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Research article

Existence and uniqueness results for mixed derivative involving fractional operators

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Abstract: In this article, we discuss the existence and uniqueness results for mix derivative involving fractional operators of order $\beta \in (1,2)$ and $\gamma \in (0,1)$. We prove some important results by using integro-differential equation of pantograph type. We establish the existence and uniqueness of the solutions using fixed point theorem. Furthermore, one application is likewise given to represent our fundamental results.

Keywords: fractional derivatives; existence conditions; integro-differential equations; pantograph equations
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1. Introduction

The pantograph equation is a special type of functional differential equations with proportional delay. The present study introduces a compound technique incorporating the perturbation method with a iteration algorithm to solve numerically the delay differential equation of pantograph type. The pantograph equations became a prime example of delay differential equation in the recent years. Over the last few years, the continuous and discrete cases of the pantograph equation have been extensively explored see [1–3].

Different authors discuss linear and non-linear pantograph equations. The solution to the simplest homogeneous linear pantograph cannot be expressed in terms of elementary functions. We can't solve even the simplest non-homogeneous linear pantograph problem using standard approaches like variation of constants and Laplace transformation. The existence and uniqueness of solutions for the linear pantograph equation's initial value problems change significantly depending on the beginning locations chosen. In general, the solution to the initial value problem may or may not exist, or may not be unique. some authors discuss linear pantograph Volterra delay-integro-differential equation and the multi-terms boundary value problem of fractional pantograph differential equations [4, 5].

It is also possible to obtain additive, multiplicative, and functional separable solutions, as well as several additional precise solutions. Nonlinear pantograph-type PDEs of a more broad form, containing one or two arbitrary functions Polyanin et al. [6] examine Nonlinear pantograph-type diffusion PDEs, exact solutions and the principle of analogy. Recently, many research on fractional-order pantograph differential equations have recently been published, involving various operators [7–9], φ-Caputo derivative [10], Atangana-Baleanu-Caputo derivative [11, 23, 24].

Furthermore, several scientific scholars have produced results regarding the existence and uniqueness of solutions for various classes of fractional pantograph equations by applying various fixed-point theorems as like Shah et al. [12] discussed the dynamics and stability of-fractional pantograph equations and Houas et al. [13] studied the existence and Ulam stability of fractional pantograph differential equations with two Caputo-Hadamard derivatives.

Different authors like [14, 15] work on the pantograph-catenary electrical contact system of high speed railways. The pantograph-catenary electrical contact system, which serves as the only power entrance, keeps the high-speed train's power transfer reliable and efficient. A pantograph-catenary system must take into account the wind, sand, rain, thunder, ice, and snow while designing it due to the rapid expansion of high-speed trains around the world. Commercialized lines are also being developed in China to cover isolated areas with severe environments. There are some recent results on the existence of solutions for fractional integro-differential and fractional differential equations [16, 17].

The following fractional integro-differential equation of pantograph type is considered in this work, along with appropriate initial conditions.

$$\begin{cases} {}^{C}D^{\beta}[{}^{C}D^{\gamma}v(t)] = \phi(t, v(t), v(\Lambda t)) + \int_0^{qt} k_1(t, s, v(s))ds + \int_0^{t} k_2(t, s, v(s))ds, \quad t \in \mathbf{J} := [0, M], \\ {}_0D_t^{\beta-1}v(0) = v_0 \quad v'(0) = v_1. \end{cases} \tag{1.1}$$

Where $0 < \gamma < 1$ and $1 < \beta < 2$ as well as $\Lambda, q < 1$, $\phi : \mathbf{J} \times \Re \times \Re \to \Re$ and $k_i : \mathbf{J} \times \mathbf{J} \times \Re \to \Re$ are continuous for $i = 1, 2$, and ${}^{C}D^{\beta}$, ${}^{C}D^{\gamma}$ are the Caputo fractional derivatives.

We shall first look into the existence and uniqueness of solutions for (1.1). To do so, we turn the original problem into an equivalent integral equation, then establish the existence and uniqueness of the solutions using fixed point theorems.

The following is the outline for this paper. We look at some essential preliminaries in section 2. We discuss the existence and uniqueness of problem (1.1) in section 3. We explore a helpful application to represent our primary finding in section 4. In section 5, we present some numerical methods. In section 6, we find some numerical results to show the applicability of our results.

2. Preliminaries

We present some well-known definitions and lemma in this part.

Lemma 2.1. *[18, 19] Suppose that $\beta > \varrho > 0$ and $\phi \in \mathcal{L}^1([b, d])$. At that point $D^{\varrho}I^{\beta}\phi(t) = I^{\beta-\varrho}\phi(t)$, $t \in [b, d]$.*

Lemma 2.2. *[18, 19] For $\beta > 0$ and $\varrho > -1$, we get*

$$I^{\beta}[(t-s)^{\varrho}] = \frac{\Gamma(\varrho+1)}{\Gamma(\beta+\varrho+1)}(t-y)^{\beta+\varrho}.$$

For $\varrho = 0$ and $y = 0$, we get

$$I^{\beta}[1] = \frac{1}{\Gamma(\beta+1)}t^{\beta}.$$

Assume that $[b, d] \in \Re$ be a finite interval as well as suppose that $\beta, \gamma, \xi \in \mathbf{C}$ and $\Re(z) = Re(z)$ for $z \in \mathbf{C}$. The RL-fractional integral and derivative of order $\beta \in \mathbf{C}$ are defined by

$$(I^{\beta}_{b^+}\phi)(y) = \frac{1}{\Gamma(\beta)}\int_b^y \frac{\phi(y)}{(y-s)^{1-\beta}}ds, \quad y > b, \quad \Re(\beta) > 0, \tag{2.1}$$

and

$$\begin{aligned}(D^{\beta}_{b^+}\phi)(y) &= \frac{1}{\Gamma(m-\beta)}\frac{d^m}{dy^m}\int_b^y \frac{\phi(s)}{(y-s)^{\beta-m+1}}ds \\ &= \frac{d^m}{dy^m}(I^{m-\beta}_{b^+}\phi)(y), \quad y > b, \quad \Re(\beta) \geq 0.\end{aligned}$$

On the interval $[b, d]$, the Caputo fractional derivative of order β is defined by

$$({}^{C}D^{\beta}_{b^+}z)(y) = \left(D^{\beta}_{b^+}\left[z(t) - \sum_{k=0}^{m-1}\frac{z^{(k)}(b)}{k!}(t-b)^k\right]\right)(y).$$

When $b = 0$, $I^{\beta}_{b^+}z$ and ${}^{C}D^{\beta}_{b^+}z$ are denoted by $I^{\beta}z$ and ${}^{C}D^{\beta}z$. The semi-group features of the fractional integral operator $I^{\beta}_{b^+}$ as well as the fractional differentiation operator $D^{\gamma}_{b^+}$ are given by [19].

Lemma 2.3. *Suppose that $\Re(\beta), \Re(\gamma) > 0$ as well as $\phi(y) \in \mathbf{C}[b, d]$. For $y \in [b, d]$ the following statements are true:*
(i) $(I^{\beta}_{b^+}I^{\gamma}_{b^+}\phi)(y) = (I^{\beta+\gamma}_{b^+}\phi)(y)$.
(ii) $(D^{\beta}_{b^+}I^{\beta}_{b^+}\phi)(y) = \phi(y)$.
(iii) If $\Re(\beta) > \Re(\gamma)$ at that point

$$(D^{\beta}_{b^+}I^{\beta}_{b^+}\phi)(y) = (I^{\beta-\gamma}_{b^+}\phi)(y).$$

(iv) Suppose that $m = [\Re(\beta) > 0] + 1$ for $\Re(\beta) \notin \mathbf{N}$ and $\phi_{m-\beta}(y) = (I^{m-\beta}_{b^+}\phi)(y) \in \mathbf{C}^m[b, d]$, then

$$(I^{\beta}_{b^+}D^{\beta}_{b^+}\phi)(y) = \phi(y) - \sum_{k=1}^{m}\frac{\phi^{(m-k)}_{m-\beta}}{\Gamma(\beta-k+1)}(y-a)^{\beta-k}.$$

Suppose that $\mathbf{C}_{\xi}[b, d]$ be the space of function ϕ defined on $(b, d]$ in such a way that $(y-b)^{\xi}\phi(y) \in \mathbf{C}[b, d]$ along the norm $\|\phi\|_{\mathbf{C}_{\xi}} = \|(y-b)^{\xi}\phi(y)\|_{\mathbf{C}} := \sup_{y\in[b,d]}|(y-b)^{\xi}\phi(y)|$. Note that for $\xi = 0$, $\mathbf{C}_{\xi}[b, d] = \mathbf{C}[b, d]$. The continuity of the fractional integral operator $I^{\beta}_{b^+}$ from the space $\mathbf{C}_{\xi}[b, d]$ into $\mathbf{C}[b, d]$ is discussed in the following lemma ([19] Lemma 2.8 (a)).

Lemma 2.4. *Suppose that $\Re(\beta) > 0$ and $1 \geq \Re(\xi) \geq 0$. If $\Re(\xi) \leq \Re(\beta)$ at that point the fractional integral operator $I^{\beta}_{b^+}$ is bounded from $\mathbf{C}_{\xi}[b, d]$ into $\mathbf{C}[b, d]$*

$$||I^{\beta}_{b^+}\phi||_{\mathbf{C}} \leq h||\phi||_{\mathbf{C}_{\xi}},\ h = (d - b)^{\Re(\beta-\xi)} \frac{\Gamma(\Re(\beta))|\Gamma(1 - \Re(\xi))|}{|\Gamma(\beta)|\Gamma(1 + \Re(\beta - \xi))}.$$

According to the following ([19] Lemma 2.21, part (a)) when $\Re(\beta) \notin \mathbf{N}_0$ the Riemann-Liouville fractional integral operator $I^{\beta}_{b^+}$ is the left inverse of the Caputo fractional differentiation operator ${}^{C}D^{\beta}_{b^+}$.

Lemma 2.5. *Suppose that $\beta \in \mathbf{C}$ with $0 < \Re(\beta)$ as well as $z(y) \in \mathbf{C}[b, d]$. If $\Re(\beta) \notin \mathbf{N}$, at that point*

$$({}^{C}D^{\beta}_{b^+} I^{\beta}_{b^+} z)(y) = z(y).$$

The fixed point theorem in [20], first presented by Krasnoselskii, is necessary to show that the existence of solution for (1.1).

Theorem 2.6. *Assume that E be a nonempty and convex closed subset of a Banach space Y. Let S as well as R be two operators such that*

(i) when $v, w \in E$ then $Sv + Rw \in E$,

(ii) S is continuous and compact,

(iii) R be a contraction mapping.

At that point $y \in E$ must exist in such a way that $y = Sy + Ry$.

3. Existence and uniqueness of solution

Consider $\mathbf{C}(\mathbf{J})$ be a Banach space along the norm $||v||_{\mathbf{C}} = \sup_{t\in\mathbf{J}} |v(t)|$. We define

$$\begin{aligned}(K_1 v)(t) &:= \int_0^{qt} k_1(t, s, v(s))ds,\\ (K_2 v)(t) &:= \int_0^{t} k_2(t, s, v(s))ds.\end{aligned}$$

In the next lemma, we present an integral equation that corresponding to Eq (1.1).

Lemma 3.1. *Suppose that $\phi : \mathbf{J} \times \Re \times \Re \to \Re$ as well as $k_i : \mathbf{J} \times \mathbf{J} \times \Re \to \Re$ are continuous functions. If and only if v is a solution of the fractional integral equation, then the function $v \in \mathbf{C}(\mathbf{J})$ fulfils problem* (1.1).

$$v(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} v_0 + \frac{v_0}{(\beta+\gamma)\Gamma(\beta)\Gamma(\gamma)} t^{\beta+\gamma} + \frac{v_1}{\Gamma(\gamma+1)} t^{\gamma} + I^{\beta+\gamma}_{0^+}(\phi(t, v(t), v(\Lambda t))) + I^{\beta+\gamma}_{0^+}((K_1 v)(t) + (K_2 v)(t)). \quad (3.1)$$

Proof. Suppose that $v \in \mathbf{C}(\mathbf{J})$ to solve the problem (1.1). Using the concept of the Caputo fractional derivative in Lemma (2.3) (d), we get

$$\begin{aligned}{}^{C}D^{\gamma} v(t) &= v(0) + v'(0)t + I^{\beta}_{0^+}(\phi(t, v(t), v(\Lambda t))) + I^{\beta}_{0^+}((K_1 v)(t) + (K_2 v)(t))\\ &= \frac{t^{\beta-1}}{\Gamma(\beta)} v_0 + v_1 t + I^{\beta}_{0^+}(\phi(t, v(t), v(\Lambda t))) + I^{\beta}_{0^+}((K_1 v)(t) + (K_2 v)(t)),\end{aligned}$$

apply $I_{0^+}^{\gamma}$ on both sides

$$
\begin{aligned}
I_{0^+}^{\gamma}\,{}^{C}D^{\gamma}v(t) &= v(0) + \frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_0^t (t-s)^{\beta+\gamma+1}v_0 ds + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}v_1 ds \\
&\quad + I_{0^+}^{\beta+\gamma}(\phi(t, v(t), v(\Lambda t))) + I_{0^+}^{\beta+\gamma}((K_1 v)(t) + (K_2 v)(t)) \\
v(t) &= \frac{t^{\beta-1}}{\Gamma(\beta)}v_0 + \frac{v_0}{\Gamma(\beta)\Gamma(\gamma)}\Big[-\frac{(t-s)^{\beta+\gamma}}{\beta+\gamma}\Big]_0^t + \frac{v_1}{\Gamma(\gamma)}\Big[-\frac{(t-s)^{\gamma}}{\gamma}\Big]_0^t \\
&\quad + I_{0^+}^{\beta+\gamma}(\phi(t, v(t), v(\Lambda t))) + I_{0^+}^{\beta+\gamma}((K_1 v)(t) + (K_2 v)(t)) \\
&= \frac{t^{\beta-1}}{\Gamma(\beta)}v_0 + \frac{v_0}{(\beta+\gamma)\Gamma(\beta)\Gamma(\gamma)}t^{\beta+\gamma} + \frac{v_1}{\gamma\Gamma(\gamma)}t^{\gamma} \\
&\quad + I_{0^+}^{\beta+\gamma}(\phi(t, v(t), v(\Lambda t))) + I_{0^+}^{\beta+\gamma}((K_1 v)(t) + (K_2 v)(t)) \\
&= \frac{t^{\beta-1}}{\Gamma(\beta)}v_0 + \frac{v_0}{(\beta+\gamma)\Gamma(\beta)\Gamma(\gamma)}t^{\beta+\gamma} + \frac{v_1}{\Gamma(\gamma+1)}t^{\gamma} \\
&\quad + I_{0^+}^{\beta+\gamma}(\phi(t, v(t), v(\Lambda t))) + I_{0^+}^{\beta+\gamma}((K_1 v)(t) + (K_2 v)(t)).
\end{aligned}
$$

□

We define

$$\Delta = \{(t, s) : 0 \le s \le t\}, \ \ \Delta_{\mathfrak{q}} = \{(t, s) : \mathfrak{q}t \ge s \ge 0\}.$$

The existence of (1.1) is investigated under the following conditions:
$\mathbf{H_1}$: $\phi : \mathbf{J} \times \Re \times \Re \times \Re \to \Re$ is continuous and a continuous function exists, $b : E \to [0, \infty)$ in such a way that $t \in [0, \mathfrak{T}]$ and $v_i, w_i \in \Re,\ i = 1, 2$.

$$|\phi(t, v_1, w_1) - \phi(t, v_2, w_2)| \le b(t)(|v_1 - w_1| + |v_2 - w_2|).$$

$\mathbf{H_2}$: $k_i : \mathbf{J} \times \mathbf{J} \times \Re \to \Re,\ i = 1, 2$ be continuous and there exist $b_1 : \Delta_{\mathfrak{q}} \to [0, \infty)$ and $b_2 : \Delta \to [0, \infty)$ in such a way that $d_1(t) := \int_0^{\mathfrak{q}t} b_1(t, s)ds \in \mathbf{C}(\mathbf{J}),\ d_2(t) := \int_0^t b_2(t, s)ds \in \mathbf{C}(\mathbf{J})$

$$|k_i(t, s, y)| \le b_i(t, s)(1 + |y|).$$

$\mathbf{H_3}$:

$$\frac{M^{\beta+\gamma}}{\Gamma(\beta+\gamma)}(\|d_1 + d_2\|_{\mathbf{C}} + 2\|b\|_{\mathbf{C}}) < 1.$$

Assume that the closed ball with radius $\mathfrak{r}_0$ and centre at 0 is $\mathfrak{B}_{\mathfrak{r}_0} \subset \mathbf{C}(\mathbf{J})$ as well as put

$$\tilde{\phi} := \sup\{|\phi(t, 0, 0)| : t \in \mathbf{J}\},$$

and

$$r_0 := \frac{1 + \frac{t^{\beta-1}}{\Gamma(\beta)}|v_0| + |v_1 t| + \frac{M^{\beta+\gamma}}{\Gamma(\beta+\gamma)}(\|d_1 + d_2\|_{\mathbf{C}} + \tilde{\phi})}{1 - \frac{M^{\beta+\gamma}}{\Gamma(\beta+\gamma)}(\|d_1 + d_2\|_{\mathbf{C}} + 2\|b\|_{\mathbf{C}})},$$

and define

$$(Sv)(t) := \Big(I_{0^+}^{\beta} I_{0^+}^{\gamma}((K_1 v)(s) + (K_2 v)(s))(t)\Big).$$

Lemma 3.2. *Suppose that* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$ *be satisfied, at that point the operator* S *maps* $\mathfrak{B}_{\mathfrak{r}_0}$ *into itself, and* $S : \mathfrak{B}_{\mathfrak{r}_0} \to \mathfrak{B}_{\mathfrak{r}_0}$ *is continuous and compact.*

Proof. Step 1: We prove that $S(\mathfrak{B}_{\mathfrak{r}_0}) \subset \mathfrak{B}_{\mathfrak{r}_0}$ where $\mathfrak{B}_{\mathfrak{r}_0} = \{v \in W : \|v\| \le \mathfrak{r}_0\}$. For $v \in \mathfrak{B}_{\mathfrak{r}_0}$, for assumption $(\mathbf{H}_1)$

$$
\begin{aligned}
|\phi(t, v(t), v(\Lambda t)| &\le |\phi(t, v(t), v(\Lambda t)) - \phi(s, 0, 0)| + |\phi(s, 0, 0)| \\
&\le 2b(t)\|v\| + \mathfrak{L}_1 \\
&\le 2b(t)\mathfrak{r}_0 + \mathfrak{L}_1 \\
|\phi_i(t, v(t), v(\Lambda t)| &\le |\phi_i(t, v(t), v(\Lambda t)) - \phi_i(s, 0, 0)| + |\phi_i(s, 0, 0)| \\
&\le 2b_i(t)\|v\| + \mathfrak{L}_i \\
&\le 2b_i(t)\mathfrak{r}_0 + \mathfrak{L}_i, \quad i = 1, 2, ... \, m.
\end{aligned}
$$

For $t \in \mathbf{J}$ using assumptions $(\mathbf{H}_2)$ and $(\mathbf{H}_3)$ we have

$$
\begin{aligned}
|(S v)(t)| &\le \Big(I_{0^+}^{\beta+\gamma}(|(K_1 v)(s)| + |(K_2 v)(s)|)\Big)(t) \\
&\le \Big(I_{0^+}^{\beta+\gamma}\Big(\int_0^{\mathfrak{q}s} b_1(s, \xi) d\xi + \int_0^{s} b_2(s, \xi) d\xi\Big)\Big) \\
&\le \frac{M^{\beta+\gamma}}{\Gamma(\beta + \gamma + 1)}(\|d_1 + d_2\|_{\mathbf{C}}) + \mathfrak{r}_0 \frac{M^{\beta+\gamma}}{\Gamma(\beta + \gamma + 1)}(\|d_1 + d_2\|_{\mathbf{C}}) \\
&\le \mathfrak{r}_0.
\end{aligned}
\tag{3.2}
$$

Step 2: $S : \mathfrak{B}_{\mathfrak{r}_0} \to \mathfrak{B}_{\mathfrak{r}_0}$ is continuous. $\epsilon > 0$ be a fixed point, choose an arbitrary $v, w \in \mathfrak{B}_{\mathfrak{r}_0}$ in such a way that $\|v - w\| \le \epsilon$. For $t \in \mathbf{J}$ we get

$$
|(K_i v)(t) - (K_i w)(t)| \le \int_0^t |k_i(t, s, v(s)) - k_i(t, s, w(s)) \le \omega_{\mathfrak{r}_0}(k_i, \epsilon) M,
$$

where

$$
\omega_{\mathfrak{r}_0}(k_i, \epsilon) = \sup\{|k_i(t, s, v_1) - k_i(t, s, v_2)| : t, s \in \mathbf{J}, \; v_1, v_2 \in [-\mathfrak{r}_0, \mathfrak{r}_0], |v_1 - v_2| \le \epsilon\}, \tag{3.3}
$$

for $i = 1, 2$ using (3.3) we have

$$
\begin{aligned}
|(S v)(t) - (S w)(t)| &\le \Big(I_{0^+}^{\beta+\gamma}(|(K_1 v)(s) - (K_1 w)(s)| + |(K_2 v)(s) - (K_2 w)(s)|)\Big)(t) \\
&\le \frac{(\omega_{\mathfrak{r}_0}(k_1, \xi) + \omega_{\mathfrak{r}_0}(k_2, \xi)) M^{\beta+\gamma+1}}{\Gamma(\beta + \gamma + 1)}.
\end{aligned}
\tag{3.4}
$$

We see that $\omega_{\mathfrak{r}_0}(k_i, \epsilon) \to 0,$ *as* $\epsilon \to 0$ from the uniform continuity of $k_i, \; i = 1, 2$ on bounded subsets of $\mathbf{J} \times \Re \times \Re$. As a result of the inequality (3.4) $S : \mathfrak{B}_{\mathfrak{r}_0} \to \mathfrak{B}_{\mathfrak{r}_0}$ is continuous.

Step 3: An equi-continuous subset of $\mathbf{C}(\mathbf{J})$ is $S(\mathfrak{B}_{\mathfrak{r}_0})$. Supposition $(\mathbf{H}_2)$ states that for any $v \in \mathfrak{B}_{\mathfrak{r}_0}$ as well as $s \in \mathbf{J}$ we have

$$\begin{aligned}|(K_1v)(s)| &\le \int_0^{qs} |k_1(s,\xi,v(\xi))|d\xi \\ &\le \int_0^{qs} b_1(s,\xi)(1+|v(\xi)|)d\xi \\ &\le (1+\mathfrak{r}_0)d_1(s), \end{aligned} \tag{3.5}$$

and similarly

$$\begin{aligned}|(K_2v)(s)| &\le \int_0^{s} |k_2(s,\xi,v(\xi))|d\xi \\ &\le \int_0^{s} b_2(s,\xi)(1+|v(\xi)|)d\xi \\ &\le (1+\mathfrak{r}_0)d_2(s). \end{aligned} \tag{3.6}$$

Now, let $t_1, t_2 \in \mathbf{J}$ and $t_1 \le t_2$.

By Eqs (3.5) and (3.6), we get

$$\begin{aligned}|(Sv)(t_1)-(Sv)(t_2)| &\le |(I_{0^+}^{\beta+\gamma}(K_1v)(s))(t_1) - (I_{0^+}^{\beta+\gamma}(K_1v)(s))(t_2)| \\ &+ |(I_{0^+}^{\beta+\gamma}(K_2v)(s))(t_1) - (I_{0^+}^{\beta+\gamma}(K_2v)(s))(t_2)| \\ &\le \frac{\mathfrak{r}_0+1}{\Gamma(\beta+\gamma)}\int_0^{t}(d_1+d_2)\Big(\frac{1}{(t_1-s)^{1-(\beta+\gamma)}} - \frac{1}{(t_2-s)^{1-(\beta+\gamma)}}\Big)ds \\ &+ \frac{\mathfrak{r}_0+1}{\Gamma(\beta+\gamma)}\int_{t_1}^{t_2}\frac{(d_1+d_2)}{(t_2-s)^{1-(\beta+\gamma)}}ds \\ &\le \frac{\|d_1+d_2\|_{\mathbf{C}}}{\Gamma(\beta+\gamma+1)}(2(t_2-t_1)^{\beta+\gamma} + t_2^{\beta+\gamma} - t_1^{\beta+\gamma})(\mathfrak{r}_0+1). \end{aligned} \tag{3.7}$$

As $t_1 \to t_2$, the right hand side of inequality (3.7) tends to zero. We can see from Steps 1–3 and the Arzela-Ascoli theorem that $S : \mathfrak{B}_{\mathfrak{r}_0} \to \mathfrak{B}_{\mathfrak{r}_0}$ is continuous and compact. □

Theorem 3.3. *In the space* $\mathbf{C}(\mathbf{J})$*, problem* (1.1) *has at least one solution with assumptions* ($\mathbf{H}_1$)–($\mathbf{H}_3$).

Proof. Define the R operator on $\mathbf{C}(\mathbf{J})$ as follows:

$$(Rv)(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}v_0 + v_1 t + (I_{0^+}^{\beta+\gamma}\phi(s, v(s), v(\Lambda s)))(t).$$

The operator R is clearly defined and $Rv \in \mathbf{C}(\mathbf{J})$ for some $v \in \mathbf{C}(\mathbf{J})$ due to the continuity of ϕ and Lemma (2.3).

For any $v, w \in \mathbf{B}_{\mathfrak{r}_0}$, and $t \in \mathbf{J}$ based on assumptions ($\mathbf{H}_1$)–($\mathbf{H}_3$) and inequality (3.2).

$$\begin{aligned}|(Sv)(t)+(Rw)(t)| &\le |(Sv)(t)| + \frac{t^{\beta-1}}{\Gamma(\beta)}|v_0| + |v_1 t| + (I_{0^+}^{\beta+\gamma}|\phi(s,v(s),v(\Lambda s)) - \phi(s,0,0)|)(t) \\ &+ (I_{0^+}^{\beta+\gamma}a|\phi(s,0,0)|)(t) \\ &\le \frac{t^{\beta-1}}{\Gamma(\beta)}|v_0| + |v_1 t| + (I_{0^+}^{\beta+\gamma}[d_1+d_2])(t) + \mathfrak{r}_0(I_{0^+}^{\beta+\gamma}[d_1+d_2])(t)\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\mathfrak{r}_0(I_{0^+}^{\beta+\gamma}b)(t) + (I_{0^+}^{\beta+\gamma}|\phi(s,0,0)|)(t) \\
&\leq \quad 1 + \frac{t^{\beta-1}}{\Gamma(\beta)}|v_0| + |v_1 t| + \frac{M^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}(\|d_1+d_2\|_{\mathbf{C}} + \tilde{\phi}) \\
&+ \quad \frac{M^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}(\|d_1+d_2\|_C + 2\|b\|_{\mathbf{C}})\mathfrak{r}_0 \\
&\leq \quad \mathfrak{r}_0.
\end{aligned}
$$

As a result, $Sv + Rw \in \mathfrak{B}_{\mathfrak{r}_0}$ for every $v, w \in \mathfrak{B}_{\mathfrak{r}_0}$ We can also using ($\mathbf{H}_1$) for some $v, w \in \mathbf{C}(\mathbf{J})$ we get

$$
\begin{aligned}
|(Rv)(t) - (Rw)(t)| &\leq (I_{0^+}^{\beta+\gamma}b(s)[|v(s) - w(s)| - |v(\Lambda s) - w(qs)|])(t) \\
&\leq 2\frac{M^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}\|b\|_{\mathbf{C}}\|v - w\|_{\mathbf{C}}, \quad t \in \mathbf{J}. \qquad (3.8)
\end{aligned}
$$

B is a contraction mapping, based on assumption ($\mathbf{H}_3$) and inequality (3.8). The assumptions of Theorem (2.6) are thus satisfied by Lemma (3.2). □

Theorem 3.4. *If* ($\mathbf{H}_1$) *and* ($\mathbf{H}_3$) *are true, then the following assumption is true.*
$\mathbf{H}_4$ *:* $k_i : \mathbf{J} \times \mathbf{J} \times \Re \to \Re, i = 1, 2$ *is continuous as well as* $b_i : \mathbf{J} \times \mathbf{J} \to [0, \infty), i = 1, 2$ *exist in such a way that*

$$
\begin{aligned}
d_1(t) &:= \int_0^{qt} b_1(t,s)ds \in \mathbf{C}(\mathbf{J}), \\
d_2(t) &:= \int_0^{t} b_2(t,s)ds \in \mathbf{C}(\mathbf{J}),
\end{aligned}
$$

and

$$
|k_i(t,s,y) - k_i(t,s,z)| \leq k_i(t,s)|y - z|.
$$

Then, for $\mathbf{J}$, *problem* (1.1) *has a unique solution.*

Proof. It is sufficient to prove that the integral equation (3.1) has a unique solution using Lemma (3.1). Define the $\mathcal{F}$ operator on $\mathbf{C}(\mathbf{J})$ as follows:

$$
(\mathcal{F}v)(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}v_0 + v_1 t + (I_{0^+}^{\beta+\gamma}\phi(s, v(s), v(\Lambda s)))(t) + (I_{0^+}^{\beta+\gamma}((K_1 v)(s) + (K_2 v)(s)))(t). \qquad (3.9)
$$

$\mathcal{F}v \in \mathbf{C}(\mathbf{J})$ for any $v \in \mathbf{C}(\mathbf{J})$ is simply found using the continuity of ϕ, k_1, k_2, and Lemma (2.2). $\mathcal{F}$ be the fixed point are the solution of (3.1). In the next section, we show that $\mathcal{F}$ is a contraction mapping, and $\mathcal{F}$ has a specific fixed point according to the Banach contraction principle. Suppose that $v, w \in \mathbf{C}(\mathbf{J})$. According to ($\mathbf{H}_1$) and ($\mathbf{H}_4$), for any $t \in \mathbf{J}$ we get

$$
\begin{aligned}
|(\mathcal{F}v)(t) - (\mathcal{F}w)(t)| &\leq \frac{1}{\Gamma(\beta+\gamma)}\int_0^t \frac{|\phi(s, v(s), v(\Lambda s)) - \phi(s, w(s), w(\Lambda s))|}{(t-s)^{1-(\beta+\gamma)}}ds \\
&+ \frac{1}{\Gamma(\beta+\gamma)}\int_0^t \frac{|(K_1 v)(s) - (K_1 w)(s)| + |(K_1 v)(s) - (K_2 w)(s)|}{(t-s)^{1-(\beta+\gamma)}}ds \\
&\leq 2\|v - w\|_{\mathbf{C}}(I_{0^+}^{\beta+\gamma}b)(t) + \|v - w\|_{\mathbf{C}}(I_{0^+}^{\beta+\gamma}(d_1 + d_2))(t).
\end{aligned}
$$

Hence

$$
\|\mathcal{F}v - \mathcal{F}w\| \leq \frac{M^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}(2\|b\|_{\mathbf{C}} + \|d_1 + d_2\|_{\mathbf{C}})\|v - w\|.
$$

By assumption ($\mathbf{H}_3$) it prove that $\mathcal{F}$ is a contraction mapping. □

4. Example

Example 4.1. *Consider the integro-differential equation given below*

$$\begin{cases} {}^{C}D^{1.5}[{}^{C}D^{0.2}v(t)] = \frac{\tanh(v(t)+v(\frac{1}{2}t))}{16(1+t^2)} + \int_0^{\frac{t}{8}} \frac{v(s)}{1+32\sqrt{t-s}}ds + \int_0^t \left(\frac{v(s)\sin(t-s)}{8} + \frac{t-s}{8}\right)ds, \ t \in [1,2], \\ {}_0D_t^{\beta-1}v(0) = v_0(0) = 2, \ \ v'(0) = 0. \end{cases}$$

Put

$$\phi(t,y,z) = \frac{\tanh(y+z)}{16(1+t^2)}, \ \ M = 2, \ \ \beta = 1.5, \ \ \Lambda = \frac{1}{2}, \ \ \mathfrak{q} = \frac{1}{8},$$

$$\begin{aligned} k_1(t,s,y) &= \frac{y}{1+32\sqrt{t-s}}, \ \ k_2(t,s,y) = \frac{y\sin(t-s)}{8} + \frac{t-s}{8}, \\ b(t) &= \frac{1}{16(1+t^2)}, \ \ b_1(t,s) = \frac{1}{1+32\sqrt{t-s}}, \ \ b_2(t,s) = \frac{t-s}{8}. \end{aligned}$$

At that point

$$\begin{aligned} |\phi(t,y,z) - \phi(t,u_1,u_2)| &\le b(t)(|y-u_1| + |z-u_2|), \\ |k_1(t,s,y)| &\le b_1(t,s)|y|, \ \ i = 1,2, \\ |k_2(t,s,y)| &\le b_2(t,s)(1+|y|), \\ d_1(t) &= \int_0^{\mathfrak{q}t} b_1(t,s)ds = \left(\frac{1-\sqrt{\frac{7}{8}}}{16}\right)\sqrt{t}, \\ d_2(t) &= \int_0^t b_1(t,s)ds = \frac{t^2}{16}, \end{aligned}$$

$$\frac{M^\beta}{\Gamma(\beta+1)}(\|d_1 + d_2\|_{\mathbf{C}} + 2\|b\|_{\mathbf{C}} \approx 0.7387 < 1.$$

All above relations shows that $(\mathbf{H}_1)$, $(\mathbf{H}_4)$ *and* $(\mathbf{H}_3)$ *are fulfilled.*

5. Numerical method

Here, we want to use the Sinc collocation method to approximation the solution of (1.1). Therefore, the Sinc basis functions must be defined. The following definition is given for the translated Sinc base functions

$$\mathfrak{S}(p,e)(t) = Sinc\left(\frac{t-pe}{e}\right), \ \ p = 0, \pm1, \pm2, ...,$$

where Sinc(t) on the complete real line $(-\infty,\infty)$ is given below

$$\begin{cases} Sinc(t) = \frac{\sin(\Pi t)}{\Pi t}, \ \ 0 \ne t, \\ 1 \ \ 0 = t. \end{cases}$$

Let q and m be the two integers, with the help of the previous basis function, we may approximate the following function $\phi(t)$ on the real line:

$$\phi(t) \approx \sum_{p=-q}^{m} \phi(pe) Sinc\Big(\frac{t - pe}{e}\Big), \quad t \in \Re, \tag{5.1}$$

where e represent as step size. Furthermore, we can calculate the integrals on $\Re$ in the following manner using the Sinc quadrature rule:

$$\int_{-\infty}^{\infty} \phi(t) dt \approx e \sum_{p=-q}^{m} \phi(pe). \tag{5.2}$$

Consider the single exponential transformation

$$\chi_{b,d}(t) = \log\Big(\frac{t - b}{d - t}\Big).$$

Consider the inverse function

$$\varphi_{b,d}(\zeta) = \chi_{b,d}^{-1}(\zeta) = \frac{b + dh^{\zeta}}{h^{\zeta} + 1},$$

creating the infinite strip

$$B = \{\zeta \in \mathbf{C} : |\Im(\zeta)| < l\},$$

the eye shaped domain

$$D = \{w : |\arg(\frac{v - b}{d - v})| < l\}.$$

By using $\varphi_{b,d}$, consider Γ be the image of the real line

$$\Gamma = \{t \in \mathbf{C} : t = \varphi_{b,d}(\zeta), \zeta \in \Re\}.$$

We shall define the collocation points in $[b, d]$ as the image of the equidistance points pe for some constant e in the following manner in order to employ the Sinc collocation method

$$\varphi_{b,d}(pe) = \frac{b + dh^{pe}}{h^{pe} + 1}, \quad p = -q, ..., m.$$

We will be able to approximate the function $\phi(t)$ at the finite interval $[b, d]$ using the transformations $\chi_{b,d}$ as well as $\varphi_{b,d}$.

$$\phi(t) \approx \sum_{p=-q}^{m} \phi\varphi_{b,d}(pe) Sinc\Big(\frac{\chi_{b,d}(t) - pe}{e}\Big), \quad t \in [b, d],$$

where e is a constant, q and m be the non-negative integers. Furthermore, the Sinc quadrature rule in the finite interval can be defined as

$$\int_{b}^{d} \phi(t) dt = \int_{-\infty}^{\infty} \phi(\varphi_{b,d}(\zeta))\varphi'_{b,d}(\zeta) d\zeta \approx e \sum_{p=-q}^{m} \phi(\varphi_{b,d}(pe))\varphi'_{b,d}(pe).$$

Since $\varphi'_{b,d}(\zeta) = \frac{(\zeta-b)(d-\zeta)}{d-b}$ and $\varphi_{b,d}(pe) = (d-b)y_e + b$, we get

$$\begin{aligned}\int_b^d \phi(y)dy &\approx e\sum_{p=-q}^{m}\phi(\varphi_{b,d}(pe))\varphi'_{b,d}(pe)\\ &= e\sum_{p=-q}^{m}\phi(\varphi_{b,d}(pe))\frac{(\varphi_{b,d}(pe)-b)(d-\varphi_{b,d}(pe))}{d-b}\\ &= e(d-b)\sum_{p=-q}^{m}\phi(\varphi_{b,d}(pe))y_e(1-y_e). \end{aligned}\tag{5.3}$$

Theorem 5.1. *Suppose that* $\phi \in \mathfrak{L}_{\rho-1,\sigma-1}(E)$ *with*

$$\int_{\varphi(t+\mathfrak{L}}|\phi(w)|dw \to 0, \ as\ t \to \infty,$$

where $\mathfrak{L} = \{aj : |a| < l\}$ *and*

$$\lim_{\omega\to\partial D}\inf_{\omega\subseteq D}\int_{\omega}|f(w)|dw < \infty.$$

Taking $e = \sqrt{\frac{2\Pi l}{\sigma m}}$ *as well as* $q = [(\frac{\sigma}{\rho})m] + 1$*, we get*

$$\left|\int_b^d \phi(t)dt - e\sum_{p=-q}^{m}\phi(\varphi_{b,d}(pe))\varphi'_{b,d}(pe)\right| \leq C_1 h^{-\sqrt{2\Pi l\sigma m}},$$

where C_1 *is a constant which depends on* $\phi, l, \rho,$ *and* σ*.*

The following theorems prove that for some constants ρ and σ, the relation (5.1) *and* (5.2) *attain exponential rates of convergence when* $\phi(t)$ *belongs to* $\mathfrak{L}_{\rho,\sigma}(E)$*. For the solution* (1.1) *using Sinc collocation method, near the boundary points b and d the solution tends to* 0*. Define* $K(t)$

$$K(t) = v(t) - \mu_b(t)v(b) - \mu_d(t)v(d), \tag{5.4}$$

where $\mu_b(t)$ *and* $\mu_d(t)$ *can be written as*

$$\mu_b(t) = \frac{1}{1+h^{\chi_{b,d}(t)}}, \quad \mu_d(t) = \frac{h^{\chi_{b,d}(t)}}{1+h^{\chi_{b,d}(t)}}.$$

We have

$$\lim_{t\to b}\mu_b(t) = 1, \quad \lim_{t\to d}\mu_b(t) = 0, \quad \lim_{t\to b}\mu_d(t) = 0, \quad \lim_{t\to d}\mu_d(t) = 1.$$

So, by using Sinc basis functions, the function $K(t)$ *can be approximated as shown below*

$$K(t) \approx \sum_{p=-q}^{m}\phi(\varphi_{b,d}(pe))Sinc\Big(\frac{\chi_{b,d}(t)-pe}{e}\Big). \tag{5.5}$$

Using (5.4) *and* (5.5) *we must define the approximate solution given below*

$$w_m(t) = c_{-q}\mu_b(t) + \sum_{p=-q+1}^{m-1}c_p Sinc\Big(\frac{\chi_{b,d}(t)-pe}{e}\Big) + c_m\mu_d(t). \tag{5.6}$$

Substituting the solution $w_m(t)$ *in* (3.9)*, we get*

$$w_m(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}v_0 + v_1 t + (I_{0^+}^{\beta+\gamma}\phi(s, v_m(s), v_m(\Lambda s)))(t) + (I_{0^+}^{\beta+\gamma}((K_1 v_m)(s) + (K_2 v_m)(s)))(t). \quad (5.7)$$

Define some operators

$$\begin{aligned}(S v)(t) &= (I_{0^+}^{\beta+\gamma}\phi(s, v_m(s), v_m(\Lambda s))(t),\\ (Rv)(t) &= (I_{0^+}^{\beta+\gamma}(K_1 v_m)(s))(t),\\ (Pv)(t) &= (I_{0^+}^{\beta+\gamma}(K_2 v_m)(s))(t),\end{aligned}$$

since, relation (5.7) *can be written as*

$$w_m(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}v_0 + v_1 t + (S v)(t) + (Rv)(t) + (Pv)(t). \quad (5.8)$$

Define $r_p \in [0, 1]$ *as*

$$r_p = \varphi_{0,1}(p\bar{e}) = \frac{h^{p\bar{e}}}{h^{p\bar{e}} + 1}, \quad p = -q, \cdots, m,$$

with $\bar{e}\sqrt{\frac{2\Pi l}{(\beta+\gamma)q}}$ *and* $m = [(\beta + \gamma)q] + 1$*, using relation* (5.3) *with* r_p*, we get*

$$\begin{aligned}(S v)(t) &= \frac{1}{\Gamma(\beta+\gamma)}\int_0^t \frac{\phi(s, v_m(s), v_m(\Lambda s))}{(t-s)^{1-(\beta+\gamma)}}ds\\ &\approx \frac{1}{\Gamma(\beta+\gamma)}\bar{e}t\sum_{p=-m}^{q}\frac{\phi\big(\varphi_{0,t}(p\bar{e}), v_m(\varphi_{0,t}(p\bar{e})), v_m(\Lambda\varphi_{0,t}(p\bar{e}))\big)}{t^{1-(\beta+\gamma)}(1-r_p)^{1-(\beta+\gamma)}}r_p(1-r_p)\\ &= \frac{1}{\Gamma(\beta+\gamma)}\bar{e}t\sum_{p=-m}^{q}\phi\big(\varphi_{0,t}(p\bar{e}), v_m(\varphi_{0,t}(p\bar{e})), v_m(\Lambda\varphi_{0,t}(p\bar{e}))\big)r_p(1-r_p)^{\beta+\gamma}, \quad (5.9)\end{aligned}$$

for $(Rv)(t)$*, we get*

$$\begin{aligned}(Rv)(t) &= (I_{0^+}^{\beta+\gamma}(K_1 v_m)(s))(t) = \frac{1}{\Gamma(\beta+\gamma)}\int_0^t \frac{K_1(v_m(s))}{(t-s)^{1-(\beta+\gamma)}}ds\\ &= \frac{1}{\Gamma(\beta+\gamma)}\int_0^t \frac{\int_0^{\mathfrak{q}s} k_1(s, w, v_m(w))dw}{(t-s)^{1-(\beta+\gamma)}}ds\\ &\simeq \frac{1}{\Gamma(\beta+\gamma)}\int_0^t \frac{\bar{e}\mathfrak{q}s\sum_{a=-m}^{q} k_1\big(s, \varphi_{0,\mathfrak{q}s}(a\bar{e}), w_m(\varphi_{0,\mathfrak{q}s}(a\bar{e}))\big)r_a(1-r_a)}{(t-s)^{1-(\beta+\gamma)}}ds\\ &\simeq \frac{1}{\Gamma(\beta+\gamma)}\sum_{p=-m}^{q}\\ &\times \frac{\bar{e}\mathfrak{q}\varphi_{0,t}(p\bar{e})\sum_{a=-m}^{q} k_1\big(\varphi_{0,t}(p\bar{e}), \varphi_{0,\mathfrak{q}\varphi_{0,t}(p\bar{e})}(a\bar{e}), w_m(\varphi_{0,\mathfrak{q}\varphi_{0,t}(p\bar{e})}(a\bar{e}))\big)r_a(1-r_a)}{t^{1-(\beta+\gamma)}(1-r_p)^{1-(\beta+\gamma)}}r_p(1-r_p)\\ &\simeq \frac{1}{\Gamma(\beta+\gamma)}\bar{e}t^{\beta+\gamma}\sum_{p=-m}^{q}\bar{e}\mathfrak{q}\varphi_{0,t}(p\bar{e})\end{aligned}$$

$$\left\{ \sum_{a=-m}^{q} k_1(\varphi_{0,t}(p\bar{e}), \varphi_{0,\mathfrak{q}\varphi_{0,t}(p\bar{e})}(a\bar{e}), w_m(\varphi_{0,\mathfrak{q}\varphi_{0,t}(p\bar{e})}(a\bar{e}))r_a(1-r_a) \right\} r_p(1-r_p)^{\beta+\gamma}. \tag{5.10}$$

For $(Pv)(t)$*, we have*

$$\begin{aligned}
(Pv)(t) &= (I_{0^+}^{\beta+\gamma}(K_2 v_m)(s))(t)\frac{1}{\Gamma(\beta+\gamma)}\int_0^t \frac{\int_0^s k_2(s,w,v_m(w))dw}{(t-s)^{1-(\beta+\gamma)}}ds \\
&\simeq \frac{1}{\Gamma(\beta+\gamma)}\int_0^t \frac{\bar{e}s\sum_{a=-m}^{q} k_2\big(s, \varphi_{0,s}(a\bar{e}), w_m(\varphi_{0,s}(a\bar{e}))\big)r_a(1-r_a)}{(t-s)^{1-(\beta+\gamma)}}ds \\
&\simeq \frac{1}{\Gamma(\beta+\gamma)}\sum_{p=-m}^{q} \\
&\times \frac{\bar{e}\varphi_{0,t}(p\bar{e})\sum_{a=-m}^{q} k_2\big(\varphi_{0,t}(p\bar{e}), \varphi_{0,\varphi_{0,t}(p\bar{e})}(a\bar{e}), w_m(\varphi_{0,\varphi_{0,t}(p\bar{e})}(a\bar{e}))\big)r_a(1-r_a)}{t^{1-(\beta+\gamma)}(1-r_p)^{1-(\beta+\gamma)}}r_p(1-r_p) \\
&\simeq \frac{1}{\Gamma(\beta+\gamma)}\bar{e}t^{\beta+\gamma}\sum_{p=-m}^{q}\bar{e}\varphi_{0,t}(p\bar{e}) \\
&\quad \left\{ \sum_{a=-m}^{q} k_1(\varphi_{0,t}(p\bar{e}), \varphi_{0,\varphi_{0,t}(p\bar{e})}(a\bar{e}), w_m(\varphi_{0,\varphi_{0,t}(p\bar{e})}(a\bar{e}))r_a(1-r_a) \right\} r_p(1-r_p)^{\beta+\gamma},
\end{aligned} \tag{5.11}$$

when (5.9)–(5.11) *are substituted into* (5.8)*, we obtain*

$$\begin{aligned}
w_m(t) &= \frac{t^{\beta-1}}{\Gamma(\beta)}v_0 + v_1 t + \frac{1}{\Gamma(\beta+\gamma)}\bar{e}t\sum_{p=-m}^{q}\phi(\varphi_{0,t}(p\bar{e}), v_m(\varphi_{0,t}(p\bar{e})), v_m(\Lambda\varphi_{0,t}(p\bar{e})))r_p(1-r_p)^{\beta+\gamma} \\
&+ \frac{1}{\Gamma(\beta+\gamma)}\bar{e}t^{\beta+\gamma}\sum_{p=-m}^{q}\bar{e}\mathfrak{q}\varphi_{0,t}(p\bar{e}) \\
&\times \left\{ \sum_{a=-m}^{q} k_1(\varphi_{0,t}(p\bar{e}), \varphi_{0,\mathfrak{q}\varphi_{0,t}(p\bar{e})}(a\bar{e}), w_m(\varphi_{0,\mathfrak{q}\varphi_{0,t}(p\bar{e})}(a\bar{e}))r_a(1-r_a) \right\} r_p(1-r_p)^{\beta+\gamma} \\
&+ \frac{1}{\Gamma(\beta+\gamma)}\bar{e}t^{\beta+\gamma}\sum_{p=-m}^{q}\bar{e}\varphi_{0,t}(p\bar{e}) \\
&\times \left\{ \sum_{a=-m}^{q} k_1(\varphi_{0,t}(p\bar{e}), \varphi_{0,\varphi_{0,t}(p\bar{e})}(a\bar{e}), w_m(\varphi_{0,\varphi_{0,t}(p\bar{e})}(a\bar{e}))r_a(1-r_a) \right\} r_p(1-r_p)^{\beta+\gamma}.
\end{aligned} \tag{5.12}$$

Define the points $t_k \in [0, \mathbf{T}]$

$$t_k = \varphi_{0,\mathbf{T}}(ke), \quad k = -q, \cdots, m,$$

where $e = \sqrt{\frac{\Pi l}{(\beta+\gamma)q}}$*. By definition of* $\varphi_{b,d}(t)$*, we obtain*

$$\begin{cases}
\varphi_{0,t_k}(p\bar{e}) = t_k r_p \\
\varphi_{0,\varphi_{0,t_k(p\bar{e})}}(a\bar{e}) = \mathfrak{q}t_k r_p r_a \\
\varphi_{0,\varphi_{0,t_k(p\bar{e})}}(a\bar{e}) = t_k r_p r_a.
\end{cases} \tag{5.13}$$

Using (5.13)*, we have*

$$\begin{aligned} w_m(t) &= \frac{t^{\beta-1}}{\Gamma(\beta)}v_0 + v_1 t + \frac{1}{\Gamma(\beta+\gamma)}\bar{e}t^{\beta+\gamma}\sum_{p=-m}^{q}\phi(t_k r_p, w_m(t_k r_p), w_m(\Lambda t_k r_p))r_p(1-r_p)^{\beta+\gamma} \\ &+ \frac{1}{\Gamma(\beta+\gamma)}\bar{e}t^{\beta+\gamma}\sum_{p=-m}^{q}\bar{e}\mathfrak{q}t_k r_p \\ &\times \Big(\sum_{a=-m}^{q}k_1(t_k r_p, \mathfrak{q}t_k r_p r_a, w_m(\mathfrak{q}t_k r_p r_a))r_a(1-r_a)\Big)r_p(1-r_p)^{\beta+\gamma} \\ &+ \frac{1}{\Gamma(\beta+\gamma)}\bar{e}t^{\beta+\gamma}\sum_{p=-m}^{q}\bar{e}\mathfrak{q}t_k r_p \\ &\times \Big(\sum_{a=-m}^{q}k_2(t_k r_p, t_k r_p r_a, w_m(t_k r_p r_a))r_a(1-r_a)\Big)r_p(1-r_p)^{\beta+\gamma}, \quad k=-q,...,m. \end{aligned}$$

Newton iteration method used to solve the above relationship.

6. Numerical results

In this part, we find the numerical results for Example (4.1) to check the applicability of Sinc collocation method. In Table 1, SE means single exponential. If we indicate by E_1 as well as E_2 the greatest absolute errors calculated with $q = q_1$ and q_2. The practical orders of convergence can be obtained by using the following formula

$$Order = \frac{\log(E_1/E_2)}{\log(q_2/q_1)}.$$

In order to compare our method with other ones given in [21, 22]. For different values of $q = 2, 4, 8, 16$, we solved Example (4.1) and tabulated the results at specific places in Table 1. Furthermore, we displayed the largest absolute errors at equidistant points in Table 2.

$$\Delta_1 = \{0.01, 0.02, ...1.99\}.$$

Table 1. Finding approximate solutions for Example (4.1).

$y\backslash q$	2	4	8	16
0.1	1.9892643553	1.9921014240	1.9915311325	1.9915647310
0.4	1.9685783192	1.9662758325	1.9665941826	1.9666284065
0.7	1.9349773042	1.9321150218	1.9314874599	1.9314539434
1.0	1.8769375177	1.8767757148	1.8766451890	1.8766312223
1.3	1.7937897759	1.7942987234	1.7942908009	1.7942769155
1.6	1.6804986790	1.6794220904	1.6789060214	1.6788558248
1.9	1.5285659506	1.5283296781	1.5275487916	1.5274909626

Table 2. Maximum absolute errors at equidistant points Δ_1.

q	SE	Order	Time
2	3.97×10^{-3}	–	0.572
4	8.42×10^{-4}	2.23	1.131
8	7.35×10^{-5}	3.51	7.247
16	2.17×10^{-7}	5.07	97.54
32	1.70×10^{-9}	6.99	2034

7. Conclusions

Our manuscript is mainly focused on mixed derivative for fractional differential equations of order $1 < \beta < 2$ and $0 < \gamma < 1$. Applying the main tools from the fractional calculus, fixed point theorem, integro-differential equation, we propose the definition of α-mild solutions and obtain the existence and uniqueness dependence of the solution. Furthermore, we construct some important supposition to prove some important results. Finally, we provide an application to show the applicability of our main points.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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