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*Research article*

## **Bipartite fixed-time output containment control of heterogeneous linear multi-agent systems**

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**Abstract:** This study researches the bipartite fixed-time output containment control problem of heterogeneous linear MASs with signed digraphs. The leaders' states can be estimated by a designed distributed bipartite compensator. Furthermore, each follower is allocated a time-varying coupling weight, an adaptive bipartite fixed-time protocol is raised which can estimate the leader's system matrix but also the leader's state. On the foundation of control protocols, followers' outputs are included by the convex hull constituted by leaders' outputs. In addition, by utilizing the Lyapunov function theory, some abundant speculative knowledges are deduced to guarantee adaptive bipartite fixed-time output containment of multi-agent systems. Finally, the feasibility of the anticipant theoretical results is verified by a set of simulation examples.

**Keywords:** adaptive control; multi-agent systems; antagonistic interactions; bipartite output containment; fixed-time control

**Mathematics Subject Classification:** 90C29, 93-08

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### **1. Introduction**

Cooperative control of multi-agent systems has aroused profound interest over the past couple of decades on account of its extensive applications in disparate fields, such as smart grids, transportation, sensor networks [1–4], etc. Consensus is a widespread study orientation of cooperative control, which has produced some classical literatures [5–8]. Thereinto, the previously mentioned pluri-plusieurs leaders control of MASs is diffusely applied during the practice, which can be regarded as the containment control problem. Containment control, as a normalized problem of distributed cooperative control, has developed into a consequential and vital area, in which the aim is to design an

appropriate distributed director that the dynamic convex hull constituted by leaders are guaranteed to include all followers in MASs. So far, fruitful results have been achieved on this topic, see, e.g., [9–15]. In addition, this multi-agent systems is also divided into homogeneity and heterogeneity. In contrast to homogeneous multi-agent systems, there are heterogeneous multi-agent systems in which the dynamics of each agent is not identical. In reality, the situation is even more complex, and it is clear that typical examples of heterogeneous multi-agent systems are much more common than the overly simple homogeneous multi-intelligent systems. The study of cooperative control of heterogeneous multi-intelligent systems is therefore more realistic and challenging, and also has more promising applications.

With respect to the research of containment control (CC) problem, the rate of convergence is the key index to judge the quality of the designed CC protocols. It's worth noting that the forthcoming CC protocols only ensure asymptotic realization of containment control, which illustrates that containment control can not be completed in finite time. Nevertheless, it is often advisable to implement CC in a limited time frame in engineering applications. As a matter of fact, finite-time CC has many other advantages besides faster convergence rate, such as higher control precision and better anti-interference. Therefore, the finite-time CC problem has been investigated [16–20]. For instance, a finite-time adaptive containment control method for a nonlinear multi-agent system with actuator failures and mismatched disturbances was raised in [19], and it is proved that the errors of the control system are stable in finite time in the presence of actuator faults. In [20], an observer-based two-layer distributed containment control protocol was raised to overcome the related finite-time containment problem.

Notice that the setup time largely rests with the agents' the premier conditions in the finite-time protocols. It's hard to calculate settling time accurately since it's often difficult to obtain the exact information on the initial state of the agents, which restrict the use of the finite-time CC protocols in practical applications. As a result, a fixed-time control method can be applied, in which an upper limit on the settling time can be confirmed independently of initial conditions. In recent years, distributed fixed-time control for nonlinear networked systems is discussed in [21] by using event/self-triggered method over directed graphs, so that the estimated settling time can be determined independently of the initial states of networked agents. In [22], the fixed-time containment control for second-order nonlinear multi-agent systems (MASs) is studied and a novel non-singular terminal sliding mode control protocol is designed to guarantee FTCC with distributed nonlinear MASs.

Beware that the communication links in the literature above are all non-negative. That is to say, all the relationships between agents are collaborative. Nevertheless, signed networks are more common than traditional multi-agent systems networks. In other words, the simultaneity between collaboration and competition relationships is more logical and appropriate. This type of problem is named the bipartite containment control problem [23], where the interaction among agents can be effectively modeled by signed graph, and the antagonistic/cooperative interaction between agents can be represented by negative and positive arcs respectively. Lately, the bipartite CC problem has been discussed [24–28]. Particularly, in [24], the bipartite containment tracking problem of a class of signed graphs leader-following networks was studied, and it was proved that leader-following networks can converge to symmetric trajectories containing the same convex hull and the same modulus but different signs of each leading trajectory. And taking [27] as an example, based on the nonlinear decomposition method of input quantization, an event-triggered control scheme was developed by

utilizing backstepping technology, which was based on a nonlinear decomposition approach of input quantization. Notwithstanding consequential achievement have been made in bipartite CC, and what is noteworthy is that little work has been done to deal with the finite/fixed-time bipartite containment control problems [29–32]. As far as we know, the system dynamics of the above problem are different from this paper, so there are problems that have not been solved.

Motivated by the aforesaid argumentation, the darrein target of this paper is to settle the problem of bipartite fixed-time output containment control for heterogeneous linear multi-agent systems with signed digraphs. Among others, the primary contributions of the article are given as follows.

(i) Inspired by [12] and [13], the text proposes a bipartite containment control protocol combined with an adaptive algorithm that estimates the system matrix of the leader and also the state of the leader. On the basis of the control protocol, the multiple agents in the system no longer depend on global information, which saves many measurement resources;

(ii) The bipartite containment control studied in this paper is achieved with a fixed time premise. A large number of results have been produced on containment control of multi-agent systems in asymptotic time. In contrast, containment control under fixed-time algorithms has many advantages, such as high accuracy and robustness of control, in addition to the fast convergence. Part of the inspiration for this thought is from [17, 20, 23];

(iii) Different from [28–32]. In this article, the object of study is linear time-invariant system. However, the problem of bipartite fixed-time output containment control on this base is comparatively few researched up till now.

The remainder of this article are the following: Section II renders preliminaries and Section III describes problem statement. The main results are shown in Section IV. The simulation results are shown in Section V. At last, in Section VI, some conclusions are presented.

## 2. Preliminaries

$\mathbb{R}$  stands for the set of real numbers.  $\mathbb{R}^N$  is the set of real  $N \times 1$  vectors, and we use  $\mathbb{R}^{N \times M}$  to denote the set of real  $N \times M$  matrices. In this paper, graph theory is utilized to signify the competitive-cooperative relationship between agents in MASs. A bunch of  $N+W$  agents as an illustration, their relationship can be represented by  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ , a weighed digraph, which is composed of a node set  $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ , an adjacency matrix  $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{(N+W) \times (N+W)}$  and an edge set  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ ,  $(v_j, v_i) \in \mathcal{E}$  denotes an edge representing agent  $i$  can acquire information from agent  $j$ , wherein, agent  $j$  and agent  $i$  are adjacent. And the in-degree matrix is denoted by  $\mathcal{D} = \text{diag} \left\{ \sum_{j=1}^N |a_{1j}|, \sum_{j=1}^N |a_{2j}|, \dots, \sum_{j=1}^N |a_{Nj}| \right\}$ . Thereby, It can be calculated that Laplacian matrix  $\mathcal{L} = \mathcal{D} - \mathcal{A}$ . In addition, there is a group  $\mathcal{V}$  consist of two subgroups  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , and define it by equations that  $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$  and  $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ , which means if  $v_j$  and  $v_i$  are existing the identical subgroup  $a_{ij} > 0$ ; or else  $a_{ij} < 0$ . In particular, the collaboration and competition are severally indicated by  $a_{ij} > 0$  and  $a_{ij} < 0$ . Furthermore, the signed digraph  $\mathcal{G}$  can also be called structural balance diagram. Finally,  $\sigma_i = 1$ , if  $v_i \in \mathcal{V}_1$ , and  $\sigma_i = -1$ , if  $v_i \in \mathcal{V}_2$ , respectively, which represents a symbolic parameter.

**Lemma 1.** [33] Let  $q = [q_1^T, q_2^T, \dots, q_N^T]^T \in \mathbb{R}^{Nn}$ , in which  $q_i \in \mathbb{R}^n$ ,  $i = 1, 2, \dots, N$ , afterwards, the inequality below holds:

$$q^T \text{sig}(q)^\alpha \geq n^{-\alpha} N^{\frac{1-\alpha}{2}} (q^T q)^{\frac{1+\alpha}{2}}, \quad (2.1)$$

where  $\alpha > 1$ .

**Lemma 2.** [33] Let  $q = [q_1^T, q_2^T, \dots, q_N^T]^T \in \mathbb{R}^{Nn}$ , in which  $q_i \in \mathbb{R}^n$ ,  $i = 1, 2, \dots, N$ , afterwards, the inequality below holds:

$$q^T \text{sig}(q)^\beta \geq (q^T q)^{\frac{1+\beta}{2}}, \quad (2.2)$$

where  $\beta \in (0, 1)$ .

**Lemma 3.** [21] There is a positive definite function continuously  $V(q) : \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ , in which  $\beta \in (0, 1), \alpha > 1$  and  $a, b, c > 0$ , we have

$$\dot{V}(q) + aV(q) + b(V(q))^\alpha + c(V(q))^\beta \leq 0, q \in \mathbb{R}^n \setminus \{0\}. \quad (2.3)$$

Afterwards, the settling time are as follows

$$T < \frac{1}{a(1-\beta)} \ln\left(\frac{a+c}{c}\right) + \frac{1}{b(\alpha-1)}. \quad (2.4)$$

**Lemma 4.** In this scenario,  $\mathcal{H}^* = \mathcal{D}\mathcal{H}\mathcal{D}$  and  $\mathcal{H}$  therein are positive definite, in which  $\mathcal{H}$  is defined later in (9).

**Lemma 5.** [9] If and only if  $\exists \mathcal{D} \in \mathcal{D}$ , i.e.,  $\mathcal{D}\mathcal{A}\mathcal{D}$ , is a non-negative matrix, the digraph  $\mathcal{G}$  is structurally equilibrium. Furthermore,  $\mathcal{D}$  can determine the bilaterality of the agents.

### 3. Problem statement

In this paper,  $\mathcal{F} = \{v_1, v_2, \dots, v_N\}$  can be considered as the followers set and the set of leaders is expressed by  $\mathcal{R} = \{v_{N+1}, v_{N+2}, \dots, v_{N+W}\}$ . On account of the graph theory, a heterogeneous linear multi-agent system is reckoned, then, the followers are expressed as hereunder mentioned:

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = C_i x_i, i \in \mathcal{F}, \end{cases} \quad (3.1)$$

where  $x_i \in \mathbb{R}^N$  and  $y_i \in \mathbb{R}^Q$  are the state and the output of the  $i$ -th follower, severally,  $u_i \in \mathbb{R}^P$  is the input of the  $i$ th follower, and  $A_i, B_i, C_i$  are the matrices with compatible dimensions. The leaders can be described by

$$\begin{cases} \dot{\omega}_k = A_0 \omega_k, \\ y_k = C_0 \omega_k, k \in \mathcal{R}, \end{cases} \quad (3.2)$$

where  $\omega_k \in \mathbb{R}^N$  is the state of the  $k$ -th leader and  $y_k \in \mathbb{R}^Q$  is the output of the  $k$ -th leader, separately.

**Definition 1.** In the case of  $(1-\lambda)x + \lambda y \in C$ , the set  $C \subseteq \mathbb{R}^N$  is convex for any  $x, y \in C$  and any  $\lambda \in [0, 1]$ . Let

$$Y_{\mathcal{L}} = \{y_{n+1}, -y_{n+1}, y_{n+2}, -y_{n+2}, \dots, y_{n+m}, -y_{n+m}\}$$

be the leaders' outputs set and the inverted sign.  $\text{Co}(Y_{\mathcal{L}})$  is the minimal convex set including the whole points in  $Y_{\mathcal{L}}$ . In other words, the convex hull of  $Y_{\mathcal{L}}$  is the combination of all convex of points.

**Definition 2.** under the circumstance of the signed graph  $\mathcal{G}$ , the above-mentioned problem of the systems (5) and (6) can be settled by the following guidelines:

First of all, to deal with the problem of bipartite output containment control, the output containment control problem need be solved foremost. No matter what the starting statuses of multi-agent system are, the convex hull that the outputs of leaders contain will embrace some followers' outputs.

$$\lim_{t \rightarrow \infty} \text{dist}(y_i, \text{Co}(Y_{\mathcal{L}}(t))) = 0, i \in \mathcal{F}.$$

Accordingly, the leaders' inverse output trajectories will include the outputs of other followers.

**Assumption 1.** The eigenvalues of the matrix  $A_0$  have zero real parts.

**Assumption 2.**  $B_i$  are of full-row ranks,  $i = 1, 2, \dots, N$ .

**Assumption 3.** For  $i = 1, 2, \dots, N$ , There are solutions  $(X_i, U_i)$ ,  $i = 1, 2, \dots, N$ , that satisfy the formulas below:

$$\begin{aligned} A_i X_i + B_i U_i &= X_i A_0, \\ C_i X_i - C_0 &= 0. \end{aligned} \quad (3.3)$$

**Assumption 4.** The  $\mathcal{G}$  is structurally equilibrium, there are at least one leader that has a directed spanning tree to it.

Ahead of researching more, the output containment error is caused by:

$$e_i = \sum_{j \in N_i} |a_{ij}| (y_i - \text{sgn}(a_{ij}) y_j) + \sum_{r=N+1}^{N+W} |a_{ik}| (y_i - \sigma_i y_k), i \in \mathcal{F}. \quad (3.4)$$

Turn the equation thereinbefore into matrix modality, hence let  $e = \text{col}(e_1, e_2, \dots, e_N)$ ,  $y = \text{col}(y_1, y_2, \dots, y_N)$ , then (8) can be represented as

$$e = (\mathcal{H} \otimes I_n) y - \sum_{r=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \bar{y}_k, \quad (3.5)$$

where  $\bar{y}_k = (\sigma_1 \sigma_2 \dots \sigma_N)^T \otimes y_k$ ,  $\mathcal{A}_{0k} = \text{diag}\{|a_{1k}|, |a_{2k}|, \dots, |a_{Nk}|\}$  and  $\mathcal{H} = \sum_{r=N+1}^{N+W} (1/W)L + \mathcal{A}_{0k}$ . In accordance with the problem in Definition 2, satisfy  $\lim_{t \rightarrow \infty} e_i = 0$ ,

$$\lim_{t \rightarrow \infty} y_i = \lim_{t \rightarrow \infty} \sum_{r=N+1}^{N+W} \zeta_{ik} y_k, k \in \mathcal{F}, \quad (3.6)$$

in which  $\zeta_{ik} \in \mathbb{R}$  indicates the element of  $\mathcal{H}^{-1} \mathcal{A}_{0k} 1_N$ . This manifests the bipartite output containment control problem can be settled by treating it as a adjustment problem of driving the  $e \rightarrow 0$ .

#### 4. Main results

In this section, we propose two main results of the article for the fixed-time bipartite containment control problem.

### 4.1. Bipartite fixed-time observer

In order to realize the above fixed-time bipartite output containment control of heterogeneous MASs, we present a protocol as follow:

$$\begin{cases} \dot{u}_i = K_i^1 x_i + K_i^2 f_i - d_1 K_i^3 \text{sig}\{x_i - X_i(t) f_i\}^{\tilde{\alpha}} - d_2 K_i^3 \text{sig}\{x_i - X_i(t) f_i\}^{\tilde{\beta}}, \\ \dot{f}_i = A_0 f_i - \mu_1 P \varsigma_i - \mu_2 \text{sig}\{P \varsigma_i\}^\alpha - \mu_3 \text{sig}\{P \varsigma_i\}^\beta, \end{cases} \quad (4.1)$$

where  $K_i^1, K_i^2, K_i^3 \in R^{p \times n}$  will be designed later in the Theorem 3.  $P \in R^{n_0 \times n_0} > 0$ ,  $d_1 > 0$ ,  $d_2 > 0$ ,  $\alpha > 0$ ,  $\beta > 0$ ,  $\mu_1 > 0$ ,  $\mu_2 > 0$ ,  $\mu_3 > 0$ ,  $\tilde{\alpha} > 1$  and  $\tilde{\beta} \in (0, 1)$ .  $\varsigma_i$  indicates the metrical information gathered by the  $i$ -th agent alternating with its neighbors, which is defined as:

$$\varsigma_i \triangleq \sum_{j=1}^N |a_{ij}| (f_i - \text{sgn}(a_{ij}) f_j) + \sum_{k=N+1}^{N+W} |a_{ik}| (f_i - \sigma_i \omega_k), \quad (4.2)$$

in which  $a_{ij}$  and  $a_{ik}$  are the the adjacency matrix  $A$ 's elements, and it can take another form as follow:

$$\varsigma = (\mathcal{H} \otimes I_n) f - \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \bar{\omega}. \quad (4.3)$$

Define the error variable  $\bar{f} = f - (\mathcal{H} \otimes I_n)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \bar{\omega}$ . Then

$$\dot{\bar{f}} = ((I_N \otimes A_0) - \mu_1 (\mathcal{H} \otimes P)) \bar{f} - \mu_2 \text{sig}\{(\mathcal{H} \otimes P) \bar{f}\}^\alpha - \mu_3 \text{sig}\{(\mathcal{H} \otimes P) \bar{f}\}^\beta. \quad (4.4)$$

Afterwards, we have the result as follow.

**Theorem 1.** Assume that Assumptions 1 and 4 are tenable for systems (6) and (11). Then,  $\lim_{t \rightarrow T} \bar{f} = 0$  holds if  $\mu_1 > 1/\lambda_1$ ,  $\mu_2 > 0$ ,  $\mu_3 > 0$ ,  $\alpha > 1$ ,  $\beta \in (0, 1)$ ,  $P$  satisfies

$$A_0^T P + P A_0 - 2P^2 + c I_{n_0} = 0, \quad (4.5)$$

where  $c > 0$ , and  $\lambda_1 = \lambda_{\min}(\mathcal{H})$ . Besides, the setup time are as follow

$$T < T^* \triangleq \frac{2\lambda_{\max}(\Gamma)}{c\lambda_1(1-\beta)} \ln \left\{ \frac{c\lambda_1}{2\mu_3\lambda_{\max}(\Gamma)} \left[ \frac{\lambda_{\min}(\Gamma^2)}{\lambda_{\max}(\Gamma)} \right]^{-\frac{1+\beta}{2}} + 1 \right\} + \frac{n_0^\alpha N^{\frac{\alpha-1}{2}}}{\mu_2(\alpha-1)} \left[ \frac{\lambda_{\min}(\Gamma^2)}{\lambda_{\max}(\Gamma)} \right]^{-\frac{1+\alpha}{2}}, \quad (4.6)$$

where  $\Gamma = \mathcal{H} \otimes P$ ,  $\Gamma^2 = \mathcal{H}^2 \otimes P^2$ .

*Proof.* On the condition of Assumption 4,  $H$  is positive definite and symmetrical. Next, there is an orthometric  $U \in R^{N \times N}$ , meeting  $UHU^T = J = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ .

Thinking about  $V = \bar{f}^T (\mathcal{H} \otimes P) \bar{f}$  as the preselected Lyapunov function for the system (14). Let  $\tilde{f} = (U \otimes I_{n_0}) \bar{f}$  and  $\xi = (\mathcal{H} \otimes P) \bar{f}$ . Take the time derivative of  $V$  with respect to (14) as follows

$$\begin{aligned} \dot{V} &= \bar{f}^T \left[ (\mathcal{H} \otimes (A_0^T P + P A_0)) \right] \bar{f} - 2\mu_1 \bar{f}^T (\mathcal{H}^2 \otimes P^2) \bar{f} \\ &\quad - 2\mu_2 \bar{f}^T (\mathcal{H} \otimes P) \text{sig}\{(\mathcal{H} \otimes P) \bar{f}\}^\alpha - 2\mu_3 \bar{f}^T (\mathcal{H} \otimes P) \text{sig}\{(\mathcal{H} \otimes P) \bar{f}\}^\beta \\ &= \tilde{f}^T \left[ J \otimes (A_0^T P + P A_0) - 2\mu_1 (J^2 \otimes P^2) \right] \tilde{f} - 2\mu_2 \xi^T \text{sig}\{\xi\}^\alpha - 2\mu_3 \xi^T \text{sig}\{\xi\}^\beta \\ &= \sum_{i=1}^N \tilde{f}_i^T \lambda_i (A_0^T P + P A_0 - 2\mu_1 \lambda_i P^2) \tilde{f}_i - 2\mu_2 \xi^T \text{sig}\{\xi\}^\alpha - 2\mu_3 \xi^T \text{sig}\{\xi\}^\beta. \end{aligned} \quad (4.7)$$

Due to  $\mu_1 > 1/\lambda_1$  and (15), one has

$$\sum_{i=1}^N \tilde{f}_i^T \lambda_i (A_0^T P + P A_0 - 2\mu_1 \lambda_i P^2) \tilde{f}_i \leq -c\lambda_1 \sum_{i=1}^N \tilde{f}_i^T \tilde{f}_i \leq -\frac{c\lambda_1}{\lambda_{\max}(\Gamma)} V. \quad (4.8)$$

In line with Lemma 1 and Lemma 2, one has

$$-2\mu_2 \xi^T \text{sig}\{\xi\}^\alpha - 2\mu_3 \xi^T \text{sig}\{\xi\}^\beta \leq -2\mu_2 n_0^{-\alpha} N^{\frac{1-\alpha}{2}} \left[ \frac{\lambda_{\min}(\Gamma^2)}{\lambda_{\max}(\Gamma)} \right]^{\frac{1+\alpha}{2}} V^{\frac{1+\alpha}{2}} - 2\mu_3 \left[ \frac{\lambda_{\min}(\Gamma^2)}{\lambda_{\max}(\Gamma)} \right]^{\frac{1+\beta}{2}} V^{\frac{1+\beta}{2}}. \quad (4.9)$$

Combining (18) and (19) with (17), it can be obtained

$$\dot{V} \leq -\frac{c\lambda_1}{\lambda_{\max}(\Gamma)} V - 2\mu_2 n_0^{-\alpha} N^{\frac{1-\alpha}{2}} \left[ \frac{\lambda_{\min}(\Gamma^2)}{\lambda_{\max}(\Gamma)} \right]^{\frac{1+\alpha}{2}} \times V^{\frac{1+\alpha}{2}} - 2\mu_3 \left[ \frac{\lambda_{\min}(\Gamma^2)}{\lambda_{\max}(\Gamma)} \right]^{\frac{1+\beta}{2}} V^{\frac{1+\beta}{2}}. \quad (4.10)$$

On account of  $\beta \in (0, 1)$  and  $\alpha > 1$ , it can be testified that  $[(1 + \beta)/2] \in (0, 1)$  and  $[(1 + \alpha)/2] > 1$  establish. In light of Lemma 3,  $\tilde{f} = 0$  is fast stable globally in fixed-time. Hence,  $\lim_{t \rightarrow T} \tilde{f} = 0$  holds, and  $T$  satisfies the (16) inequality. The proof is done.  $\square$

Afterwards, we will demonstrate that the control protocol designed on the base of the fixed-time observers is workable. The proof is similar to the demonstration of Theorem 3, so the procedure will be omitted.

#### 4.2. Adaptive bipartite fixed-time observer

In the previous section, we propose the fixed-time protocol to achieve the bipartite containment problem by a distributed bipartite compensator. Nonetheless, take note to the bipartite fixed-time observer, which is related to the leader's matrix  $A_0$  and the overall agents topology. In fact, it's not practical in many aspects that every follower needs to get  $A_0$ . In another aspect, followers do not know the global information in effect, in particular with grand MASs scale.

Based on the above reasons, we devise an adaptive bipartite fixed-time protocol ulteriorly, satisfying  $\lim_{t \rightarrow T_s} (A_{0i} - A_0) = 0$  and  $\lim_{t \rightarrow T} \tilde{f} = 0$ . Thereinto, the bipartite fixed-time observer can not merely complete the estimation of leader state and matrix  $A_0$  simultaneously, but also avert relying on the global information.

The design form of the adaptive bipartite fixed-time protocol is written as:

$$\begin{cases} u_i = K_i^1 x_i + K_i^2(t) f_i - d_1 K_i^3 \text{sig}\{x_i - X_i f_i\}^{\tilde{\alpha}} - d_2 K_i^3 \text{sig}\{x_i - X_i f_i\}^{\tilde{\beta}}, \\ \dot{A}_{0i} = -\kappa_1 \text{sig}\{\vartheta_i\}^\gamma - \kappa_2 \text{sig}\{\vartheta_i\}^\delta, \\ \dot{f}_i = A_{0i} f_i - c_i \mathcal{S}_i - \mu_2 \text{sig}\{\mathcal{S}_i\}^\alpha - \mu_3 \text{sig}\{\mathcal{S}_i\}^\beta, \\ \dot{c}_i = \mathcal{S}_i^T \mathcal{S}_i, c_i(0) = c_{i0}, \end{cases} \quad (4.11)$$

where  $A_{0i} \in R^{n_0 \times n_0}$  is the estimation of  $A_0$  in the protocol above,  $\vartheta_i = \sum_{j=1}^N |a_{ij}| (A_{0i} - A_{0j}) + a_{i0} (A_{0i} - A_0)$ , and  $c_i$  is coupling gain.  $\kappa_1 > 0, \kappa_2 > 0, \mu_2 > 0, \mu_3 > 0, \gamma > 0, \delta > 0, \alpha > 0$  and  $\beta > 0$  are the parameters which will be confirmed soon. Similarly,  $K_i^1, K_i^2(t), K_i^3 \in R^{P \times N}$  will be designed later in the Theorem 3.

Afterwards, we have the result as follow.

**Theorem 2.** Assume that Assumptions 1 and 4 are tenable for systems (6) and (21). Next, it gets: i)  $\lim_{t \rightarrow T_s} (A_{0i} - A_0) = 0$  holds if  $\kappa_1 > 0$ ,  $\kappa_2 > 0$ ,  $\gamma > 1$  and  $\delta \in (0, 1)$ ; ii) The setup times upper limit  $T$  and  $T_s$  are stationary, which have no connection with the initial conditions. Furthermore, each  $c_i$  converges to a certain bounded value.

*Proof.* We prove the above separately.

i) Provide a matrix error  $\bar{A}_{0i} = A_{0i} - A_0$ , and let  $\bar{A}_0 = [\bar{A}_{01}^T, \bar{A}_{02}^T, \dots, \bar{A}_{0N}^T]^T$ . The following form can be readily obtained:

$$\dot{\bar{A}}_0 = -\kappa_1 \text{sig}\{(\mathcal{H}^* \otimes I_{n_0}) \bar{A}_0\}^\gamma - \kappa_2 \text{sig}\{(\mathcal{H}^* \otimes I_{n_0}) \bar{A}_0\}^\delta, \quad (4.12)$$

in which  $\mathcal{H}^*$  is positive and symmetric, as described in Lemma 5.

The operation  $\langle \bar{A}_0 \rangle_j$ ,  $j = 1, 2, \dots, n_0$ , is defined to represent the  $j$ th column of  $\bar{A}_0$ . Next, it get

$$\dot{\langle \bar{A}_0 \rangle}_j = -\kappa_1 \text{sig}\{(\mathcal{H}^* \otimes I_{n_0}) \langle \bar{A}_0 \rangle_j\}^\gamma - \kappa_2 \text{sig}\{(\mathcal{H}^* \otimes I_{n_0}) \langle \bar{A}_0 \rangle_j\}^\delta. \quad (4.13)$$

Let  $V(\langle \bar{A}_0 \rangle_j) = \langle \bar{A}_0 \rangle_j^T (\mathcal{H}^* \otimes I_{n_0}) \langle \bar{A}_0 \rangle_j$ , and  $\xi_j = (\mathcal{H}^* \otimes I_{n_0}) \langle \bar{A}_0 \rangle_j$ . The time derivative of  $V(\langle \bar{A}_0 \rangle_j)$  with respect to (23) can be expressed

$$\begin{aligned} \dot{V}(\langle \bar{A}_0 \rangle_j) &= -2\kappa_1 \langle \bar{A}_0 \rangle_j^T (\mathcal{H}^* \otimes I_{n_0}) \text{sig}\{(\mathcal{H}^* \otimes I_{n_0}) \langle \bar{A}_0 \rangle_j\}^\gamma \\ &\quad - 2\kappa_2 \langle \bar{A}_0 \rangle_j^T (\mathcal{H}^* \otimes I_{n_0}) \text{sig}\{(\mathcal{H}^* \otimes I_{n_0}) \langle \bar{A}_0 \rangle_j\}^\delta \\ &= -2\kappa_1 \xi_j^T \text{sig}\{\xi_j\}^\gamma - 2\kappa_2 \xi_j^T \text{sig}\{\xi_j\}^\delta \leq -2\kappa_1 n_0^{-\gamma} N^{\frac{1-\gamma}{2}} (\xi_j^T \xi_j)^{\frac{1+\gamma}{2}} - 2\kappa_2 (\xi_j^T \xi_j)^{\frac{1+\delta}{2}} \\ &\leq -2\kappa_1 n_0^{-\gamma} N^{\frac{1-\gamma}{2}} \left[ \frac{\lambda_{\min}(\mathcal{H}^{*2})}{\lambda_{\max}(\mathcal{H}^*)} \right]^{\frac{1+\gamma}{2}} V(\langle \bar{A}_0 \rangle_j)^{\frac{1+\gamma}{2}} - 2\kappa_2 \left[ \frac{\lambda_{\min}(\mathcal{H}^{*2})}{\lambda_{\max}(\mathcal{H}^*)} \right]^{\frac{1+\delta}{2}} V(\langle \bar{A}_0 \rangle_j)^{\frac{1+\delta}{2}}, \end{aligned} \quad (4.14)$$

where Lemmas 1 and 2 are utilized.

By the above,  $\langle \bar{A}_0 \rangle_j = 0$  is stable globally in fixed-time. The setup time

$$T_{s(j)} < T_{s(j)}^* \triangleq \frac{1}{\kappa_2(1-\delta)} \left[ \frac{\lambda_{\min}(\mathcal{H}^{*2})}{\lambda_{\max}(\mathcal{H}^*)} \right]^{-\frac{1+\delta}{2}} + \frac{n_0^\gamma N^{\frac{\gamma-1}{2}}}{\kappa_1(\gamma-1)} \left[ \frac{\lambda_{\min}(\mathcal{H}^{*2})}{\lambda_{\max}(\mathcal{H}^*)} \right]^{-\frac{1+\gamma}{2}}. \quad (4.15)$$

In addition,  $\bar{A}_0$  is stable globally in fixed-time with the settling time  $T_s < T_s^* \triangleq \max\{T_{s(1)}^*, T_{s(2)}^*, \dots, T_{s(n_0)}^*\}$ . Hence,  $\lim_{t \rightarrow T_s} (A_{0i} - A_0) = 0$  holds. ii) Define the error  $\bar{f} = f -$

$(\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \omega$ , and let  $\bar{f} = [\bar{f}_1^T, \bar{f}_2^T, \dots, \bar{f}_N^T]^T$ . The augmented system can be obtained

$$\begin{aligned} \dot{\bar{f}} &= \dot{f} - (\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) (I_n \otimes A_0) \omega \\ &= (I_N \otimes A_0 - c_i (\mathcal{H} \otimes I_N)) \bar{f} - \mu_2 \text{sig}\{(\mathcal{H} \otimes I_N) \bar{f}\}^\alpha - \mu_3 \text{sig}\{(\mathcal{H} \otimes I_N) \bar{f}\}^\beta \\ &\quad + \tilde{A}_{0i} \bar{f} + (\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \tilde{A}_{0i} (I_N \otimes \omega), \end{aligned} \quad (4.16)$$



where  $\tilde{A}_{0i} = \text{blockdiag}(\tilde{A}_{01}, \tilde{A}_{02}, \dots, \tilde{A}_{0N})$ . Let  $V_1 = \bar{f}^T (\mathcal{H} \otimes I_{n_0}) \bar{f}$  and  $V_2 = \sum_{i=1}^N (c_i - \theta)^2$ , in which  $\theta$  is a positive constant to be confirmed. Afterwards, for systems (21) and (26),  $V = V_1 + V_2$  can be regarded as an alternative Lyapunov function.

The time derivative of  $V_1$  with respect to (26) are following when  $t \in (0, T_s)$

$$\begin{aligned} \dot{V}_1 &= 2\bar{f}^T (\mathcal{H} \otimes I_{n_0}) \dot{\bar{f}} \\ &= \bar{f}^T \left[ H \otimes (A_0^T + A_0) - 2c_i (\mathcal{H}^2 \otimes I_{n_0}) \right] \bar{f} + 2\bar{f}^T (\mathcal{H} \otimes I_{n_0}) \tilde{A}_{0i} \bar{f} \\ &\quad - 2\mu_2 \bar{f}^T (\mathcal{H} \otimes I_{n_0}) \text{sig}\{(\mathcal{H} \otimes I_{n_0}) \bar{f}\}^\alpha - 2\mu_3 \bar{f}^T (\mathcal{H} \otimes I_{n_0}) \text{sig}\{(\mathcal{H} \otimes I_{n_0}) \bar{f}\}^\delta \\ &\quad + 2\bar{f}^T \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \tilde{A}_{0i} (1_N \otimes \omega). \end{aligned} \quad (4.17)$$

The time derivative of  $V_2$  with respect to (21) can be indicated as

$$\dot{V}_2 = 2 \sum_{i=1}^N (c_i - \theta) \varsigma_i^T \varsigma_i. \quad (4.18)$$

Let  $\tilde{f} = (U \otimes I_{n_0}) \bar{f}$  and  $\varsigma = (\mathcal{H} \otimes I_{n_0}) \bar{f}$ . In accordance with Lemma 1 and Lemma 2, the following can be obtained:

$$\begin{aligned} \dot{V} &= \dot{V}_1 + \dot{V}_2 \\ &= \bar{f}^T \left[ J \otimes (A_0^T + A_0) - 2\theta (J^2 \otimes I_{n_0}) \right] \bar{f} + 2\bar{f}^T (\mathcal{H} \otimes I_{n_0}) \tilde{A}_{0i} \bar{f} \\ &\quad - 2\mu_2 \varsigma^T \text{sig}\{\varsigma\}^\alpha - 2\mu_3 \varsigma^T \text{sig}\{\varsigma\}^\beta + 2\bar{f}^T \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \tilde{A}_{0i} (1_N \otimes \omega) \\ &\leq \sum_{i=1}^N \tilde{f}_i^T \lambda_i (A_0^T + A_0 - 2\theta \lambda_i I_{n_0}) \tilde{f}_i - 2\mu_2 n_0^{-\alpha} N^{\frac{1-\alpha}{2}} (\varsigma^T \varsigma)^{\frac{1+\alpha}{2}} \\ &\quad - 2\mu_3 (\varsigma^T \varsigma)^{\frac{1+\beta}{2}} + 2\bar{f}^T (\mathcal{H} \otimes I_{n_0}) \tilde{A}_{0i} \bar{f} + (1_N \otimes \omega)^T (1_N \otimes \omega) \\ &\quad + \bar{f}^T \times \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \tilde{A}_{0i} \tilde{A}_{0i}^T \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n)^T \bar{f}. \end{aligned} \quad (4.19)$$

There exists a  $\theta > 0$  that  $A_0^T + A_0 - 2\theta \lambda_i I_{n_0}$  is Hurwitz. In light of the proof of i), it is understood that  $\tilde{A}_{0i} = 0$  is fixed-time stable globally. Let  $\chi_1 = (\mathcal{H} \otimes I_{n_0}) \tilde{A}_{0i}$  and  $\chi_2 = \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \tilde{A}_{0i}$ . In finite time, both  $\chi_1$  and  $\chi_2$  converge to 0. After that, there are bounded constants  $c_0$ ,  $c_1$  and  $c_2$ , thus making the following fact true:

$$\begin{aligned} \dot{V} &\leq -2\mu_2 n_0^{-\alpha} N^{\frac{1-\alpha}{2}} (\varsigma^T \varsigma)^{\frac{1+\alpha}{2}} - 2\mu_3 (\varsigma^T \varsigma)^{\frac{1+\beta}{2}} + \frac{c_1}{\lambda_{\min}(\mathcal{H})} V_1 + \frac{c_2}{\lambda_{\min}(\mathcal{H})} V_1 + c_0 \\ &\leq -2\mu_2 n_0^{-\alpha} N^{\frac{1-\alpha}{2}} \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\min}(\mathcal{H})} \right]^{\frac{1+\alpha}{2}} V_1^{\frac{1+\alpha}{2}} - 2\mu_3 \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\min}(\mathcal{H})} \right]^{\frac{1+\beta}{2}} V_1^{\frac{1+\beta}{2}} + \frac{c_1+c_2}{\lambda_{\min}(\mathcal{H})} V_1 + c_0, \end{aligned} \quad (4.20)$$

where  $c_0 \geq (1_N \otimes x_0)^T (1_N \otimes x_0)$ . On account of the fact that  $\frac{c_1+c_2}{\lambda_{\min}(\mathcal{H})}$ ,  $V$  and  $c_0$  are bounded. Hence,  $\bar{f}$  and  $c_i$  are bounded. Therefore,  $V(T_s)$  is also bounded.

On the condition of  $t \in [T_s, \infty]$ ,  $\tilde{A}_{0i} = 0$ . The time derivative of  $V_1$  with respect to (26) can be equal to

$$\begin{aligned} \dot{V}_1 &= \bar{f}^T \left[ \mathcal{H} \otimes (A_0^T + A_0) - 2c_i (\mathcal{H}^2 \otimes I_{n_0}) \right] \bar{f} \\ &\quad - 2\mu_2 \bar{f}^T (\mathcal{H} \otimes I_{n_0}) \text{sig}\{(\mathcal{H} \otimes I_{n_0}) \bar{f}\}^\alpha \\ &\quad - 2\mu_3 \bar{f}^T (\mathcal{H} \otimes I_{n_0}) \text{sig}\{(\mathcal{H} \otimes I_{n_0}) \bar{f}\}^\beta. \end{aligned} \quad (4.21)$$

The time derivative of  $V_2$  along (21) is (28). We have

$$\begin{aligned} \dot{V} &= \dot{V}_1 + \dot{V}_2 = \bar{f}^T \left[ \mathcal{H} \otimes (A_0^T + A_0) \right] \bar{f} \\ &\quad - 2\theta \sum_{i=1}^N \varsigma_i^T \varsigma_i - 2\mu_2 \zeta^T \text{sig}\{\zeta\}^\alpha - 2\mu_3 \zeta^T \text{sig}\{\zeta\}^\beta \\ &\leq -2\mu_2 n_0^{-\alpha} N^{\frac{1-\alpha}{2}} \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\min}(\mathcal{H})} \right]^{\frac{1+\alpha}{2}} V_1^{\frac{1+\alpha}{2}} - 2\mu_3 \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\min}(\mathcal{H})} \right]^{\frac{1+\beta}{2}} V_1^{\frac{1+\beta}{2}}. \end{aligned} \quad (4.22)$$

Distinctly,  $\bar{f}$ ,  $c_i$  and  $V$  are bounded. There are  $\theta > 0$  up to  $\Delta_c = \max\{\theta - c_i, i = 1, 2, \dots, N\} > 0$ . Next

$$\begin{aligned} \dot{V}_1 &= \dot{V} - \dot{V}_2 \leq -2\mu_2 n_0^{-\alpha} N^{\frac{1-\alpha}{2}} \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\max}(\mathcal{H})} \right]^{\frac{1+\alpha}{2}} V_1^{\frac{1+\alpha}{2}} \\ &\quad - 2\mu_3 \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\max}(\mathcal{H})} \right]^{\frac{1+\beta}{2}} V_1^{\frac{1+\beta}{2}} + \frac{2\Delta_c \lambda_{\max}(\mathcal{H}^2)}{\lambda_{\min}(\mathcal{H})} V_1. \end{aligned} \quad (4.23)$$

Let  $\varpi_1 = \left[ \frac{\mu_3(1-\psi_1)\lambda_{\min}(\mathcal{H})}{\Delta_c \lambda_{\max}(\mathcal{H}^2)} \right]^{(2/1-\beta)} \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\max}(\mathcal{H})} \right]^{[(1+\beta)/1-\beta]}$ , where  $\psi_1 \in (0, 1)$ . Define a bounded set  $\Xi_1 = \{\bar{f}(T_s) \mid \bar{f}(T_s)^T (\mathcal{H} \otimes I_{n_0}) \bar{f}(T_s) \leq \varpi_1\}$ .

If  $\bar{f}(T_s) \in \Xi_1$ , then

$$\dot{V}_1 \leq -2\mu_2 n_0^{-\alpha} N^{\frac{1-\alpha}{2}} \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\max}(\mathcal{H})} \right]^{\frac{1+\alpha}{2}} V_1^{\frac{1+\alpha}{2}} - 2\psi_1 \mu_3 \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\max}(\mathcal{H})} \right]^{\frac{1+\beta}{2}} V_1^{\frac{1+\beta}{2}}. \quad (4.24)$$

Thus,  $\bar{f} = 0$  is fixed-time stable globally. If  $\bar{f}(T_s) \notin \Xi_1$ , then  $V_1(T_s) > \varpi_1$ . There is a bounded  $\tau > T_s$ , so that for  $t \geq \tau$ ,  $\bar{f}(t) \in \Xi_1$ . It is annotated by reducing it to fallacy. Assume the mentioned conclusion is invalid. So the following inequality is true for all  $\tau$ :

$$\begin{aligned} V(T_s) &\geq V(T_s) - V(\tau) \\ &\geq \int_{T_s}^{\tau} \left\{ 2\mu_2 n_0^{-\alpha} N^{\frac{1-\alpha}{2}} \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\max}(\mathcal{H})} \right]^{\frac{1+\alpha}{2}} V_1^{\frac{1+\alpha}{2}} + 2\mu_3 \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\max}(\mathcal{H})} \right]^{\frac{1+\beta}{2}} V_1^{\frac{1+\beta}{2}} \right\} ds \\ &> \left\{ 2\mu_2 n_0^{-\alpha} N^{\frac{1-\alpha}{2}} \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\max}(\mathcal{H})} \right]^{\frac{1+\alpha}{2}} \varpi_1^{\frac{1+\alpha}{2}} + 2\mu_3 \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\max}(\mathcal{H})} \right]^{\frac{1+\beta}{2}} \varpi_1^{\frac{1+\beta}{2}} \right\} (\tau - T_s) \\ &\stackrel{\Delta}{=} \rho(\tau - T_s). \end{aligned} \quad (4.25)$$

From (35),  $V(T_s)$  has no bound as  $\tau \rightarrow \infty$ , which contradicts the truth that  $V(T_s)$  is bounded. Hence, the result is correct. The time for  $\bar{f}(T_s)$  to enter the set  $\Xi_1$  is calculated as

$$\tau = \frac{V(T_s)}{\rho} + T_s. \quad (4.26)$$

Let  $\varpi_2 = \left[ \frac{\Delta_c \lambda_{\max}(\mathcal{H}^2) n_0^\alpha N^{[(\alpha-1)/2]}}{(\mu_2(1-\psi_2)\lambda_{\min}(\mathcal{H}))} \right]^{(2/\alpha-1)} \left[ \frac{\lambda_{\max}(\mathcal{H})}{\lambda_{\min}(\mathcal{H}^2)} \right]^{[(1+\alpha)/(\alpha-1)]}$ , where  $\psi_2 \in (0, 1)$ . Define a bounded  $\Xi_2 = \{\bar{f}(T_s) \mid \bar{f}(T_s)^T (\mathcal{H} \otimes I_{n_0}) \bar{f}(T_s) \geq \varpi_2\}$ . If  $\bar{f}(T_s) \in \Xi_2$ , we have

$$\dot{V}_1 \leq -2\psi_2 \mu_2 n_0^{-\alpha} N^{\frac{1-\alpha}{2}} \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\max}(\mathcal{H})} \right]^{\frac{1+\alpha}{2}} V_1^{\frac{1+\alpha}{2}} - 2\mu_3 \left[ \frac{\lambda_{\min}(\mathcal{H}^2)}{\lambda_{\max}(\mathcal{H})} \right]^{\frac{1+\beta}{2}} V_1^{\frac{1+\beta}{2}}. \quad (4.27)$$

It's known from the above proof that  $\lim_{t \rightarrow T} \left[ f_i - (\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \omega_k \right] = 0$  holds. It can be seen from (21) that  $c_i$  is increasing monotonically. Combining the boundedness of  $c_i$  and the global fixed-time stability of  $\bar{f}$  in the above analysis, each coupling gain converges to a bounded value. The final demonstration is done.  $\square$

### 4.3. Control protocol design

Then, we will show that control protocol designed according to adaptive fixed-time observers is feasible.

**Theorem 3.** For MASs (5) and (6), assume that Assumptions 1–4 hold and that an adaptive bipartite fixed-time observer is designed via Theorem 2. If  $K_i^1$  satisfies  $A_i + B_i K_i^1$  is Hurwitz,  $K_i^2(t) = U_i(t) - K_i^1 X_i(t)$ , and  $K_i^3$  satisfies  $B_i K_i^3 = I_{n_i \times n_i}$ , the bipartite output containment problem can be solved by the control protocol (21).

*Proof.* Let  $\bar{K}_i^2(t) = K_i^2(t) - K_i^1$ ,  $\bar{X}_i(t) = X_i(t) - X_i$ ,  $\bar{x}_i = x_i - (\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) X_i \bar{\omega}$ , and  $\hat{x}_i = x_i - X_i f_i$ . Thus, we have

$$\begin{aligned}
 \dot{\hat{x}}_i &= \dot{x}_i - (\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) X_i \dot{\bar{\omega}} \\
 &= (A_i + B_i K_i^1) x_i + B_i K_i^2(t) \left( \bar{f}_i + (\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \bar{\omega} \right) \\
 &\quad - d_1 \text{sig}\{x_i - (\bar{X}_i(t) + X_i) f_i\}^{\bar{\alpha}} - d_2 \text{sig}\{x_i - (\bar{X}_i(t) + X_i) f_i\}^{\bar{\beta}} \\
 &\quad - (\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) X_i (I_n \otimes A_0) \bar{\omega} \\
 &= (A_i + B_i K_i^1) x_i + B_i K_i^2(t) \bar{f}_i + B_i \bar{K}_i^2(t) (\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \bar{\omega} \\
 &\quad + B_i (U_i - K_i^1 X_i) (\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \bar{\omega} - d_1 \text{sig}\{\bar{x}_i - X_i \bar{f}_i - \bar{X}_i(t) f_i\}^{\bar{\alpha}} \\
 &\quad - d_2 \text{sig}\{\bar{x}_i - X_i \bar{f}_i - \bar{X}_i(t) f_i\}^{\bar{\beta}} - (\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) X_i (I_n \otimes A_0) \bar{\omega} \\
 &= (A_i + B_i K_i^1) x_i - d_1 \text{sig}\{\bar{x}_i - X_i \bar{f}_i - \bar{X}_i(t) f_i\}^{\bar{\alpha}} - d_2 \text{sig}\{\bar{x}_i - X_i \bar{f}_i - \bar{X}_i(t) f_i\}^{\bar{\beta}} \\
 &\quad + B_i K_i^2(t) \bar{f}_i + B_i \bar{K}_i^2(t) (\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) \bar{\omega}
 \end{aligned} \tag{4.28}$$

Due to the boundedness of  $\bar{K}_i^2(t)$ ,  $\bar{X}_i(t)$ , and  $\bar{f}_i$ , one easily knows that  $\bar{x}_i$  when  $t \in (0, \max\{T, T_{\max}\})$ . According to the previous analysis, the first three variables are all 0 when  $t \in (\max\{T, T_{\max}\}, \infty)$ . Let  $A_{i\sigma} = A_i + B_i K_i^1$ . Then, we have

$$\dot{\hat{x}}_i = A_{i\sigma} \bar{x}_i - d_1 \text{sig}\{\bar{x}_i\}^{\bar{\alpha}} - d_2 \text{sig}\{\bar{x}_i\}^{\bar{\beta}} \tag{4.29}$$

A candidate Lyapunov function  $V_{i\sigma} = \bar{x}_i^T \bar{x}_i$  for the system (39) is considered. The time derivative of

$V_{i\sigma}$  along (39) can be acquired

$$\begin{aligned} \dot{V}_{i\sigma} &= \bar{x}_i^T (A_{i\sigma}^T + A_{i\sigma}) \bar{x}_i - 2d_1 \bar{x}_i^T \text{sig}\{\bar{x}_i\}^{\tilde{\alpha}} - 2d_2 \bar{x}_i^T \text{sig}\{\bar{x}_i\}^{\tilde{\beta}} \\ &\leq \lambda_{\max}(A_{i\sigma}^T + A_{i\sigma}) V_{i\sigma} - 2d_1 \sum_{m=1}^{n_i} |\bar{x}_{i(m)}|^{1+\tilde{\alpha}} - 2d_2 \sum_{m=1}^{n_i} |\bar{x}_{i(m)}|^{1+\tilde{\beta}} \\ &\leq \lambda_{\max}(A_{i\sigma}^T + A_{i\sigma}) V_{i\sigma} - 2d_1 n_i^{-\tilde{\alpha}} V_{i\sigma}^{\frac{1+\tilde{\alpha}}{2}} - 2d_2 n_i^{-\tilde{\beta}} V_{i\sigma}^{\frac{1+\tilde{\beta}}{2}}. \end{aligned} \quad (4.30)$$

In accordance with Lemma 3,  $\bar{x}_i = 0$  has global fast fixed time stability. It can be derived that  $\bar{x} = [\bar{x}_1^T, \bar{x}_2^T, \dots, \bar{x}_N^T]^T = 0$  is fast fixed-time stable globally. Thus, we have

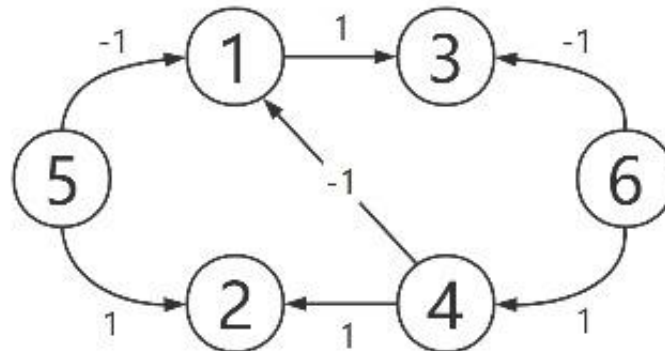
$$\lim_{t \rightarrow T_c} \left( y_i - (\mathcal{H} \otimes I_N)^{-1} \sum_{k=N+1}^{N+W} (\mathcal{A}_{0k} \otimes I_n) y_k \right) = \lim_{t \rightarrow T_c} C_i \bar{x}_i = 0. \quad (4.31)$$

The proof is done.  $\square$

**Corollary 1.** For (5) and (6), assume that Assumptions 1-4 hold and that a bipartite fixed-time observer is designed via Theorem 1. If  $K_i^1$  satisfies  $A_i + B_i K_i^1$  is Hurwitz,  $K_i^2 = U_i - K_i^1 X_i$ , and  $K_i^3$  satisfies  $B_i K_i^3 = I_{n_i \times n_i}$ , the bipartite output containment problem can be solved by the control protocol (11). The solution of the regulator (7) is  $(X_i, U_i)$ , and the same control parameters as in Theorem 3.

## 5. Numerical example

In this section, the validity of Theorem 2 is substantiated by two sets of numerical simulation. Consider the MASs in Figure 1, which includes four followers and two leaders. It is obvious that the digraph accords with Assumption 4 and is signed.



**Figure 1.** Signed digraph.

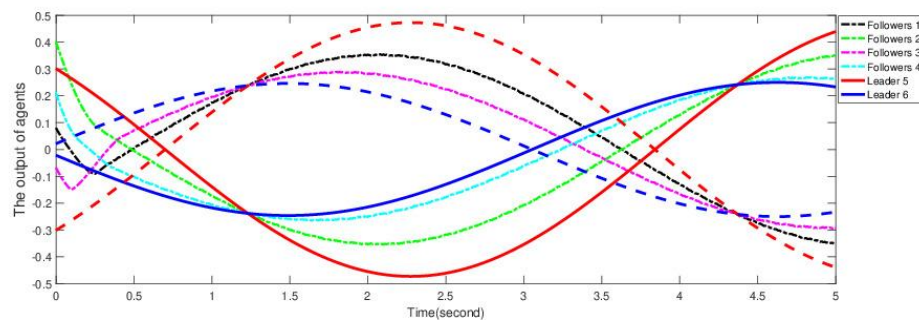
It's observed that 1,2,3,4 represent followers and the others are leaders in Figure 1. In addition, it's revealed that the digraph  $\mathcal{G}$  is structurally balanced and has two competing subgroup  $\mathcal{V}_1 = 1, 3$  and  $\mathcal{V}_2 = 2, 4$ . Choose the relevant matrices as follows:

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & 1 \\ -2 & -0.8 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ -1.5 & -1 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1.2 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 1 \\ -0.5 & -1.4 \end{pmatrix}, \\ A_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B_1 = B_2 = B_3 = B_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C_1 = C_2 = C_3 = C_4 = \begin{pmatrix} 1 & 0 \end{pmatrix}. \end{aligned}$$

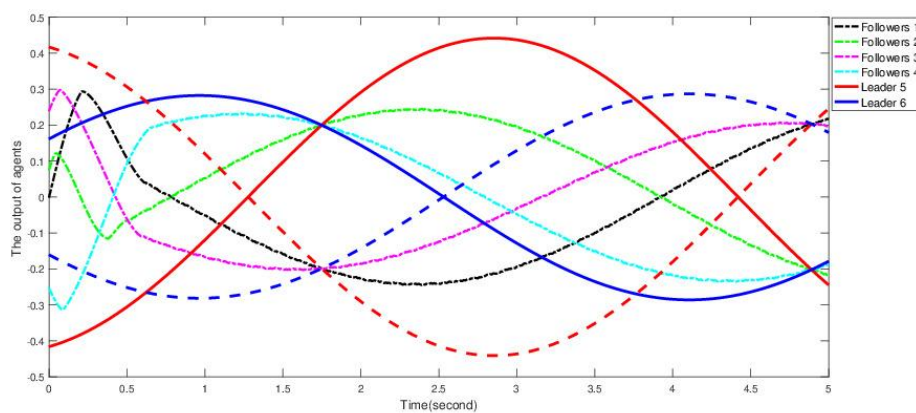
$$\begin{aligned}
K_1^1 &= \begin{pmatrix} 0 & 0 \\ 0.5 & 0.4 \end{pmatrix}, K_1^2 = \begin{pmatrix} 0 & 0 \\ 0.5 & 0.4 \end{pmatrix}, K_2^1 = \begin{pmatrix} 0 & 0 \\ 0.25 & 0.4 \end{pmatrix}, K_2^2 = \begin{pmatrix} 0 & 0 \\ 0.25 & 0.6 \end{pmatrix}, \\
K_3^1 &= \begin{pmatrix} 0 & 0 \\ -1 & 0.6 \end{pmatrix}, K_3^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0.6 \end{pmatrix}, K_4^1 = \begin{pmatrix} 0 & 0 \\ -0.3 & 0.7 \end{pmatrix}, K_4^2 = \begin{pmatrix} 0 & 0 \\ -0.2 & 0.7 \end{pmatrix}, \\
K_1^3 &= K_2^3 = K_3^3 = K_4^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

It can be testified that Assumptions 1–4 hold. In a general way, since the agents' initial parameters are selected at random, two sets of simulation diagrams are shown here to demonstrate generality.

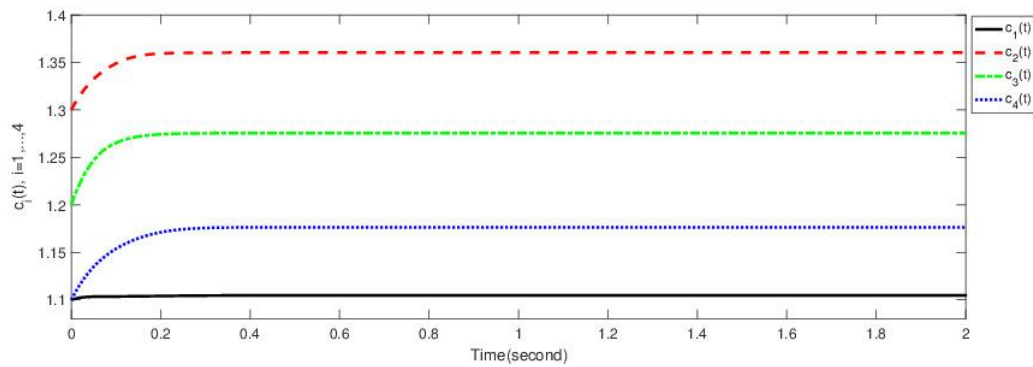
According to Theorem 1, correlation parameters are selected as  $d_1 = d_2 = 1$ ,  $\mu_1 = \mu_2 = 1.5$ ,  $\alpha = 1.2$ ,  $\beta = 0.5$ ,  $\tilde{\alpha} = 1.3$ ,  $\tilde{\beta} = 0.3$ . Besides,  $K_i^2$  are confirmed by  $U_i - K_i^1 X_i$ , and the solution of the regulator equations (7) is  $(X_i, U_i)$ . The evolutions in the agents' output  $y_i$  over time are plotted in Figures 2 and 3. It is obvious to see that the two followers' output tracks (light blue and green lines) extend to the interior of the range invested by the leaders' output trajectories. Conversely, the outputs of the remaining two followers (purple and black lines) are opposite to the inverse tracks of the leaders' outputs. Thus, the adaptive protocol (21) supports the implementation of bipartite output containment control. In the end, the variations of adaptive coupling weights  $c_i(t)$  assigned to each follower are shown in Figures 4 and 5. In addition, the bipartite containment output errors are represented in Figures 6 and 7, which can converge quickly to zero.



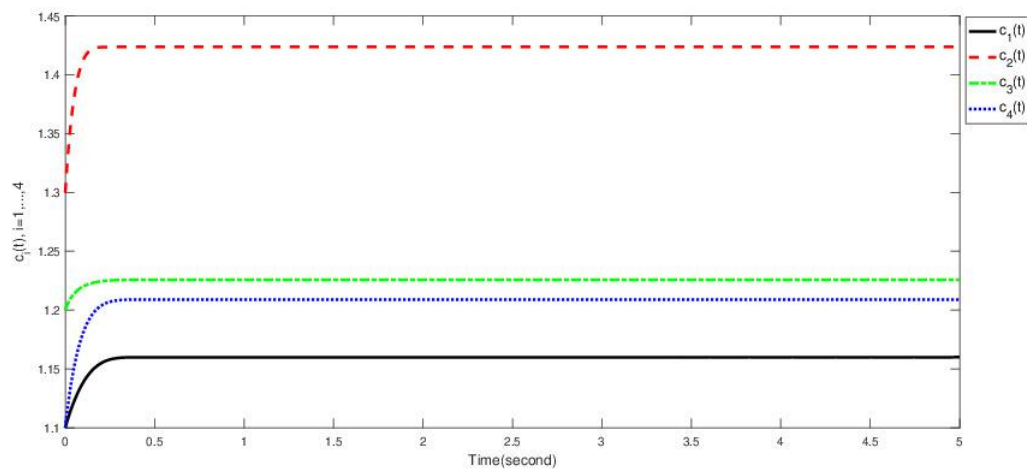
**Figure 2.** First set of agents' outputs  $y_i(t)$ .



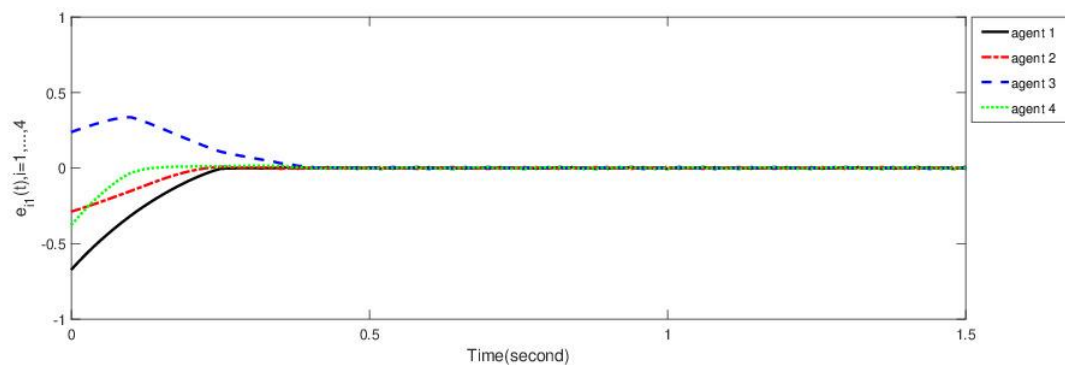
**Figure 3.** Second set of agents' outputs  $y_i(t)$ .



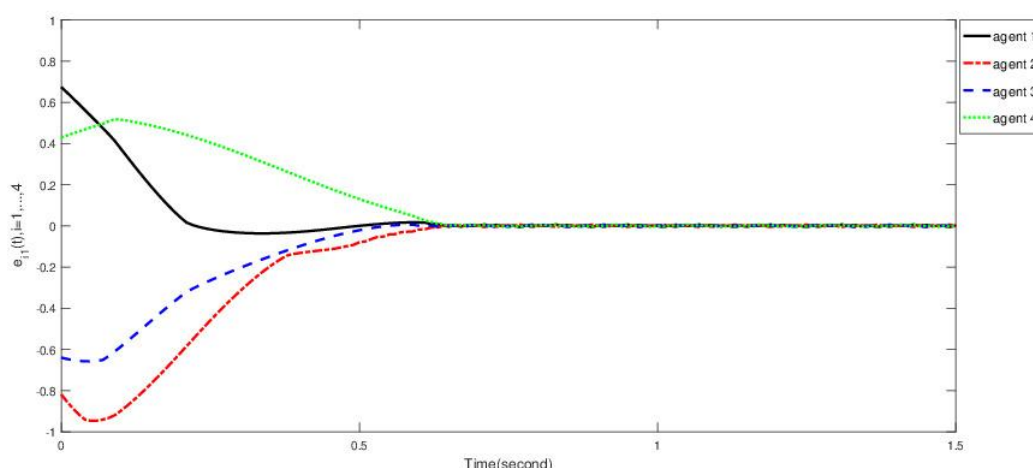
**Figure 4.** First set of adaptive coupling weights  $c_i(t)$ .



**Figure 5.** Second set of adaptive coupling weights  $c_i(t)$ .



**Figure 6.** First set of output errors of agents  $e_i(t)$ .



**Figure 7.** Second set of output errors of agents  $e_i(t)$ .

## 6. Conclusions

In this paper, the discussion and design of bipartite fixed-time output containment control for a class of linear time-invariant system is investigated. By constructing a bipartite compensator distributively. The problem of bipartite output containment is treated as the escalation of adjustment problem of multiagent systems. Two protocols are proposed in order to realize bipartite fixed-time output containment control. Using the Lyapunov function theory and the descriptor systems stability method, some abundant criteria are deduced to guarantee adaptive bipartite fixed-time output containment of multi-agent systems. In the end, the feasibility of the anticipated theoretical results is verified by a set of simulation examples. In our prospective work, we are willing to study the bipartite predefined-time containment problem of more sophisticated MASs.

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## Conflict of interest

The authors declare no conflict of interest.

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