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Research article

Fixed point theorems via auxiliary functions with applications to two-term fractional differential equations with nonlocal boundary conditions

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Abstract: In this study, the $(h\text{-}\varphi)_R$ and $(h\text{-}\varphi)_M$-contractions with two metrics endowed with a directed graph are examined using auxiliary functions. We propose a set of criteria that guarantees the existence of common fixed points for our contractions. This leads to a generalization of previous results in the literature. Towards our accomplishments, we establish affirmative results that demonstrate solutions to a class of nonlinear two-term fractional differential equations with nonlocal boundary conditions. To further corroborate our major findings, we also provide instances.

Keywords: fixed point; contraction; two-term fractional differential equation
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1. Introduction

One of the most interesting areas in scientific study is regarding natural phenomena, which many researchers have previously investigated in mathematical models through differential operators, see for instance [1–6]. Currently, mathematicians are paying a lot of attention to the fractional differential operator since it has been widely used in a variety of fields, including the risk-controlled financial market [7–9], engineering and scientific fields [10–14]. To be more precise, recent advances in multi-term fractional differential equations can be found in [15–23]. Moreover, research in fractional derivatives of type Caputo and integral operators was performed in [24]. Additionally, the nonlinear two-term fractional differential equations were intensively studied with some scientific publications related to the nonlocal BVPs equations, which are pertinent to the developing topic in [19, 20, 25–27]. It is worth mentioning that the existence and uniqueness of a solution to a differential equation are frequently obtained by using the concepts of fixed point theorem, as seen in [28–33]. Therefore, fixed point theory has played an important role in the study of fractional differential operators.

The idea of a fixed point theorem for metric spaces endowed with graphs was initially proposed by Jachymski [34] in 2008. Since then, other researchers have focused on this concept in a variety of spaces endowed with graphs, see [35–39] for instance. One of the most significant consequences of this generalization is the extension of the well-known Banach contraction principle to the case of metric spaces endowed with graphs, see [35, 38].

There are various ways that mathematicians could investigate fixed point theory. One can consider contractions with Geraghty functions, which are among the most influential ideas in this area, see [40–48] for further information. In 2017, Charoensawan and Atiponrat [47] introduced a new class of contractions, namely θ-ϕ-contractions, which are of Geraghty's type. Now, let us recall essential concepts that will be considered throughout this paper below.

Definition 1.1. *[47] Suppose that (X, d) is a metric space endowed with a directed graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$, and $\mu, \delta : X \to X$ are functions. Let us define the following sets:*

$$\begin{aligned} X(\mu, \delta) &:= \{u \in X : (\mu u, \delta u) \in E(\mathcal{G})\}, \\ C(\mu, \delta) &:= \{u \in X : \mu u = \delta u\}, \end{aligned}$$

and

$$Cm(\mu, \delta) := \{u \in X : \mu u = \delta u = u\}.$$

We note that $C(\mu, \delta)$ is the set of all coincidence points of μ and δ. Additionally, $Cm(\mu, \delta)$ is the set of all common fixed points of μ and δ.

Lemma 1.2. *[47] Let (X, d) be a metric space endowed with a directed graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$, and let $\mu, \delta : X \to X$ be functions. If $C(\mu, \delta) \neq \emptyset$, then $X(\mu, \delta) \neq \emptyset$.*

Definition 1.3. *[47] Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a directed graph, and let $\mu, \delta : X \to X$ be functions. We say that μ is δ-edge preserving with respect to $\mathcal{G}$ whenever for each $x \in X$,*

$$\text{if } (\delta x, \delta y) \in E(\mathcal{G}), \text{ then } (\mu x, \mu y) \in E(\mathcal{G}).$$

Definition 1.4. *[47] Let (X, d) and (Y, d') be metric spaces, and let $\mu : X \to Y$ and $\delta : X \to X$ be functions. We say that μ is δ-Cauchy on X whenever for any sequence $\{x_n\}$ in X with $\{\delta x_n\}$ being Cauchy in (X, d), the sequence $\{\mu x_n\}$ is Cauchy in (Y, d').*

On the other hand, Martinez-Moreno et al. [48] demonstrated fascinating results on common fixed point theorems for Geraghty's type contraction mappings employing the monotone property with two metrics as a consequence of d-compatibility and δ-uniform continuity. This motivates us to investigate metric spaces equipped with two distance functions in our work.

Due to their numerous scientific applications, fractional differential equations have garnered a lot of attention from mathematicians in recent years. As can be seen, for example, in [49–52], that fixed point theory has strongly contributed to the knowledge of fractional differential equations. In addition, it is worth emphasizing that our recent work is inspired by Karapınar's investigation of the fixed point theorem using auxiliary functions in [52], which provided insight into its usefulness for fractional differential equations. Therefore, in this research, we replace θ-ϕ contraction mappings with auxiliary functions to improve the outcomes in [31,47]. This enables us to derive actual criteria for the existence of common fixed points in the setting of auxiliary functions endowed with two metrics and a directed graph. Subsequently, we provide the applications for a class of nonlinear two-term fractional differential equations in our third section.

2. Main results

Study results about the existence of common fixed points for auxiliary functions with two metrics endowed with a directed graph are presented in this section. Let us first define the classes of functions that will be taken into account in this task.

Assume that $\varphi : [0, \infty) \to [0, \infty)$ is a function with the properties listed below:

- φ is increasing and continuous,
- $\varphi(r) = 0$ if and only if $r = 0$.

The set of all functions φ satisfying the aforementioned constraints will be referred to as Φ going forward.

In spired by [52], we define the class $\mathcal{A}(X)$ consisting all auxiliary functions $h : X \times X \to [0, 1]$ such that

$$\text{if } \lim_{n\to\infty} h(x_n, y_n) = 1, \text{ then } \lim_{n\to\infty} d(x_n, y_n) = 0 \tag{2.1}$$

for all sequences $\{x_n\}$ and $\{y_n\}$ in X with $\{d(x_n, y_n)\}$ is decreasing, where (X, d) is a metric space.

Example 2.1. *[52] Let $h_1, h_2 : \mathbb{R} \times \mathbb{R} \to [0, 1]$, for all $x, y \in \mathbb{R}$, defined by*

(1) $h_1(x, y) = c$ for some $c \in (0, 1)$,
(2) $h_2(x, y) = \dfrac{1}{1 + x + y}$.

Then $h_1, h_2 \in \mathcal{A}(X)$.

Lemma 2.2. *For a sequence $\{x_n\}$ in a metric space (X, d) and a function $\delta : X \to X$ such that*

$$\lim_{n\to\infty} d(\delta x_n, \delta x_{n+1}) = 0,$$

if $\{\delta x_n\}$ is not a Cauchy sequence, then there exists $\epsilon > 0$ such that, for all $k \in \mathbb{N}$, there are $n_k, m_k \in \mathbb{N}$ with $n_k > m_k \geq k$ satisfying n_k is the smallest number such that

$$d(\delta x_{n_k}, \delta x_{m_k}) \geq \epsilon \text{ and } d(\delta x_{n_k-1}, \delta x_{m_k}) < \epsilon.$$

Then, we obtain

$$\varepsilon = \lim_{k\to\infty} d(\delta x_{m_k}, \delta x_{n_k}) = \lim_{k\to\infty} d(\delta x_{m_k+1}, \delta x_{n_k+1}).$$

Proof. Suppose that $\{\delta x_n\}$ is not Cauchy. By definition, there is a positive number $\epsilon > 0$ such that for all $k \in \mathbb{N}$, there are $n_k, m_k \in \mathbb{N}$ with $n_k > m_k \geq k$ satisfying n_k is the smallest number such that

$$d(\delta x_{n_k}, \delta x_{m_k}) \geq \epsilon \text{ and } d(\delta x_{n_k-1}, \delta x_{m_k}) < \epsilon.$$

This means

$$\begin{aligned}\epsilon &\le d(\delta x_{m_k}, \delta x_{n_k}) \\ &\le d(\delta x_{m_k}, \delta x_{n_k-1}) + d(\delta x_{n_k-1}, \delta x_{n_k}) \\ &< \epsilon + d(\delta x_{n_k-1}, \delta x_{n_k}).\end{aligned} \tag{2.2}$$

Letting $k \to \infty$ and applying the fact that $\lim_{n\to\infty} d(\delta x_n, \delta x_{n+1}) = 0$, we receive

$$\lim_{k\to\infty} d(\delta x_{m_k}, \delta x_{n_k}) = \epsilon > 0. \tag{2.3}$$

Consider, by the triangle inequality, that

$$d(\delta x_{m_k}, \delta x_{n_k}) \le d(\delta x_{m_k}, \delta x_{m_k+1}) + d(\delta x_{m_k+1}, \delta x_{n_k+1}) + d(\delta x_{n_k+1}, \delta x_{n_k}),$$

and

$$d(\delta x_{m(k+1)}, \delta x_{n_k+1}) \le d(\delta x_{m(k+1)}, \delta x_{m_k}) + d(\delta x_{m_k}, \delta x_{n_k}) + d(\delta x_{n_k}, \delta x_{n_k+1}).$$

Then, we obtain

$$\begin{aligned}&d(\delta x_{m_k}, \delta x_{n_k}) - d(\delta x_{m_k}, \delta x_{m_k+1}) - d(\delta x_{n_k+1}, \delta x_{n_k}) \\ &\le d(\delta x_{m_k+1}, \delta x_{n_k+1}) \\ &\le d(\delta x_{m_k+1}, \delta x_{m_k}) + d(\delta x_{m_k}, \delta x_{n_k}) + d(\delta x_{n_k}, \delta x_{n_k+1}).\end{aligned}$$

By

$$\lim_{n\to\infty} d(\delta x_n, \delta x_{n+1}) = 0,$$

and taking k to ∞ in (2.2), we obtain

$$\varepsilon = \lim_{k\to\infty} d(\delta x_{m_k}, \delta x_{n_k}) \le \lim_{k\to\infty} d(\delta x_{m_k+1}, \delta x_{n_k+1}) \le \lim_{k\to\infty} d(\delta x_{m_k}, \delta x_{n_k}) = \varepsilon.$$

Thus,

$$\lim_{k\to\infty} d(\delta x_{m_k+1}, \delta x_{n_k+1}) = \lim_{k\to\infty} d(\delta x_{m_k}, \delta x_{n_k}) = \varepsilon.$$

□

Here, we are ready to define a new category of contractions defined as follows. Let $(X, d, \mu, \delta, \mathcal{G})$ refer to a structure throughout this work that has the properties listed below:

- $X \neq \emptyset$ and (X, d) is a metric space,
- X is endowed with a directed grahp $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$,
- μ and δ are self mappings,
- μ is δ-edge preserving with respect to $\mathcal{G}$.

Lemma 2.3. *On $(X, d, \mu, \delta, \mathcal{G})$. Let a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} \delta x_n = \lim_{n\to\infty} \mu x_n = u,$$

where $u \in X$ and $(\delta x_{n-1}, \delta x_n) \in E(\mathcal{G})$. If μ is $\mathcal{G}$-continuous, with μ and δ being d-compatible, then $u \in C(\mu, \delta)$.

Proof. Let $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \delta x_n = \lim_{n\to\infty} \mu x_n = u,$$

where $u \in X$, and $(\delta x_{n-1}, \delta x_n) \in E(\mathcal{G})$. Additionally, we conclude that

$$\lim_{n\to\infty} d(\delta\mu x_n, \mu\delta x_n) = 0 \tag{2.4}$$

due to μ and δ being d-compatible. Finally, we consider

$$d(\delta u, \mu u) \leq d(\delta u, \delta\mu x_n) + d(\delta\mu x_n, \mu\delta x_n) + d(\mu\delta x_n, \mu u).$$

By combining the continuity of δ with the notion that μ is $\mathcal{G}$-continuous, $(\delta x_{n-1}, \delta x_n) \in E(\mathcal{G})$ and using (2.4), it can be concluded that $d(\delta u, \mu u) = 0$ when $n \to \infty$. As a result, $\delta u = \mu u$, which indicates that u is a coincidence point of μ and δ. Thus, $u \in C(\mu, \delta)$. □

Definition 2.4. *On $(X, d, \mu, \delta, \mathcal{G})$. If the following criteria are satisfied, the pair (μ, δ) will be referred to as an $(h\text{-}\varphi)_R$-contraction with regard to d. There exists $h \in \mathcal{A}(X)$ and $\varphi \in \Phi$ with $(\delta x, \delta y) \in E(\mathcal{G})$ for $x, y \in X$, we have*

$$\varphi(d(\mu x, \mu y)) \leq h(\delta x, \delta y)\varphi(R(\delta x, \delta y)),$$

where $R : X \times X \to [0, \infty)$ is defined by

$$\begin{aligned} R(\delta x, \delta y) = \max \Bigg\{ & \frac{d(\delta x, \mu x) d(\mu y, \delta y)}{d(\delta x, \delta y)} + |d(\delta x, \delta y) - d(\delta x, \mu x)|, \\ & d(\delta x, \delta y) + |d(\delta x, \mu x) - d(\delta y, \mu y)|, \\ & d(\delta x, \mu x) + |d(\delta x, \delta y) - d(\delta y, \mu y)|, \\ & d(\delta y, \mu y) + |d(\delta x, \delta y) - d(\delta x, \mu x)|, \\ & \frac{d(\delta x, \mu y) + d(\delta y, \mu x) + |d(\delta x, \delta y) - d(\mu x, \mu y)|}{2} \Bigg\}, \end{aligned}$$

for $x, y \in X$.

The result in [47] can be applied to the case of auxiliary functions owing to the aforementioned definition. In actuality, we are now prepared to demonstrate and present our key findings. The motivation for the following theorem comes from [53] and incorporates two metrics.

Theorem 2.5. *On $(X, d', \mu, \delta, \mathcal{G})$, let (X, d') be a complete metric space, and let d be another metric on X. Assume that (μ, δ) is an $(h\text{-}\varphi)_R$-contraction with respect to d and that the following criteria are satisfied.*

(1) $\delta : (X, d') \to (X, d')$ is continuous, and $\delta(X)$ is d'-closed,
(2) $\mu(X) \subseteq \delta(X)$,
(3) The transitivity property of $E(\mathcal{G})$ holds,
(4) If $d \ngeq d'$, suppose that $\mu : (X, d) \to (X, d')$ is δ-Cauchy on X,
(5) $\mu : (X, d') \to (X, d')$ is $\mathcal{G}$-continuous, and μ and δ are d'-compatible.

Consequently, it can be seen that

$$X(\mu, \delta) \neq \emptyset \iff C(\mu, \delta) \neq \emptyset.$$

Proof. (⇐) This derives from Lemma 1.2.

(⇒) Assume that $X(\mu,\delta) \neq \emptyset$ and $x_0 \in X$ with $(\delta x_0, \mu x_0) \in E(\mathcal{G})$. According to the assumption that $\mu(X) \subseteq \delta(X)$ and $\mu(x_0) \in X$, we could establish a sequence $\{x_n\}$ in X such that $\delta x_n = \mu x_{n-1}$ for every $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $\delta x_{n_0} = \delta x_{n_0-1}$, then x_{n_0-1} is a coincidence point of μ and δ. We may therefore now assume that $\delta x_n \neq \delta x_{n-1}$ for all $n \in \mathbb{N}$.

Because $(\delta x_0, \mu x_0) = (\delta x_0, \delta x_1) \in E(\mathcal{G})$ and μ is δ-edge preserving with respect to $\mathcal{G}$, it is precise to state that $(\mu x_0, \mu x_1) = (\delta x_1, \delta x_2) \in E(\mathcal{G})$. We obtain $(\delta x_{n-1}, \delta x_n) \in E(\mathcal{G})$ for every $n \in \mathbb{N}$ through mathematical induction. As (μ,δ) is an $(h\text{-}\varphi)_R$-contraction with respect to d, for each $n \geq 0$,

$$\begin{aligned}\varphi(d(\delta x_{n+1}, \delta x_{n+2})) &= \varphi(d(\mu x_n, \mu x_{n+1}))\\ &\leq h(\delta x_n, \delta x_{n+1})\varphi(R(\delta x_n, \delta x_{n+1}))\\ &\leq \varphi(R(\delta x_n, \delta x_{n+1})). \qquad (2.5)\end{aligned}$$

Additionally, a straightforward calculation demonstrates that

$$\begin{aligned}R(gx_n, gx_{n+1}) = \max\Bigg\{ &\frac{d(\delta x_n, \mu x_n) d(\mu x_{n+1}, \delta x_{n+1})}{d(\delta x_n, \delta x_{n+1})} + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_n, \mu x_n)|,\\ &d(\delta x_n, \delta x_{n+1}) + |d(\delta x_n, \mu x_n) - d(\delta x_{n+1}, \mu x_{n+1})|,\\ &d(\delta x_n, \mu x_n) + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_{n+1}, \mu x_{n+1})|,\\ &d(\delta x_{n+1}, \mu x_{n+1}) + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_n, \mu x_n)|,\\ &\frac{d(\delta x_n, \mu x_{n+1}) + d(\delta x_{n+1}, \mu x_n) + |d(\delta x_n, \delta x_{n+1}) - d(\mu x_n, \mu x_{n+1})|}{2}\Bigg\}\\ = \max\Bigg\{ &\frac{d(\delta x_n, \delta x_{n+1}) d(\delta x_{n+2}, \delta x_{n+1})}{d(\delta x_n, \delta x_{n+1})} + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_n, \delta x_{n+1})|,\\ &d(\delta x_n, \delta x_{n+1}) + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_{n+1}, \delta x_{n+2})|,\\ &d(\delta x_n, \delta x_{n+1}) + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_{n+1}, \delta x_{n+2})|,\\ &d(\delta x_{n+1}, \delta x_{n+2}) + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_n, \delta x_{n+1})|,\\ &\frac{d(\delta x_n, \delta x_{n+2}) + d(\delta x_{n+1}, \delta x_{n+1}) + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_{n+1}, \delta x_{n+2})|}{2}\Bigg\}\\ = \max\Bigg\{ &d(\delta x_{n+2}, \delta x_{n+1}),\\ &d(\delta x_n, \delta x_{n+1}) + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_{n+1}, \delta x_{n+2})|,\\ &d(\delta x_n, \delta x_{n+1}) + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_{n+1}, \delta x_{n+2})|,\\ &d(\delta x_{n+1}, \delta x_{n+2}),\\ &\frac{d(\delta x_n, \delta x_{n+2}) + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_{n+1}, \delta x_{n+2})|}{2}\Bigg\}.\end{aligned}$$

If we denote by

$$g_n = d(\delta x_n, \delta x_{n+1}).$$

As $d(\delta x_n, \delta x_{n+2}) \leq d(\delta x_n, \delta x_{n+1}) + d(\delta x_{n+1}, \delta x_{n+2})$, we get

$$R(\delta x_n, \delta x_{n+1}) \leq \max\left\{\delta_{n+1}, \delta_n + |\delta_n - \delta_{n+1}|, \frac{\delta_n + \delta_{n+1} + |\delta_n - \delta_{n+1}|}{2}\right\}.$$

Now, suppose that δ_n is not decreasing, then there exists $C \in \mathbb{N}$ such that $\delta_C \leq \delta_{C+1}$ so we have

$$R(\delta x_C, \delta x_{C+1}) \leq \delta_{C+1}.$$

By the inequality (2.5) and the property of φ, we obtain

$$\varphi(\delta_{C+1}) \leq h(\delta x_C, \delta x_{C+1})\varphi(R(\delta x_C, \delta x_{C+1})) \leq h(\delta x_C, \delta x_{C+1})\varphi(\delta_{C+1}) \leq \varphi(\delta_{C+1}).$$

Since for every $n \in \mathbb{N}$, $\delta x_n \neq \delta x_{n-1}$, we have $\delta_{C+1} = d(\delta x_{C+1}, \delta x_{C+2}) > 0$ which follows from the above inequality, we get $h(\delta x_C, \delta x_{C+1}) = 1$. By the fact that $h \in \mathcal{A}(X)$, we obtain $d(\delta x_C, \delta x_{C+1}) = 0$. It is a contradiction. Therefore, δ_n is decreasing, $\delta_n > \delta_{n+1}$ for all $n \geq 0$, we have

$$R(\delta x_n, \delta x_{n+1}) \leq \max\{2\delta_n - \delta_{n+1}, \delta_n, \delta_{n+1}\} = R^*(n).$$

Since δ_n is bounded below, the sequence converges. Let

$$\lim_{n\to\infty} \delta_n = L \geq 0.$$

Contrarily, suppose that $L > 0$. Due to the property of φ, $\lim_{n\to\infty} \varphi(\delta_n) = \varphi(L) > 0$. By (2.5), it is demonstrated that

$$\begin{aligned}\varphi(\delta_{n+1}) &= \varphi(d(\mu x_n, \mu x_{n+1})) \\ &\leq h(\delta x_n, \delta x_{n+1})\varphi(R(\delta x_n, \delta x_{n+1})) \\ &\leq h(\delta x_n, \delta x_{n+1})\varphi(R^*(n)) \\ &\leq \varphi(R^*(n)).\end{aligned}$$

In the inequality above, when we take $n \to \infty$, we obtain

$$1 = \lim_{n\to\infty} \frac{\varphi(\delta_{n+1})}{\varphi(R^*(n))} \leq \lim_{n\to\infty} h(\delta x_n, \delta x_{n+1}) \leq 1.$$

Therefore, $\lim_{n\to\infty} h(\delta x_n, \delta x_{n+1}) = 1$. According to the notion of auxiliary functions,

$$\lim_{n\to\infty} d(\delta x_n, \delta x_{n+1}) = \lim_{n\to\infty} \delta_n = 0,$$

which contradicts to the assumption. So, $\lim_{n\to\infty} d(\delta x_n, \delta x_{n+1}) = 0$.

The sequence $\{\delta x_n\}$ has to be Cauchy, as we will demonstrate next. Contrarily, suppose that $\{\delta x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ such that, for all $k \in \mathbb{N}$, there are $n_k, m_k \in \mathbb{N}$ with $n_k > m_k \geq k$ satisfying n_k is the smallest number such that

$$d(\delta x_{n_k}, \delta x_{m_k}) \geq \epsilon \text{ and } d(\delta x_{n_k-1}, \delta x_{m_k}) < \epsilon.$$

By Lemma 2.2, this implies

$$\lim_{k\to\infty} d(\delta x_{m_k+1}, \delta x_{n_k+1}) = \lim_{k\to\infty} d(\delta x_{m_k}, \delta x_{n_k}) = \varepsilon.$$

We determine that $(\delta x_{m_k}, \delta x_{n_k}) \in E(\mathcal{G})$ for each $k \in \mathbb{N}$ owing to the transitivity property of $E(\mathcal{G})$. As a result,

$$\begin{aligned}\varphi(d(\delta x_{m_k+1}, \delta x_{n_k+1})) &= \varphi(d(\mu x_{m_k}, \mu x_{n_k})) \\ &\leq h(\delta x_{m_k}, \delta x_{n_k})\varphi(R(\delta x_{m_k}, \delta x_{n_k})),\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
&R(\delta x_{m_k}, \delta x_{n_k}) \\
&= \max\Bigg\{\frac{d(\delta x_{m_k}, \mu x_{m_k})d(\mu x_{n_k}, \delta x_{n_k})}{d(\delta x_{m_k}, \delta x_{n_k})} + |d(\delta x_{m_k}, \delta x_{n_k}) - d(\delta x_{m_k}, \mu x_{m_k})|, \\
&\quad d(\delta x_{m_k}, \delta x_{n_k}) + |d(\delta x_{m_k}, \mu x_{m_k}) - d(\delta x_{n_k}, \mu x_{n_k})|, \\
&\quad d(\delta x_{m_k}, \mu x_{m_k}) + |d(\delta x_{m_k}, \delta x_{n_k}) - d(\delta x_{n_k}, \mu x_{n_k})|, \\
&\quad d(\delta x_{n_k}, \mu x_{n_k}) + |d(\delta x_{m_k}, \delta x_{n_k}) - d(\delta x_{m_k}, \mu x_{m_k})|, \\
&\quad \frac{d(\delta x_{m_k}, \mu x_{n_k}) + d(\delta x_{n_k}, \mu x_{m_k}) + |d(\delta x_{m_k}, \delta x_{n_k}) - d(\mu x_{m_k}, \mu x_{n_k})|}{2}\Bigg\} \\
&= \max\Bigg\{\frac{d(\delta x_{m_k}, \delta x_{m_k+1})d(\delta x_{n_k+1}, \delta x_{n_k})}{d(\delta x_{m_k}, \delta x_{n_k})} + |d(\delta x_{m_k}, \delta x_{n_k}) - d(\delta x_{m_k}, \delta x_{m_k+1})|, \\
&\quad d(\delta x_{m_k}, \delta x_{n_k}) + |d(\delta x_{m_k}, \delta x_{m_k+1}) - d(\delta x_{n_k}, \delta x_{n_k+1})|, \\
&\quad d(\delta x_{m_k}, \delta x_{m_k+1}) + |d(\delta x_{m_k}, \delta x_{n_k}) - d(\delta x_{n_k}, \delta x_{n_k+1})|, \\
&\quad d(\delta x_{n_k}, \delta x_{n_k+1}) + |d(\delta x_{m_k}, \delta x_{n_k}) - d(\delta x_{m_k}, \delta x_{m_k+1})|, \\
&\quad \frac{d(\delta x_{m_k}, \delta x_{n_k+1}) + d(\delta x_{n_k}, \delta x_{m_k+1}) + |d(\delta x_{m_k}, \delta x_{n_k}) - d(\delta x_{m_k+1}, \delta x_{n_k+1})|}{2}\Bigg\}.
\end{aligned}$$

Since $\lim_{n\to\infty} d(\delta x_n, \delta x_{n+1}) = 0$, applying the preceding equality with $k \to \infty$ means that

$$\lim_{k\to\infty} R(\delta x_{m_k}, \delta x_{n_k}) = \lim_{k\to\infty} d(\delta x_{m_k}, \delta x_{n_k}) = \epsilon > 0.$$

Combining the aforementioned fact with the inequality (2.6), we obtain

$$1 = \lim_{k\to\infty} \frac{\varphi(d(\delta x_{m_k}, \delta x_{n_k}))}{\varphi(R(\delta x_{m_k}, \delta x_{n_k}))} \leq \lim_{k\to\infty} h(\delta x_{m_k}, \delta x_{n_k}) \leq 1.$$

As a result, $\lim_{k\to\infty} h(\delta x_{m_k}, \delta x_{n_k}) = 1$. Then, $\lim_{k\to\infty} d(\delta x_{m_k}, \delta x_{n_k}) = 0$, which contradicts to (2.3). That $\{\delta x_n\}$ is Cauchy in (X, d) must therefore be true.

In the following part, we demonstrate that in the metric space (X, d'), $\{\delta x_n\}$ is also Cauchy. The proof is simple when $d \geq d'$. The case $d \ngeq d'$ is therefore taken into consideration. Let $\varepsilon > 0$. We conclude that $\{\mu x_n\}$ is Cauchy in (X, d') since $\{\delta x_n\}$ is Cauchy in (X, d) and μ is δ-Cauchy on X. So, there exists $N_0 \in \mathbb{N}$ such that

$$d'(\delta x_{n+1}, \delta x_{m+1}) = d'(\mu x_n, \mu x_m) < \varepsilon, \quad \forall n, m \geq N_0.$$

The sequence $\{\delta x_n\}$ is therefore Cauchy in (X, d').

Since $\delta(X)$ is a d'-closed subset of (X, d'), which is complete, it follows that $u = \delta x \in \delta(X)$ exists satisfying

$$\lim_{n\to\infty} \delta x_n = \lim_{n\to\infty} \mu x_n = u.$$

Since assumption (5), by Lemma 2.3 which indicates that u is a coincidence point of μ and δ. Thus, $u \in C(\mu, \delta)$. □

We analyze the scenario in which the two metrics d and d' coincide in our next theorem.

Definition 2.6. *On $(X, d, \mu, \delta, \mathcal{G})$. If the following criteria are satisfied, the pair (μ, δ) will be referred to as an $(h\text{-}\varphi)_M$-contraction with regard to d. There exists $h \in \mathcal{A}(X)$ and $\varphi \in \Phi$ with $(\delta x, \delta y) \in E(\mathcal{G})$ for $x, y \in X$, we have*

$$\varphi(d(\mu x, \mu y)) \leq h(\delta x, \delta y)\varphi(M(\delta x, \delta y)),$$

where $M : X \times X \to [0, \infty)$ is defined by

$$M(\delta x, \delta y) = \max\left\{\frac{d(\delta x, \mu x)[1 + d(\delta y, \mu y)]}{1 + d(\delta x, \delta y)} + |d(\delta x, \delta y) - d(\delta x, \mu x)|,\right.$$
$$\frac{d(\delta y, \mu y)[1 + d(\delta x, \mu x)]}{1 + d(\delta x, \delta y)} + |d(\delta x, \delta y) - d(\delta x, \mu x)|,$$
$$\left. d(\delta x, \delta y) + |d(\delta x, \mu x) - d(\delta y, \mu y)|\right\},$$

for $x, y \in X$.

Theorem 2.7. *On $(X, d, \mu, \delta, \mathcal{G})$, let (X, d) be a complete metric space with an $(h\text{-}\varphi)_M$-contraction (μ, δ). Suppose that the following criteria are satisfied.*

(1) δ is continuous, and $\delta(X)$ is closed.
(2) $\mu(X) \subseteq \delta(X)$.
(3) The transitivity property $E(\mathcal{G})$ holds.
(4) At least one of the statements below is satisfied.
(a) μ is $\mathcal{G}$-continuous, and μ and δ are d-compatible,
(b) $(X, d, \mathcal{G})$ has the property A in [34], and

$$\text{if } \lim_{n\to\infty} h(\delta x_n, \delta y_n) = 1, \text{ then } \lim_{n\to\infty} d(\mu x_n, \mu y_n) = 0.$$

Consequently, we obtain that

$$X(\mu, \delta) \neq \emptyset \iff C(\mu, \delta) \neq \emptyset.$$

Proof. ($\Leftarrow$) This derives from Lemma 1.2.

($\Rightarrow$) Assume that $X(\mu, \delta) \neq \emptyset$ and $x_0 \in X$ with $(\delta x_0, \mu x_0) \in E(\mathcal{G})$. According to the assumption that $\mu(X) \subseteq \delta(X)$ and $\mu(x_0) \in X$, we could establish a sequence $\{x_n\}$ in X such that $\delta x_n = \mu x_{n-1}$ for every $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $\delta x_{n_0} = \delta x_{n_0-1}$, then x_{n_0-1} is a coincidence point of μ and δ. We may therefore now assume that $\delta x_n \neq \delta x_{n-1}$ for all $n \in \mathbb{N}$.

Because $(\delta x_0, \mu x_0) = (\delta x_0, \delta x_1) \in E(\mathcal{G})$ and μ is δ-edge preserving with respect to $\mathcal{G}$, it is precise to state that $(\mu x_0, \mu x_1) = (\delta x_1, \delta x_2) \in E(\mathcal{G})$. We obtain $(\delta x_{n-1}, \delta x_n) \in E(\mathcal{G})$ for every $n \in \mathbb{N}$ through mathematical induction. As (μ, δ) is an $(h\text{-}\varphi)_M$-contraction with respect to d, for each $n \geq 0$,

$$\begin{aligned} \varphi(d(\delta x_{n+1}, \delta x_{n+2})) &= \varphi(d(\mu x_n, \mu x_{n+1})) \\ &\leq h(\delta x_n, \delta x_{n+1})\varphi(M(\delta x_n, \delta x_{n+1})) \\ &\leq \varphi(M(\delta x_n, \delta x_{n+1})). \end{aligned} \tag{2.7}$$

Additionally, a straightforward calculation demonstrates that

$$\begin{aligned} M(\delta x_n, \delta x_{n+1}) = \max\Bigg\{ & \frac{d(\delta x_n, \mu x_n)[1 + d(\delta x_{n+1}, \mu x_{n+1})]}{1 + d(\delta x_n, \delta x_{n+1})} + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_n, \mu x_n)|, \\ & \frac{d(\delta x_{n+1}, \mu x_{n+1})[1 + d(\delta x_n, \mu x_n)]}{1 + d(\delta x_n, \delta x_{n+1})} + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_n, \mu x_n)|, \\ & d(\delta x_n, \delta x_{n+1}) + |d(\delta x_n, \mu x_n) - d(\delta x_{n+1}, \mu x_{n+1})| \Bigg\} \\ = \max\Bigg\{ & \frac{d(\delta x_n, \delta x_{n+1})[1 + d(\delta x_{n+1}, \delta x_{n+2})]}{1 + d(\delta x_n, \delta x_{n+1})} + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_n, \delta x_{n+1})|, \\ & \frac{d(\delta x_{n+1}, \delta x_{n+2})[1 + d(\delta x_n, \delta x_{n+1})]}{1 + d(\delta x_n, \delta x_{n+1})} + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_n, \delta x_{n+1})|, \\ & d(\delta x_n, \delta x_{n+1}) + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_{n+1}, \delta x_{n+2})| \Bigg\} \\ = \max\Bigg\{ & \frac{d(\delta x_n, \delta x_{n+1})[1 + d(\delta x_{n+1}, \delta x_{n+2})]}{1 + d(\delta x_n, \delta x_{n+1})}, \\ & d(\delta x_{n+1}, \delta x_{n+2}), d(\delta x_n, \delta x_{n+1}) + |d(\delta x_n, \delta x_{n+1}) - d(\delta x_{n+1}, \delta x_{n+2})| \Bigg\}. \end{aligned}$$

If we denote by

$$g_n = d(\delta x_n, \delta x_{n+1}).$$

We have

$$M(\delta x_n, \delta x_{n+1}) \leq \max\left\{ \frac{\delta_n(1 + \delta_{n+1})}{1 + \delta_n}, \delta_{n+1}, \delta_n + |\delta_n - \delta_{n+1}| \right\}.$$

Now, suppose that δ_n is not decreasing, then there exists $C \in \mathbb{N}$ such that $\delta_C \leq \delta_{C+1}$, we have

$$M(\delta x_C, \delta x_{C+1}) = \delta_{C+1},$$

then by the inequality (2.7), we have

$$\varphi(\delta_{C+1}) \leq h(\delta x_C, \delta x_{C+1})\varphi(M(\delta x_C, \delta x_{C+1})) = h(\delta x_C, \delta x_{C+1})\varphi(\delta_{C+1}) \leq \varphi(\delta_{C+1}).$$

Since for every $n \in \mathbb{N}$, $\delta x_n \neq \delta x_{n-1}$, we have $\delta_{C+1} = d(\delta x_{C+1}, \delta x_{C+2}) > 0$ which follows from the above inequality, we get $h(\delta x_C, \delta x_{C+1}) = 1$. By the fact that $h \in \mathcal{A}(X)$, we obtain $d(\delta x_C, \delta x_{C+1}) = 0$. It is a contradiction. Therefore, δ_n is decreasing, $\delta_n > \delta_{n+1}$ for all $n \geq 0$, we have

$$M(\delta x_n, \delta x_{n+1}) = \max\{2\delta_n - \delta_{n+1}, \delta_n, \delta_{n+1}\}.$$

Since δ_n is bounded below, the sequence converges. Let

$$\lim_{n\to\infty} \delta_n = C \geq 0.$$

Contrarily, suppose that $C > 0$. Due to the property of φ, $\lim_{n\to\infty} \varphi(\delta_n) = \varphi(C) > 0$. By (2.7), it is demonstrated that

$$\begin{aligned}\phi(\delta_{n+1}) &= \phi(d(\mu x_n, \mu x_{n+1})) \\ &\leq h(\delta x_n, \delta x_{n+1})\phi(M(\delta x_n, \delta x_{n+1})) \\ &\leq \phi(M(\delta x_n, \delta x_{n+1})).\end{aligned}$$

In the inequality above, when we take $n \to \infty$, we obtain

$$1 = \lim_{n\to\infty} \frac{\phi(\delta_{n+1})}{\phi(M(\delta x_n, \delta x_{n+1}))} \leq \lim_{n\to\infty} h(\delta x_n, \delta x_{n+1}) \leq 1.$$

Therefore, $\lim_{n\to\infty} h(\delta x_n, \delta x_{n+1}) = 1$. According to the notion of auxiliary functions,

$$\lim_{n\to\infty} d(\delta x_n, \delta x_{n+1}) = \lim_{n\to\infty} \delta_n = 0,$$

which contradicts to the assumption. So, $\lim_{n\to\infty} d(\delta x_n, \delta x_{n+1}) = 0$.

The sequence $\{\delta x_n\}$ has to be Cauchy, as we will demonstrate next. Contrarily, suppose that $\{\delta x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ such that, for all $k \in \mathbb{N}$, there are $n_k, m_k \in \mathbb{N}$ with $n_k > m_k \geq k$ satisfying n_k is the smallest number such that

$$d(\delta x_{n_k}, \delta x_{m_k}) \geq \epsilon \text{ and } d(\delta x_{n_k-1}, \delta x_{m_k}) < \epsilon.$$

By Lemma 2.2, this implies

$$\lim_{k\to\infty} d(\delta x_{m_k+1}, \delta x_{n_k+1}) = \lim_{k\to\infty} d(\delta x_{m_k}, \delta x_{n_k}) = \varepsilon.$$

We determine that $(\delta x_{m_k}, \delta x_{n_k}) \in E(\mathcal{G})$ for each $k \in \mathbb{N}$ owing to the transitivity property of $E(\mathcal{G})$. As a result,

$$\begin{aligned}\varphi(d(\delta x_{m_k+1}, \delta x_{n_k+1})) &= \varphi(d(\mu x_{m_k}, \mu x_{n_k})) \\ &\leq h(\delta x_{m_k}, \delta x_{n_k})\varphi(M(\delta x_{m_k}, \delta x_{n_k})),\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}&M(\delta x_{m_k}, \delta x_{n_k}) \\ &= \max\Bigg\{\frac{d(\delta x_{m_k}, \mu x_{m_k})[1 + d(\delta x_{n_k}, \mu x_{n_k})]}{1 + d(\delta x_{m_k}, \delta x_{n_k})} + |d(\delta x_{m_k}, \delta x_{n_k}) - d(\delta x_{m_k}, \mu x_{m_k})|, \\ &\quad \frac{d(\delta x_{n_k}, \mu x_{n_k})[1 + d(\delta x_{m_k}, \mu x_{m_k})]}{1 + d(\delta x_{m_k}, \delta x_{n_k})} + |d(\delta x_{m_k}, \delta x_{n_k}) - d(\delta x_{m_k}, \mu x_{m_k})|, \\ &\quad d(\delta x_{m_k}, \delta x_{n_k}) + |d(\delta x_{m_k}, \mu x_{m_k}) - d(\delta x_{n_k}, \mu x_{n_k})|\Bigg\}\end{aligned}$$

$$= \max\left\{\frac{d(\delta x_{m_k},\delta x_{m_k+1})[1+d(\delta x_{n_k},\delta x_{n_k+1})]}{1+d(\delta x_{m_k},\delta x_{n_k})} + |d(\delta x_{m_k},\delta x_{n_k}) - d(\delta x_{m_k},\delta x_{m_k+1})|,\right.$$
$$\frac{d(\delta x_{n_k},\delta x_{n_k+1})[1+d(\delta x_{m_k},\delta x_{m_k+1})]}{1+d(\delta x_{m_k},\delta x_{n_k})} + |d(\delta x_{m_k},\delta x_{n_k}) - d(\delta x_{m_k},\delta x_{m_k+1})|,$$
$$\left. d(\delta x_{m_k},\delta x_{n_k}) + |d(\delta x_{m_k},\delta x_{m_k+1}) - d(\delta x_{n_k},\delta x_{n_k+1})|\right\}.$$

Since $\lim_{n\to\infty} d(\delta x_n,\delta x_{n+1}) = 0$, applying the preceding equality with $k\to\infty$ means that

$$\lim_{k\to\infty} M(\delta x_{m_k},\delta x_{n_k}) = \lim_{k\to\infty} d(\delta x_{m_k},\delta x_{n_k}) = \epsilon > 0.$$

Combining the aforementioned fact with the inequality (2.8), we obtain

$$1 = \lim_{k\to\infty}\frac{\varphi(d(\delta x_{m_k+1},\delta x_{n_k+1}))}{\varphi(M(\delta x_{m_k},\delta x_{n_k}))} \le \lim_{k\to\infty} h(\delta x_{m_k},\delta x_{n_k}) \le 1.$$

As a result, $\lim_{k\to\infty} h(\delta x_{m_k},\delta x_{n_k}) = 1$. Then, $\lim_{k\to\infty} d(\delta x_{m_k},\delta x_{n_k}) = 0$, which contradicts to (2.3). That $\{\delta x_n\}$ is Cauchy in (X,d) must therefore be true.

Since $\delta(X)$ is a d-closed subset of (X,d), which is complete, it follows that there exists $x,u\in X$ such that $u=\delta x\in\delta(X)$ satisfying

$$\lim_{n\to\infty}\delta x_n = \lim_{n\to\infty}\mu x_n = u. \tag{2.9}$$

Since assumption (a), by Lemma 2.3 which indicates that u is a coincidence point of μ and δ. Thus $u\in C(\mu,\delta)$.

Assume that the statement (b) is satisfied. Because of (2.9), we assert that x must be a coincidence point of μ and δ. On the other hand, suppose that x is not a coincidence point of μ and δ. As a result, $\mu x\neq\delta x$ and thus $d(\mu x,\delta x)>0$. Since the triple $(X,d,\mathcal{G})$ has the property A, $(\delta x_n,\delta x)\in E(\mathcal{G})$ for all $n\in\mathbb{N}$. Consequently,

$$d(\delta x,\mu x)\le d(\delta x,\mu x_{n_k}) + d(\mu x_{n_k},\mu x).$$

Hence,

$$d(\delta x,\mu x) - d(\delta x,\mu x_{n_k})\le d(\mu x_{n_k},\mu x).$$

The definition of φ actually proves that

$$\begin{aligned}\varphi(d(\delta x,\mu x) - d(\delta x,\mu x_{n_k})) &\le \varphi(d(\mu x_{n_k},\mu x))\\ &\le h(\delta x_{n_k},\delta x)\varphi(M(\delta x_{n_k},\delta x))\\ &< \varphi(M(\delta x_{n_k},\delta x)),\end{aligned} \tag{2.10}$$

where

$$M(\delta x_{n_k},\delta x) = \max\left\{\frac{d(\delta x_{n_k},\mu x_{n_k})[1+d(\delta x,\mu x)]}{1+d(\delta x_{n_k},\delta x)} + |d(\delta x_{n_k},\delta x) - d(\delta x_{n_k},\mu x_{n_k})|,\right.$$
$$\frac{d(\delta x,\mu x)[1+d(\delta x_{n_k},\mu x_{n_k})]}{1+d(\delta x_{n_k},\delta x)} + |d(\delta x_{n_k},\delta x) - d(\delta x_{n_k},\mu x_{n_k})|,$$

$$d(\delta x_{n_k}, \delta x) + |d(\delta x_{n_k}, \mu x_{n_k}) - d(\delta x, \mu x)|\Big\}.$$

In the equation above, when we take $n \to \infty$ and use (2.9), we obtain

$$\lim_{k\to\infty} M(\delta x_{n_k}, \delta x) = d(\delta x, \mu x) > 0.$$

By the attribute of φ, we get

$$\lim_{k\to\infty} \varphi(M(\delta x_{n_k}, \delta x)) = \varphi(d(\delta x, \mu x)) > 0.$$

Then, taking $k \to \infty$ in (2.10) gives us that $\lim_{k\to\infty} h(\delta x_{n_k}, \delta x) = 1$. This implies

$$d(\delta x, \mu x) = \lim_{k\to\infty} d(\mu x_{n_k}, \mu x) = 0,$$

which is a contradiction. As a result, $\mu x = \delta x$, and we can derive that μ and δ have x as one of their coincidence points. □

By applying an additional assumption, as in the following theorem, we could reach a stronger conclusion on the presence of a common fixed point.

Theorem 2.8. *Let us apply all the notations and requirements from Theorem 2.5. Moreover, suppose additionally that*

(6) It is precise to state that $(\delta x, \delta y)$ is in $E(\mathcal{G})$ for any $x, y \in C(\mu, \delta)$ with $\delta x \neq \delta y$.

Consequently, we obtain

$$X(\mu, \delta) \neq \emptyset \text{ if and only if } Cm(\mu, \delta) \neq \emptyset.$$

Proof. By proving Theorem 2.5, it is sufficient to account for the only if case with the assumption that the statement (6) above holds. There exists an element $x \in X$ such that $\delta x = \mu x$, according to Theorem 2.5.

Initially, let us assume that $y \in X$ is also a coincidence point, i.e., $\delta y = \mu y$. We will show that $\delta x = \delta y$. Contrarily, suppose that $\delta x \neq \delta y$, we have $d(\delta x, \delta y) > 0$. By statement (6) above, $(\delta x, \delta y) \in E(\mathcal{G})$, which concludes

$$\varphi(d(\mu x, \mu y)) \leq h(\delta x, \delta y)\varphi(R(\delta x, \delta y)) \leq \varphi(R(\delta x, \delta y)) = \varphi(d(\mu x, \mu y)).$$

Due to the property of φ, $h(\delta x, \delta y) = 1$, we have $d(\delta x, \delta y) = 0$. It is a contradiction. Therefore, $\delta x = \delta y$.

The next step is to put $x_0 = x$ and utilize the statement (2) from Theorem 2.5 to establish a sequence $\{x_n\}$ such that $\delta x_n = \mu x_{n-1}$ for every $n \in \mathbb{N}$. As x is a coincidence point, we could suppose that $x_n = x$, then $\delta x_n = \mu x$ for each $n \in \mathbb{N}$.

In order for $\delta z = \delta\delta x = \delta\mu x$, allow $z = \delta x$. Note also that $\delta x_n = \mu x = \mu x_{n-1}$ for any $n \in \mathbb{N}$. Therefore,

$$\lim_{n\to\infty} \mu x_n = \lim_{n\to\infty} \delta x_n = \mu x$$

in (X, d'). Furthermore,

$$\lim_{n\to\infty} d'(\delta\mu x_n, \mu\delta x_n) = 0,$$

because μ and δ are d'-compatible. This indicates that $\delta\mu x = \mu\delta x$. Thus, $\delta z = \delta\mu x = \mu\delta x = \mu z$ so $z \in C(\mu, \delta)$. Following the proof above, we have $\mu z = \delta z = \delta x = z$. Hence, $z \in Cm(\mu, \delta)$. □

Theorem 2.9. *Let us apply all the notations and requirements from Theorem 2.7. Moreover, suppose additionally that*

(6) It is precise to state that $(\delta x, \delta y)$ is in $E(\mathcal{G})$ for any $x, y \in C(\mu, \delta)$ with $\delta x \neq \delta y$.

Consequently, we obtain

$$X(\mu, \delta) \neq \emptyset \text{ if and only if } Cm(\mu, \delta) \neq \emptyset.$$

To support our main findings, we provide an example.

Example 2.10. *Let $X = [0, \infty) \subseteq \mathbb{R}$, and $d, d' : X \times X \to [0, \infty)$ be such that*

$$d(x, y) = |x - y| \text{ and } d'(x, y) = L|x - y|,$$

for all $x, y \in X$ with a constant $L \in (1, \infty)$. We note that d and d' are metrics. Additionally, it is obvious that $d < d'$ by the way we specify our metrics. Then, assume

$$E(\mathcal{G}) = \{(x, y) : x = y \text{ or } x, y \in [0, 1] \text{ with } x \leq y\}.$$

Moreover, let $\mu : X \to X$ and $\delta : X \to X$ be given by

$$\delta x = x^2 \text{ and } \mu x = \ln\left(1 + \frac{x^2}{7}\right),$$

for all $x \in X$. In order for the pair (μ, δ) to be an $(h\text{-}\varphi)_R$-contraction with regard to d, the conditions (1) and (2) must be satisfied, which we shall demonstrate.

First, let $(\delta x, \delta y) \in E(\mathcal{G})$. It is noticeable that $(\mu x, \mu y) \in E(\mathcal{G})$ if $x = y$. In contrast, if $(\delta x, \delta y) \in E(\mathcal{G})$ and $\delta x \leq \delta y$, then $\delta x = x^2$, $\delta y = y^2 \in [0, 1]$ and $x^2 = \delta x \leq \delta y = y^2$. Hence,

$$\mu x = \ln\left(1 + \frac{x^2}{7}\right) \leq \ln\left(1 + \frac{y^2}{7}\right) = \mu y$$

and $\mu x, \mu y \in [0, 1]$. Thus, $(\mu x, \mu y) \in E(\mathcal{G})$.

Second, set $\phi(t) = 7t$, and define $h : X \times X \to [0, 1)$ by the following equation.

$$h(x, y) = \begin{cases} \dfrac{\arctan\left(\frac{|x-y|}{7}\right)}{\psi(x, y)} & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

where

$$\psi(x, y) = 2\,|x - y| + \ln\left(\frac{7 + x}{7 + y}\right).$$

We first note that the function

$$\psi(x, y) = 2\,|x - y| + \ln\left(\frac{7 + x}{7 + y}\right)$$

is positive for $x, y > 0$. In the case of $x > y$, it is easy to see that $\psi(x, y) > 0$. On the other hand, we observe that

$$\psi(x, y) = 2(y - x) + \ln\left(\frac{7 + x}{7 + y}\right) = (y - x) + y - \ln\left(1 + \frac{y}{7}\right) - x + \ln\left(1 + \frac{x}{7}\right).$$

Since, the function $\gamma(x) = x - \ln\left(1 + \frac{x}{7}\right)$ *is an increasing function. As a result, we conclude that* $\psi(x, y)$ *is a positive function, therefore,* $h(x, y)$ *is also a positive function. It is straightforward to prove that* $\phi \in \Phi$ *and* $h \in \mathcal{A}(X)$*. The profile of the function* $h(x, y)$ *is plotted in Figure 1.*

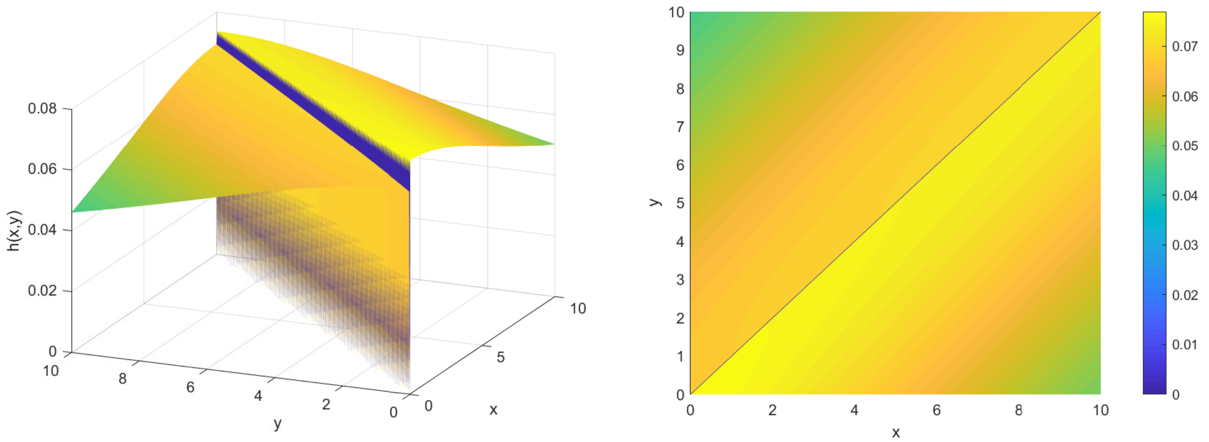

Figure 1. The profile of the function $h(x, y)$.

Next, let $x, y \in X$ *such that* $(\delta x, \delta y) \in E(\mathcal{G})$*. If* $\delta x = \delta y$*, then* $x = y$ *so that the requirement (2) holds. In the case of* $x^2 = \delta x < \delta y = y^2$*, it follows that*

$$
\begin{aligned}
\phi(d(\mu x, \mu y)) = 7d(\mu x, \mu y) &= 7\left|\ln\left(1 + \frac{x^2}{7}\right) - \ln\left(1 + \frac{y^2}{7}\right)\right| \\
&= 7\ln\left(\frac{1 + \frac{y^2}{7}}{1 + \frac{x^2}{7}}\right) \\
&= 7\ln\left(1 + \frac{\frac{y^2}{7} - \frac{x^2}{7}}{1 + \frac{x^2}{7}}\right) \\
&\le 7\ln\left(1 + \left|\frac{x^2}{7} - \frac{y^2}{7}\right|\right) \\
&\le 7\arctan\left(\left|\frac{x^2}{7} - \frac{y^2}{7}\right|\right) \\
&\le \frac{7\arctan\left(|x^2 - y^2|\right)}{\psi(x^2, y^2)}\psi(x^2, y^2).
\end{aligned}
$$

To obtain the requirements, using $x < y$*, we see that*

$$
\begin{aligned}
\psi(x^2, y^2) &= 2(y^2 - x^2) - \ln\left(1 + \frac{y^2}{7}\right) + \ln\left(1 + \frac{x^2}{7}\right) \\
&= d(x, y) + \left[y^2 - \ln\left(1 + \frac{y^2}{7}\right) - x^2 + \ln\left(1 + \frac{x^2}{7}\right)\right] \\
&= d(x, y) + \left|d(\delta y, \mu y) - d(\delta x, \hat{f} y)\right|,
\end{aligned}
$$

which yields

$$\phi(d(\mu x,\mu y)) \le h(x,y)\,(d(x,y) + |d(\delta y,\mu y) - d(\delta x,\mu y)|) \le h(x,y)R(\delta x,\delta y).$$

Consequently, the pair (μ,δ) satisfies condition (2).

We will demonstrate that the requirements (1) through (5) of Theorem 2.5 attained in the final part of this example.

(1) $\delta : (X,d') \to (X,d')$ is obviously continuous, and $\delta(X) = [0,\infty)$ is also d'-closed,

(2) It is observable that $\mu(X) = \delta(X) = X$,

(3) The transitivity property $E(\mathcal{G})$ holds,

(4) Because $d < d'$, we shall demonstrate that $\mu : (X,d) \to (X,d')$ is δ-Cauchy. Assuming $\epsilon > 0$ and a sequence $\{x_n\}$ in X where $\{\delta x_n\}$ is Cauchy in (X,d), there exists $N \in \mathbb{N}$ such that $d(\delta x_n,\delta x_m) < \dfrac{\epsilon}{L}$ for any $n,m \ge N$. Therefore,

$$\begin{aligned}
d'(\mu x_n,\mu x_m) &= L|\mu x_n - \mu y_m| \\
&= L\left|\ln\left(1+\frac{(x_n)^2}{7}\right) - \ln\left(1+\frac{(x_m)^2}{7}\right)\right| \\
&= L\left|\ln\left(\frac{1+\frac{(x_m)^2}{7}}{1+\frac{(x_n)^2}{7}}\right)\right| \\
&= L\left|\ln\left(1+\frac{\frac{(x_m)^2}{7}-\frac{(x_n)^2}{7}}{1+\frac{(x_n)^2}{7}}\right)\right| \\
&\le L\left[\ln\left(1+\left|\frac{(x_n)^2}{7}-\frac{(x_m)^2}{7}\right|\right)\right] \\
&\le L\left|\frac{(x_n)^2}{7}-\frac{(x_m)^2}{7}\right| \\
&< L\left|(x_n)^2-(x_m)^2\right| \\
&= Ld(\delta x_n,\delta x_m) \\
&< L\left(\frac{\epsilon}{L}\right) \\
&= \epsilon.
\end{aligned}$$

This pertains to $\mu : (X,d) \to (X,d')$ being δ-Cauchy.

(5) $\mu : (X,d') \to (X,d')$ is obviously $\mathcal{G}$-continuous. In addition, μ and δ are d'-compatible since for every sequence $\{x_n\}$ in X with

$$\lim_{n\to\infty}\delta x_n = \lim_{n\to\infty}\mu x_n = x,$$

it has the consequence that $\ln\left(1+\dfrac{x}{7}\right) = x$. This concludes $x = 0$. As $n \to \infty$,

$$d'(\delta\mu x_n,\mu\delta x_n) = L\left|\left(\ln\left(1+\frac{(x_n)^2}{7}\right)\right)^2 - \ln\left(1+\frac{(x_n)^4}{7}\right)\right| \to 0.$$

Finally, it is noticeable that $(\delta 0,\mu 0) = (0,0) \in E(\mathcal{G})$ so $X(\mu,\delta)$ is nonempty. From Theorem 2.5, $C(\mu,\delta)$ is nonempty. In actuality, it is clear that $0 \in C(\mu,\delta)$.

3. Application to nonlinear two-term fractional differential equations with nonlocal boundary conditions

Numerous scientific studies state that the theory of fractional differential equations has become more popular as a result of its applications in a variety of engineering and scientific fields, for instance, see [10–14]. Therefore, we apply our findings in this section to investigate the existence of any solutions to curtain Caputo fractional boundary value problems with nonlocal boundary conditions. Multi-term fractional differential equations have recently made significant contributions [15–23]. Motivated by [21–23], we study the nonlinear two-term fractional differential equations in the following form:

$$^{c}D^{\alpha}y(t) + b^{c}D^{\beta}y(t) = f(t, y(t)), \quad t \in [0, 1], \tag{3.1}$$

with the nonlocal boundary conditions

$$y(0) = 0, \quad \text{and} \quad y(1) = y(\eta), \quad \eta \in (0, 1) \tag{3.2}$$

where α, β are arbitrary real constants with $0 \leq \beta \leq 1 < \alpha \leq 2$, and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous. It is worth noting that nonlocal BVPs appear to be more intriguing than local ones due to their greater naturalness and the variety of applications they offer. Additionally, the local conditions $y(0) = 0$ and $y'(1) = 0$ can be considered as the limit case of (3.2) when $\eta \to 1^{-}$. Here, we provide some scientific publications related to the nonlocal BVPs equations, which are relevant to the developing topic in [19, 20, 25–27].

Recalling the definition of the Caputo fractional derivative and its related definitions is necessary before moving on to the outcomes of existence. Let α be a positive real number. The Caputo derivative of fractional order α is defined as follows for a continuous function $y(t)$:

$$^{c}D^{\alpha}y = I^{\lceil\alpha\rceil-\alpha}D^{\lceil\alpha\rceil}y,$$

where $\lceil\alpha\rceil$ is the smallest integer which is greater than α and I^{α} is the Riemann-Liouville integral operator of order $\alpha \geq 0$ defined by

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) ds.$$

Noting that when $\alpha = 0$, the operator I^0 is referred to the identity operator and the gamma function Γ is defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt.$$

The fractional integral satisfies the following equalities:

$$I^{\alpha}I^{\beta}y(t) = I^{\alpha+\beta}y(t), \quad \alpha, \beta \geq 0,$$

$$I^{\alpha}t^{k} = \frac{\Gamma(k + 1)}{\Gamma(\alpha + k + 1)} t^{\alpha+k}, \quad \alpha, k \geq -1.$$

Additionally, according to the $\alpha-$order Caputo fractional derivative and its integer-ordered, we get

$$I^{\alpha}D^{\alpha}y(t) = y(t) - \sum_{k=0}^{m-1} y^{(k)}(0)\frac{t^k}{k!}, \quad m - 1 < m \leq \alpha. \tag{3.3}$$

In order to obtain our goal, we suppose that $y : [0,1] \to \mathbb{R}$ is a solution of the systems (3.1) and (3.2). We see that

$$\begin{aligned} I^\alpha f(t,y(t)) &= I^\alpha[{}^cD^\alpha y(t)] + bI^\alpha[{}^cD^\beta y(t)] \\ &= y(t) + a_0 + a_1 t + bI^{\alpha-\beta} I^\beta[{}^cD^\beta y(t)] \\ &= y(t) + a_0 + a_1 t + bI^{\alpha-\beta}[y(t) + a_0] \\ &= y(t) + a_0 + a_1 t + bI^{\alpha-\beta} y(t) + \frac{a_0 b}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta}. \end{aligned}$$

Consequently, we have

$$y(t) = a_0 + \frac{a_0 b}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta} + a_1 t + bI^{\alpha-\beta} y(t) - I^\alpha f(t,y(t)). \tag{3.4}$$

This implies that the initial value problem (BVP) (3.1) and (3.2) is equivalent to the Volterra integral equation in a specific type. We have $a_0 = 0$ by applying the boundary conditions $y(0) = 0$. The solution is consequently condensed to

$$y(t) = a_1 t + bI^{\alpha-\beta} y(t) - I^\alpha f(t,y(t)).$$

Applying the boundary condition $y(1) = y(\eta)$ allows us to have the coefficient

$$a_1 = \frac{1}{1-\eta}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 G_\alpha(s;\eta) f(s,y(s))ds - \frac{b}{\Gamma(\alpha-\beta)}\int_0^1 G_{\alpha-\beta}(s;\eta) f(s,y(s))ds\right],$$

where the function $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is defined by

$$G_\gamma(s;\eta) = \begin{cases} (1-s)^{\gamma-1} - (\eta-s)^{\gamma-1}, & 0 \le s \le \eta, \\ (1-s)^{\gamma-1}, & s > \eta. \end{cases}$$

Substituting the value of a_1 in the expressions for $y(t)$, we get the solution of the BVP (3.1) and (3.2) as the solution of the Volterra integral equation in the following form:

$$\begin{aligned} y(t) = \frac{t}{1-\eta}&\left[\frac{1}{\Gamma(\alpha)}\int_0^1 G_\alpha(s;\eta) f(s,y(s))ds - \frac{b}{\Gamma(\alpha-\beta)}\int_0^1 G_{\alpha-\beta}(s;\eta) f(s,y(s))ds\right] \\ &+ \frac{b}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} y(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds. \end{aligned} \tag{3.5}$$

Next, an integral operator is typically used to establish a fixed point problem. In our case, we consider the integral operator $T : C[0,1] \to C[0,1]$ defined by

$$\begin{aligned} T(y)(t) = \frac{t}{1-\eta}&\left[\frac{1}{\Gamma(\alpha)}\int_0^1 G_\alpha(s;\eta) f(s,y(s))ds - \frac{b}{\Gamma(\alpha-\beta)}\int_0^1 G_{\alpha-\beta}(s;\eta) f(s,y(s))ds\right] \\ &+ \frac{b}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} y(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds. \end{aligned} \tag{3.6}$$

We can observe that the solution of BVP (3.1) and (3.2) is given by $Ty = y$. In order to achieve the existence of the solutions, we let $\xi : \mathbb{R}^2 \to \mathbb{R}$, $E(\mathcal{G}) = \{(u, v) \in \mathbb{R}^2 : \xi(u, v) \geq 0\}$ and consider the following conditions:

(H_1) There exists $u_0 \in C[0, 1]$ such that $\xi(u_0, T(u_0)) \geq 0$ for all $t \in [0, 1]$.

(H_2) For all $t \in [0, 1]$ and $u, v \in C[0, 1]$,

$$\xi(u, v) \geq 0 \Longrightarrow \xi(Tu(t), Tv(t)) \geq 0.$$

(H_3) For all $v, u, w \in C[0, 1]$ and $t \in [0, 1]$,

$$\xi(u(t), v(t)) \geq 0 \;\text{ and }\; \xi(v(t), w(t)) \geq 0 \;\text{ together imply }\; \xi(u(t), w(t)) \geq 0.$$

(H_4) For any $t \in [0, 1]$ and for all $u, v \in \mathbb{R}$ with $\xi(u, v) \geq 0$, there is a positive constant L such that

$$|f(t, u) - f(t, v)| \leq L|u - v|.$$

Here, we provide the following useful lemma related to the conditions that appeared in the main theorem of this section. The results can be verified straightforwardly, therefore, we leave the proof.

Lemma 3.1. *Assume that (H_1)–(H_3) hold. If $E(\mathcal{G}) = \{(u, v) \in \mathbb{R}^2 : \xi(u, v) \geq 0\}$, then we have the following:*

(1) There exists $u_0 \in C[0, 1]$ such that $(u_0, T(u_0)) \in E(\mathcal{G})$ for all $t \in [0, 1]$,

(2) For all $t \in [0, 1]$ and $u, v \in C[0, 1]$,

$$(u(t), v(t)) \in E(\mathcal{G}) \Longrightarrow (Tu(t), Tv(t)) \in E(\mathcal{G}),$$

(3) The transitivity property of $E(\mathcal{G})$ holds.

Before going though the existence theorem of the BVP (3.1) and (3.2), we introduce the solution space $C([0, 1])$ equipped by the metric

$$d_\sigma(u, v) = \max_{t \in [0,1]} \frac{|u(t) - v(t)|}{e^{\sigma t}}, \quad u, v \in C([0, 1]).$$

We note that the metric space $(C[0, 1], d_\sigma)$ is complete.

Theorem 3.2. *Assume that the conditions (H_1)–(H_4) hold. If σ is sufficiently large such that*

$$\frac{2}{(1-\eta)}\left[\frac{(3-\eta)L}{\sigma^\alpha} + \frac{(3-\eta)|b|}{\sigma^{\alpha-\beta}}\right] < 1,$$

then T has at least one fixed point $u^* \in (C[0, 1], d_\sigma)$, which means the BVP (3.1) *and* (3.2) *has at least one solution $u^* \in (C[0, 1], d_\sigma)$.*

Proof. Let $E(\mathcal{G}) = \{(u, v) \in \mathbb{R}^2 : \xi(u, v) \geq 0\}$. From Lemma 3.1, We have $X(f, g) \neq \emptyset$, T is edge-preserving with regard to $\mathcal{G}$ and $E(\mathcal{G})$ satisfies the transitivity property. Now, in order to demonstrate this, we concentrate on the actual contraction property of T. As a result, we begin by the condition (H_4) that, for $u, v \in C[0, 1]$ such that $(u, v) \in E(\mathcal{G})$,

$$
\begin{aligned}
|T(u)(t)-T(v)(t)| &= \Big|\frac{t}{1-\eta}\frac{1}{\Gamma(\alpha)}\int_0^1 G_\alpha(s;\eta)(f(s,u(s))-f(s,v(s)))ds \\
&\qquad - \frac{t}{1-\eta}\frac{b}{\Gamma(\alpha-\beta)}\int_0^1 G_{\alpha-\beta}(s;\eta)(f(s,u(s))-f(s,v(s)))ds \\
&\qquad + \frac{b}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}(u(s)-v(s))ds \\
&\qquad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(f(s,u(s))-f(s,v(s)))ds\Big| \\
&\le \frac{L}{(1-\eta)\Gamma(\alpha)}\int_0^1 |G_\alpha(s;\eta)|\,|u(s)-v(s)|ds \\
&\qquad + \frac{L|b|}{(1-\eta)\Gamma(\alpha-\beta)}\int_0^1 |G_{\alpha-\beta}(s;\eta)||u(s)-v(s)|ds \\
&\qquad + \frac{|b|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|u(s)-v(s)|ds \\
&\qquad + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u(s)-v(s)|ds \\
&= \frac{L}{(1-\eta)\Gamma(\alpha)}\int_0^1 |G_\alpha(s;\eta)|\,e^{\sigma s}\frac{|u(s)-v(s)|}{e^{\sigma s}}ds \\
&\qquad + \frac{Lb}{(1-\eta)\Gamma(\alpha-\beta)}\int_0^1 |G_{\alpha-\beta}(s;\eta)|e^{\sigma s}\frac{|u(s)-v(s)|}{e^{\sigma s}}ds \\
&\qquad + \frac{|b|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}e^{\sigma s}\frac{|u(s)-v(s)|}{e^{\sigma s}}ds \\
&\qquad + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{\sigma s}\frac{|u(s)-v(s)|}{e^{\sigma s}}ds \\
&= \Big[\frac{L}{(1-\eta)\Gamma(\alpha)}\int_0^1 |G_\alpha(s;\eta)|\,e^{\sigma s}ds + \frac{Lb}{(1-\eta)\Gamma(\alpha-\beta)}\int_0^1 |G_{\alpha-\beta}(s;\eta)|e^{\sigma s}ds \\
&\qquad + \frac{|b|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}e^{\sigma s}ds + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{\sigma s}ds\Big]d_\sigma(u,v).
\end{aligned}
$$

By applying the fact that

$$
\int_0^t (t-s)^{\gamma-1}e^{\sigma s}ds \le \frac{\Gamma(\gamma)}{\sigma^\gamma}, \quad t\ge 0, \quad \sigma>0,
$$

for $\gamma > 0$, we have

$$
\int_0^1 |G_\gamma(s;\eta)|e^{\sigma s}ds \le \int_0^1 (1-s)^{\gamma-1}e^{\sigma s}ds + \int_0^\eta (\eta-s)^{\gamma-1}e^{\sigma s}ds \le 2\frac{\Gamma(\gamma)}{\sigma^\gamma},
$$

Consequently, we get

$$
|T(u)(t)-T(v)(t)| \le \frac{1}{(1-\eta)}\left[\frac{(3-\eta)L}{\sigma^\alpha} + \frac{(3-\eta)|b|}{\sigma^{\alpha-\beta}}\right]d_\sigma(u,v),
$$

which implies

$$\frac{|T(u)(t)-T(v)(t)|}{e^{\sigma t}} \leq \frac{1}{(1-\eta)}\left[\frac{(3-\eta)L}{\sigma^{\alpha}}+\frac{(3-\eta)|b|}{\sigma^{\alpha-\beta}}\right]d_{\sigma}(u,v), \quad t\in[0,1].$$

Here, we let $\varphi(t)=t/2$ which is $\varphi\in\Phi$. Hence,

$$d_{\sigma}(Tu,Tv) \leq \frac{2}{(1-\eta)}\left[\frac{(3-\eta)L}{\sigma^{\alpha}}+\frac{(3-\eta)|b|}{\sigma^{\alpha-\beta}}\right]\varphi(d_{\sigma}(u,v)). \tag{3.7}$$

Therefore, by applying the σ is sufficiently large such that

$$\frac{2}{(1-\eta)}\left[\frac{(3-\eta)L}{\sigma^{\alpha}}+\frac{(3-\eta)|b|}{\sigma^{\alpha-\beta}}\right] < 1,$$

then we reach

$$d_{\sigma}(Tu,Tv) \leq \phi(d_{\sigma}(u,v)).$$

To this end, we define $h: C[0,1]\times C[0,1]\to[0,1]$ by

$$h(u,v)=\begin{cases}\frac{\phi(d_{\sigma}(u,v))}{d_{\sigma}(u,v)} & \text{if } u\neq v,\\ 0 & \text{if } u=v.\end{cases}$$

Finally, by utilizing Theorem 2.7, we then have T is $(h,\phi)_M$ contraction. It follows that u^* exists in $C[0,1]$ such that $Tu^*=u^*$ as desired. □

Additionally, one can observe the following for $E(\mathcal{G})=\mathbb{R}^2$:
(H_1^*) There is a positive constant L such that

$$|f(t,u)-f(t,v)| \leq L|u-v|,$$

for each $t\in[0,1]$ and $u,v\in\mathbb{R}$.

The following corollary is provided by Theorem 3.2.

Corollary 3.3. *If (H_1^*) holds, then the BVP* (3.1) *and* (3.2) *has at least one solution $u^*\in C[0,1]$.*

Example 3.4. *For $0\leq\beta\leq 1<\alpha\leq 2$, we consider the following fractional differential equation*

$${}^cD^{\alpha}y(t)+b\,{}^cD^{\beta}y(t)=L\sqrt{t}\,(\arctan(y(t))-g(t)), \quad t\in[0,1], \tag{3.8}$$

with the boundary conditions

$$y(0)=0, \quad \text{and} \ \ y(1)=y(\eta). \tag{3.9}$$

Observe that $f(t,y(t))=L\sqrt{t}\,(y(t)-g(t))$, we can have

$$|f(t,u(t))-f(t,v(t))| = L\sqrt{t}\,|\arctan(u(t))-\arctan(v(t))| \leq L\,|u(t)-v(t)|, \quad t\in[0,1],$$

which yields the confirmation of the condition (H_1^*). Consequently, Corollary 3.3 conclusion is applicable, and then the BVP (3.8) *and* (3.9) *has at least one solution on* $(C[0,1],d_{\sigma})$*, where*

$$\frac{2}{(1-\eta)}\left[\frac{(3-\eta)L}{\sigma^{\alpha}}+\frac{(3-\eta)|b|}{\sigma^{\alpha-\beta}}\right] < 1.$$

4. Conclusions

In this study, we investigated the $(h\text{-}\varphi)_R$ and $(h\text{-}\varphi)_M$ contractions with two metrics endowed with a directed graph and established the requirements that guarantee the existence of some common fixed points. The obtained results extend and generalize the theorems given in the literature, including [31, 47, 52]. Furthermore, by applying our main results, the existence of solutions to a class of nonlinear two-term fractional differential equations is successfully acquired. The nonlocal boundary conditions are used in the problems, giving new consequences to study and analyze the existence of a solution to the fractional BVPs. Additionally, some examples pertaining to the fixed point theorems and the nonlocal BVPs equations are provided to support our theoretical results. Based on these findings, we shall extend the fixed-point techniques and use them to investigate the existence of solutions to nonlinear fractional equations in other types.

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Conflict of interest

No conflict of interest exists.

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