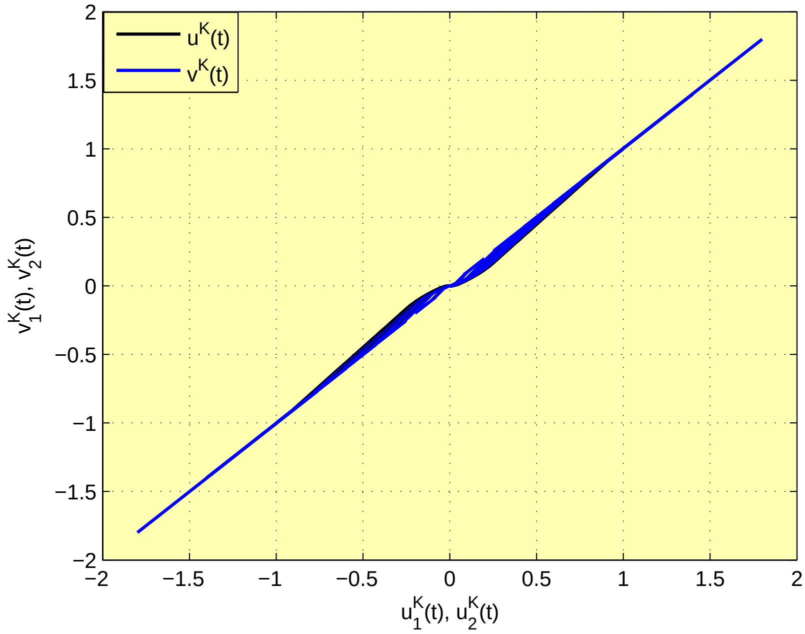

Figure 8. Phase representation of NNs (4.2) between the imaginary subspace $u(t)^K$, $v(t)^K$.

5. Conclusions

This paper investigated the GAS problem for a class of quaternion-valued T-S fuzzy BAM NNs with time-varying delays. By applying T-S fuzzy models, we first considered a general form of quaternion-valued T-S fuzzy BAM NNs with time-varying delays. Then, we applied the Cauchy-Schwarz algorithm and homeomorphism principle to obtain sufficient conditions for the existence and uniqueness of the equilibrium point. By utilizing suitable LKFs and newly developed quaternion-valued WBII, some sufficient criteria are obtained to guarantee the GAS of the considered networks. Further, the results of this paper are presented in terms of quaternion-valued LMIs, which can be solved using the MATLAB YALMIP toolbox. Two numerical examples are presented with their simulations to demonstrate the validity of the theoretical analysis.

The proposed results of this paper can be used to analyze various dynamics of quaternion-valued T-S fuzzy BAM NNs such as finite-time stability, dissipativity, state estimation, synchronization and so on. Thus, we will soon examine the finite-time stability of the following quaternion-valued fractional-order

T-S fuzzy BAM NNs with impulses.

$$\begin{cases} {}_0^C D_t^{\alpha} p_r(t) = \sum_{h=1}^{m} \psi_h(\vartheta(t)) \Big\{ -d_{1r}^{\ h} p_r(t) + \sum_{s=1}^{m} a_{1rs}^{\ h} f_{1s}(q_s(t)) + \sum_{s=1}^{m} b_{1rs}^{\ h} f_{1s}(q_s(t-\ell(t))) \\ +J_{1r} \Big\}, t \neq t_k,\ r = 1, 2..., n, \\ \triangle p_r(t_k) = \alpha_k(p_r(t_k)),\ t = t_k,\ k = 1, 2, ..., \\ p_r(t) = \varphi_{1r}(t),\ t \in [-\ell, 0], \\ {}_0^C D_t^{\alpha} q_s(t) = \sum_{h=1}^{m} \psi_h(\vartheta(t)) \Big\{ -d_{2s}^{\ h} q_s(t) + \sum_{r=1}^{n} a_{2sr}^{\ h} f_{2r}(p_r(t)) + \sum_{r=1}^{n} b_{2sr}^{\ h} f_{2r}(p_r(t-\ell(t))) \\ +J_{2s} \Big\}, t \neq t_k,\ s = 1, 2, ..., m, \\ \triangle q_s(t_k) = \beta_k(q_s(t_k)),\ t = t_k,\ k = 1, 2, ..., \\ q_s(t) = \varphi_{2s}(t),\ t \in [-\ell, 0]. \end{cases}$$

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Conflict of interest

The authors declare no conflict of interest.

References

1. J. J. Hopfield, Neural networks and physical systems with emergent collective computational abilities, *Proc. Natl. Acad. Sci. U.S.A*, **79** (1982), 2554–2558. https://doi.org/10.1073/pnas.79.8.2554
2. M. A. Cohen, S. Grossberg, Absolute stability of global pattern formation and parallel memory storage by competitive neural networks, *IEEE Trans. Syst. Man Cybern.*, **13** (1983), 815–826. https://doi.org/10.1109/TSMC.1983.6313075
3. L. Chua, L. Yang, Cellular neural networks: Applications, *IEEE Trans. Circuits Syst.*, **35** (1988), 1273-1290. https://doi.org/10.1109/31.7601
4. Q. Song, J. Cao, Stability analysis of impulsive Cohen-Grossberg neural network with unbounded discrete time-varying delays, *Int. J. Neural Syst.*, **17** (2007), 407–417. https://doi.org/10.1142/S012906570700124X
5. B. Kosko, Bidirectional associative memories, *IEEE Trans. Syst. Man Cybern.*, **18** (1988), 49–60. https://doi.org/10.1109/21.87054

6. K. Gopalsamy, X. Z. He, Delay-independent stability in bi-directional associative memory networks, *IEEE Trans. Neural Netw.*, **5** (1994), 998–1002. https://doi.org/10.1109/72.329700

7. Q. Song, J. Cao, Global exponential stability and existence of periodic solutions in BAM networks with delays and reaction-diffusion terms, *Chaos Soliton. Fract.*, **23** (2005), 421–430. https://doi.org/10.1016/j.chaos.2004.04.011

8. J. H. Park, C. H. Park, O. M. Kwon, S. M. Lee, A new stability criterion for bidirectional associative memory neural networks of neutral-type, *Appl. Math. Comput.*, **199** (2008), 716–722. https://doi.org/10.1016/j.amc.2007.10.032

9. J. H. Park, O. M. Kwon, Delay-dependent stabilility criterion for bidirectional associative memory neural networks with interval time-varying delays, *Modern Phys. Lett. B*, **23** (2009), 35–46. https://doi.org/10.1142/S0217984909017807

10. Q. Zhu, C. Huang, X. Yang, Exponential stability for stochastic jumping BAM neural networks with time-varying and distributed delays, *Nonlinear Anal. Hybrid Syst.*, **5** (2011), 52–57. https://doi.org/10.1016/j.nahs.2010.08.005

11. J. Li, W. Zhou, Z. Yang, State estimation and input-to-state stability of impulsive stochastic BAM neural networks with mixed delays, *Neurocomputing*, **227** (2017), 37–45. https://doi.org/10.1016/j.neucom.2016.08.101

12. Y. Zhao, J. Kurths, L. Duan, Input-to-state stability analysis for memristive BAM neural networks with variable time delays, *Phys. Lett. A*, **383** (2019), 1143–1150. https://doi.org/10.1016/j.physleta.2019.01.015

13. Z. Wang, L. Huang, Global stability analysis for delayed complex-valued BAM neural networks, *Neurocomputing*, **173** (2016), 2083–2089. https://doi.org/10.1016/j.neucom.2015.09.086

14. R. Guo, Z. Zhang, X. Liu, C. Lin, Existence, uniqueness, and exponential stability analysis for complex-valued memristor-based BAM neural networks with time delays, *Appl. Math. Comput.*, **311** (2017), 100–117. https://doi.org/10.1016/j.amc.2017.05.021

15. J. Liu, J. Jian, Global dissipativity of a class of quaternion-valued BAM neural networks with time delay, *Neurocomputing*, **349** (2019), 123–132. https://doi.org/10.1016/j.neucom.2019.03.026

16. R. Zhao, B. Wang, J. Jian, Lagrange stability of BAM quaternion-valued inertial neural networks via auxiliary function-based integral inequalities, *Neural Process. Lett.*, **54** (2022), 1351–1369. https://doi.org/10.1007/s11063-021-10685-6

17. Z. Zhang, Z. Quan, Global exponential stability via inequality technique for inertial BAM neural networks with time delays, *Neurocomputing*, **151** (2015), 1316–1326. https://doi.org/10.1016/j.neucom.2014.10.072

18. J. Zhang, S. Zhu, N. Lu, S. Wen, Multistability of state-dependent switching neural networks with discontinuous nonmonotonic piecewise linear activation functions, *Neurocomputing*, **437** (2021), 300–311. https://doi.org/10.1016/j.neucom.2021.01.046

19. S. Zhu, Y. Shen, Robustness analysis of global exponential stability of neural networks with Markovian switching in the presence of time-varying delays or noises, *Neural Comput. Appl.*, **23** (2013), 1563–1571. https://doi.org/10.1007/s00521-012-1105-0

20. R. Sriraman, R. Samidurai, Global asymptotic stability analysis for neutral-type complex-valued neural networks with random time-varying delays, *Int. J. Syst. Sci.*, **50** (2019), 1742–1756. https://doi.org/10.1080/00207721.2019.1623340

21. R. Vijay Aravind, P. Balasubramaniam, Stochastic stability of fractional-order Markovian jumping complex-valued neural networks with time-varying delays, *Neurocomputing*, **439** (2021), 122–133. https://doi.org/10.1016/j.neucom.2021.01.053

22. R. Sriraman, Y. Cao, R. Samidurai, Global asymptotic stability of stochastic complex-valued neural networks with probabilistic time-varying delays, *Math. Comput. Simul.*, **171** (2020), 103–118. https://doi.org/10.1016/j.matcom.2019.04.001

23. Z. Zhang, T. Zheng, Global asymptotic stability of periodic solutions for delayed complex-valued Cohen-Grossberg neural networks by combining coincidence degree theory with LMI method, *Neurocomputing*, **289** (2018), 220–230. https://doi.org/10.1016/j.neucom.2018.02.033

24. Y. Liu, Y. Zheng, J. Lu, J. Cao, L. Rutkowski, Constrained quaternion-variable convex optimization: A quaternion-valued recurrent neural network approach, *IEEE Trans. Neural Netw. Learn. Syst.*, **31** (2020), 1022–1035. https://doi.org/10.1109/TNNLS.2019.2916597

25. A. Sudbery, Quaternionic analysis, *Math. Proc. Camb. Philos. Soc.*, **85** (1979), 199–225. https://doi.org/10.1017/S0305004100055638

26. T. Isokawa, T. Kusakabe, N. Matsui, F. Peper, Quaternion neural network and its application, *Knowl.-Based Intell. Eng. Syst.*, **2774** (2003), 318–324. https://doi.org/10.1007/978-3-540-45226-3-44

27. T. Parcollet, M. Morchid, G. Linares, A survey of quaternion neural networks, *Artif. Intell. Rev.*, **53** (2020), 2957–2982. https://doi.org/10.1007/s10462-019-09752-1

28. R. Sriraman, G. Rajchakit, C. P. Lim, P. Chanthorn, R. Samidurai, Discrete-time stochastic quaternion-valued neural networks with time delays: An asymptotic stability analysis, *Symmetry*, **12** (2020), 936. https://doi.org/10.3390/sym12060936

29. X. You, Q. Song, J. Liang, Y. Liu, F. E. Alsaadi, Global μ-stability of quaternion-valued neural networks with mixed time-varying delays, *Neurocomputing*, **290** (2018), 12–25. https://doi.org/10.1016/j.neucom.2018.02.030

30. Q. Song, L. Long, Z. Zhao, Y. Liu, F. E. Alsaadi, Stability criteria of quaternion-valued neutral-type delayed neural networks, *Neurocomputing*, **412** (2020), 287–294. https://doi.org/10.1016/j.neucom.2020.06.086

31. X. You, S. Dian, R. Guo, S. Li, Exponential stability analysis for discrete-time quaternion-valued neural networks with leakage delay and discrete time-varying delays, *Neurocomputing*, **430** (2021), 71–81. https://doi.org/10.1016/j.neucom.2020.12.021

32. T. Takagi, M. Sugeno, Fuzzy identification of systems and its applications to modeling and control, *IEEE T. Syst. Man CY-S.*, **15** (1985), 116–132. https://doi.org/10.1109/TSMC.1985.6313399

33. L. Wang, H. K. Lam, New stability criterion for continuous-time Takagi-Sugeno fuzzy systems with time-varying delay, *IEEE T. Cybernetics.*, **49** (2019), 1551–1556. https://doi.org/10.1109/TCYB.2018.2801795

34. Y. Y. Hou, T. L. Liao, J. J. Yan, Stability analysis of Takagi-Sugeno fuzzy cellular neural networks with time-varying delays, *IEEE T. Syst. Man CY-S.*, **37** (2007), 720–726. https://doi.org/10.1109/TSMCB.2006.889628

35. C. K. Ahn, Delay-dependent state estimation for T-S fuzzy delayed Hopfield neural networks, *Nonlinear Dyn.*, **61** (2010), 483–489. https://doi.org/10.1007/s11071-010-9664-z

36. J. Jian, P. Wan, Global exponential convergence of fuzzy complex-valued neural networks with time-varying delays and impulsive effects, *Fuzzy Sets Syst.*, **338** (2018), 23–39. https://doi.org/10.1016/j.fss.2017.12.001

37. Z. Zhang, J. Cao, Finite-time synchronization for fuzzy inertial neural networks by maximum value approach, *IEEE Trans. Fuzzy Syst.*, **30** (2022), 1436–1446. https://doi.org/10.1109/TFUZZ.2021.3059953

38. S. Zhu, Y. Shen, Robustness analysis for connection weight matrices of global exponential stable time varying delayed recurrent neural networks, *Neurocomputing*, **113** (2013), 220–226. https://doi.org/10.1016/j.neucom.2013.01.006

39. G. Rajchakit, P. Chanthorn, P. Kaewmesri, R. Sriraman, C. P. Lim, Global Mittag-Leffler stability and stabilization analysis of fractional-order quaternion-valued memristive neural networks, *Mathematics*, **8** (2020), 422. https://doi.org/10.3390/math8030422

40. U. Humphries, G. Rajchakit, P. Kaewmesri, P. Chanthorn, R. Sriraman, R. Samidurai, C. P. Lim, Global stability analysis of fractional-order quaternion-valued bidirectional associative memory neural networks, *Mathematics*, **8** (2020), 801. https://doi.org/10.3390/math8050801

41. J. Wang, H. Jiang, T. Ma, C. Hu, Delay-dependent dynamical analysis of complex-valued memritive neural networks: Continuous-time and discrete-time cases, *Neural Netw.*, **101** (2018), 33–46. https://doi.org/10.1016/j.neunet.2018.01.015

42. C. A. Popa, Global μ-stability of neutral-type impulsive complex-valued BAM neural networks with leakage delay and unbounded time-varying delays, *Neurocomputing*, **376** (2020), 73–94. https://doi.org/10.1016/j.neucom.2019.09.008

43. P. G. Park, J. W. Ko, C. Jeong, Reciprocally convex approach to stability of systems with time-varying delays, *Automatica*, **47** (2011), 235–238. https://doi.org/10.1016/j.automatica.2010.10.014

44. A. Seuret, F. Gouaisbaut, Wirtinger-based integral inequality: Application to time-delay systems, *Automatica*, **49** (2013), 2860–2866. https://doi.org/10.1016/j.automatica.2013.05.030

45. Z. Zhang, J. Cao, Novel finite-time synchronization criteria for inertial neural networks with time delays via integral inequality method, *IEEE Trans. Neural Netw. Learn. Syst.*, **30** (2019), 1476–1485. https://doi.org/10.1109/TNNLS.2018.2868800

46. Z. Zhang, L. Ren, New sufficient conditions on global asymptotic synchronization of inertial delayed neural networks by using integrating inequality techniques, *Nonlinear Dyn.*, **95** (2018), 905–917. https://doi.org/10.1007/s11071-018-4603-5

47. X. Chen, Z. Li, Q. Song, J. Hu, Y. Tan, Robust stability analysis of quaternion-valued neural networks with time delays and parameter uncertainties, *Neural Netw.*, **91** (2017), 55–65. https://doi.org/10.1016/j.neunet.2017.04.006

48. F. Zhang, Quaternions and matrices of quaternions, *Linear Algebra Appl.*, **251** (1997), 21–57. https://doi.org/10.1016/0024-3795(95)00543-9
49. X. Chen, Q. Song, State estimation for quaternion-valued neural networks with multiple time delays, *IEEE T. Syst. Man Cy-S.*, **49** (2019), 2201–2209. https://doi.org/10.1109/TSMC.2017.2776940
50. Z. Tu, Y. Zhao, N. Ding, Y. Feng, W. Zhang, Stability analysis of quaternion-valued neural networks with both discrete and distributed delays, *Appl. Math. Comput.*, **343** (2019), 342–353. https://doi.org/10.1016/j.amc.2018.09.049