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Research article

A novel structure of q-rung orthopair fuzzy sets in ring theory

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Abstract: The q-rung orthopair fuzzy atmosphere is an innovative approach for handling unclear circumstances in a range of decision making problems. As compare to intuitionistic fuzzy sets, this one is more appropriate and adaptable because it evaluates the significance of ring theory while retaining the features of q-rung orthopair fuzzy sets. In this study, we characterize q-rung orthopair fuzzy subring as a modification of the pythagorean fuzzy subring. We introduce the novel idea of q-rung orthopair fuzzy subring and investigate the algebraic characteristics for the q-rung orthopair fuzzy subrings. Furthermore, we establish the concept of q-rung orthopair fuzzy quotient ring and q-rung orthopair fuzzy left and right ideals. Also, we describe the q-rung orthopair fuzzy level subring and associate axioms. Finally, we investigate how ring homomorphism influences the q-rung orthopair fuzzy subring and investigate there pre-images homomorphism on q-ROFSR and different aspects of images.

Keywords: fuzzy ideal; orthopair; quotient ring; fuzzy level subring
Mathematics Subject Classification: 05C25, 11E04, 20G15

1. Introduction

The typical application of fuzzy set is to providing of membership values. One area of abstract algebra which has seen extensive use in visual media is ring theory. Unfortunately, there is not much of a connection between ring theory and picture classification. It is exceedingly important in numerous aspects of science, such as algebraic geometry, cryptography and various applications for computer vision [8]. Using algebra to predict desired results in a variety of fields of study, including mathematics, chemistry, computer science, graphic design and physics. Rings and group theory carry the secrets to patterns that are difficult to decode, even with the aid of computing power. In 1965, Zadeh [1] proposed a novel model that explains obscurity, vagueness, and unpredictability in data and described how these issues arise in practical situations. He explained the fundamental foundational findings of this theory, termed as fuzzy sets theory. A function from $\mathbb{T} \to [0,1] \subseteq R$ is used in crisp fuzzy set theory to express a fuzzy subset of a classical set $\mathbb{T}$. Rosenfeld [2] established the fuzzy theory notion for the first time in group theory. Also covered fuzzy normal subgroups, fuzzy quotient and fuzzy subgroups in 1971. Hanan *et al.* [37] discussed the advance algebraic structure of (α, β)-complex fuzzy subgroups. Gulzar *et al.* [38, 39] introduce t-intuitionistic fuzzy subgroups and novel idea of Q-complex fuzzy sub-rings.

Fuzzy subgroup and fuzzy ideals were examined by Liu [5] in 1982. In 1992, Dixit [11] made the initial suggestion for fuzzy subring, which results from fuzzy subset. Malik and Mordeson [12] introduced the notion of extended fuzzy subrings and ideals. Anthony and Sherwood [3] modify the definition of fuzzy subgroup. The idea of level subgroups was defined by Das [4] in 1981, using the notion of fuzzy subgroups and also provided a classification of all finite cyclic fuzzy subgroups. Atanassov [6] published intuitionistic fuzzy sets for the first time in 1986. Important algebraic aspects of this situation were examined as it was widely embraced. Because this idea has proven to be more effective in the scientific community, it is being investigated with membership and non-membership degrees in $[0,1]$, where for membership values $\lambda(u) : \mathbb{T} \to [0,1]$ and for non-membership values $\overline{\lambda}(u) : \mathbb{T} \to [0,1]$ such that $\lambda_T(u) + \overline{\lambda}_T(u) \leq 1$. Supporting and non supporting membership functions of intuitionistic fuzzy sets suggest that they may handle ambiguous and uncertain situations in physical problems, particularly in the domain of decision-making, in contrast to classical fuzzy sets [7, 9, 10]. In 2003, Banerjee and Basnet [31] created a new hypothesis of a fuzzy ring theory with intuitionistic attributes. Bhakat and Das [13] explain the principles of a ring and redefine the concept of fuzzy subrings for fuzzy prime, semi-prime, maximal ideals and characterization of such fuzzy ideals in 1996. Hur *et al.* [30] examined intuitionist fuzzy subring, intuitionist fuzzy ideals, created connection between intuitionist fuzzy completely prime ideals and weakly prime ideals and also provide grading. A novel sort of fuzzy ring, fuzzy subring, fuzzy ideal and discussion of fuzzy ring homomorphism were investigated by Aktas and Cagman [14] in 2007. They also looked at the fundamental properties of these new fuzzy rings, which are similar to those of regular rings. Masarwah and Ghafur [23] established connections between doubt bipolar fuzzy subalgebras, doubt bipolar fuzzy H-ideals, and doubt bipolar fuzzy ideals in BCK-BCI algebras. Masarwah introduces the concepts of m-polar fuzzy subalgebras and m-polar fuzzy ideals, and related features are examined. Additionally, they concentrated on merging the ideas of m-polar fuzzy sets and m-polar fuzzy points to form the concept of m-polar (α, β)-fuzzy ideals in BCK/BCI algebraic structures in [24, 25]. In [26, 27] the concepts of closed cubic intuitionistic ideals, closed cubic intuitionistic p-ideals, and closed cubic intuitionistic a-ideals are presented. Related features are also examined.

Yager [15, 16] introduced the idea of a Pythagorean fuzzy subset in 2013, where the sum of square of true value and square of falls degrees corresponds to the range [0,1]. Therefore, $\lambda(u) : \mathbb{T} \to [0,1]$, $\overline{\lambda}(u) : \mathbb{T} \to [0,1]$, $u \in \mathbb{T}$, $(\lambda_{\mathbb{T}}(u))^2 + (\overline{\lambda}_{\mathbb{T}}(u))^2 \leq 1$. Pythagorean fuzzy subset has greatly contributed to the understanding of decision-making problems. Peng and Yang [17] used Pythagorean fuzzy aggregation operators to define division and subtraction and examined their features such as boundedness, idempotency, monotonicity. To demonstrate the created methodology, an example of analyzing internet stock performance is provided. In 2018, Zeng [18] examined four different existing ranked techniques and compared them with Pythagorean fuzzy set. They suggested the Pythagorean fuzzy group decision-making technique with numerous criteria, including a numerical illustration, and concluded that Pythagorean fuzzy set is more effective. The Pythagorean fuzzy set has different varieties of applications in our daily life, In order to deal with imprecision. Ejegwa [19] adopted Pythagorean fuzzy sets because they generalized intuitionistic fuzzy sets with a broad variety of applications, it is important to examined how inventive these sets are in tackling the problem of career placements. Furthermore, he discussed the method for choosing careers based on academic ability and demonstrated it by utilizing the suggested Pythagorean fuzzy set. Li and Lu [20] introduced Pythagorean fuzzy mechanism for sorting priorities by similarity to an ideal solution to implement. The study investigated multi-criteria decision-making and gave a real world example of how to handle a pressing social sector issue. Ejegwa [21] established the method of the max-min-max composite relation for Pythagorean fuzzy sets and used it to solve the problem of medical diagnostics.The application of the enhanced composite relation for Pythagorean fuzzy sets in medical diagnostics is studied using a fictitious medical database. Zhou *et al.* [22] established a novel divergence measure for Pythagorean fuzzy sets that is predicated on the Dempster-Shafer evidence theory's belief function and has a stronger response than other techniques currently in use to identify medical diagnoses. This measure can help prevent results that are counterintuitive, particularly when there is a data dispute. Tchier *et al.* [40] described the group decision-making technique under picture fuzzy soft perimeters. Shit [41] establish harmonic aggregation operator with trapezoidal picture fuzzy numbers. and Its Application in a Multiple Attribute Decision-Making Problem.

In 2017, Yager [32] introduced the novel concept of q-rung orthopair fuzzy set (q-ROFS) . These are more generalized versions of the Pythagorean fuzzy set and the intuitionistic fuzzy set. The total sum of the qth powers of belonging and not belonging is bounded by one in the q-ROFS. A wider variety of eligible orthopairs becomes available as q increases, providing the user a more broad conceptual framework to express their view on the belonging score. To improve the ability of 2-tuple linguistic terms to describe complex information, Sumera et al. develop a multiple attribute group decision-making strategy based on q-rung orthopair fuzzy 2-tuple linguistic set [28]. M. Akram et al. [29] extend the range of applications of the Einstein operators to the q-rung picture fuzzy environment and construct unique notions of q-rung picture fuzzy aggregate operators under the Einstein operators and describe their use in multi-attribute decision-making. The q-ROFS is redefined by Ali [33] by using different important tools, discussed the basic algebraic operation under action and fetchers of q-ROFS, presented the orbit based q-ROFS with help of graphically explanation and examples. If belonging score $\lambda(u)$ and not belonging score $\overline{\lambda}(u)$ to the range [0,1]. Therefore, $\lambda(u)$: $\mathbb{T} \rightarrow [0,1]$, $\overline{\lambda}(u) : \mathbb{T} \rightarrow [0,1]$, $u \in \mathbb{T}$ $(\lambda_{\mathbb{T}}(u))^q + (\overline{\lambda}_{\mathbb{T}}(u))^q \leq 1$. Wang *et al.* [34] introduced the ten similarity measures by examining the function of supporting degree, non-supporting degree, and indeterminacy supporting degree among the rung orthopair fuzzy sets relying on cotangent and the conventional cosine similarity measurements and also q-rung orthopair fuzzy sets were used to handle multiple criteria decision. Peng and Liu [35] demonstrated the accuracy of the similarity measure under investigation by using it for pattern classification, clustering, and medical problems. To validate the conclusions and show the applicability and availability of similarity measures between q-ROFS, a few illustrative cases were discussed. Asima and Razaq [36] introduced the novel concept of the q-rung orthopair fuzzy subgroup and established some basic findings . Moreover, examined the q-rung orthopair fuzzy left and right cosets, q-rung orthopair fuzzy level subgroups, and also discussed the basic definitions and homorphism under q-rung orthopair fuzzy. The significance of the observations utilized throughout the article is described in the following table. Also included the page where each one is introduced or used for the first time.

q-ROFS	q-rung orthopair fuzzy set
q-ROFSR	q-rung orthopair fuzzy subring
q-ROFQR	q-rung orthopair fuzzy quotient ring
q-ROFLI	q-rung orthopair fuzzy left ideal
q-ROFRI	q-rung orthopair fuzzy right ideal

This article describes the study of q-rung orthopair fuzzy subring (q-ROFSR). The key objective of this study is to investigate a number of fundamental mathematical characteristics of q-ROFSRs. In Section 2 we establish the novel concepts of q-ROFSR, the criteria of q-ROFSR and describe their fundamental attributes. In Section 3 we introduce left ideals, right ideals, q-rung orthopair fuzzy quotient ring (q-ROFQR) and discuss some important basic theorems. The idea of q-ROFSR and also discuss some fundamental findings in Section 4. In Section 5 we prove that the image and inverse image of q-ROFSR are q-rung orthopair fuzzy subring under ring homomorphism.

2. The q-rung orthopair fuzzy subring

In this section, we define the q-rung orthopair fuzzy subring (q-ROFSR) and investigate some of its basic algebraic characteristics. We further explore various examples and demonstrate the sufficient and necessary conditions of the q-ROFSR.

Definition 2.1. Let $\mathbb{Q}$ and $\mathbb{S}$ be any two sets. Then the mapping $A : \mathbb{S} \times \mathbb{Q} \longrightarrow [0, 1]$ is called a $\mathbb{Q}$-fuzzy set in $\mathbb{S}$.

Definition 2.2. Assume that Γ be a subring, then a q-rung orthopair fuzzy set (q-ROFS) $\mathbb{M} = \{j, \delta_M(j), \Psi_M(j) : r \in \Gamma,\ (\delta_M(r))^q + (\Psi_M(r))^q \leq 1\}$ of Γ is known as q-ROFSR of Γ if the given axioms are hold:

(1) $(\delta_{\mathbb{M}}(j - r))^q \geq \{(\delta_{\mathbb{M}}(j))^q \wedge (\delta_{\mathbb{M}}(r))^q\}$,

(2) $(\Psi_{\mathbb{M}}(j - r))^q \leq \{(\Psi_{\mathbb{M}}(j))^q \vee (\Psi_{\mathbb{M}}(r))^q\}$,

(3) $(\delta_{\mathbb{M}}(jr))^q \geq \{(\delta_{\mathbb{M}}(j))^q \wedge (\delta_{\mathbb{M}}(r))^q\}$,

(4) $(\Psi_{\mathbb{M}}(jr))^q \leq \{(\Psi_{\mathbb{M}}(j))^q \vee (\Psi_{\mathbb{M}}(r))^q\}$ for all $j, r \in \Gamma$.

Theorem 2.3. *Suppose* $\mathbb{M} = \{(r,\ \delta_M(r),\ \Psi_M(r)) : r \in \Gamma,\ (\delta_M(r))^q + (\Psi_M(r))^q \leq 1\}$ *be a q-ROFSR of* Γ *then the following axioms are hold:*

(1) $(\delta_M(0))^q \geq (\delta_M(r))^q$ *and* $(\Psi_M(0))^q \leq (\Psi_M(r))^q$ *for all* $r \in \Gamma$.

(2) $(\delta_M(-r))^q = (\delta_M(r))^q$ *and* $(\Psi_M(r))^q = (\Psi_M(r))^q$ *for all* $r \in \Gamma$.

Proof. (1) Suppose that $r \in \Gamma$, then

$$\begin{aligned}
(\delta_M(r-r))^q &\geq \{(\delta_M(r))^q \wedge (\delta_M(-r))^q\} \\
(\delta_M(0))^q &\geq \{(\delta_M(-r))^q \wedge (\delta_M(r))^q\} \\
&= (\delta_M(r))^q \\
\Rightarrow & \quad (\delta_M(0))^q \geq \{(\delta_M(r))^q\}. \\
\text{In same way,} \quad (\Psi_M(r-r))^q &\leq \{(\Psi_M(r))^q \wedge (\Psi_M(-r))^q\} \\
\Rightarrow (\Psi_M(0))^q &\leq \{(\Psi_M(-r))^q \wedge (\Psi_M(r))^q\} \\
&= (\Psi_M(r))^q \\
&\Rightarrow (\Psi_M(0))^q \leq \{(\Psi_M(r))^q\}.
\end{aligned}$$

(2) $(\delta_M(-r))^q \leq ((\delta_M(r))^q$ and $(\Psi_M(-r))^q \leq (\Psi_M(r))^q$ for all $r \in \Gamma$. So $(\delta_M(-(-r)))^q \geq ((\delta_M(-r))^q$ and $(\Psi_M(-(-r)))^q \leq (\Psi_M(-r))^q$. In other words $(\delta_M(-r))^q \geq ((\delta_M(r))^q$ and $(\Psi_M(-r))^q \leq ((\Psi_M(r))^q$. Thus, $(\delta_M(-r))^q = ((\delta_M(r))^q$ and $(\Psi_M(-r))^q = ((\Psi_M(r))^q$ for all $r \in \Gamma$.

This theorem demonstrates that every pythagorean fuzzy subring of Γ is q-ROFSR of Γ. □

Theorem 2.4. *Suppose that* Γ *be a ring and* $\mathbb{L} = \{(j,\ \delta_L(j),\ \Psi_L(j)) : j \in \Gamma,\ \{\delta_L(j)\}^2 + \{\Psi_L(j)\}^2 \leq 1\}$ *be a pythagorean fuzzy subring of* Γ *then* $\mathbb{L}$ *is q-ROFSR of* Γ.

Proof. Assume that $j, r \in \Gamma$, then

$$\begin{aligned}
(\delta_L(j-r))^2 &\geq \{(\delta_L(j))^2 \wedge (\delta_L(r))^2\} \\
(\Psi_L(j-r))^2 &\leq \{(\Psi_L(j))^2 \vee (\Psi_L(r))^2\} \\
(\delta_L(jr))^2 &\geq \{(\delta_L(j))^2 \wedge (\delta_L(r))^2\} \\
\text{and} \quad (\Psi_L(jr))^2 &\leq \{(\Psi_L(j))^2 \vee (\Psi_L(r))^2\}\ for\ all \ \ j, r \in \Gamma.
\end{aligned}$$

This implies that

$$\begin{aligned}
(\delta_L(j-r))^q &\geq \{(\delta_L(j))^q \wedge (\delta_L(r))^q\} \\
(\Psi_L(j-r))^q &\leq \{(\Psi_L(j))^q \vee (\Psi_L(r))^q\} \\
(\delta_L(jr))^q &\geq \{(\delta_L(j))^q \wedge (\delta_L(r))^q\} \\
\text{and} \quad (\Psi_L(jr))^q &\leq \{(\Psi_L(j))^q \vee (\Psi_L(r))^q\}\ for\ all \ \ j, r \in \Gamma.
\end{aligned}$$

Since $(\delta_L(j))^2, (\delta_L(r))^2, (\delta_L(r))^2, (\Psi_L(j))^2, (\Psi_L(r))^2, (\Psi_L(r))^2 \in [0,1]$. So for all $q > 1$ by using $(\delta_L(j))^q \leq (\delta_L(j))^2, (\delta_L(r))^q \leq (\delta_L(r))^2, (\delta_L(r))^q \leq (\delta_L(r))^2, (\Psi_L(j))^q \geq (\Psi_L(j))^2, (\Psi_L(r))^q \geq (\Psi_L(r))^2$ and $(\Psi_L(r))^q \geq (\Psi_L(r))^2$. Thus,

$$\{(\delta_M(j))^2 + (\Psi_M(j))^2 \leq 1\} \tag{2.1}$$

$$\{(\delta_M(r))^2 + (\Psi_M(r))^2 \leq 1\} \tag{2.2}$$

$$\{(\delta_M(r))^q + (\Psi_M(r))^q \leq 1\} \tag{2.3}$$

$$\{(\delta_M(j))^q + (\Psi_M(j))^q \leq 1\}. \tag{2.4}$$

These inequalities show that L is a q-ROFSR of Γ. □

The negation of the above theorem is shown to be invalid in the following example.

Example 2.5. Let $(Z_4, +, .)$ be ring, where $Z_4 = \{\overline{0},\ \overline{1},\ \overline{2},\ \overline{3}\}$ and $L = \{< j,\ \delta(l),\ \Psi(l) > j \in Z_4\}$ be q-ROFS over Z_4 for $q \geq 3$ define as

$$\delta_{\mathrm{L}}(\mathrm{j}) = \begin{cases} 0 & \text{if j} \in \{\overline{1},\ \overline{3}\} \\ 0.75 & \text{if j} \in Z_4 \backslash \{\overline{0},\ \overline{1},\ \overline{3}\} \\ 1 & \text{if j} \in Z_4 \backslash \{\overline{1},\ \overline{2},\ \overline{3}\} \end{cases}$$

and

$$\Psi_{\mathrm{L}}(\mathrm{j}) = \begin{cases} 1 & \text{if j} \in Z_4 \backslash \{\overline{0},\ \overline{2}\} \\ 0.75 & \text{if j} \in Z_4 \backslash \{\overline{0},\ \overline{1},\ \overline{3}\} \\ 0 & \text{if j} \in Z_4 \backslash \{\overline{1},\ \overline{2},\ \overline{3}\}. \end{cases}$$

Clearly, L is 3-ROFSR of Z_4 but it is not pythagorean fuzzy subring of Z_4 as $(0.75)^2 + (0.75)^2 > 1$.

Theorem 2.6. *The q-ROFS* $\mathbb{M} = \{r, \delta_M(r), \Psi_M(r) : r \in \Gamma,\ (\delta_M(r))^q + (\Psi_M(r))^q \leq 1\}$ *of* Γ *is a q-ROFSR of* Γ *if and only if* $(\delta_M(l-r))^q \geq \{(\delta_M(l))^q \wedge (\delta_M(r))^q\}$ *and* $(\Psi_M(l-r))^q \leq \{(\Psi_M(l))^q \vee (\Psi_M(r))^q\}$ *for all* $l, r \in \Gamma$.

Proof. Let $\mathbb{M} = \{(r,\ \delta_M(r),\ \Psi_M(r)) : r \in \Gamma, (\delta_M(r))^q + (\Psi_M(r))^q \leq 1\}$ be a q-ROFSR of Γ. Then for all $l, r \in \Gamma$, $(\delta_M(l-r))^q \geq \{(\delta_M(l))^q \wedge (\delta_M(-r))^q\} = \{(\delta_M(l))^q \wedge (\delta_M(r))^q\}$ and $(\Psi_M(l-r))^q \leq \{(\Psi_M(l))^q \vee (\Psi_M(-r))^q\} = \{(\Psi_M(l))^q \vee (\Psi_M(r))^q\}$. Obviously, $(\delta_M(lr))^q \geq \{(\delta_M(l))^q \wedge (\delta_M(r))^q\}$ and $(\Psi_M(lr))^q \leq \{(\Psi_M(l))^q \vee (\Psi_M(r))^q\}$ are also hold.
Conversly, assume that $(\delta_M(l-r))^q \geq \{(\delta_M(l))^q \wedge (\delta_M(r))^q\}$ and $(\Psi_M(l-r))^q \leq \{(\Psi_M(l))^q \vee (\Psi_M(r))^q\}$ for all $l, r \in \Gamma$. Then $(\delta_M(l-r))^q = (\delta_M(l-(-(-r))))^q \geq \{(\delta_M(l))^q \wedge (\delta_M(-r))^q\} = \{(\delta_M(l))^q \wedge (\delta_M(r))^q\}$. So, $(\delta_M(l-r))^q \geq \{(\delta_M(l))^q \wedge (\delta_M(r))^q\}$. Similarly, $(\Psi_M(l-r))^q = (\Psi_M(l-(-(-r))))^q \leq \{(\Psi_M(l))^q \vee (\Psi_M(-r))^q\} = \{(\Psi_M(l))^q \vee (\Psi_M(r))^q\}$. So, $(\Psi_M(l-r))^q \leq \{(\Psi_M(l))^q \vee (\delta_M(r))^q\}$. Next, $(\delta_M(-r))^q = (\delta_M(0-r))^q \geq \{(\delta_M(0))^q \wedge (\delta_M(r))^q\} = \{(\delta_M(r))^q\}$ which means that $\{(\delta_M(-r))^q\} \geq \{(\delta_M(r))^q\}$. In similar fashion, $(\Psi_M(-r))^q = (\delta_M(0-r))^q \leq \{(\Psi_M(0))^q \vee (\Psi_M(r))^q\} = \{(\Psi_M(r))^q\}$. This conclude that $\{(\Psi_M(-r))^q\} \leq \{(\Psi_M(r))^q\}$. So, all above expressions prove that $\mathbb{M}$ is a q-ROFSR of Γ. □

Theorem 2.7. *Let* $\mathbb{F}_1 = \{j, \delta_{F1}(j), \Psi_{F1}(j) : j \in \Gamma,\ (\delta_{F1}(j))^q + (\Psi_{F1}(j))^q \leq 1\}$ *and* $\mathbb{F}_2 = \{j, \delta_{F2}(j), \Psi_{F2}(j) : j \in \Gamma,\ (\delta_M(j))^q + (\Psi_M(j))^q \leq 1\}$ *be two q-ROFSR of* Γ. *Then* $\mathbb{F}_1 \cap \mathbb{F}_2$ *be q-ROFSR of* Γ.

Proof. Assume that $\mathbb{F}_1$ and $\mathbb{F}_2$ be two q-ROFSR of Γ. Then for all $u_1, u_2 \in \Gamma$, we have

$$\begin{aligned}
(\delta_{(F1\cap F2)}(j-r))^q &= \{(\delta_{F1}(j-r))^q \wedge (\delta_{F2}(j-r))^q\} \\
&\geq \{(\delta_{F1}(j))^q \wedge (\delta_{F1}(r))^q\} \wedge \{(\delta_{F2}(j))^q \wedge (\delta_{F2}(r))^q\} \\
&= ((\delta_{F1}(j))^q \wedge (\delta_{F2}(j))^q) \wedge ((\delta_{F1}(r))^q \wedge (\delta_{F2}(r))^q) \\
&= \{(\delta_{(F1\cap F2)}(j))^q \wedge (\delta_{(F1\cap F2)}(r))^q\} \\
(\delta_{(F1\cap F2)}(j-r))^q &\geq (\delta_{(F1\cap F2)}(j))^q \wedge (\delta_{(F1\cap F2)}(r))^q.
\end{aligned}$$

Moreover,

$$(\Psi_{(F1\cap F2)}(j-r))^q = \{(\Psi_{F1}(j-r))^q \vee (\Psi_{F2}(j-r))^q\}$$

$$
\begin{aligned}
&\leq \{\{(\Psi_{F1}(j))^q \vee (\Psi_{F1}(r))^q\} \vee \{(\Psi_{F2}(j))^q \vee (\Psi_{F2}(r))^q\}\} \\
&= \{((\Psi_{F1}(j))^q \vee (\Psi_{F2}(j))^q) \vee ((\Psi_{F1}(r))^q \vee (\Psi_{F2}(r))^q)\} \\
&= \{(\Psi_{(F1\cap F2)}(j))^q \vee (\Psi_{(F1\cap F2)}(r))^q\} \\
(\Psi_{(F1\cap F2)}(j-r))^q &\leq \{(\Psi_{(F1\cap F2)}(j))^q \vee (\Psi_{(F1\cap F2)}(r))^q\}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
(\delta_{(F1\cap F2)}(jr))^q &= \{(\delta_{F1}(jr))^q \wedge (\delta_{F2}(jr))^q\} \\
&\geq \{(\delta_{F1}(j))^q \wedge (\delta_{F1}(r))^q\} \wedge \{(\delta_{F2}(j))^q \wedge (\delta_{F2}(r))^q\} \\
&= ((\delta_{F1}(j))^q \wedge (\delta_{F2}(j))^q) \wedge ((\delta_{F1}(r))^q \wedge (\delta_{F2}(r))^q) \\
&= \{(\delta_{(F1\cap F2)}(j))^q \wedge (\delta_{(F1\cap F2)}(r))^q\} \\
(\delta_{(F1\cap F2)}(jr))^q &\geq (\delta_{(F1\cap F2)}(j))^q \wedge (\delta_{(F1\cap F2)}(r))^q.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(\Psi_{(F1\cap F2)}(jr))^q &= \{(\Psi_{F1}(jr))^q \vee (\Psi_{F2}(jr))^q\} \\
&\leq \{\{(\Psi_{F1}(j))^q \vee (\Psi_{F1}(r))^q\} \vee \{(\Psi_{F2}(j))^q \vee (\Psi_{F2}(r))^q\}\} \\
&= \{((\Psi_{F1}(j))^q \vee (\Psi_{F2}(j))^q) \vee ((\Psi_{F1}(r))^q \vee (\Psi_{F2}(r))^q)\} \\
&= \{(\Psi_{(F1\cap F2)}(j))^q \vee (\Psi_{(F1\cap F2)}(r))^q\} \\
(\Psi_{(F1\cap F2)}(j-r))^q &\leq \{(\Psi_{(F1\cap F2)}(j))^q \vee (\Psi_{(F1\cap F2)}(r))^q\}.
\end{aligned}
$$

Hence, this shows that $F_1 \cap F_2$ is q-ROFSR of Γ. □

Theorem 2.8. *Let* $\mathbb{M} = \{j,\ \delta_M(j),\ \Psi_M(j) : j \in \Gamma,\ (\delta_M(j))^q + (\Psi_M(j))^q \leq 1\}$ *be a q-ROFSR of* Γ. *Then* $(\delta_M(j^m))^q \geq (\delta_M(j))^q$ *and* $(\Psi_M(j^m))^q \leq (\Psi_M(j)^q)$ *for all* $j \in \Gamma$ *and* $m \in \mathbb{N}$.

Proof. To prove this theorem, we use mathematical induction technique. Consider, $j \in \Gamma$, then $(\delta_M(j-r)(j-r))^q = (\delta_M(j-r)(j-r))^q \geq \{(\delta_M(j-r))^q \wedge (\delta_M(j-r))^q\} = (\delta_M(j-r))^q$. As a result, this inequality is true for $m = 2$. Let assume that this inequality exist the for $\overline{m} = n-1$, such that $(\delta_M(j-r)^{n-1})^q \geq (\delta_M(j-r))^q$. Then $(\delta_M(j-r)^n)^q = (\delta_M(j-r)^{n-1}(j-r))^q \geq \{(\delta_M(j-r))^q \wedge (\delta_M(j-r)^{n-1})^q\} = (\delta_M(j-r))^q$. Thus by mathematical induction, we obtain $(\delta_M(j-r)^m)^q \geq \{(\delta_M(j-r))^q\}$ for all $\overline{m} \in \mathbb{N}$. In same fashion, $(\Psi_M(j-r)^2)^q = (\Psi_M(j-r)(j-r))^q \leq \{(\Psi_M(j-r))^q \vee (\Psi_M(j-r))^q\} = (\Psi_M(j-r))^q$. Result is valid for $m = 2$ so, suppose that it is also true for $\overline{m} = n-1$, such that$(\Psi_M(j^{n-1}))^q \leq (\Psi_M(j))^q$. Then $(\Psi_M(j^n))^q = (\Psi_M(j^{n-1}j))^q \leq \{(\Psi_M(j))^q \vee (\Psi_M(j^{n-1}))^q\} = (\Psi_M(j))^q$. Thus by mathematical induction, we have $(\Psi_M(j^m))^q \leq (\Psi_M(j))^q$ for all $m \in \mathbb{N}$. Moreover, $(\delta_M(j^{n-1}))^q \geq (\delta_M(j))^q$. Then $(\delta_M(j^n))^q = (\delta_M(j^{n-1}j))^q \geq \{(\delta_M(j))^q \wedge (\delta_M(j^{n-1}))^q\} = (\delta_M(j))^q$. Now, we have $(\delta_M(j^m))^q \geq \{(\delta_M(j))^q\}$ for all $m \in \mathbb{N}$. We conclude that $(\Psi_M(j^2))^q = (\Psi_M(jj))^q \leq \{(\Psi_M(j))^q \vee (\Psi_M(j))^q\} = (\Psi_M(j))^q$. In same fashion, we obtain following result $(\Psi_M(j^{n-1}))^q \leq (\Psi_M(j))^q$. Then $(\Psi_M(j^n))^q = (\Psi_M(j^{n-1}j))^q \leq \{(\Psi_M(j))^q \vee (\Psi_M(j^{n-1}))^q\} = (\Psi_M(j))^q$. Thus by mathematical induction, we get $(\Psi_M(j^m))^q \leq (\Psi_M(j))^q$ for all $m \in \mathbb{N}$. □

Theorem 2.9. *Suppose* $\mathbb{M} = \{j,\ \delta_M(j),\ \Psi_M(j) : j \in \Gamma,\ (\delta_M(j))^q + (\Psi_M(j))^q \leq 1\}$ *be a q-ROFSR of* Γ. *If* $\delta_M(j) \neq \delta_M(r)$ *and* $\Psi_M(j) \neq \Psi_M(r)$ *for some* $j, r \in \Gamma$, *than* $(\delta_M(j-r))^q = (\delta_M(j))^q \wedge (\delta_M(r))^q$ *and* $(\delta_M(j-r))^q = (\Psi_M(j))^q \vee (\Psi_M(r))^q$.

Proof. To prove this results we take arbitrary elements $j,\ r \in \Gamma$, therefore $\delta_M(j) > \delta_M(r)$ then clearly $(\delta_M(j))^q > (\delta_M(r))^q$. Consider,

$$\begin{aligned}(\delta_M(r))^q &= \delta_M(j - j + r)^q \\ &\geq \{(\delta_M(-j))^q \wedge (\delta_M(j + r))^q\} \\ &= \{(\delta_M(j))^q \wedge (\delta_M(j + r))^q\}. \end{aligned} \tag{2.5}$$

Since $(\delta_M(j))^q > (\delta_M(r))^q$, now from expression (2.5) we get

$$(\delta_M(r))^q \geq (\delta_M(j + r))^q \text{Also,} \tag{2.6}$$

$$(\delta_M(j - r))^q \geq \{(\delta_M(j))^q \wedge (\delta_M(r))^q\} = (\delta - K(r))^q \tag{2.7}$$

$$\text{thus,} \quad (\delta_M(j + r))^q \geq (\delta_M(r))^q \tag{2.8}$$

by Eqs (2.6) and (2.8) we obtain

$$(\delta_M(j - r))^q = (\delta_K(r))^q = \{(\delta_M(j))^q \wedge (\delta_M(r))^q\}. \tag{2.9}$$

Furthermore, we get the results if $\Psi_M(j) > \Psi_M(r)$.

$$\begin{aligned}\text{Suppose,} \quad (\Psi_M(r))^q &= \Psi_M(j - j + r)^q \\ &\leq \{(\Psi_M(-j))^q \vee (\Psi_M(j + r))^q\} \\ &= \{(\Psi_M(j))^q \vee (\Psi_M(j + r))^q\}. \end{aligned} \tag{2.10}$$

Since $(\Psi_M(j))^q > (\Psi_M(r))^q$, now from expression (2.10) we get

$$(\Psi_M(r))^q \leq (\Psi_M(j + r))^q \tag{2.11}$$

$$\begin{aligned}\text{Also, } (\Psi_M(j - r))^q &\leq \{(\Psi_M(j))^q \vee (\Psi_M(r))^q\} = (\Psi - K(r))^q \text{that is} \\ &(\Psi_M(j + r))^q \leq (\Psi_M(r))^q \end{aligned} \tag{2.12}$$

by Eqs (2.11) and (2.12) we obtain

$$(\Psi_M(j - r))^q = (\Psi_M(r))^q = \{(\Psi_M(j))^q \vee (\Psi_M(r))^q\}.$$

□

Theorem 2.10. *Suppose* 0 *represent the identity element of* Γ *and* $\mathbb{M} = \{j,\ \delta_M(j),\ \Psi_M(j) : j \in \Gamma,\ (\delta_M(j))^q + (\Psi_M(j))^q \leq 1\}$ *be a q-ROFSR of* Γ. *Therefore,*

(1) If $(\delta_M(j))^q = (\delta_M(0))^q$ *for some* $j \in \Gamma$, *then* $(\delta_M(j - r))^q = (\delta_M(r))^q$ *for all* $r \in \Gamma$,

(2) If $(\Psi_M(j))^q = (\Psi_M(0))^q$ *for some* $j \in \Gamma$, *then* $(\Psi_M(j - r))^q = (\Psi_M(r))^q$ *for all* $r \in \Gamma$,

(3) If $(\delta_M(j))^q = (\delta_M(0))^q$ *for some* $j \in \Gamma$, *then* $(\delta_M(jr))^q = (\delta_M(r))^q$, *for all* $r \in \Gamma$,

(4) If $(\Psi_M(j))^q = (\Psi_M(0))^q$ *for some* $j \in \Gamma$, *then* $(\Psi_M(jr))^q = (\Psi_M(r))^q$ *for all* $r \in \Gamma$.

Proof. Assume that $\mathbb{M} = \{u, \delta_M(u), \Psi_M(u)\}$ be a q-ROFSR of Γ. As given $(\delta_M(j))^q = (\delta_M(0))^q$.

$$\begin{aligned}(\delta_M(r))^q &= \delta_M(j - j + r)^q \\ &\geq \{(\delta_M(j))^q \wedge (\delta_M(r - j))^q\} \end{aligned}$$

$$\begin{aligned} &= \{(\delta_M(j))^q \wedge (\delta_M(r-j))^q\} \\ &= \{(\delta_M(0))^q \wedge (\delta_M(r-j))^q\}. \end{aligned} \tag{2.13}$$

As, $(\delta_M(0))^q \geq (\delta_M(r))^q$ by using the expression (2.13) we get

$$(\delta_M(r))^q \geq \delta_M(r-j)^q. \tag{2.14}$$

Thus, $(\delta_M(r-j))^q \geq \{(\delta_M(j))^q \wedge (\delta_M(r))^q\} = (\delta_K(r))^q$. re have

$$(\delta_M(r-j))^q \geq (\delta_M(r))^q. \tag{2.15}$$

From Eqs (2.14)and (2.15) we obtain, $(\delta_M(r-j))^q = (\delta_M(r))^q$. Now, $(\Psi_M(j))^q = (\Psi_M(0))^q$.

$$\begin{aligned} (\Psi_M(r))^q &= \Psi_M(j-j+r)^q \\ &\leq \{(\Psi_M(j))^q \vee (\Psi_M(r-j))^q\} \\ &= \{(\delta_M(j))^q \vee (\delta_M(r-j))^q\} \\ &= \{(\delta_M(0))^q \vee (\delta_M(r-j))^q\}. \end{aligned} \tag{2.16}$$

Because, $(\Psi_M(0))^q \leq (\Psi_M(r))^q$ form the expression (2.16) we get

$$(\Psi_M(r))^q \leq \Psi_M(r-j)^q. \tag{2.17}$$

Thus, $(\Psi_M(r-j))^q \leq \{(\Psi_M(j))^q \vee (\Psi_M(r))^q\} = (\Psi_K(r))^q$, then we have

$$(\Psi_M(r-j))^q \leq (\Psi_M(r))^q. \tag{2.18}$$

From Eqs (2.17) and (2.18), we obtain, $(\Psi_M(r-j))^q = (\Psi_M(r))^q$.

$$\begin{aligned} (\delta_M(r))^q &= \delta_M(jj^{-1}r)^q \\ &\geq \{(\delta_M(j^{-1}))^q \wedge (\delta_M(rj))^q\} \\ &\geq \{(\delta_M(j))^q \wedge (\delta_M(rj))^q\} \\ &= \{(\delta_M(0))^q \wedge (\delta_M(rj))^q\}. \end{aligned} \tag{2.19}$$

As, $(\delta_M(0))^q \geq (\delta_M(r))^q$ by using the expression (2.19) we get

$$(\delta_M(r))^q \geq \delta_M(rj)^q. \tag{2.20}$$

Thus, $(\delta_M(rj))^q \geq \{(\delta_M(j))^q \wedge (\delta_M(r))^q\} = (\delta_K(r))^q$. we have

$$(\delta_M(rj))^q \geq (\delta_M(r))^q. \tag{2.21}$$

From Eqs (2.20) and (2.21) we obtain, $(\delta_M(rj))^q = (\delta_M(r))^q$.
Now, $(\Psi_M(j))^q = (\Psi_M(0))^q$.

$$\begin{aligned} (\Psi_M(r))^q &= \Psi_M(jj^{-1}r)^q \\ &\leq \{(\Psi_M(j^{-1}))^q \vee (\Psi_M(rj))^q\} \\ &= \{(\delta_M(j))^q \vee (\delta_M(rj))^q\} \\ &= \{(\delta_M(0))^q \vee (\delta_M(rj))^q\}. \end{aligned} \tag{2.22}$$

Because, $(\Psi_M(0))^q \leq (\Psi_M(r))^q$ form the expression (2.22) we get

$$(\Psi_M(r))^q \leq \Psi_M(rj)^q. \tag{2.23}$$

Thus, $(\Psi_M(rj))^q \leq \{(\Psi_M(j))^q \vee (\Psi_M(r))^q\} = (\Psi_K(r))^q$, then we have

$$(\Psi_M(rj))^q \leq (\Psi_M(r))^q. \tag{2.24}$$

From Eqs (2.23) and (2.24), we obtain, $(\Psi_M(rj))^q = (\Psi_M(r))^q$. □

Theorem 2.11. *Suppose* 0 *represent the identity element of* Γ *and* $\mathbb{M} = \{r, \delta_M(r), \Psi_M(r) : r \in \Gamma,\ (\delta_M(r))^q + (\Psi_M(r))^q \leq 1\}$ *be a q-ROFSR of* Γ. *Then* $\mathbb{H} = \{r \in \Gamma : (\delta_M(r))^q = (\delta_M(0))^q$ *and* $(\Psi_M(r))^q = (\Psi_M(0))^q\}$ *is a subring of* Γ.

Proof. By definition of $\mathbb{H}$, $e \in \mathbb{H}$, so $\mathbb{H}$ is non-empty subset of Γ. Suppose $j, r \in \mathbb{H}$, then $(\delta_M(j))^q = (\delta_M(e))^q = (\delta_M(r))^q$ and $(\Psi_M(j))^q = (\Psi_M(e))^q = (\Psi_M(r))^q$. Now,

$$\begin{aligned}
(\delta_M(j-r))^q &\geq \{(\delta_M(j))^q \wedge (\delta_M(j))^q\} \\
&= \{(\delta_M(j))^q \wedge (\delta_M(r))^q\} \\
&= \{(\delta_M(0))^q \wedge (\delta_M(0))^q\} \\
&= (\delta_M(0))^q.
\end{aligned}$$

By Theorem (2.3), we have $(\delta_M(0))^q \geq (\delta_M(j-r))^q$. Then, it is clearly $(\delta_M(0))^q = (\delta_M(j-r))^q$. Now we show that $(\Psi_M(0))^q = (\Psi_M(j-r))^q$ for which

$$\begin{aligned}
(\Psi_M(j-r))^q &\leq \{(\Psi_M(j))^q \vee (\Psi_M(j))^q\} \\
&= [(\Psi_M(j))^q \vee (\Psi_M(r))^q] \\
&= [(\Psi_M(0))^q \vee (\Psi_M(0))^q] \\
&= (\Psi_M(0))^q.
\end{aligned}$$

By Theorem (2.3) we have $(\Psi_M(0))^q \leq (\Psi_M(j-r))^q$. Then it is clearly $(\Psi_M(0))^q = (\Psi_M(j-r))^q$. Hence, $j - r \in \Gamma$. Furtherore,

$$\begin{aligned}
(\delta_M(jr))^q &\geq \{(\delta_M(j))^q \wedge (\delta_M(j))^q\} \\
&= \{(\delta_M(j))^q \wedge (\delta_M(r))^q\} \\
&= \{(\delta_M(0))^q \wedge (\delta_M(0))^q\} \\
&= (\delta_M(0))^q.
\end{aligned}$$

By Theorem (2.3), we have $(\delta_M(0))^q \geq (\delta_M(jr))^q$. Then, it is clearly $(\delta_M(0))^q = (\delta_M(jr))^q$. Now we show that $(\Psi_M(0))^q = (\Psi_M(jr))^q$ for which

$$\begin{aligned}
(\Psi_M(jr))^q &\leq \{(\Psi_M(j))^q \vee (\Psi_M(j))^q\} \\
&= [(\Psi_M(j))^q \vee (\Psi_M(r))^q] \\
&= [(\Psi_M(0))^q \vee (\Psi_M(0))^q] \\
&= (\Psi_M(0))^q.
\end{aligned}$$

By Theorem (2.3) we have $(\Psi_M(0))^q \leq (\Psi_M(j-r))^q$. Then it is clearly $(\Psi_M(0))^q = (\Psi_M(jr))^q$. Hence, $j - r \in \Gamma$. This illustrates the proof. □

3. q-Rung orthopair fuzzy ideal and q-rung orthopair fuzzy quotient ring

Now, In this section we define q-rung orthopair fuzzy left, right ideal (q-ROFLI), (q-ROFRI) and q-ROFQR. Moreover, q-rung orthopair fuzzy quotient ring, some of its basic algebraic results, properties and examples are also discuss.

Definition 3.1. Assume $\mathbb{I} = \{r,\ \delta_I(r),\ \Psi_I(r) : r \in \Gamma,\ (\delta_M(r))^q + (\Psi_M(r))^q \leq 1\}$ be a non empty q-ROFS of ring Γ is said to be left ideal if following conditions are hold:
if $j, r \in \Gamma$ then,

(1) $(\delta_M(r - j))^q \geq \{(\delta_I(r))^q \wedge (\delta_I(j))^q\}$,

(2) $(\delta_I(jr))^q \geq (\delta_I(r))^q$,

(3) $(\Psi_I(r - j))^q \leq \{(\Psi_I(r))^q \vee (\Psi_I(j))^q\}$,

(4) $(\Psi_I(jr))^q \leq (\Psi_I(r))^q$.

Example 3.2. Let $(Z_3, +, .)$ be ring, where $Z_3 = \{\overline{0},\ \overline{1},\ \overline{2}, \}$ and $L = \{< j,\ \delta(j),\ \Psi(j) > j \in Z_3,\ (\delta_L(j))^q + (\Psi_L(j))^q \leq 1\}$ be q-ROFS over Z_3, where $q \geq 3$ define as

$$\delta_{\mathrm{L}}(\mathrm{j}) = \begin{cases} 0.75 & \text{if } \mathrm{j} \in \{\overline{1},\ \overline{2}\} \\ 0.8 & \text{if } \mathrm{j} \in \{\overline{0}\} \end{cases}$$

and

$$\Psi_{\mathrm{L}}(\mathrm{j}) = \begin{cases} 0.8 & \text{if } \mathrm{j} \in \{\overline{1},\ \overline{2}\} \\ 0.75 & \text{if } \mathrm{j} \in \{\overline{0}\}. \end{cases}$$

Clearly, L is 3-ROFLI of Z_3.

Definition 3.3. Assume $\mathbb{I} = \{j, \delta_I(j), \Psi_I(j) : r \in \Gamma,\ (\delta_M(r))^q + (\Psi_M(r))^q \leq 1\}$ be a non empty q-ROFS of ring Γ is said to be right ideal if following conditions are hold:
if $r, j \in \Gamma$ then,

(1) $(\delta_M(j - r))^q \geq \{(\delta_I(j))^q \wedge (\delta_I(r))^q\}$,

(2) $(\delta_I(jr))^q \geq (\delta_I(j))^q$,

(3) $(\Psi_I(j - r))^q \leq \{(\Psi_I(j))^q \vee (\Psi_I(r))^q\}$,

(4) $(\Psi_I(jr))^q \leq (\Psi_I(j))^q$.

A q-ROFS $\mathbb{I}$ of ring Γ have both left and right ideal is called q-ROFI.

Definition 3.4. Assume $\mathbb{I} = \{r,\ \delta_I(r),\ \Psi_I(r) : r \in \Gamma,\ (\delta_M(r))^q + (\Psi_M(r))^q \leq 1\}$ be a non empty q-ROFS of ring Γ is said to be ideal if following conditions are hold:
if $j, r \in \Gamma$ then,

(1) $(\delta_M(r - j))^q \geq \{(\delta_I(l))^q \wedge (\delta_I(j))^q\}$,

(2) $(\delta_I(rj))^q \geq \max\{(\delta_I(r))^q \wedge (\delta_I(j))^q\}$,

(3) $(\Psi_I(r-j))^q \leq \{(\Psi_I(r))^q \vee (\Psi_I(j))^q\}$,

(4) $(\Psi_I(rj))^q \leq \min\{(\delta_I(r))^q \vee (\delta_I(j))^q\}$.

Theorem 3.5. *Consider* $\mathbb{M} = \{r, \delta_M(r), \Psi_M(r) : r \in \Gamma,\ (\delta_M(r))^q + (\Psi_M(r))^q \leq 1\}$ *be a q-ROFSR of* Γ. *If* $\mathbb{M}$ *is a q-RPFI of* Γ *then* $(\delta_M(j-r))^q = (\delta_M(j-r))^q$ *and* $(\Psi_M(j-r))^q = (\Psi_M(j-r))^q$ *for all* $j, r \in \Gamma$.

This theorem's proof is simple to understand.

Theorem 3.6. *Let* $\mathbb{I}$ *be q-ROFI. Then* $\mathbb{M} = \{j \in \Gamma | (\delta_I(0))^q = (\delta_I(j))^q$ *and* $(\Psi_I(0))^q = (\Psi_I(w))^q\}$ *is a q-ROFI of a ring* Γ *and 0 denotes the identity element in* Γ.

Proof. Assume $\mathbb{M}$ is q-ROFS of a ring Γ also $0 \in \mathbb{M}$ $\mathbb{M}$ is q-ROFI of a ring Γ. Let $j, r \in \Gamma$. $(\delta_I(r))^q = (\delta_I(0))^q$ and $(\Psi_I(r))^q = (\Psi_I(0))^q$ such that,

$$\begin{aligned}
(\delta_I(j-r))^q &\geq \{(\delta_I(j))^q \wedge (\delta_I(r))^q\} \\
&= \{(\delta_I(0))^q \wedge (\delta_I(0))^q\}. \\
(\delta_I(j-r))^q &\geq (\delta_I(0))^q. && (3.1) \\
\text{But we know that } (\delta_I(j-r))^q &\leq (\delta_I(0))^q. && (3.2) \\
\text{By using Eqs (3.1)and (3.2) } (\delta_I(j-r))^q &= (\delta_I(0))^q. \\
\text{Similarly, } (\delta_I(jr))^q &\geq \{(\delta_I(j))^q \wedge (\delta_I(r))^q\} \\
&\geq \{(\delta_I(0))^q \wedge (\delta_I(0))^q\} \\
(\delta_I(jr))^q &\geq (\delta_I(0))^q && (3.3) \\
\text{But we know that } (\delta_I(jr))^q &\leq (\delta_I(0))^q. && (3.4) \\
\text{By using Eqs (3.3)and (3.4) } (\delta_I(j-r))^q &= (\delta_I(0))^q.
\end{aligned}$$

Since: $\mathbb{I}$ be q-ROFI so,
Furthermore,

$$\begin{aligned}
(\Psi_I(j-r))^q &\leq \{(\Psi_I(j))^q \vee (\Psi_I(r))^q\} \\
&= \{(\Psi_I(0))^q \vee (\Psi_I(0))^q\}. \\
(\Psi_I(j-r))^q &\leq (\Psi_I(0))^q. && (3.5) \\
\text{But we know that } (\Psi_I(j-r))^q &\geq (\Psi_I(0))^q. && (3.6) \\
\text{By using Eqs (3.5)and (3.6) } (\Psi_I(j-r))^q &= (\Psi_I(0))^q. \\
\text{Similarly, } (\Psi_I(jr))^q &\leq \{(\Psi_I(j))^q \vee (\Psi_I(r))^q\} \\
&\leq \{(\Psi_I(0))^q \vee (\Psi_I(0))^q\} \\
(\Psi_I(jr))^q &\leq (\Psi_I(0))^q. && (3.7) \\
\text{But we know that } (\Psi_I(jr))^q &\geq (\Psi_I(0))^q. && (3.8) \\
\text{By using Eqs (3.7)and (3.8) } (\Psi_I(j-r))^q &= (\Psi_I(0))^q.
\end{aligned}$$

Conditions are hold. So, $\mathbb{M}$ is a q-ROFI of a ring Γ. □

4. q-Rung orthopair fuzzy level subring

In this portion, we study the idea of q-rung orthopair fuzzy level subring (q-ROFLSR) and also discuss some important results.

Definition 4.1. Let $\mathbb{F} = \{r,\ \delta_F(r),\ \Psi_F(r)\}$ be a q-ROFS of crisp set of Γ and $(\xi, \tau) \in [0, 1]$ such that $0 \leq (\xi)^q + (\tau)^q \leq 1$. Then $\mathbb{F}_{(\xi,\tau)} = \{r \in \Gamma : (\delta_F(r))^q \geq \xi \ \ and \ \ (\Psi_F(r))^q \leq \tau\}$ is known as q-rung orthopair fuzzy level subset of q-ROFS $\mathbb{W}$ of Γ.

Theorem 4.2. *If* $\mathbb{F} = \{u, \delta_F(r),\ \Psi_F(r)\}$ *be a q-ROFS of* Γ *and* $\xi, \tau, \acute{\xi}, \acute{\tau} \in [0, 1]$. *Then*

(1) $\mathbb{F}_{(\xi,\tau)} \subseteq \mathbb{F}_{(\acute{\xi},\acute{\tau})}$ *if* $\acute{\xi} \leq \xi$ *and* $\tau \leq \acute{\tau}$,

(2) $\mathbb{F}_{(\xi,\tau)} \subseteq \mathbb{F}_{(\acute{\xi},\acute{\tau})}$ *if* $\mathbb{F} \subseteq \acute{\mathbb{F}}$.

Proof. (1) Let $l \in \mathbb{F}_{(\xi,\tau)}$ then $(\delta_F(l))^q \geq \xi$ and $(\Psi_F(l))^q \leq \tau$ such that $\acute{\xi} \leq \xi$ and $\tau \leq \acute{\tau}$, thus $(\delta_F(l))^q \geq \xi \geq \acute{\xi}$ and $(\Psi_F(l))^q \leq \tau \leq \acute{\tau}$. This implies that $l \in \mathbb{F}_{(\acute{\xi},\acute{\tau})}$. So, $\mathbb{F}_{(\xi,\tau)} \subseteq \mathbb{F}_{(\acute{\xi},\acute{\tau})}$.
(2) Let $l \in \mathbb{F}_{(\xi,\tau)}$ then $(\delta_F(l))^q \geq \xi$ and $(\Psi_F(l))^q \leq \tau$ and $\mathbb{F} \subseteq \acute{\mathbb{F}}$ therefore $(\delta_F(l))^q \leq (\delta_{\acute{F}(l)})^q$ and $(\Psi_F(l))^q \geq (\Psi_{\acute{F}(l)})^q$. Then consequently, $\xi \leq (\delta_F(l))^q \leq (\delta_{\acute{F}(l)})^q$ and $\tau \geq (\Psi_F(l))^q \geq (\Psi_{\acute{F}(l)})^q$, after this implementations we get $l \in \acute{\mathbb{F}}_{(\xi,\tau)}$. Hence, $\mathbb{F}_{(\xi,\tau)} \subseteq \acute{\mathbb{F}}_{(\xi,\tau)}$. □

Theorem 4.3. *If* $\mathbb{F} = \{r, \delta_F(r),\ \Psi_F(r)\}$ *be a q-ROFS of* Γ *and* $\xi, \tau, \acute{\xi}, \acute{\tau} \in [0, 1]$. *Then*

(1) $\mathbb{F}_{(\xi,\tau)} \subseteq \mathbb{F}_{(\acute{\xi},\acute{\tau})}$ *if* $\acute{\xi} \leq \xi$ *and* $\tau \leq \acute{\tau}$,

(2) $\mathbb{F}_{(\xi,\tau)} \subseteq \mathbb{F}_{(\acute{\xi},\acute{\tau})}$ *if* $\mathbb{F} \subseteq \acute{\mathbb{F}}$.

Proof. (1) Let $l \in \mathbb{F}_{(\xi,\tau)}$ then $(\delta_F(l))^q \geq \xi$ and $(\Psi_F(l))^q \leq \tau$ such that $\acute{\xi} \leq \xi$ and $\tau \leq \acute{\tau}$, thus $(\delta_F(l))^q \geq \xi \geq \acute{\xi}$ and $(\Psi_F(l))^q \leq \tau \leq \acute{\tau}$. This implies that $l \in \mathbb{F}_{(\acute{\xi},\acute{\tau})}$. So, $\mathbb{F}_{(\xi,\tau)} \subseteq \mathbb{F}_{(\acute{\xi},\acute{\tau})}$.
(2) Let $l \in \mathbb{F}_{(\xi,\tau)}$ then $(\delta_F(l))^q \geq \xi$ and $(\Psi_F(l))^q \leq \tau$ and $\mathbb{F} \subseteq \acute{\mathbb{F}}$ therefore $(\delta_F(l))^q \leq (\delta_{\acute{F}(l)})^q$ and $(\Psi_F(l))^q \geq (\Psi_{\acute{F}(l)})^q$. Then consequently, $\xi \leq (\delta_F(l))^q \leq (\delta_{\acute{F}(l)})^q$ and $\tau \geq (\Psi_F(l))^q \geq (\Psi_{\acute{F}(l)})^q$, after this implementations we get $l \in \acute{\mathbb{F}}_{(\xi,\tau)}$. Hence, $\mathbb{F}_{(\xi,\tau)} \subseteq \acute{\mathbb{F}}_{(\xi,\tau)}$. □

Theorem 4.4. *A q-ROFS* $\mathbb{F}$ *of ring* Γ *is a q-ROFSR of* Γ *if and only if q-rung orthopair fuzzy level set* $\mathbb{F}_{(\xi,\tau)}$ *of subring* Γ *is a subring of* Γ.

Proof. By the definition of q-rung orthopair fuzzy level set $\mathbb{F}_{(\xi,\tau)} = \{r \in \Gamma : (\delta_F(r))^q \geq \xi and (\Psi_F(r))^q \leq \tau\}$. For all entries $(\xi, \tau) \in [0, 1]$ we have $(\delta_F(l))^q \geq \xi$ and $(\Psi_F(l))^q \leq \tau$. As a result, at least $0 \in \mathbb{F}_{(\xi,\tau)}$ which is show that $\mathbb{F}_{(\xi,\tau)}$ is non-empty.

Let $l, r \in \mathbb{F}_{(\xi,\tau)}$ then by definition of q-rung orthopair fuzzy level set $\mathbb{F}_{(\xi,\tau)}$ we have $(\delta_F(l))^q \geq \xi,\ \delta_F(w))^q \geq \xi$ and $(\Psi_F(l))^q \leq \tau,\ (\Psi_F(r))^q \leq \tau$. Here, $\mathbb{F}$ is a q-ROFSR of Γ. So,

$$\begin{aligned}
(\delta_F(l-r))^q &\geq \{(\delta_F(l))^q \wedge (\delta_F(r))^q\} \\
&= \{(\delta_F(l))^q \wedge (\delta_F(-r))^q\} \\
&= \{(\delta_F(l))^q \wedge (\delta_F(r))^q\} \\
&\geq \{\xi \wedge \xi\}
\end{aligned}$$

$$
\begin{aligned}
(\delta_F(l-r))^q &\geq \{\xi\}. \\
(\delta_F(lr))^q &\geq \{(\delta_F(l))^q \wedge (\delta_F(r))^q\} \\
&= \{\xi \wedge \xi\} \\
(\delta_F(lr))^q &\geq \xi.
\end{aligned}
$$

$$
\begin{aligned}
\text{and } (\Psi_F(l-r))^q &\leq \{(\Psi_F(l))^q \vee (\Psi_F(r))^q\} \\
&= \{(\Psi_F(l))^q \vee (\Psi_F(-r))^q\} \\
&= \{(\Psi_F(l))^q \vee (\Psi_F(r))^q\} \\
&\leq \{\tau \vee \tau\} \\
(\Psi_F(l-r))^q &\leq \{\tau\}. \\
(\Psi_F(lr))^q &\leq \{(\Psi_F(l))^q \vee (\Psi_F(r))^q\} \\
&= \{\tau \vee \tau\} \\
(\Psi_F(lu))^q &\leq \tau.
\end{aligned}
$$

Thus, $(l-r), (lr) \in \mathbb{F}_{(\xi,\tau)}$ by subring criteria $\mathbb{F}_{(\xi,\tau)}$ is a subring of Γ.
Conversely, suppose $\mathbb{F}$ q-ROFS Γ and $\forall\ \xi, \tau \in [0,1]$, $\mathbb{F}_{(\xi,\tau)}$ be subring of Γ. Let $r, l \in \Gamma$ such that $(\delta_F(l))^q = \xi_1$, $(\delta_F(r))^q = \xi_2$, $(\Psi_F(l))^q = \tau_1$, $(\Psi_F(r))^q = \tau_2$. Then $l, r \in \mathbb{F}_{((\xi_1\wedge,\xi_2),\ (\tau_1,\wedge\tau_2))}$ and $\mathbb{F}_{((\xi_1\wedge,\xi_2),\ (\tau_1,\wedge\tau_2))}$ is q-ROFSR of Γ, therefore $(lr), (l-r) \in \mathbb{F}_{((\xi_1\wedge,\xi_2),\ (\tau_1,\wedge\tau_2))}$. By using this

$$
\begin{aligned}
(\delta_F(l-r))^q &\geq \{\xi_1 \wedge \xi_2\} \\
&\geq \{(\delta_F(l))^q \wedge (\delta_F(r))^q\} \\
(\delta_F(l-r))^q &= \{(\delta_F(l))^q \wedge (\delta_F(-r))^q\}.
\end{aligned}
$$

Since, $\qquad (\delta_F(l))^q \leq \{(\delta_F(-r))^q\}$.

$$
\begin{aligned}
(\delta_F(lr))^q &\geq \{\xi_1 \wedge \xi_2\} \\
&= \{(\delta_F(l))^q \wedge (\delta_F(r))^q\}.
\end{aligned}
$$

$$
\begin{aligned}
(\Psi_F(l-r))^q &\leq \{\tau_1 \vee \tau_2\} \\
&\leq \{(\Psi_F(l))^q \vee (\Psi_F(r))^q\} \\
(\Psi_F(j-w))^q &= \{(\Psi_F(l))^q \vee (\Psi_F(-r))^q\}.
\end{aligned}
$$

Since, $(\Psi_F(r))^q \geq \{(\Psi_F(-r))^q\}$.

$$
\begin{aligned}
(\Psi_F(lr))^q &\leq \{\tau_1 \vee \tau_2\} \\
&= \{(\Psi_F(l))^q \vee (\Psi_F(r))^q\}.
\end{aligned}
$$

Hence, $\mathbb{F}$ is a q-ROFSR of Γ. □

5. Homomorphism on q-rung orthopair fuzzy subring

The effect of ring homomorphism on q-ROFSR is examined in this section. We examine the q-ROFSR's pre-image and image homomorphism properties. Additionally, investigate the pre-image homomorphism on q-ROFSR and different aspects of images.

Theorem 5.1. *Let Γ_1 and Γ_2 are two subrings. Let κ be surjective homomorphism from Γ_1 to Γ_2 and $\mathbb{I} = \{w_1,\ \delta_I(w_1),\ \Psi_I(w_1) : w_1 \in \Gamma_1\}$ be q-ROFSR of Γ_1. Then $\kappa(\mathbb{I}) = \{w,\ \delta_I(w),\ \Psi_I(w) : w \in \Gamma_2\}$ is a q-ROFSR of Γ_2.*

Proof. Let $\kappa : \Gamma_1 \to \Gamma_2$ is a onto homomorphism, then $\kappa(\Gamma_1) = \Gamma_2$. Let w_2, j_2 be elements of Γ_2. Suppose $j_2 = \kappa(j_1)$, $w_2 = \Gamma(w_1)$, $\kappa(j_1w_1) = \kappa(j_1)\kappa(w_1) = j_2w_2$ and $\kappa(j_1 - w_1) = \kappa(j_1) - \kappa(w_1) = j_2 - w_2$.

$$\begin{aligned}
(\delta_{\kappa(I)}(j_1 - w_1))^q &= \{(\delta_I(z)|z \in \Gamma_1 \vee \kappa(z) = (j_1 - w_1))^q\}\\
&= \{((\delta_I(z)))^q|z \in \Gamma_1 \vee \kappa(z) = (j_1 - w_1)\}\\
&= \{(\delta_I(j_1 - w_1))^q|z \in \Gamma_1, \{\kappa(j_1) = j_2, \kappa(w_1) = w_2\} \vee \{\kappa(z) = (j_1 - w_1)\\
&\quad \text{and } \kappa(j_1 - w_1) = \kappa(j_1) - \kappa(w_1) = j_2 - w_2\}\}. \ (\textit{Because } \kappa \textit{ is homomorphism})\\
&\geq \max\{\{(\delta_I(j_1))^q \ \wedge\ (\delta_I(j_2))^q\}|j_1, w_1 \in \Gamma_1, \{\kappa(j_1) = j_2, \kappa(w_1) = w_2\}\}.\\
&\qquad \text{I is } q\text{-ROFSR } \Gamma_1.\\
&= \{\max((\delta_I(j_1))^q : j_1 \in \Gamma_1, \kappa(j_1) = j_2) \wedge \max((\delta_I(j_2))^q : j_1 \in \Gamma_1, \kappa(j_1) = j_2)\}\\
&= \{(\delta_{\kappa(I)}(j_2))^q \wedge (\delta_{\kappa(I)}(w_2))^q\}.\\
\text{So, } (\delta_{\kappa(I)}(j_1 - w_1))^q &\geq \{(\delta_{\kappa(I)}(j_2))^q \wedge (\delta_{\kappa(I)}(w_2))^q\} \text{ for all } j_2, w_2 \in \Gamma_2.
\end{aligned}$$

$$\begin{aligned}
(\delta_{\kappa(I)}(j_1w_1))^q &= (\delta_I(z)|z \in \Gamma_1 \vee \kappa(z) = (j_1w_1))^q\\
&= \{((\delta_I(z)))^q|z \in \Gamma_1 \vee \kappa(z) = (j_1w_1)\}\\
&= \{(\delta_I(j_1w_1))^q|z \in \Gamma_1, \{\kappa(j_1) = j_2, \kappa(w_1) = w_2\} \vee \{\kappa(z) = (j_1w_1)\\
&\quad \text{and } \kappa(j_1w_1) = \kappa(j_1)\kappa(w_1) = j_2w_2\}\}. \ \textit{Because } \kappa \textit{ is homomorphism}\\
&\geq \max\{\{(\delta_I(j_1))^q \ \wedge\ (\delta_I(j_2))^q\}|j_1, w_1 \in \Gamma_1, \{\kappa(j_1) = j_2, \kappa(w_1) = w_2\}\}\\
&\qquad \text{I is } q\text{-ROFSR of } \Gamma_1\\
&= \{\max((\delta_I(j_1))^q : j_1 \in \Gamma_1, \kappa(j_1) = j_2) \wedge \max((\delta_I(j_2))^q : j_1 \in \Gamma_1, \kappa(j_1) = j_2)\}\\
&= \{(\delta_{\kappa(I)}(j_2))^q \wedge (\delta_{\kappa(I)}(w_2))^q\}.\\
\text{Therefore, } (\delta_{\kappa(I)}(j_1w_1))^q &\geq \{(\delta_{\kappa(I)}(j_2))^q \wedge (\delta_{\kappa(I)}(w_2))^q\} \text{ for all } j_2, w_2 \in \Gamma_2.
\end{aligned}$$

Similar, for non membership $\Psi_{\kappa(I)}$.

$$\begin{aligned}
(\Psi_{\kappa(I)}(j_1 - w_1))^q &= (\Psi_I(z)|z \in \Gamma_1 \wedge \kappa(z) = (j_1 - w_1))^q\\
&= \{((\Psi_I(z)))^q|z \in \Gamma_1 \wedge \kappa(z) = (j_1 - w_1)\}\\
&= \{(\Psi_I(j_1 - w_1))^q|z \in \Gamma_1,\ \{\kappa(j_1) = j_2, \kappa(w_1) = w_2\} \wedge \{\kappa(z) = (j_1 - w_1)\\
&\quad \text{and } \kappa(j_1 - w_1) = \kappa(j_1) - \kappa(w_1) = j_2 - w_2\}\}. \ \textit{Because } \kappa \textit{ is homomorphism}
\end{aligned}$$

$$
\begin{aligned}
&\leq && \min\{\{(\Psi_I(j_1))^q \ \vee \ (\Psi_I(j_2))^q\}|\ j_1,\ w_1 \in \Gamma_1, \{\kappa(j_1) = j_2,\ \kappa(w_1) = w_2\}\} \\
& && \ \text{I is } q\text{-ROFSR of } \Gamma_1 \\
&= && \{\min((\Psi_I(j_1))^q : j_1 \in \Gamma_1, \kappa(j_1) = j_2) \vee \min((\Psi_I(j_2))^q : j_1 \in \Gamma_1,\ \kappa(j_1) = j_2)\} \\
&= && \{(\Psi_{\kappa(I)}(j_2))^q \vee (\Psi_{\kappa(I)}(w_2))^q\}. \\
\text{So, } (\Psi_{\kappa(I)}(j_1 - w_1))^q &\leq && \{(\Psi_{\kappa(I)}(j_2))^q \vee (\Psi_{\kappa(I)}(w_2))^q\} \text{ for all } j_2,\ w_2 \in \Gamma_2.
\end{aligned}
$$

$$
\begin{aligned}
(\Psi_{\kappa(I)}(j_1 w_1))^q &= (\Psi_I(z)|z \in \Gamma_1 \wedge \kappa(z) = (j_1 w_1))^q \\
&= \{((\Psi_I(z)))^q|z \in \Gamma_1 \wedge \kappa(z) = (j_1 w_1)\} \\
&= \{(\Psi_I(j_1 w_1))^q|z \in \Gamma_1,\ \{\kappa(j_1) = j_2,\ \kappa(w_1) = w_2\} \wedge \{\kappa(z) = (j_1 w_1) \\
&\quad \text{and } \kappa(j_1 w_1) = \kappa(j_1)\kappa(w_1) = j_2 w_2\}\}. \textit{ Because } \kappa \textit{ is homomorphism} \\
&\leq \min\{\{(\Psi_I(j_1))^q \ \vee \ (\Psi_I(j_2))^q\}|\ j_1,\ w_1 \in \Gamma_1,\ \{\kappa(j_1) = j_2,\ \kappa(w_1) = w_2\}\} \\
&\quad \ \text{I is } q\text{-ROFSR of } \Gamma_1. \\
&= \{\min((\Psi_I(j_1))^q : j_1 \in \Gamma_1,\ \kappa(j_1) = j_2) \vee \min((\Psi_I(j_2))^q : j_1 \in \Gamma_1,\ \kappa(j_1) = j_2)\} \\
&= \{(\Psi_{\kappa(I)}(j_2))^q \vee (\Psi_{\kappa(I)}(w_2))^q\}. \\
\text{Thus, } (\Psi_{\kappa(I)}(j_1 w_1))^q &\leq \{(\Psi_{\kappa(I)}(j_2))^q \vee (\Psi_{\kappa(I)}(w_2))^q\} \text{ for all } j_2,\ w_2 \in \Gamma_2.
\end{aligned}
$$

$\kappa(\mathbb{I}) = \{w_2, \delta_I(w), \Psi_I(w) : w_2 \in \Gamma\}$ is a q-ROFSR of Γ_2. □

The converse of Theorem (5.1) is hold in the form of Theorem (5.2).

Theorem 5.2. *Let $\kappa : \Gamma_1 \to \Gamma_2$ is a bijective homomorphism and $\mathbb{T} = \{w_2,\ \delta_T(w_2),\ \Psi_T(w_2) : w_2 \in \Gamma_2\}$ is a q-ROFSR of Γ_2 such that $\kappa^{-1}(\mathbb{I}) = \{w_1,\ \delta_{\kappa^{-1}T}(w_1),\ \Psi_{\kappa^{-1}T}(w_1) : w_1 \in \Gamma_1\}$ is a q-ROFSR of Γ_1.*

Proof. Consider $j_1,\ w_1 \in \Gamma_1$, then $j_1 w_1 \in \Gamma_2$. Now,

$$
\begin{aligned}
\delta_{\kappa^{-1}(T)}(j_1 - w_1))^q &= (\delta_{(T)}\kappa(j_1 - w_1))^q \\
&= (\delta_{(T)}\kappa(j_1) - \kappa(w_1))^q \ (\kappa \text{ is homomorphism}) \\
&\geq \{(\delta_T(j_1))^q \wedge (\delta_T(w_1))^q\}(\text{T is } q\text{-ROFSR of } \Gamma_2). \\
&\geq \{\delta_{\kappa^{-1}(T)}(j_1))^q \wedge \delta_{\kappa^{-1}(T)}(w_1))^q\}.
\end{aligned}
$$

Similarly in case of non membership,

$$
\begin{aligned}
\Psi_{\kappa^{-1}(T)}(j_1 - w_1))^q &= (\Psi_{(T)}\kappa(j_1 - w_1))^q \\
&= (\Psi_{(T)}\kappa(j_1)\kappa(w_1))^q \ (\kappa \text{ is homomorphism}) \\
&\leq \{(\Psi_T(j_1))^q \vee (\Psi_T(w_1))^q\}(\text{T is } q\text{-ROFSR of } \Gamma_2). \\
&\leq \{\Psi_{\kappa^{-1}(T)}(j_1))^q \vee \Psi_{\kappa^{-1}(T)}(w_1))^q\}
\end{aligned}
$$

$$
\begin{aligned}
\delta_{\kappa^{-1}(T)}(j_1 w_1))^q &= (\delta_{(T)}\kappa(j_1 w_1))^q \\
&= (\delta_{(T)}\kappa(j_1)\kappa(w_1))^q \ (\kappa \text{ is homomorphism}) \\
&\geq \{(\delta_T(j_1))^q \wedge (\delta_T(w_1))^q\}(\text{T is } q\text{-ROFSR of } \Gamma_2). \\
&\geq \{\delta_{\kappa^{-1}(T)}(j_1))^q \wedge \delta_{\kappa^{-1}(T)}(w_1))^q\}.
\end{aligned}
$$

In other words,

$$\begin{aligned}
\Psi_{\kappa^{-1}(T)}(j_1w_1))^q &= (\Psi_{(T)}\kappa(j_1w_1))^q \\
&= (\Psi_{(T)}\kappa(j_1)\kappa(w_1))^q \ (\kappa \text{ is homomorphism}) \\
&\leq \{(\Psi_T(j_1))^q \vee (\Psi_T(w_1))^q\}(\text{T is } q\text{-ROFSR of } \Gamma_2). \\
&\leq \{\Psi_{\kappa^{-1}(T)}(j_1))^q \vee \Psi_{\kappa^{-1}(T)}(w_1))^q\}.
\end{aligned}$$

□

Theorem 5.3. *If* $\kappa : \Gamma_1 \to \Gamma_2$ *is surjective homomorphism,* Γ_1, Γ_2 *are two subrings and* $\mathbb{I}$ *be q-ROFI of* Γ_1 *then* $\kappa(\mathbb{I})$ *is q-ROFI of* Γ_2.

Proof. Suppose $\mathbb{I}$ be q-ROFI of Γ_1, then $(\delta_I(j_1-w_1))^q = (\delta_I(w_1-j_1))^q$ and $(\Psi_I(j_1-w_1))^q = (\Psi_M(w_1-j_1))^q$ for all $j_1,\ w_1 \in \Gamma_1$. Let $j_2,\ w_2 \in \Gamma_2$. Then there exist some elements $j_1,\ w_1 \in \Gamma_1$ for this $w_2 = \kappa(w_1)$ *and* $j_2 = \kappa(j_1)$. Then,

$$\begin{aligned}
(\delta_{\kappa(I)}(j_2 - w_2))^q &= \{(\delta_I(z)|z \in \Gamma_1 \vee \kappa(z) = (j_2 - w_2))^q\} \\
&= \{((\delta_I(z)))^q|z \in \Gamma_1 \vee \kappa(z) = (j_2 - w_2)\} \\
&= \{(\delta_I(j_2 - w_2))^q|z \in \Gamma_1, \{\kappa(j_1) = j_2, \kappa(w_1) = w_2\} \vee \{\kappa(z) = (j_2 - w_2) \\
&\quad \text{and } \kappa(j_2 - w_2) = \kappa(j_1) - \kappa(w_1) = j_2 - w_2\}\}. \textit{ Because } \kappa \textit{ is homomorphism} \\
&\geq \max\{\{(\delta_I(j_1))^q \ \wedge \ (\delta_I(j_2))^q\}|j_1,\ w_1 \in \Gamma_1,\ \{\kappa(j_1) = j_2,\ \kappa(w_1) = w_2\}\} \\
&= \{\max((\delta_I(j_1))^q : j_1 \in \Gamma_1,\ \kappa(j_1) = j_2) \wedge \max((\delta_I(j_2))^q : j_1 \in \Gamma_1,\ \kappa(j_1) = j_2)\} \\
&= \{(\delta_{\kappa(I)}(j_2))^q \wedge (\delta_{\kappa(I)}(w_2))^q\}.
\end{aligned}$$

$$\text{So, } (\delta_{\kappa(I)}(j_2 - w_2))^q \geq \{(\delta_{\kappa(I)}(j_2))^q \wedge (\delta_{\kappa(I)}(w_2))^q\} \text{ for all } j_2,\ w_2 \in \Gamma_2.$$

$$\begin{aligned}
(\delta_{\kappa(I)}(j_1w_1))^q &= \{(\delta_I(z)|z \in \Gamma_1 \vee \kappa(z) = (j_1w_1))^q\} \\
&= \{((\delta_I(z)))^q|z \in \Gamma_1 \vee \kappa(z) = (j_1w_1)\} \\
&= \{(\delta_I(j_1w_1))^q|z \in \Gamma_1,\ \{\kappa(j_1) = j_2,\ \kappa(w_1) = w_2\} \vee \{\kappa(z) = (j_1w_1) \\
&\quad \text{and } \kappa(j_1w_1) = \kappa(j_1)\kappa(w_1) = j_2w_2\}\}. \textit{ Because } \kappa \textit{ is homomorphism} \\
&\geq \max\{\{(\delta_I(j_1))^q \ \wedge \ (\delta_I(j_2))^q\}|j_1, w_1 \in \Gamma_1,\ \{\kappa(j_1) = j_2, \kappa(w_1) = w_2\}\} \\
&= \{\max((\delta_I(j_1))^q : j_1 \in \Gamma_1,\ \kappa(j_1) = j_2) \wedge \max((\delta_I(j_2))^q : j_1 \in \Gamma_1,\ \kappa(j_1) = j_2)\} \\
&= \{(\delta_{\kappa(I)}(j_2))^q \wedge (\delta_{\kappa(I)}(w_2))^q\}.
\end{aligned}$$

$$\text{So, } (\delta_{\kappa(I)}(j_1w_1))^q \geq \{(\delta_{\kappa(I)}(j_2))^q \wedge (\delta_{\kappa(I)}(w_2))^q\} \text{ for all } j_2,\ w_2 \in \Gamma_2.$$

In same way we can show that $(\Psi_{\kappa(I)}(j_2 - w_2))^q \leq \{(\Psi_{\kappa(I)}(j_2))^q \vee (\Psi_{\kappa(I)}(w_2))^q\}$ and $(\Psi_{\kappa(I)}(j_1w_1))^q \leq \{(\Psi_{\kappa(I)}(j_2))^q \vee (\Psi_{\kappa(I)}(w_2))^q\}$ for all $j_2, w_2 \in \Gamma_2$. Hence, all the axiom of q-ROFI are hold. So, $\kappa(\mathbb{I})$ is q-ROSI of Γ_2. □

Theorem 5.4. *Let* Γ_1, Γ_2 *are two subrings. Let* κ *be bijective homomorphism* $\kappa : \Gamma_1 \to \Gamma_2$ *and* $\mathbb{I} = \{w_1,\ \delta_I(w_1),\ \Psi_I(w_1) : w_1 \in \Gamma_1\}$ *be q-ROFI of* Γ_2. *Then* $\kappa^{-1}(\mathbb{I}) = \{w_2,\ \delta_I(w_2),\ \Psi_I(w) : w_1 \in \Gamma_1\}$ *is a q-ROFSR of* Γ_1.

Proof. $\mathbb{I}$ be q-ROFI of Γ_2 then $(\delta_I(j_2 - w_2))^q = (\delta_I(w_2 - j_2))^q$ and $(\Psi_I(j_2 - w_2))^q = (\Psi_M(w_2 - j_2))^q$ for all $j_2, w_2 \in \Gamma_2$. Suppose that $j_1,\ w_1 \in \Gamma_1$.then there exist some elements $j_1,\ w_1 \in \Gamma_1$ for this $w_2 = \kappa(w_1)$ *and* $j_2 = \kappa(j_1)$. Let w_2, j_2 be elements of Γ_2. Suppose $j_2 = \kappa(j_1)$, $w_2 = \Gamma(w_1), \kappa(j_1 w_1) = \kappa(j_1)\kappa(w_1) = j_2 w_2$ and $\kappa(j_1 - w_1) = \kappa(j_1) - \kappa(w_1) = j_2 - w_2$.

$$
\begin{aligned}
(\delta_{\kappa^{-1}(I)}(j_1 - w_1))^q &= (\kappa^{-1}(\delta_I)((j_1 - w_1))^q \\
(\delta_{\kappa^{-1}(I)}(j_1 - w_1))^q &= (\delta_I(\kappa(j_1 - w_1)))^q \\
&= \{(\delta_I(j_1 - w_1))^q | (j_1 - w_1) \in \Gamma_1, \{\kappa(j_1) = j_2, \kappa(w_1) = w_2\} \vee \{\kappa(z) = (j_1 - w_1) \\
&\quad\ \text{and } \kappa(j_1 - w_1) = \kappa(j_1) - \kappa(w_1) = j_2 - w_2\}\}. \\
&\geq \{\max((\delta_I(j_1))^q : j_1 \in \Gamma_1, \kappa(j_1) = j_2) \wedge \max((\delta_I(j_2))^q : j_1 \in \Gamma_1, \kappa(j_1) = j_2)\} \\
&= \{(\delta_{\kappa(I)}(j_2))^q \wedge (\delta_{\kappa(I)}(w_2))^q\}. \\
\text{So, } (\delta_{\kappa(I)}(j_2 - w_2))^q &\geq \{(\delta_{\kappa(I)}(j_2))^q \wedge (\delta_{\kappa(I)}(w_2))^q\} \text{ for all } j_2, w_2 \in \Gamma_2.
\end{aligned}
$$

Similarly, $(\Psi_{\kappa(I)}(j_2 - w_2))^q \leq \{(\Psi_{\kappa(I)}(j_2))^q \vee (\Psi_{\kappa(I)}(w))^q\}$ for all $j_2, w_2 \in \Gamma_2$.

$$
\begin{aligned}
(\delta_{\kappa^{-1}(I)}(j_1 w_1))^q &= (\kappa^{-1}(\delta_I)((j_1 w_1))^q \\
(\delta_{\kappa^{-1}(I)}(j_1 w_1))^q &= (\delta_I(\kappa(j_1 w_1)))^q \\
&= \{(\delta_I(j_1 w_1))^q | (j_1 w_1) \in \Gamma_1, \{\kappa(j_1) = j_2, \kappa(w_1) = w_2\} \vee \{\kappa(z) = (j_1 w_1) \\
&\quad\ \text{and } \kappa(j_1 w_1) = \kappa(j_1)\kappa(w_1) = j_2 w_2\}\}. \textit{ Because } \kappa \textit{ is homomorphism} \\
&\geq \{\max((\delta_I(j_1))^q : j_1 \in \Gamma_1, \kappa(j_1) = j_2) \wedge \max((\delta_I(j_2))^q : j_1 \in \Gamma_1, \kappa(j_1) = j_2)\} \\
&= \{(\delta_{\kappa(I)}(j_2))^q \wedge (\delta_{\kappa(I)}(w_2))^q\}. \\
\text{So, } (\delta_{\kappa(I)}(j_2 w_2))^q &\geq \{(\delta_{\kappa(I)}(j_2))^q \wedge (\delta_{\kappa(I)}(w_2))^q\} \text{ for all } j_2, w_2 \in \Gamma_2.
\end{aligned}
$$

Hence, $\kappa^{-1}(\mathbb{I})$ is a q-ROFSR of Γ_1. □

6. Conclusions

The Objectives of this article is to introduce q-ROFSR and their operations. We have examined certain algebraic features of q-ROFSR. We generated the essential and appropriate parameters for q-ROFSR. Every Pythagorean fuzzy subring is q-ROFSR, like we have shown, but the converse is not true. We have introduced q-rung orthopair fuzzy quotient ring and q-rung orthopair fuzzy left ideal right ideals. We got q-rung orthopair fuzzy level subring is an ideal. Moreover, we discussed the consequence of homomorphism on q-ROFSR. In the future, we intend to define q-ROFSR on various algebraic structures, such as isomorphism on subrings and subfields under the action of q-ROFSR. We will also apply the Q-ROFS to various complex algebraic structures.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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