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Research article

The dual fuzzy matrix equations: Extended solution, algebraic solution and solution

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Abstract: In this paper, we propose a direct method to solve the dual fuzzy matrix equation of the form $\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{\mathbf{X}} + \widetilde{\mathbf{D}}$ with $\mathbf{A}$, $\mathbf{C}$ matrices of crisp coefficients and $\widetilde{\mathbf{B}}$, $\widetilde{\mathbf{D}}$ fuzzy number matrices. Extended solution and algebraic solution of the dual fuzzy matrix equations are defined and the relationship between them is investigated. This article focuses on the algebraic solution and a necessary and sufficient condition for the unique algebraic solution existence is given. By algebraic methods we not need to transform a dual fuzzy matrix equation into two crisp matrix equations to solve. In addition, the general dual fuzzy matrix equations and dual fuzzy linear systems are investigated based on the generalized inverses of the matrices. Especially, the solution formula and calculation method of the dual fuzzy matrix equation with triangular fuzzy number matrices are given and discussed. The effectiveness of the proposed method is illustrated with examples.

Keywords: fuzzy number; dual fuzzy matrix equation; extended solution; algebraic solution
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1. Introduction

The theory of systems of simultaneous linear equations has a wide range of applications in all branches of mathematics and in many other fields such as physics, transportation planning [25], optimization [29], business, finance, management [8], current flow and control theory [10]. In many applications, some of the systems have uncertainty in parameters and measurements that are represented by fuzzy numbers rather than crisp numbers. Therefore, it is immensely important to develop and improve the problem of solving fuzzy matrix equations.

A general method for solving the fuzzy linear system $\mathbf{A}\widetilde{X} = \widetilde{B}$, where $\mathbf{A}$ is a crisp-valued matrix and $\widetilde{B}$ is a fuzzy number-valued vector, was first proposed by Fridman et al. [3,14]. Various methods emerged later to solve such fuzzy linear system [2,4,5,16,18,22].

In recent years, the fuzzy linear systems in dual form are developing rapidly and it has a wide range of applications in various branches of science such as economics, finance, engineering and physics [21]. In 2000, Ma et al. [20] firstly proposed an embedding method for solving the dual fuzzy linear system $\mathbf{A}\widetilde{X} = \mathbf{B}\widetilde{X} + \widetilde{Y}$, in which $\mathbf{A}$ and $\mathbf{B}$ are two crisp matrices and $\widetilde{Y}$ is a fuzzy number vector. In addition, they illustrated that the system $\mathbf{A}\widetilde{X} = \mathbf{B}\widetilde{X} + \widetilde{Y}$ is not equivalent to the system $(\mathbf{A} - \mathbf{B})\widetilde{X} = \widetilde{Y}$, since there does not exist an element $\widetilde{y}$ such that $\widetilde{x} + \widetilde{y} = 0$ for an arbitrary fuzzy number $\widetilde{x}$. Also, Wang et al. [28] presented an iterative algorithms for solving dual fuzzy linear systems of the form $\widetilde{X} = \mathbf{A}\widetilde{X} + \widetilde{Y}$, where $\mathbf{A}$ is a real $n \times n$ matrix, $\widetilde{X}$ and $\widetilde{Y}$ are fuzzy number vectors. In 2006, Muzziloi et al. [21] considered fuzzy linear systems of the form $\mathbf{A}_1 x + b_1 = \mathbf{A}_2 x + b_2$ with $\mathbf{A}_1$, $\mathbf{A}_2$ square matrices of fuzzy coefficients and b_1, b_2 fuzzy number vectors. In 2008, Abbasbandy et al. [6] proposed a numerical method to obtain the minimal solution of the $m \times n$ general dual fuzzy linear systems $\mathbf{A}\widetilde{X} + \widetilde{Y} = \mathbf{B}\widetilde{X} + \widetilde{Z}$ based on pseudo-inverse calculation. Later, Ezzati [12] investigated the non-square symmetric dual fuzzy linear system of the form $\mathbf{A}\widetilde{X} = \mathbf{B}\widetilde{X} + \widetilde{Y}$. In 2009, Sun and Guo [24] solved a non-square dual fuzzy linear systems $\mathbf{A}\widetilde{X} + \widetilde{Y} = \mathbf{B}\widetilde{X} + \widetilde{Z}$, in which $\mathbf{A}$ and $\mathbf{B}$ are non-full rank matrices. In 2012, Fariborzi Araghi et al. [13] solved a non-square dual fuzzy linear system $\mathbf{A}\widetilde{X} + \widetilde{Y} = \mathbf{B}\widetilde{X} + \widetilde{Z}$, by applied a special algorithm of the class of ABS algorithms. In 2013, Otadi proposed a new model for solving the dual fuzzy linear system $\mathbf{A}\widetilde{X} = \mathbf{B}\widetilde{X} + \widetilde{Y}$ [23]. In 2013, Gong et al. [17] obtained a simple and practical method to solve the dual fuzzy matrix equation $\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{\mathbf{X}} + \widetilde{\mathbf{D}}$, in which $\mathbf{A}$, $\mathbf{C}$ are $m \times n$ matrices and $\widetilde{\mathbf{B}}$, $\widetilde{\mathbf{D}}$ are $m \times p$ LR fuzzy numbers matrices. By the arithmetic operations on LR fuzzy numbers space, they fined that the above dual fuzzy matrix equation could be converted into two classical matrix equations to solve, and the LR minimal fuzzy solution and the strong(weak) LR minimal fuzzy solutions of the dual fuzzy matrix equation are derived based on the generalized inverses of matrices. In 2019, Gong et al. investigated the solution of $m \times n$ fuzzy linear system $\mathbf{A}\widetilde{x} = \widetilde{y}$ based on LR-trapezoidal fuzzy numbers and its numerical calculation. In 2021, M. Ghanbari et al. [15] proposed a straightforward approach for solving dual fuzzy linear systems of the form $\mathbf{A}\widetilde{X} + \widetilde{Y} = \mathbf{B}\widetilde{X} + \widetilde{Z}$, where $\mathbf{A}$ and $\mathbf{B}$ are crisp-valued matrices and $\widetilde{Y}$ and $\widetilde{Z}$ are fuzzy number vectors. The benefits of this method is that it does not need to transformed into two crisp linear systems. For more research results see [7,9, 26].

The main purpose of this paper is to explore how to solve the dual fuzzy matrix equations $\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{\mathbf{X}} + \widetilde{\mathbf{D}}$ algebraically, in which $\mathbf{A}$, $\mathbf{C}$ are $n \times n$ matrices and $\widetilde{\mathbf{B}}$, $\widetilde{\mathbf{D}}$ are $n \times n$ fuzzy number matrices. First, we define the extended and algebraic solutions of the dual fuzzy matrix equations $\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{\mathbf{X}} + \widetilde{\mathbf{D}}$. Meanwhile, the relationship between them is investigated. Second, a necessary and sufficient condition for the unique algebraic solution existence is given. Unlike the existing methods, the main advantage of our method is that there is no need to convert a dual fuzzy matrix equation to two crisp matrix equations to solve. Finally, by the generalized inverses of the matrices we solve the general dual fuzzy matrix equations and dual fuzzy linear systems.

The rest of the paper is organized as follows: Section 2 reviews some basic concepts associated with fuzzy numbers and establishes several useful results. The definition of dual fuzzy matrix equation is given. Then, two types of solutions for a dual fuzzy matrix equation are presented and the relationship between them is investigated. In Section 3, our method is explained by presenting a theorem. Numerical examples are given in Section 4. Finally, we conclude the paper in Section 5.

2. Preliminaries

At first, we will recall some basic concepts associated with fuzzy numbers.

Definition 2.1. *(see [5, 27]) A fuzzy set $\widetilde{x}$ with the membership function $\mu_{\widetilde{x}} : \mathbb{R} \to [0, 1]$ is a fuzzy number if*
(1) *There exists $t_0 \in \mathbb{R}$ such that $\mu_{\widetilde{x}}(t_0) = 1$, i.e., $\widetilde{x}$ is normal;*
(2) *For any $\lambda \in [0, 1]$ and $s, t \in \mathbb{R}$, we have $\mu_{\widetilde{x}}(\lambda s + (1 - \lambda)t) \geqslant \min\{\mu_{\widetilde{x}}(s), \mu_{\widetilde{x}}(t)\}$, i.e., $\tilde{x}$ is a convex fuzzy set;*
(3) *For any $s \in \mathbb{R}$, the set $\{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) > s\}$ is an open set in $\mathbb{R}$, i.e., $\mu_{\widetilde{x}}$ is upper semi-continuous on $\mathbb{R}$;*
(4) *The set $\overline{\{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) > 0\}}$ is compact set in $\mathbb{R}$, where $\overline{A}$ denotes the closure of A.*

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the space of fuzzy numbers. It immediately follows that $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$ because $\mathbb{R} = \{x_t : t \text{ is real number}\}$. For $0 < \alpha \leqslant 1$, we denote $[\widetilde{x}]_\alpha = \{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) \geqslant \alpha\}$ and $[\widetilde{x}]_0 = \overline{\{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) > 0\}}$. Then $[\widetilde{x}]_\alpha$ will be called the $\alpha-$level set of the fuzzy number $\widetilde{x}$. The 1−level is called the core of the fuzzy number, while the 0−level is called the support of the fuzzy number. Usually, the support of the fuzzy number $\widetilde{x}$ is defined as $supp(\widetilde{x}) = [\widetilde{x}]_0 = \overline{\{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) > 0\}}$.

Lemma 2.1. *(see [19]) If $\widetilde{x} \in \mathbb{R}_{\mathcal{F}}$ is a fuzzy number and $[\widetilde{x}]_\alpha$ are its $\alpha-$levels then*
(1) $[\widetilde{x}]_\alpha = [\underline{x}(\alpha), \overline{x}(\alpha)]$ *is a bounded closed interval, for each $\alpha \in [0, 1]$;*
(2) $[\underline{x}(\alpha_1), \overline{x}(\alpha_1)] \supseteq [\underline{x}(\alpha_2), \overline{x}(\alpha_2)]$ *for all $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$;*
(3) $[\lim_{k\to\infty} \underline{x}(\alpha_k), \lim_{k\to\infty} \overline{x}(\alpha_k)] = [\underline{x}(\alpha), \overline{x}(\alpha)]$ *whenever α_k is a non-decreasing sequence in $[0, 1]$ converging to α.*

Remark 2.1. *(see [22]) By lemma 2.1 it is conclude that if the family $\{[\underline{x}(\alpha), \overline{x}(\alpha)] : 0 \leqslant \alpha \leqslant 1\}$, presents the $\alpha-$levels of a fuzzy number, then*
(1) *The condition (1) implies the functions $\underline{x}$ and $\overline{x}$ are bounded over $[0, 1]$ and $\underline{x} \leqslant \overline{x}$ for each $\alpha \in [0, 1]$;*
(2) *The condition (2) implies the functions $\underline{x}$ and $\overline{x}$ are non-decreasing and non-increasing over $[0, 1]$, respectively;*
(3) *The condition (3) implies the functions $\underline{x}$ and $\overline{x}$ are left-continuous over $[0, 1]$.*

For $x, y \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, based on the extension principle, arithmetic operations on the fuzzy numbers are presented using the concept of $\alpha-$levels of fuzzy numbers and interval arithmetic. Then the $\alpha-$levels of the sum $\widetilde{x} + \widetilde{y}$ and the product $\lambda \cdot \widetilde{x}$ are obtained as follows

$$[\widetilde{x} + \widetilde{y}]_\alpha = [\widetilde{x}]_\alpha + [\widetilde{y}]_\alpha = \{s + t : s \in [\widetilde{x}]_\alpha, t \in [\widetilde{y}]_\alpha\} = [\underline{x}(\alpha) + \underline{y}(\alpha), \overline{x}(\alpha) + \overline{y}(\alpha)],$$

$$[\lambda \cdot \widetilde{x}]_\alpha = \lambda \cdot [\widetilde{x}]_\alpha = \{\lambda t : t \in [\widetilde{x}]_\alpha\} = \begin{cases} [\lambda\underline{x}(\alpha), \lambda\overline{x}(\alpha)], & \lambda \geqslant 0, \\ [\lambda\overline{x}(\alpha), \lambda\underline{x}(\alpha)], & \lambda < 0. \end{cases}$$

Definition 2.2. *We say that two fuzzy numbers $\widetilde{x}$ and $\widetilde{y}$ are equal, if and only if for any $t \in \mathbb{R}$, $\mu_{\widetilde{x}}(t) = \mu_{\widetilde{y}}(t)$, i.e. $[\widetilde{x}]_\alpha = [\widetilde{y}]_\alpha$, for any $\alpha \in [0, 1]$. Also $\widetilde{x} \subseteq \widetilde{y} \Leftrightarrow [\widetilde{x}]_\alpha \subseteq [\widetilde{y}]_\alpha$, for any $\alpha \in [0, 1]$.*

Definition 2.3. *(see [11]) A triangular fuzzy number $\widetilde{x} = (x^l, x^m, x^u)$ is a fuzzy set defined on the set $\mathbb{R}$ of real numbers, whose membership function is defined as follows*

$$\mu_{\widetilde{x}}(t) = \begin{cases} (t - x^l)/(x^m - x^l), & if x^l \leqslant t \leqslant x^m, \\ (x^u - t)/(x^u - x^m), & if x^m \leqslant t \leqslant x^u, \\ 0, & others, \end{cases}$$

where x^l, x^m and x^u are called the lower bound, the mode and the upper bound of the triangular fuzzy number $\widetilde{x} = (x^l, x^m, x^u)$, respectively, and $x^l \leqslant x^m \leqslant x^u$. If $x^l \geqslant 0$, then the triangular fuzzy number $\widetilde{x} = (x^l, x^m, x^u)$ is called a positive triangular fuzzy number.

If $x^u \leqslant 0$, then the triangular fuzzy number $\widetilde{x} = (x^l, x^m, x^u)$ is called a negative triangular fuzzy number. The $\alpha-$levels of $\widetilde{x} = (x^l, x^m, x^u)$ is denoted as $[\widetilde{x}]_\alpha = [x^l + \alpha(x^m - x^l), x^u - \alpha(x^u - x^m)]$.

Definition 2.4. *Let $\widetilde{x} = (x^l, x^m, x^u)$, $\widetilde{y} = (y^l, y^m, y^u)$ be two triangular fuzzy numbers, λ a real number. Then the arithmetic operations of $\widetilde{x}$ and $\widetilde{y}$ are defined as follows:*
(1) $\widetilde{x} \oplus \widetilde{y} = (x^l, x^m, x^u) \oplus (y^l, y^m, y^u) = (x^l + y^l, x^m + y^m, x^u + y^u)$;
(2) $\widetilde{x} \ominus \widetilde{y} = (x^l, x^m, x^u) \ominus (y^l, y^m, y^u) = (x^l - y^u, x^m - y^m, x^u - y^l)$;
(3) $\lambda \otimes \widetilde{x} = \lambda \otimes (x^l, x^m, x^u) \approx \begin{cases} (\lambda x^l, \lambda x^m, \lambda x^u), & if \lambda \geqslant 0, \\ (\lambda x^u, \lambda x^m, \lambda x^l), & if \lambda < 0. \end{cases}$

In continuation, we define two concepts namely $\alpha-$center and $\alpha-$radius of an arbitrary fuzzy number.

Definition 2.5. *(see [1]) $x^C(\alpha)$ is called the $\alpha-$center of the fuzzy number $\widetilde{x}$ if $x^C(\alpha) = \frac{\overline{x}(\alpha)+\underline{x}(\alpha)}{2}$, for any $\alpha \in [0, 1]$.*

Definition 2.6. *(see [1]) $x^R(\alpha)$ is called the α-radius of the fuzzy number $\widetilde{x}$ if $x^R(\alpha) = \frac{\overline{x}(\alpha)-\underline{x}(\alpha)}{2}$, for any $\alpha \in [0, 1]$.*

Obviously, the $\alpha-$center and $\alpha-$radius of an arbitrary fuzzy number are crisp real functions of α.

Remark 2.2. *Let $\widetilde{u} = \sum_{i=1}^n \lambda_i \widetilde{x}_i$, $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n \in \mathbb{R}_\mathcal{F}$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$. then*

$$u^C(\alpha) = \sum_{i=1}^n \lambda_i x_i^C(\alpha),\ u^R(\alpha) = \sum_{i=1}^n |\lambda_i| x_i^R(\alpha).$$

Definition 2.7. *Let $\widetilde{x}_i$ be a fuzzy number $(i = 1, 2, \ldots, n)$, then we say that $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$ is a fuzzy number-valued vector. The $\alpha-$center and $\alpha-$radius of $[\widetilde{X}]_\alpha = ([\widetilde{x}_1]_\alpha, [\widetilde{x}_2]_\alpha, \ldots, [\widetilde{x}_n]_\alpha)^T$ can be defined by $X^C(\alpha) = (x_1^C(\alpha), x_2^C(\alpha), \ldots, x_n^C(\alpha))^T$ and $X^R(\alpha) = (x_1^R(\alpha), x_2^R(\alpha), \ldots, x_n^R(\alpha))^T$.*

Therefore, we can obtain

$$\begin{aligned} \widetilde{X} \subseteq \widetilde{Y} &\Leftrightarrow [\widetilde{X}]_\alpha \subseteq [\widetilde{Y}]_\alpha, \forall \alpha \in [0, 1] \\ &\Leftrightarrow [\widetilde{x}_i]_\alpha \subseteq [\widetilde{y}_i]_\alpha, i = 1, 2, \ldots, n, \alpha \in [0, 1], \end{aligned}$$

where $\widetilde{X}$ and $\widetilde{Y}$ are two fuzzy number-valued vectors.

Theorem 2.1. *Let matrix $\mathbf{A} = (a_{ij})_{n\times n}$ be a crisp-valued matrix, vector $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \cdots, \widetilde{x}_n)^\top$ be a fuzzy number-valued vector. We have*

$$(\mathbf{A} \cdot \widetilde{X})^C(\alpha) = \mathbf{A} \cdot X^C(\alpha),\ (\mathbf{A} \cdot \widetilde{X})^R(\alpha) = |\mathbf{A}| \cdot X^R(\alpha).$$

Based on the results obtained above, in the following comment we will consider a dual fuzzy matrix equation.

Definition 2.8. *The matrix equation*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} & \cdots & \widetilde{x}_{1n} \\ \widetilde{x}_{21} & \widetilde{x}_{22} & \cdots & \widetilde{x}_{2n} \\ \vdots & \vdots & & \vdots \\ \widetilde{x}_{n1} & \widetilde{x}_{n2} & \cdots & \widetilde{x}_{nn} \end{pmatrix} + \begin{pmatrix} \widetilde{b}_{11} & \widetilde{b}_{12} & \cdots & \widetilde{b}_{1n} \\ \widetilde{b}_{21} & \widetilde{b}_{22} & \cdots & \widetilde{b}_{2n} \\ \vdots & \vdots & & \vdots \\ \widetilde{b}_{n1} & \widetilde{b}_{n2} & \cdots & \widetilde{b}_{nn} \end{pmatrix}$$
$$= \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}\begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} & \cdots & \widetilde{x}_{1n} \\ \widetilde{x}_{21} & \widetilde{x}_{22} & \cdots & \widetilde{x}_{2n} \\ \vdots & \vdots & & \vdots \\ \widetilde{x}_{n1} & \widetilde{x}_{n2} & \cdots & \widetilde{x}_{nn} \end{pmatrix} + \begin{pmatrix} \widetilde{d}_{11} & \widetilde{d}_{12} & \cdots & \widetilde{d}_{1n} \\ \widetilde{d}_{21} & \widetilde{d}_{22} & \cdots & \widetilde{d}_{2n} \\ \vdots & \vdots & & \vdots \\ \widetilde{d}_{n1} & \widetilde{d}_{n2} & \cdots & \widetilde{d}_{nn} \end{pmatrix}, \tag{2.1}$$

where a_{ij}, c_{ij}, $1 \leqslant i, j \leqslant n$ are real numbers and elements $\widetilde{b}_{ij}$, $\widetilde{d}_{ij}$, $1 \leqslant i, j \leqslant n$ are fuzzy numbers, is called a dual fuzzy matric equation. Using matric notation, we have

$$\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{\mathbf{X}} + \widetilde{\mathbf{D}}.$$

A fuzzy number matric $\widetilde{\mathbf{X}} = (\widetilde{x}_{ij})_{n\times n}$ is called the solution of the dual fuzzy matric Eq (2.1) *if satisfies*

$$\mathbf{A}\widetilde{x}_j + \widetilde{\mathbf{B}}_j = \mathbf{C}\widetilde{x}_j + \widetilde{\mathbf{D}}_j, \ j = 1, 2, \ldots, n,$$

where $\widetilde{x}_j = (\widetilde{x}_{1j}, \widetilde{x}_{2j}, \ldots, \widetilde{x}_{nj})^T$, $\widetilde{\mathbf{B}}_j = (\widetilde{b}_{1j}, \widetilde{b}_{2j}, \ldots, \widetilde{b}_{nj})^T$, $\widetilde{\mathbf{D}}_j = (\widetilde{d}_{1j}, \widetilde{d}_{2j}, \ldots, \widetilde{d}_{nj})^T$, $j = 1, 2, \ldots, n$ are $j-$th column of fuzzy matrices $\widetilde{\mathbf{X}}$, $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{D}}$, respectively.

It is well known that for an arbitrary fuzzy number $\widetilde{x}$, there exists no element $\widetilde{y} \in \mathbb{R}_{\mathcal{F}}$ such that $\widetilde{x} + \widetilde{y} = 0$. Consequently, we cannot equivalently replace the dual fuzzy matric equation (2.1) by the fuzzy matric equation $(\mathbf{A} - \mathbf{C})\widetilde{\mathbf{X}} = \widetilde{\mathbf{D}} - \widetilde{\mathbf{B}}$. Therefore, it is crucial to develop mathematical methods that can solve the dual fuzzy matric equation (2.1).

Example 2.1. Consider the problem of the classical coordinate rotation and shift in Cartesian coordinate systems: a point $P(x, y)$ rotates $\theta_i (i = 1, 2)$ in counterclockwise, and then shifts the origin of the coordinate to the point $P_i(x_i, y_i)$, $(i = 1, 2)$, and we obtains $P'_i(x'_i, y'_i)$, $(i = 1, 2)$ in new coordinate system. The relationship between $P(x, y)$, $P'_i(x'_i, y'_i)$, $(i = 1, 2)$ and $P_i(x_i, y_i)$, $(i = 1, 2)$ as follows.

$$\begin{pmatrix} x'_i \\ y'_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x_i \\ -y_i \end{pmatrix}.$$

In some sense, we need to calculate those points $P(x, y)$ such that $P'_i(x'_i, y'_i)$, $(i = 1, 2)$ are equal to each other at least in quantitative values or in a specific functions in an engineering modeling. In fact, the problem will be converted to solving a two dimensional matrix linear system as follows.

$$\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x_1 \\ -y_1 \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x_2 \\ -y_2 \end{pmatrix}.$$

As we all know, in the classical theory of the matrix linear system, it equivalent to the following linear system

$$\begin{pmatrix} \cos\theta_i - \cos\theta_2 & \sin\theta_i - \sin\theta_2 \\ -\sin\theta_i + \sin\theta_2 & \cos\theta_i - \cos\theta_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}.$$

However, there are often uncertainty of parameters in the process of mathematical modeling in a concrete engineering, and the parameter with uncertainty is easy to describe whether it can be written as fuzzy number in some sense. When $x_i, y_i (i = 1, 2)$ are fuzzy numbers. We need to solve the following two dimensional fuzzy matrix linear system.

$$\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x_1 \\ -y_1 \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x_2 \\ -y_2 \end{pmatrix}.$$

For above two dimensional fuzzy matrix linear system, the system $\mathbf{A}\widetilde{X} + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{X} + \widetilde{\mathbf{D}}$ is not equivalent to the system $(\mathbf{A} - \mathbf{C})\widetilde{X} = (\widetilde{\mathbf{B}} - \widetilde{\mathbf{D}})$ since there does not exist an element fuzzy number $\widetilde{v}$ such that $\widetilde{u} + \widetilde{v} = \widetilde{0}$ for any fuzzy number $\widetilde{u}$.

The following definition gives two types of the solutions to the dual fuzzy matrix equation (2.1).

Definition 2.9. *Let det*$(\mathbf{A} - \mathbf{C}) \neq 0$ *for the Eq* (2.1). *The extended solution of the Eq* (2.1) *is defined as follows*

$$\widetilde{\mathbf{X}}_E = (\mathbf{A} - \mathbf{C})^{-1}(\widetilde{\mathbf{D}} - \widetilde{\mathbf{B}}).$$

Remark 2.3. *Consider Eq* (2.1). *If* $\widetilde{x}_{ij}$, $\widetilde{b}_{ij}$ *and* $\widetilde{d}_{ij}$ *are triangular fuzzy numbers and also det*$(\mathbf{A}-\mathbf{C}) \neq 0$, *then the extended solution* $\widetilde{\mathbf{X}}_E = (x^l_E, x^m_E, x^u_E)$ *is equivalent to the matrix system*

$$\begin{aligned} \widetilde{\mathbf{X}}_E &= (\mathbf{A} - \mathbf{C})^{-1}((\mathbf{D}^l, \mathbf{D}^m, \mathbf{D}^u) - (\mathbf{B}^l, \mathbf{B}^m, \mathbf{B}^u)) \\ &= (\mathbf{A} - \mathbf{C})^{-1}(\mathbf{D}^l - \mathbf{B}^u, \mathbf{D}^m - \mathbf{B}^m, \mathbf{D}^u - \mathbf{B}^l). \end{aligned}$$

Definition 2.10. *Let* $\widetilde{x}_{j_A} = (\widetilde{x}_{1j_A}, \widetilde{x}_{2j_A}, \ldots, \widetilde{x}_{nj_A})^\top$ *denote the jth column of the* $\widetilde{\mathbf{X}}_A$. *Then* $\widetilde{\mathbf{X}}_A$ *is said to be an algebraic solution of equation* (2.1) *equivalent to*

$$\mathbf{A}\widetilde{\mathbf{X}}_A + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{\mathbf{X}}_A + \widetilde{\mathbf{D}},$$

or

$$\sum_{k=1}^{n} a_{ik}\widetilde{x}_{kj_A} + \widetilde{b}_{ij} = \sum_{k=1}^{n} c_{ik}\widetilde{x}_{kj_A} + \widetilde{d}_{ij},\ i, j = 1, 2, \ldots, n.$$

Remark 2.4. *By Definition* 2.9 *and Theorem* 2.1 *we have*

$$\begin{aligned} \mathbf{X}^C_E(\alpha) &= (\mathbf{A} - \mathbf{C})^{-1}(\mathbf{D}^C(\alpha) - \mathbf{B}^C(\alpha)),\ \alpha \in [0, 1], \\ \mathbf{X}^R_E(\alpha) &= |(\mathbf{A} - \mathbf{C})^{-1}|(\mathbf{D}^R(\alpha) + \mathbf{B}^R(\alpha)),\ \alpha \in [0, 1]. \end{aligned}$$

Remark 2.5. *By Definition* 2.5, *Definition* 2.6, *Remark* 2.3 *and Remark* 2.4, *if* $\widetilde{x}_{ij}$, $\widetilde{b}_{ij}$ *and* $\widetilde{d}_{ij}$ *are triangular fuzzy numbers and also det*$(\mathbf{A} - \mathbf{C}) \neq 0$, *we have*

$$\begin{aligned} \mathbf{X}^C_E(\alpha) &= (\mathbf{A} - \mathbf{C})^{-1}(\alpha(\mathbf{D}^m - \mathbf{B}^m) + \frac{1-\alpha}{2}(\mathbf{D}^l + \mathbf{D}^u - \mathbf{B}^l - \mathbf{B}^u)), \\ \mathbf{X}^R_E(\alpha) &= \frac{1-\alpha}{2}|(\mathbf{A} - \mathbf{C})^{-1}|(\mathbf{D}^u - \mathbf{D}^l + \mathbf{B}^u - \mathbf{B}^l). \end{aligned}$$

The following result provide the relationship between the extended solution $\widetilde{\mathbf{X}}_E$ and the algebraic solution $\widetilde{\mathbf{X}}_A$.

Theorem 2.2. *Suppose that both the extended and the algebraic solutions of the dual fuzzy matrix equation* (2.1) *exist, we have* $\mathbf{X}_A^C(\alpha) = \mathbf{X}_E^C(\alpha)$.

Proof. Let $\widetilde{\mathbf{X}}_A$ is an algebraic solution of Eq (2.1), we have $\mathbf{A}\widetilde{\mathbf{X}}_A + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{\mathbf{X}}_A + \widetilde{\mathbf{D}}$. According to the matrix theory, we know that the above matrix equation is equivalent to $\mathbf{A}\widetilde{x}_{j_A} + \widetilde{\mathbf{B}}_j = \mathbf{C}\widetilde{x}_{j_A} + \widetilde{\mathbf{D}}_j$, $j = 1, 2, \ldots, n$. From the above conclusions, we have $\mathbf{A} \cdot x_{j_A}^C(\alpha) + \mathbf{B}_j^C(\alpha) = \mathbf{C} \cdot x_{j_A}^C(\alpha) + \mathbf{D}_j^C(\alpha)$. Since the extended solution $\widetilde{\mathbf{X}}_E$ exists, $det(\mathbf{A} - \mathbf{C}) \neq 0$. In addition, since the $\alpha-$center of an arbitrary fuzzy number is a crisp function in terms of α, we obtain $x_{j_A}^C(\alpha) = (\mathbf{A} - \mathbf{C})^{-1} \cdot (\mathbf{D}_j^C(\alpha) - \mathbf{B}_j^C(\alpha)) = x_{j_E}^C(\alpha)$, $j = 1, 2, \ldots, n$. It is easy to verify that $\mathbf{X}_A^C(\alpha) = \mathbf{X}_E^C(\alpha)$.

3. The discussion of the solution for dual fuzzy matrix equations

In this section, we give a method for obtaining an algebraic solution of the dual fuzzy matrix equation (2.1) by first giving the following theorem.

Theorem 3.1. *The Eq* (2.1) *has a unique algebraic solution if and only if* $det(\mathbf{A}-\mathbf{C}) \neq 0$, $det(|\mathbf{C}|-|\mathbf{A}|) \neq 0$ *and the family of sets* $[\underline{\mathbf{X}}_E(\alpha) + \mathbf{F}(\alpha), \overline{\mathbf{X}}_E(\alpha) - \mathbf{F}(\alpha)]$*constructs the* $\alpha-$*levels of a fuzzy number-valued matrix for any* $\alpha \in [0, 1]$. *Where* $[\widetilde{\mathbf{X}}_E]_\alpha = [\underline{\mathbf{X}}_E(\alpha), \overline{\mathbf{X}}_E(\alpha)]$ *and*

$$\mathbf{F}(\alpha) = \mathbf{X}_E^R(\alpha) + (|\mathbf{C}| - |\mathbf{A}|)^{-1}(\mathbf{D}^R(\alpha) - \mathbf{B}^R(\alpha)). \tag{3.1}$$

Thus, the unique algebraic solution of Eq (2.1) *is expressed in terms of the* $\alpha-$*levels as*

$$[\widetilde{\mathbf{X}}_A]_\alpha = [\underline{\mathbf{X}}_E(\alpha) + \mathbf{F}(\alpha), \overline{\mathbf{X}}_E(\alpha) - \mathbf{F}(\alpha)], \tag{3.2}$$

for any $\alpha \in [0, 1]$.

Proof. Suppose that the Eq (2.1) has a unique algebraic solution $\widetilde{\mathbf{X}}_A$. firstly, the fuzzy matrix equation (2.1) can be written in the block forms

$$\mathbf{A}(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n) + (\widetilde{\mathbf{B}}_1, \widetilde{\mathbf{B}}_2, \ldots, \widetilde{\mathbf{B}}_n) = \mathbf{C}(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n) + (\widetilde{\mathbf{D}}_1, \widetilde{\mathbf{D}}_2, \ldots, \widetilde{\mathbf{D}}_n),$$

where $\widetilde{x}_j = (\widetilde{x}_{1j}, \widetilde{x}_{2j}, \ldots, \widetilde{x}_{nj})^\top$, $\widetilde{\mathbf{B}}_j = (\widetilde{b}_{1j}, \widetilde{b}_{2j}, \ldots, \widetilde{b}_{nj})^\top$ and $\widetilde{\mathbf{D}}_j = (\widetilde{d}_{1j}, \widetilde{d}_{2j}, \ldots, \widetilde{d}_{nj})^\top$, $j = 1, 2, \ldots, n$ denote the jth column of unknown matrix $\widetilde{\mathbf{X}}$ and fuzzy numbers matrices $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{D}}$, respectively. Thus the original Eq (2.1) is equivalent to the following dual fuzzy linear systems

$$\mathbf{A}\widetilde{x}_j + \widetilde{\mathbf{B}}_j = \mathbf{C}\widetilde{x}_j + \widetilde{\mathbf{D}}_j, \quad j = 1, 2, \ldots, n.$$

Based on Definition 2.7 and Eq (3.2), then we can obtain

$$\begin{aligned} &\sum_{k=1}^{n} a_{ik}[\underline{x}_{kj_E}(\alpha) + f_{kj}(\alpha), \overline{x}_{kj_E}(\alpha) - f_{kj}(\alpha)] + [\underline{b}_{ij}(\alpha), \overline{b}_{ij}(\alpha)] \\ &= \sum_{k=1}^{n} c_{ik}[\underline{x}_{kj_E}(\alpha) + f_{kj}(\alpha), \overline{x}_{kj_E}(\alpha) - f_{kj}(\alpha)] + [\underline{b}_{ij}(\alpha), \overline{d}_{ij}(\alpha)]. \end{aligned}$$

Note that for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
&\sum_{k:a_{ik}\geqslant 0} a_{ik}(\underline{x}_{kj_E}(\alpha) + f_{kj}(\alpha)) + \sum_{k:a_{ik}<0} a_{ik}(\overline{x}_{kj_E}(\alpha) - f_{kj}(\alpha)) \\
&- \sum_{k:c_{ik}\geqslant 0} c_{ik}(\underline{x}_{kj_E}(\alpha) + f_{kj}(\alpha)) - \sum_{k:c_{ik}<0} c_{ik}(\overline{x}_{kj_E}(\alpha) - f_{kj}(\alpha)) \\
&= \underline{d}_{ij}(\alpha) - \underline{b}_{ij}(\alpha)
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\sum_{k:a_{ik}\geqslant 0} a_{ik}(\overline{x}_{kj_E}(\alpha) - f_{kj}(\alpha)) + \sum_{k:a_{ik}<0} a_{ik}(\underline{x}_{kj_E}(\alpha) + f_{kj}(\alpha)) \\
&- \sum_{k:c_{ik}\geqslant 0} c_{ik}(\overline{x}_{kj_E}(\alpha) - f_{kj}(\alpha)) - \sum_{k:c_{ik}<0} c_{ik}(\underline{x}_{kj_E}(\alpha) + f_{kj}(\alpha)) \\
&= \overline{d}_{ij}(\alpha) - \overline{b}_{ij}(\alpha).
\end{aligned} \tag{3.4}$$

By Eq (3.1), Definitions 2.5 and 2.6, $i = 1, 2, \ldots, n$, we have

$$(|\mathbf{C}| - |\mathbf{A}|)F_j(\alpha) = (|\mathbf{C}| - |\mathbf{A}|)\left(\frac{\overline{x}_{j_E}(\alpha) - \underline{x}_{j_E}(\alpha)}{2}\right) + \frac{\overline{\mathbf{D}}_j(\alpha) - \underline{\mathbf{D}}_j(\alpha)}{2} - \frac{\overline{\mathbf{B}}_j(\alpha) - \underline{\mathbf{B}}_j(\alpha)}{2}$$

and

$$\sum_{k=1}^{n}(|c_{ik}| - |a_{ik}|)f_{kj}(\alpha) = \sum_{k=1}^{n}(|c_{ik}| - |a_{ik}|)\frac{\overline{x}_{kj_E}(\alpha) - \underline{x}_{kj_E}(\alpha)}{2} + \frac{\overline{d}_{ij}(\alpha) - \underline{d}_{ij}(\alpha)}{2} - \frac{\overline{b}_{ij}(\alpha) - \underline{b}_{ij}(\alpha)}{2}. \tag{3.5}$$

By the $det(\mathbf{A} - \mathbf{C}) \neq 0$, we know that $\widetilde{x}_{j_E} = (\mathbf{A} - \mathbf{C})^{-1}(\widetilde{\mathbf{D}}_j - \widetilde{\mathbf{B}}_j)$, thus

$$(\mathbf{A} - \mathbf{C})\widetilde{x}_{j_E} = (\widetilde{\mathbf{D}}_j - \widetilde{\mathbf{B}}_j).$$

According to Theorem 2.1, for any $\alpha \in [0, 1]$, we have

$$(\mathbf{A} - \mathbf{C})x^C_{j_E}(\alpha) = \mathbf{D}^C_j(\alpha) - \mathbf{B}^C_j(\alpha).$$

This implies

$$\sum_{k=1}^{n}(a_{ik} - b_{ik})\left(\underline{x}_{kj_E}(\alpha) + \overline{x}_{kj_E}(\alpha)\right) = (\underline{d}_{ij_E}(\alpha) + \overline{d}_{ij_E}(\alpha)) - (\underline{b}_{ij_E}(\alpha) + \overline{b}_{ij_E}(\alpha)), \tag{3.6}$$

for any $i = 1, 2, \ldots, n$.

By Eqs (3.5) and (3.6), we have

$$\begin{aligned}
&\sum_{k:a_{ik}\geqslant 0} a_{ik}(\underline{x}_{kj_E}(\alpha) + f_{kj}(\alpha)) + \sum_{k:a_{ik}<0} a_{ik}(\overline{x}_{kj_E}(\alpha) - f_{kj}(\alpha)) \\
&- \sum_{k:c_{ik}\geqslant 0} c_{ik}(\underline{x}_{kj_E}(\alpha) + f_{kj}(\alpha)) - \sum_{k:c_{ik}<0} c_{ik}(\overline{x}_{kj_E}(\alpha) - f_{kj}(\alpha))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k:a_{ik}\geqslant 0} a_{ik}\underline{x}_{kj_E}(\alpha) + \sum_{k:a_{ik}<0} a_{ik}\overline{x}_{kj_E}(\alpha) - \sum_{k:c_{ik}\geqslant 0} c_{ik}\underline{x}_{kj_E}(\alpha) - \sum_{k:c_{ik}<0} c_{ik}\overline{x}_{kj_E}(\alpha) + \sum_{k=1}^{n}|a_{ik}|f_{kj}(\alpha) - \sum_{k=1}^{n}|c_{ik}|f_{kj}(\alpha)\\
&= \sum_{k:a_{ik}\geqslant 0} a_{ik}\underline{x}_{kj_E}(\alpha) + \sum_{k:a_{ik}<0} a_{ik}\overline{x}_{kj_E}(\alpha) - \sum_{k:c_{ik}\geqslant 0} c_{ik}\underline{x}_{kj_E}(\alpha) - \sum_{k:c_{ik}<0} c_{ik}\overline{x}_{kj_E}(\alpha) - \sum_{k=1}^{n}(|c_{ik}| - |a_{ik}|)f_{kj}(\alpha)\\
&= \sum_{k:a_{ik}\geqslant 0} a_{ik}\underline{x}_{kj_E}(\alpha) + \sum_{k:a_{ik}<0} a_{ik}\overline{x}_{kj_E}(\alpha) - \sum_{k:c_{ik}\geqslant 0} c_{ik}\underline{x}_{kj_E}(\alpha) - \sum_{k:c_{ik}<0} c_{ik}\overline{x}_{kj_E}(\alpha)\\
&- \sum_{k=1}^{n}(|c_{ik}| - |a_{ik}|)\frac{\overline{x}_{kj_E}(\alpha) - \underline{x}_{kj_E}(\alpha)}{2} - \frac{\overline{d}_{ij_E}(\alpha) - \underline{d}_{ij_E}(\alpha)}{2} + \frac{\overline{b}_{ij_E}(\alpha) - \underline{b}_{ij_E}(\alpha)}{2}\\
&= \sum_{k:a_{ik},c_{ik}\geqslant 0}(a_{ik} - c_{ik})\left(\frac{\overline{x}_{kj_E}(\alpha) - \underline{x}_{kj_E}(\alpha)}{2} + \underline{x}_{kj_E}(\alpha)\right) + \sum_{k:a_{ik},c_{ik}<0}(a_{ik} - c_{ik})\left(\overline{x}_{kj_E}(\alpha) + \frac{\underline{x}_{kj_E}(\alpha) - \overline{x}_{kj_E}(\alpha)}{2}\right)\\
&- \frac{\overline{d}_{ij_E}(\alpha) - \underline{d}_{ij_E}(\alpha)}{2} + \frac{\overline{b}_{ij_E}(\alpha) - \underline{b}_{ij_E}(\alpha)}{2}\\
&= \frac{1}{2}\sum_{k:a_{ik},c_{ik}\geqslant 0}(a_{ik} - c_{ik})(\overline{x}_{kj_E}(\alpha) + \underline{x}_{kj_E}(\alpha)) + \frac{1}{2}\sum_{k:a_{ik},c_{ik}<0}(a_{ik} - c_{ik})(\overline{x}_{kj_E}(\alpha) + \underline{x}_{kj_E}(\alpha))\\
&- \frac{1}{2}(\overline{d}_{ij_E}(\alpha) - \underline{d}_{ij_E}(\alpha)) + \frac{1}{2}(\overline{b}_{ij_E}(\alpha) - \underline{b}_{ij_E}(\alpha))\\
&= \frac{1}{2}\sum_{k=1}^{n}(a_{ik} - c_{ik})(\overline{x}_{kj_E}(\alpha) + \underline{x}_{kj_E}(\alpha)) - \frac{1}{2}(\overline{d}_{ij_E}(\alpha) - \underline{d}_{ij_E}(\alpha)) + \frac{1}{2}(\overline{b}_{ij_E}(\alpha) - \underline{b}_{ij_E}(\alpha))\\
&= \frac{1}{2}(\underline{d}_{ij_E}(\alpha) + \overline{d}_{ij_E}(\alpha)) - \frac{1}{2}(\underline{b}_{ij_E}(\alpha) + \overline{b}_{ij_E}(\alpha)) - \frac{1}{2}(\overline{d}_{ij_E}(\alpha) - \underline{d}_{ij_E}(\alpha)) + \frac{1}{2}(\overline{b}_{ij_E}(\alpha) - \underline{b}_{ij_E}(\alpha))\\
&= \underline{d}_{ij_E}(\alpha) - \underline{b}_{ij_E}(\alpha),
\end{aligned}
$$

for any $\alpha \in [0, 1]$ and $i = 1, 2, \ldots, n$, the proof of Eq (3.3) is complete. Similarly,

$$
\begin{aligned}
&\sum_{k:a_{ik}\geqslant 0} a_{ik}(\overline{x}_{kj_E}(\alpha) - f_{kj}(\alpha)) + \sum_{k:a_{ik}<0} a_{ik}(\underline{x}_{kj_E}(\alpha) + f_{kj}(\alpha))\\
&- \sum_{k:c_{ik}\geqslant 0} c_{ik}(\overline{x}_{kj_E}(\alpha) - f_{kj}(\alpha)) - \sum_{k:c_{ik}<0} c_{ik}(\underline{x}_{kj_E}(\alpha) + f_{kj}(\alpha))\\
&= \sum_{k:a_{ik}\geqslant 0} a_{ik}\overline{x}_{kj_E}(\alpha) + \sum_{k:a_{ik}<0} a_{ik}\underline{x}_{kj_E}(\alpha) - \sum_{k:c_{ik}\geqslant 0} c_{ik}\overline{x}_{kj_E}(\alpha) - \sum_{k:c_{ik}<0} c_{ik}\underline{x}_{kj_E}(\alpha) + \sum_{k=1}^{n}(|c_{ik}| - |a_{ik}|)f_{kj}(\alpha)\\
&= \frac{1}{2}\sum_{k=1}^{n}(a_{ik} - c_{ik})(\overline{x}_{kj_E}(\alpha) + \underline{x}_{kj_E}(\alpha)) + \frac{1}{2}(\overline{d}_{ij_E}(\alpha) - \underline{d}_{ij_E}(\alpha)) - \frac{1}{2}(\overline{b}_{ij_E}(\alpha) - \underline{b}_{ij_E}(\alpha))\\
&= \frac{1}{2}(\underline{d}_{ij_E}(\alpha) + \overline{d}_{ij_E}(\alpha)) - \frac{1}{2}(\underline{b}_{ij_E}(\alpha) + \overline{b}_{ij_E}(\alpha)) + \frac{1}{2}(\overline{d}_{ij_E}(\alpha) - \underline{d}_{ij_E}(\alpha)) - \frac{1}{2}(\overline{b}_{ij_E}(\alpha) - \underline{b}_{ij_E}(\alpha))\\
&= \overline{d}_{ij_E}(\alpha) - \overline{b}_{ij_E}(\alpha),
\end{aligned}
$$

for any $\alpha \in [0, 1]$ and $i = 1, 2, \ldots, n$, the proof of Eq (3.4) is complete. This prove that $\widetilde{\mathbf{X}}_A$ is an algebraic solution of dual fuzzy matrix Eq (2.1). In the following we verify the uniqueness of this solution. Suppose that $\widetilde{\mathbf{W}}$ is another algebraic solution of the equation (2.1). Using Theorem 2.1 we have

$$\mathbf{A}x_{j_A}^C(\alpha) + \mathbf{B}_j^C(\alpha) = \mathbf{C}x_{j_A}^C(\alpha) + \mathbf{D}_j^C(\alpha),$$
$$\mathbf{A}w_j^C(\alpha) + \mathbf{B}_j^C(\alpha) = \mathbf{C}w_j^C(\alpha) + \mathbf{D}_j^C(\alpha).$$

Hence, we obtain

$$|\mathbf{A}|x_{j_A}^R(\alpha) + \mathbf{B}_j^R(\alpha) = |\mathbf{C}|x_{j_A}^R(\alpha) + \mathbf{D}_j^R(\alpha),$$
$$|\mathbf{A}|w_j^R(\alpha) + \mathbf{B}_j^R(\alpha) = |\mathbf{C}|w_j^R(\alpha) + \mathbf{D}_j^R(\alpha).$$

This implies that $x_{j_A}^C(\alpha) = w_j^C(\alpha)$, $x_{j_A}^R(\alpha) = w_j^R(\alpha)$, we have $\widetilde{w}_j = \widetilde{x}_{j_A}$.

Now we say that the fuzzy number-valued vector $\widetilde{w}_j$ is unique algebraic solution of the system (2.1), Also, the crisp vectors $w_j^C(\alpha)$ and $w_j^R(\alpha)$ are unique solution of the following crisp linear systems

$$\mathbf{A}x_j^C(\alpha) + \mathbf{B}_j^C(\alpha) = \mathbf{C}x_j^C(\alpha) + \mathbf{D}_j^C(\alpha)$$

and

$$|\mathbf{A}|x_j^R(\alpha) + \mathbf{B}_j^R(\alpha) = |\mathbf{C}|x_j^R(\alpha) + \mathbf{D}_j^R(\alpha).$$

These imply that $det(\mathbf{A} - \mathbf{C}) \neq 0$ and $det(|\mathbf{C}| - |\mathbf{A}|) \neq 0$ and consequently $[\widetilde{x}_{j_A}]_\alpha$ can be obtained by Eq (3.2) for any $\alpha \in [0, 1]$. In the first part of the proof, we obtain

$$\mathbf{A}[\widetilde{x}_{j_A}]_\alpha + [\widetilde{\mathbf{B}}_j]_\alpha = \mathbf{C}[\widetilde{x}_{j_A}]_\alpha + [\widetilde{\mathbf{D}}_j]_\alpha.$$

Obviously, $[\widetilde{x}_{j_A}]_\alpha = [\widetilde{w}_j]_\alpha$, for any $\alpha \in [0, 1]$, $j = 1, 2, \ldots, n$. Thus, Eq (3.2) constructs the $\alpha-levels$ of a fuzzy number-matrix, the proof is complete.

Corollary 3.1. *Let $\widetilde{\mathbf{X}} = (x^l, x^m, x^u)$, $\widetilde{\mathbf{B}} = (\mathbf{B}^l, \mathbf{B}^m, \mathbf{B}^u)$ and $\widetilde{\mathbf{D}} = (\mathbf{D}^l, \mathbf{D}^m, \mathbf{D}^u)$ are triangular fuzzy numbers matrices for the Eq* (2.1)*. According to Remark* 2.5 *and Theorem* 3.1*, we have*

$$\mathbf{F}(\alpha) = \frac{1-\alpha}{2}(|(\mathbf{A} - \mathbf{C})^{-1}|(\mathbf{D}^u - \mathbf{D}^l + \mathbf{B}^u - \mathbf{B}^l) + (|\mathbf{C}| - |\mathbf{A}|)^{-1}(\mathbf{D}^u - \mathbf{D}^l - \mathbf{B}^u + \mathbf{B}^l)),$$

$$\underline{\mathbf{X}}_E(\alpha) = \mathbf{X}_E^C(\alpha) - \mathbf{X}_E^R(\alpha) = (\mathbf{A}-\mathbf{C})^{-1}(\alpha(\mathbf{D}^m-\mathbf{B}^m) + \frac{1-\alpha}{2}(\mathbf{D}^l+\mathbf{D}^u-\mathbf{B}^l-\mathbf{B}^u)) - \frac{1-\alpha}{2}|(\mathbf{A}-\mathbf{C})^{-1}|(\mathbf{D}^u-\mathbf{D}^l+\mathbf{B}^u-\mathbf{B}^l),$$

$$\overline{\mathbf{X}}_E(\alpha) = \mathbf{X}_E^C(\alpha) + \mathbf{X}_E^R(\alpha) = (\mathbf{A}-\mathbf{C})^{-1}(\alpha(\mathbf{D}^m-\mathbf{B}^m) + \frac{1-\alpha}{2}(\mathbf{D}^l+\mathbf{D}^u-\mathbf{B}^l-\mathbf{B}^u)) + \frac{1-\alpha}{2}|(\mathbf{A}-\mathbf{C})^{-1}|(\mathbf{D}^u-\mathbf{D}^l+\mathbf{B}^u-\mathbf{B}^l).$$

Hence,

$$\underline{\mathbf{X}}_A(\alpha) = \underline{\mathbf{X}}_E(\alpha) + \mathbf{F}(\alpha) = (\mathbf{A}-\mathbf{C})^{-1}(\alpha(\mathbf{D}^m-\mathbf{B}^m) + \frac{1-\alpha}{2}(\mathbf{D}^l+\mathbf{D}^u-\mathbf{B}^l-\mathbf{B}^u)) + \frac{1-\alpha}{2}(|\mathbf{C}|-|\mathbf{A}|)^{-1}(\mathbf{D}^u-\mathbf{D}^l-\mathbf{B}^u+\mathbf{B}^l),$$

$$\overline{\mathbf{X}}_A(\alpha) = \overline{\mathbf{X}}_E(\alpha) - \mathbf{F}(\alpha) = (\mathbf{A}-\mathbf{C})^{-1}(\alpha(\mathbf{D}^m-\mathbf{B}^m) + \frac{1-\alpha}{2}(\mathbf{D}^l+\mathbf{D}^u-\mathbf{B}^l-\mathbf{B}^u)) - \frac{1-\alpha}{2}(|\mathbf{C}|-|\mathbf{A}|)^{-1}(\mathbf{D}^u-\mathbf{D}^l-\mathbf{B}^u+\mathbf{B}^l).$$

Thus, we obtain the unique algebraic solution $\widetilde{\mathbf{X}}_A = (x_A^l, x_A^m, x_A^u)$ to the dual fuzzy matrix equation (2.1)*, where*

$$x_A^l = \frac{1}{2}(\mathbf{A} - \mathbf{C})^{-1}(\mathbf{D}^l + \mathbf{D}^u - \mathbf{B}^l - \mathbf{B}^u) + \frac{1}{2}(|\mathbf{C}| - |\mathbf{A}|)^{-1}(\mathbf{D}^u - \mathbf{D}^l - \mathbf{B}^u + \mathbf{B}^l),$$
$$x_A^u = \frac{1}{2}(\mathbf{A} - \mathbf{C})^{-1}(\mathbf{D}^l + \mathbf{D}^u - \mathbf{B}^l - \mathbf{B}^u) - \frac{1}{2}(|\mathbf{C}| - |\mathbf{A}|)^{-1}(\mathbf{D}^u - \mathbf{D}^l - \mathbf{B}^u + \mathbf{B}^l),$$
$$x_A^m = (\mathbf{A} - \mathbf{C})^{-1}(\mathbf{D}^m - \mathbf{B}^m).$$

Remark 3.1. *The condition of det*$(\mathbf{A}-\mathbf{C}) \neq 0$ *and det*$(|\mathbf{C}|-|\mathbf{A}|) \neq 0$ *in Theorem* 3.1 *is a necessary condition for the existence of unique algebraic solution of the dual fuzzy matrix equation* (2.1). *In other words, if det*$(\mathbf{A}-\mathbf{C}) = 0$ *or det*$(|\mathbf{C}|-|\mathbf{A}|) = 0$*, then the Eq* (2.1) *may not have an algebraic solution or it may have infinite algebraic solutions.*

Remark 3.2. *Let* $\mathbf{F}(\alpha) = (f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha))$, $f_j(\alpha) = (f_{1j}(\alpha), f_{2j}(\alpha), \ldots, f_{nj}(\alpha))^\top$*, by Theorem* 3.1*, if* $\mathbf{F}(\alpha) \geqslant 0$*, i.e.* $f_{ij}(\alpha) \geqslant 0$*, then* $\widetilde{\mathbf{X}}_A \subseteq \widetilde{\mathbf{X}}_E$*, for each* $\alpha \in [0,1]$*. Also, if* $\mathbf{F}(\alpha) \leqslant 0$*, i.e.* $f_{ij}(\alpha) \leqslant 0$*, then* $\widetilde{\mathbf{X}}_E \subseteq \widetilde{\mathbf{X}}_A$*.*

We use the same method to discuss fuzzy matrix equations

$$\mathbf{A}\widetilde{\mathbf{X}} = \mathbf{C}\widetilde{\mathbf{X}} + \widetilde{\mathbf{D}}, \tag{3.7}$$

$$\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{B}} = \widetilde{\mathbf{D}}, \tag{3.8}$$

and

$$\mathbf{A}\widetilde{\mathbf{X}} = \widetilde{\mathbf{D}}. \tag{3.9}$$

(1) Let $\mathbf{B} = \mathbf{O}$, then the dual fuzzy matrix equation (2.1) is reduced to the fuzzy matrix equation

$$\mathbf{A}\widetilde{\mathbf{X}} = \mathbf{C}\widetilde{\mathbf{X}} + \widetilde{\mathbf{D}}.$$

According to our proposed method, the following results are obvious:

The dual fuzzy matrix equation (3.7) has a unique algebraic solution if and only if $det(\mathbf{A}-\mathbf{C}) \neq 0$, $det(|\mathbf{C}|-|\mathbf{A}|) \neq 0$ and the family of sets $[\underline{\mathbf{X}}_E(\alpha) + \mathbf{F}(\alpha), \overline{\mathbf{X}}_E(\alpha) - \mathbf{F}(\alpha)]$ constructs the $\alpha-$levels of a fuzzy number-valued matrix and $[\widetilde{\mathbf{X}}_A]_\alpha = [\underline{\mathbf{X}}_E(\alpha) + \mathbf{F}(\alpha), \overline{\mathbf{X}}_E(\alpha) - \mathbf{F}(\alpha)]$. Where

$$\widetilde{\mathbf{X}}_E = (\mathbf{A}-\mathbf{C})^{-1}\widetilde{\mathbf{D}},\ \ [\widetilde{\mathbf{X}}_E]_\alpha = (\mathbf{A}-\mathbf{C})^{-1}[\widetilde{\mathbf{D}}]_\alpha,\ \ \mathbf{X}_E^R(\alpha) = |(\mathbf{A}-\mathbf{C})^{-1}|\mathbf{D}^R(\alpha),$$

and the parameter matrix

$$\mathbf{F}(\alpha) = \mathbf{X}_E^R(\alpha) + (|\mathbf{C}|-|\mathbf{A}|)^{-1}\mathbf{D}^R(\alpha),\ \alpha \in [0,1].$$

(2) In the dual fuzzy matrix equation (2.1), let $\mathbf{C} = \mathbf{O}$ and have

$$\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{B}} = \widetilde{\mathbf{D}}.$$

Then the dual fuzzy matrix equation (3.8) has a unique algebraic solution if and only if $det(\mathbf{A}) \neq 0$, $det((-1)\cdot|\mathbf{A}|) \neq 0$ and the family of sets $[\underline{\mathbf{X}}_E(\alpha) + \mathbf{F}(\alpha), \overline{\mathbf{X}}_E(\alpha) - \mathbf{F}(\alpha)]$ constructs the $\alpha-$levels of a fuzzy number-valued matrix and $[\widetilde{\mathbf{X}}_A]_\alpha = [\underline{\mathbf{X}}_E(\alpha) + \mathbf{F}(\alpha), \overline{\mathbf{X}}_E(\alpha) - \mathbf{F}(\alpha)]$. Where

$$\widetilde{\mathbf{X}}_E = \mathbf{A}^{-1}(\widetilde{\mathbf{D}} - \widetilde{\mathbf{B}}),\ \ [\widetilde{\mathbf{X}}_E]_\alpha = \mathbf{A}^{-1}([\widetilde{\mathbf{D}}]_\alpha - [\widetilde{\mathbf{B}}]_\alpha),\ \ \mathbf{X}_E^R(\alpha) = |\mathbf{A}^{-1}|(\mathbf{D}^R(\alpha) + \mathbf{B}^R(\alpha)),$$

and the parameter matrix

$$\mathbf{F}(\alpha) = \mathbf{X}_E^R(\alpha) + ((-1)\cdot|\mathbf{A}|)^{-1}(\mathbf{D}^R(\alpha) + \mathbf{B}^R(\alpha)).$$

(3) In the dual fuzzy matrix equation (2.1), let $\mathbf{B} = \mathbf{O}$, $\mathbf{C} = \mathbf{O}$, there is a fuzzy matrix equation

$$\mathbf{A}\widetilde{\mathbf{X}} = \widetilde{\mathbf{D}}.$$

Then the dual fuzzy matrix equation (3.9) has a unique algebraic solution if and only if $det(\mathbf{A}) \neq 0$, $det((-1)\cdot|\mathbf{A}|) \neq 0$ and the family of sets $[\underline{\mathbf{X}}_E(\alpha)+\mathbf{F}(\alpha), \overline{\mathbf{X}}_E(\alpha)-\mathbf{F}(\alpha)]$ constructs the $\alpha-$levels of a fuzzy number-valued matrix and $[\widetilde{\mathbf{X}}_A]_\alpha = [\underline{\mathbf{X}}_E(\alpha)+\mathbf{F}(\alpha), \overline{\mathbf{X}}_E(\alpha)-\mathbf{F}(\alpha)]$. Where

$$\widetilde{\mathbf{X}}_E = \mathbf{A}^{-1}\widetilde{\mathbf{D}}, \ \ [\widetilde{\mathbf{X}}_E]_\alpha = \mathbf{A}^{-1}[\widetilde{\mathbf{D}}]_\alpha, \ \ \mathbf{X}^R_E(\alpha) = |\mathbf{A}^{-1}|\mathbf{D}^R(\alpha),$$

and the parameter matrix

$$\mathbf{F}(\alpha) = \mathbf{X}^R_E(\alpha) + ((-1)\cdot|\mathbf{A}|)^{-1}\mathbf{D}^R(\alpha).$$

Theorem 3.2. *Let* $det(\mathbf{A}-\mathbf{C}) \neq 0$ *and* $det(|\mathbf{C}|-|\mathbf{A}|) \neq 0$ *for the dual fuzzy matrix equation* (2.1). *If* $\widetilde{\mathbf{B}}$ *and* $\widetilde{\mathbf{D}}$ *are crisp-valued matrices, then for any* $\alpha \in [0,1]$, $\mathbf{F}(\alpha) = 0$ *and*

$$\widetilde{\mathbf{X}}_A = \widetilde{\mathbf{X}}_E = (\mathbf{A}-\mathbf{C})^{-1}(\widetilde{\mathbf{D}}-\widetilde{\mathbf{B}}).$$

Proof. Since the matrices $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{D}}$ are crisp-valued matrices, obviously, $\mathbf{B}^R(\alpha) = 0$ and $\mathbf{D}^R(\alpha) = 0$. We infer that

$$\mathbf{X}^R_E(\alpha) = |(\mathbf{A}-\mathbf{C})^{-1}|(\mathbf{D}^R(\alpha)+\mathbf{B}^R(\alpha)) = 0.$$

This follows that

$$\mathbf{F}(\alpha) = \mathbf{X}^R_E(\alpha) + (|\mathbf{C}|-|\mathbf{A}|)^{-1}\left(\mathbf{D}^R(\alpha)-\mathbf{B}^R(\alpha)\right) = 0,$$

then

$$\widetilde{\mathbf{X}}_A = \widetilde{\mathbf{X}}_E = (\mathbf{A}-\mathbf{C})^{-1}(\widetilde{\mathbf{D}}-\widetilde{\mathbf{B}}).$$

Remark 3.3. *Let* $\widetilde{\mathbf{X}} = (x^l, x^m, x^u)$, $\widetilde{\mathbf{B}} = (\mathbf{B}^l, \mathbf{B}^m, \mathbf{B}^u)$ *and* $\widetilde{\mathbf{D}} = (\mathbf{D}^l, \mathbf{D}^m, \mathbf{D}^u)$ *are triangular fuzzy numbers for the Eq* (2.1). *By Theorem* 3.2, *we have* $\mathbf{B}^l = \mathbf{B}^m = \mathbf{B}^u$ *and* $\mathbf{D}^l = \mathbf{D}^m = \mathbf{D}^u$, *according to Corollary* 3.1

$$\mathbf{F}(\alpha) = 0$$

and

$$\underline{\mathbf{X}}_E(\alpha) = \underline{\mathbf{X}}_A(\alpha) = (\mathbf{A}-\mathbf{C})^{-1}(\alpha(\mathbf{D}^m-\mathbf{B}^m) + \frac{1-\alpha}{2}(\mathbf{D}^l+\mathbf{D}^u-\mathbf{B}^l-\mathbf{B}^u)),$$
$$\overline{\mathbf{X}}_E(\alpha) = \overline{\mathbf{X}}_A(\alpha) = (\mathbf{A}-\mathbf{C})^{-1}(\alpha(\mathbf{D}^m-\mathbf{B}^m) + \frac{1-\alpha}{2}(\mathbf{D}^l+\mathbf{D}^u-\mathbf{B}^l-\mathbf{B}^u)).$$

Remark 3.4. *For Theorem* 3.2, *the condition that* $\widetilde{\mathbf{B}}$ *and* $\widetilde{\mathbf{D}}$ *are crisp-valued matrices is a sufficient but not a necessary condition. That is, even if* $\widetilde{\mathbf{B}}$ *(or* $\widetilde{\mathbf{D}}$*) is not a crisp-valued matrices, it is possible to obtain* $\mathbf{F}(\alpha) = 0$ *and consequently* $\widetilde{\mathbf{X}}_A = \widetilde{\mathbf{X}}_E$.

For example, consider the dual fuzzy matrix equation

$$\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{\mathbf{X}} + \widetilde{\mathbf{D}},$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and also

$$\widetilde{\mathbf{B}} = \begin{pmatrix} (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{pmatrix}, \widetilde{\mathbf{D}} = \begin{pmatrix} (3,4,7) & (3,4,7) \\ (0,0,0) & (0,0,0) \end{pmatrix}.$$

Not that $det(\mathbf{A}-\mathbf{C})=-1$ and $det(|\mathbf{C}|-|\mathbf{A}|)=1$, then

$$\mathbf{F}(\alpha)=\frac{1-\alpha}{2}(|(\mathbf{A}-\mathbf{C})^{-1}|(\mathbf{D}^u-\mathbf{D}^l+\mathbf{B}^u-\mathbf{B}^l)+(|\mathbf{C}|-|\mathbf{A}|)^{-1}(\mathbf{D}^u-\mathbf{D}^l-\mathbf{B}^u+\mathbf{B}^l))=\begin{pmatrix}(0,0,0) & (0,0,0)\\ (0,0,0) & (0,0,0)\end{pmatrix}.$$

Therefore

$$\widetilde{\mathbf{X}}_A=\widetilde{\mathbf{X}}_E=(\mathbf{A}-\mathbf{C})^{-1}(\widetilde{\mathbf{D}}-\widetilde{\mathbf{B}}).$$

The following theorem presents a sufficient condition for the existence and uniqueness of the algebraic solution of Eq (2.1).

Theorem 3.3. *In the fuzzy matrix equation* (2.1)*, let*
(1) *both* $det(\mathbf{A}-\mathbf{C})\neq 0$ *and* $det(|\mathbf{C}|-|\mathbf{A}|)\neq 0$*;*
(2) $\mathbf{F}(\alpha)$ *is a bounded left-continuous nondecreasing matrix function over* $[0,1]$*, i.e.,* $f_{ij}(\alpha)$ *for* $i=1,2,\ldots,n$*, are bounded left-continuous nondecreasing functions over* $[0,1]$*. Where* $f_j(\alpha)=(f_{1j}(\alpha),f_{2j}(\alpha),\ldots,f_{nj}(\alpha))^\top$ *is the jth column of* $F(\alpha)$*,* $j=1,2,\ldots,n$*;*
(3) $\mathbf{F}(\alpha)\leqslant\mathbf{X}^R_E(\alpha)$*, i.e.* $f_{ij}(\alpha)\leqslant(\overline{x}_{ij_E}(\alpha)-\underline{x}_{ij_E}(\alpha))/2$*, for* $i=1,2,\ldots,n$*; Where* $F_j(\alpha)=(f_{1j}(\alpha),f_{2j}(\alpha),\ldots,f_{nj}(\alpha))^\top$ *and* $x^R_{j_E}(\alpha)=(x^R_{1j_E}(\alpha),x^R_{2j_E}(\alpha),\ldots,x^R_{nj_E}(\alpha))^\top$ *denote the jth column of* $\mathbf{F}(\alpha)$ *and* $\mathbf{X}^R_E(\alpha)$*, respectively.*

Then, the dual fuzzy matrix equation (2.1) *has a unique algebraic solution with the* $\alpha-$*levels indicated by Eq* (3.2)*.*

Proof. Suppose $\widetilde{\mathbf{X}}_A$ is an algebraic solution of (2.1), we have

$$\mathbf{A}\widetilde{\mathbf{X}}_A+\widetilde{\mathbf{B}}=\mathbf{C}\widetilde{\mathbf{X}}_A+\widetilde{\mathbf{D}}.$$

Since matrix $det(\mathbf{A}-\mathbf{C})\neq 0$, the extended solution $\widetilde{\mathbf{X}}_E$ exists and the $\alpha-$levels expressed as

$$[\widetilde{\mathbf{X}}_E]_\alpha=[\underline{\mathbf{X}}_E(\alpha),\overline{\mathbf{X}}_E(\alpha)],$$

since$\mathbf{F}(\alpha)$ is a bounded left-continuous non-decreasing matrix function over $[0,1]$, $-\mathbf{F}(\alpha)$ is a bounded left-continuous non-increasing matrix function over $[0,1]$; Also because $\mathbf{F}(\alpha)\leqslant\mathbf{X}^R_E(\alpha)$, i.e.

$$\overline{\mathbf{X}}_E(\alpha)-\underline{\mathbf{X}}_E(\alpha)-2\mathbf{F}(\alpha)\geqslant 0,$$

then $[\underline{\mathbf{X}}_E(\alpha)+\mathbf{F}(\alpha),\overline{\mathbf{X}}_E(\alpha)-\mathbf{F}(\alpha)]$ satisfies the required conditions of Definition 2.1, Lemma 2.1 and Remark 2.1 and also constructs the α-levels of a fuzzy number-valued matrix, since $|\mathbf{C}|-|\mathbf{A}|$ is invertible, $[\widetilde{\mathbf{X}}_A]_\alpha$ can be obtained by Eq (3.2), i.e. $[\widetilde{\mathbf{X}}_A]_\alpha=[\underline{\mathbf{X}}_E(\alpha)+\mathbf{F}(\alpha),\overline{\mathbf{X}}_E(\alpha)-\mathbf{F}(\alpha)]$, for any $\alpha\in[0,1]$.

The proof is complete.

Theorem 3.4. *Let* $\widetilde{x}_{j_E}(\alpha)=(\widetilde{x}_{1j_E}(\alpha),\widetilde{x}_{2j_E}(\alpha),\ldots,\widetilde{x}_{nj_E}(\alpha))^\top$*,* $F_j(\alpha)=(f_{1j}(\alpha),f_{2j}(\alpha),\ldots,f_{nj}(\alpha))^\top$*, for the dual fuzzy matrix equation* (2.1)*, and there exist at least* $\alpha_0\in[0,1]$ *and* $i^*\in 1,2,\ldots,n$*, such that*

$$f_{i^*j}(\alpha_0)>x^R_{i^*j_E}(\alpha_0)=\frac{\overline{x}_{i^*j_E}(\alpha_0)-\underline{x}_{i^*j_E}(\alpha_0)}{2}.$$

then the dual fuzzy matrix equation (2.1) *does not has a unique algebraic solution.*

Proof. Suppose that the dual fuzzy matrix equation (2.1) has a unique algebraic solution, then by Theorem (3.1), it follows that $[\underline{x}_{ij_E}(\alpha)+f_{ij_E}(\alpha), \overline{x}_{ij_E}(\alpha)-f_{ij_E}(\alpha)]$ constructs $\alpha-$levels of a fuzzy number, for any $i \in 1, 2, \ldots, n$ and $\alpha \in [0, 1]$. Since it is a closed interval, then we have

$$\underline{x}_{i^* j_E}(\alpha_0) + f_{i^* j_E}(\alpha_0) \leqslant \overline{x}_{i^* j_E}(\alpha_0) - f_{i^* j_E}(\alpha_0)$$

or

$$f_{i^* j_E}(\alpha_0) \leqslant \frac{\overline{x}_{i^* j_E}(\alpha_0) - \underline{x}_{i^* j_E}(\alpha_0)}{2}.$$

Clearly, this is a contradiction.

Definition 3.1. *(see [17]) Let* $\mathbf{A}$ *be a* $m \times n$ *matrix and* $(\cdot)^T$ *denote the transpose of the matrix* $(\cdot)$*. We recall that a generalized inverse* $\mathbf{G}$ *of* $\mathbf{A}$ *is an* $n \times m$ *matrix which satisfies one or more of Penrose equations*
(1) $\mathbf{AGA} = \mathbf{A}$,
(2) $\mathbf{GAG} = \mathbf{G}$,
(3) $(\mathbf{AG})^T = \mathbf{AG}$,
(4) $(\mathbf{GA})^T = \mathbf{GA}$.

The matrix $\mathbf{G}$ *is called a* $g-$*inverse of* $\mathbf{A}$ *if it satisfies* (1)*. As usual, the* $g-$*inverse of* $\mathbf{A}$ *is denoted by* $\mathbf{A}^-$*. If* $\mathbf{G}$ *satisfies* (1) *and* (2)*, then it is called a reflexive inverse of* $\mathbf{A}$*. When the matrix* G *satisfies* $(1) - (4)$*, it is called the Moore-Penrose inverse of* $\mathbf{A}$*. Any matrix* $\mathbf{A}$ *admits a unique Moore-Penrose inverse, denoted by* $\mathbf{A}^\dagger$*.*

By our proposed method, the following result is obvious.

Corollary 3.2. *If for the dual fuzzy matrix equation* $\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{\mathbf{X}} + \widetilde{\mathbf{D}}$*, where* $\mathbf{A}$*,* $\mathbf{C}$ *are* $m \times n$ *matrices and* $\widetilde{\mathbf{B}}$*,* $\widetilde{\mathbf{D}}$ *are* $m \times p$ *fuzzy numbers matrices. Note that the* $\alpha-$*levels of the unique algebraic solution of equation is expressed by*

$$[\widetilde{\mathbf{X}}_A]_\alpha = [\underline{\mathbf{X}}_E(\alpha) + \mathbf{F}(\alpha), \overline{\mathbf{X}}_E(\alpha) - \mathbf{F}(\alpha)],\ \forall \alpha \in [0, 1],$$

where $\widetilde{\mathbf{X}}_E = (\mathbf{A} - \mathbf{C})^\dagger(\widetilde{\mathbf{D}} - \widetilde{\mathbf{B}})$ *and* $\mathbf{F}(\alpha) = \mathbf{X}_E^R(\alpha) + (|\mathbf{C}| - |\mathbf{A}|)^\dagger(\mathbf{D}^R(\alpha) - \mathbf{B}^R(\alpha))$.

Remark 3.5. *Let* $\widetilde{\mathbf{X}} = (x^l, x^m, x^u)$*,* $\widetilde{\mathbf{B}} = (\mathbf{B}^l, \mathbf{B}^m, \mathbf{B}^u)$ *and* $\widetilde{\mathbf{D}} = (\mathbf{D}^l, \mathbf{D}^m, \mathbf{D}^u)$ *are triangular fuzzy numbers matrices. By Corollary* 3.2*,*

$$\widetilde{\mathbf{X}}_E = (x_E^l, x_E^m, x_E^u) = (\mathbf{A} - \mathbf{C})^\dagger(\mathbf{D}^l - \mathbf{B}^u, \mathbf{D}^m - \mathbf{B}^m, \mathbf{D}^u - \mathbf{B}^l)$$

and

$$\widetilde{\mathbf{X}}_A = (x_A^l, x_A^m, x_A^u),$$

where

$$\begin{aligned} x_A^l &= \frac{1}{2}(\mathbf{A} - \mathbf{C})^\dagger(\mathbf{D}^l + \mathbf{D}^u - \mathbf{B}^l - \mathbf{B}^u) + \frac{1}{2}(|\mathbf{C}| - |\mathbf{A}|)^\dagger(\mathbf{D}^u - \mathbf{D}^l - \mathbf{B}^u + \mathbf{B}^l), \\ x_A^u &= \frac{1}{2}(\mathbf{A} - \mathbf{C})^\dagger(\mathbf{D}^l + \mathbf{D}^u - \mathbf{B}^l - \mathbf{B}^u) - \frac{1}{2}(|\mathbf{C}| - |\mathbf{A}|)^\dagger(\mathbf{D}^u - \mathbf{D}^l - \mathbf{B}^u + \mathbf{B}^l), \\ x_A^m &= (\mathbf{A} - \mathbf{C})^\dagger(\mathbf{D}^m - \mathbf{B}^m). \end{aligned}$$

Corollary 3.3. *As a special case of the dual fuzzy matrix equation, if for the dual fuzzy linear system* $\mathbf{A}\widetilde{X} + \widetilde{Y} = \mathbf{B}\widetilde{X} + \widetilde{Z}$, *where* $\mathbf{A}$, $\mathbf{B}$ *are* $m \times n$ *matrices and* $\widetilde{Y}$, $\widetilde{Z}$ *are fuzzy numbers vectors. We can obtain the* $\alpha-$*levels of the unique algebraic solution of equation is expressed by*

$$[\widetilde{X}_A]_\alpha = [\underline{X}_E(\alpha) + F(\alpha), \overline{X}_E(\alpha) - F(\alpha)], \forall \alpha \in [0,1],$$

where $\widetilde{X}_E = (\mathbf{A} - \mathbf{B})^\dagger(\widetilde{Z} - \widetilde{Y})$ *and* $F(\alpha) = X_E^R(\alpha) + (|\mathbf{B}| - |\mathbf{A}|)^\dagger(Z^R(\alpha) - Y^R(\alpha))$.

Remark 3.6. *Let* $\widetilde{X} = ((x_1^l, x_1^m, x_1^u), \cdots, (x_n^l, x_n^m, x_n^u))^\top$, $\widetilde{Y} = ((y_1^l, y_1^m, y_1^u), \cdots, (y_n^l, y_n^m, y_n^u))^\top$, $\widetilde{Z} = ((z_1^l, z_1^m, z_1^u), \cdots, (z_n^l, z_n^m, z_n^u))^\top$ *are triangular fuzzy numbers vectors. By Corollary 3.3,* $\widetilde{X}_E = (x_E^l, x_E^m, x_E^u) = (\mathbf{A} - \mathbf{B})^\dagger(Z^l - Y^u, Z^m - Y^m, Z^u - Y^l)$ *and* $\widetilde{X}_A = (x_A^l, x_A^m, x_A^u)$, *where*

$$\begin{aligned}
x_A^l &= \frac{1}{2}(\mathbf{A} - \mathbf{B})^\dagger(Z^l + Z^u - Y^l - Y^u) + \frac{1}{2}(|\mathbf{B}| - |\mathbf{A}|)^\dagger(Z^u - Z^l - Y^u + Y^l),\\
x_A^u &= \frac{1}{2}(\mathbf{A} - \mathbf{B})^\dagger(Z^l + Z^u - Y^l - Y^u) - \frac{1}{2}(|\mathbf{B}| - |\mathbf{A}|)^\dagger(Z^u - Z^l - Y^u + Y^l),\\
x_A^m &= (\mathbf{A} - \mathbf{B})^\dagger(Z^m - Y^m).
\end{aligned}$$

4. Numerical examples

Example 4.1. Consider the 2×2 dual fuzzy matrix equation

$$\begin{pmatrix} 1 & -2 \\ 10 & 1 \end{pmatrix}\begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{pmatrix} + \begin{pmatrix} \widetilde{b}_{11} & \widetilde{b}_{12} \\ \widetilde{b}_{21} & \widetilde{b}_{22} \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{pmatrix} + \begin{pmatrix} \widetilde{d}_{11} & \widetilde{d}_{12} \\ \widetilde{d}_{21} & \widetilde{d}_{22} \end{pmatrix},$$

where

$$\widetilde{\mathbf{B}} = \begin{pmatrix} \widetilde{b}_{11} & \widetilde{b}_{12} \\ \widetilde{b}_{21} & \widetilde{b}_{22} \end{pmatrix} = \begin{pmatrix} (-5,1,2) & (1,3,4) \\ (-3,1,4) & (2,5,6) \end{pmatrix}$$

and

$$\widetilde{\mathbf{D}} = \begin{pmatrix} \widetilde{d}_{11} & \widetilde{d}_{12} \\ \widetilde{d}_{21} & \widetilde{d}_{22} \end{pmatrix} = \begin{pmatrix} (2,4,12) & (1,5,7) \\ (-2,3,6) & (-7,-5,-2) \end{pmatrix}.$$

We first compute the extended solution $\widetilde{\mathbf{X}}_E$ as follows

$$\begin{aligned}
\widetilde{\mathbf{X}}_E =& (\mathbf{A} - \mathbf{C})^{-1}(\mathbf{D}^l - \mathbf{B}^u, \mathbf{D}^m - \mathbf{B}^m, \mathbf{D}^u - \mathbf{B}^l)\\
=& \begin{pmatrix} (-\frac{19}{6}, -\frac{7}{18}, \frac{1}{2}) & (-\frac{31}{18}, -\frac{8}{9}, \frac{5}{18}) \\ (-\frac{37}{3}, -\frac{20}{9}, \frac{2}{3}) & (-\frac{32}{9}, -\frac{2}{9}, \frac{31}{9}) \end{pmatrix}.
\end{aligned}$$

Now, we compute the x_A^l, x_A^m, x_A^u as follows

$$x_A^l = \begin{pmatrix} -\frac{23}{6} & -\frac{29}{9} \\ -\frac{37}{3} & -\frac{59}{9} \end{pmatrix}, \quad x_A^m = \begin{pmatrix} -\frac{7}{18} & -\frac{8}{9} \\ -\frac{20}{9} & -\frac{2}{9} \end{pmatrix}, \quad x_A^u = \begin{pmatrix} \frac{7}{6} & \frac{16}{9} \\ \frac{2}{3} & \frac{58}{9} \end{pmatrix}.$$

Finally, we can obtain the unique algebraic solution as follows

$$\widetilde{\mathbf{X}}_A = \begin{pmatrix} (-\frac{23}{6}, -\frac{7}{18}, \frac{7}{6}) & (-\frac{29}{9}, -\frac{8}{9}, \frac{16}{9}) \\ (-\frac{37}{3}, -\frac{20}{9}, \frac{2}{3}) & (-\frac{59}{9}, -\frac{2}{9}, \frac{58}{9}) \end{pmatrix}.$$

Note that each element in $\widetilde{\mathbf{X}}_A$ is a triangular fuzzy number, so the algebraic solution is acceptable.

Example 4.2. Consider the 3×3 dual fuzzy matrix equation

$$\begin{pmatrix} 10 & 2 & 0 \\ 3 & 24 & 0 \\ 0 & 0 & 16 \end{pmatrix}\begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} & \widetilde{x}_{13} \\ \widetilde{x}_{21} & \widetilde{x}_{22} & \widetilde{x}_{23} \\ \widetilde{x}_{31} & \widetilde{x}_{32} & \widetilde{x}_{33} \end{pmatrix} + \begin{pmatrix} \widetilde{b}_{11} & \widetilde{b}_{12} & \widetilde{b}_{13} \\ \widetilde{b}_{21} & \widetilde{b}_{22} & \widetilde{b}_{13} \\ \widetilde{b}_{31} & \widetilde{b}_{32} & \widetilde{b}_{33} \end{pmatrix}$$
$$= \begin{pmatrix} -12 & 2 & 0 \\ 3 & -26 & 0 \\ 0 & 0 & -18 \end{pmatrix}\begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} & \widetilde{x}_{13} \\ \widetilde{x}_{21} & \widetilde{x}_{22} & \widetilde{x}_{23} \\ \widetilde{x}_{31} & \widetilde{x}_{32} & \widetilde{x}_{33} \end{pmatrix} + \begin{pmatrix} \widetilde{d}_{11} & \widetilde{d}_{12} & \widetilde{d}_{13} \\ \widetilde{d}_{21} & \widetilde{d}_{22} & \widetilde{d}_{23} \\ \widetilde{d}_{31} & \widetilde{d}_{32} & \widetilde{d}_{33} \end{pmatrix},$$

where

$$\widetilde{\mathbf{B}} = \begin{pmatrix} \widetilde{b}_{11} & \widetilde{b}_{12} & \widetilde{b}_{13} \\ \widetilde{b}_{21} & \widetilde{b}_{22} & \widetilde{b}_{23} \\ \widetilde{b}_{31} & \widetilde{b}_{32} & \widetilde{b}_{33} \end{pmatrix} = \begin{pmatrix} (-4,-3,7) & (10,20,30) & (-10,-1,2) \\ (-10,-9,1) & (-1,6,8) & (-11,-10,1) \\ (-14,-2,2) & (8,12,16) & 9-8,-7,1) \end{pmatrix}$$

and

$$\widetilde{\mathbf{D}} = \begin{pmatrix} \widetilde{d}_{11} & \widetilde{d}_{12} & \widetilde{d}_{13} \\ \widetilde{d}_{21} & \widetilde{d}_{22} & \widetilde{d}_{23} \\ \widetilde{d}_{31} & \widetilde{d}_{32} & \widetilde{d}_{33} \end{pmatrix} = \begin{pmatrix} (-8,-7,2) & (-4,-2,6) & (-11,-8,-7) \\ (-1,0,4) & (0,2,7) & (-1,5,10) \\ (3,5,6) & (2,4,5) & 5,7,12) \end{pmatrix}.$$

Similarly, first, we get

$$\begin{aligned} \widetilde{\mathbf{X}}_E =& (\mathbf{A}-\mathbf{C})^{-1}(\mathbf{D}^l - \mathbf{B}^u, \mathbf{D}^m - \mathbf{B}^m, \mathbf{D}^u - \mathbf{B}^l) \\ =& \begin{pmatrix} (-\frac{15}{22}, -\frac{2}{11}, \frac{3}{11}) & (-\frac{17}{11}, -1, -\frac{2}{11}) & (-\frac{13}{22}, -\frac{7}{22}, \frac{3}{22}) \\ (-\frac{1}{25}, \frac{9}{50}, \frac{7}{25}) & (-\frac{4}{25}, -\frac{2}{25}, \frac{4}{25}) & (-\frac{1}{25}, \frac{3}{10}, \frac{21}{50}) \\ (\frac{1}{34}, \frac{7}{34}, \frac{10}{17}) & (-\frac{7}{17}, -\frac{4}{17}, -\frac{3}{34}) & (\frac{2}{17}, \frac{7}{17}, \frac{10}{17}) \end{pmatrix}. \end{aligned}$$

Next, we compute the x_A^l, x_A^m, x_A^u as follows

$$x_A^l = \begin{pmatrix} -\frac{5}{11} & -\frac{37}{11} & -\frac{49}{22} \\ -\frac{69}{50} & -\frac{1}{2} & -\frac{3}{50} \\ -\frac{50}{17} & -\frac{51}{34} & -\frac{5}{34} \end{pmatrix}, \quad x_A^m = \begin{pmatrix} -\frac{2}{11} & -1 & -\frac{7}{22} \\ \frac{9}{50} & -\frac{2}{25} & \frac{3}{10} \\ \frac{7}{34} & -\frac{4}{17} & \frac{7}{17} \end{pmatrix}, \quad x_A^u = \begin{pmatrix} \frac{1}{22} & \frac{18}{11} & \frac{39}{22} \\ \frac{81}{50} & \frac{1}{2} & \frac{11}{25} \\ \frac{121}{34} & 1 & \frac{29}{34} \end{pmatrix}.$$

Finally, we can obtain the unique algebraic solution as follows

$$\widetilde{\mathbf{X}}_A = \begin{pmatrix} (-\frac{5}{11}, -\frac{2}{11}, \frac{1}{22}) & (-\frac{37}{11}, -1, \frac{18}{11}) & (-\frac{49}{22}, -\frac{7}{22}, \frac{39}{22}) \\ (-\frac{69}{50}, \frac{9}{50}, \frac{81}{50}) & (-\frac{1}{2}, -\frac{2}{25}, \frac{1}{2}) & (-\frac{3}{50}, \frac{3}{10}, \frac{11}{25}) \\ (-\frac{50}{17}, \frac{7}{34}, \frac{121}{34}) & (-\frac{51}{34}, -\frac{4}{17}, 1) & (-\frac{5}{34}, \frac{7}{17}, \frac{29}{34}) \end{pmatrix}.$$

Obviously every element in $\widetilde{\mathbf{X}}_A$ is a triangular fuzzy number, so the algebraic solution is acceptable.

Example 4.3. Consider the dual fuzzy linear system

$$\begin{pmatrix} -4 & -6 \\ 1 & 0 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{pmatrix} + \begin{pmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \\ \widetilde{y}_3 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 1 & 0 \\ -6 & 7 \end{pmatrix}\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{pmatrix} + \begin{pmatrix} \widetilde{z}_1 \\ \widetilde{z}_2 \\ \widetilde{z}_3 \end{pmatrix}.$$

where

$$\widetilde{Z} = \begin{pmatrix} \widetilde{z}_1 \\ \widetilde{z}_2 \\ \widetilde{z}_3 \end{pmatrix} = \begin{pmatrix} (1,2,3) \\ (1,3,5) \\ (-3,-4,-2) \end{pmatrix}, \quad \widetilde{Y} = \begin{pmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \\ \widetilde{y}_3 \end{pmatrix} = \begin{pmatrix} (1,4,7) \\ (-2,1,3) \\ (-2,0,2) \end{pmatrix}.$$

Then, we can obtain the extended solution $\widetilde{x}_E$ as follows

$$\begin{aligned}\widetilde{X}_E &= (\mathbf{A}-\mathbf{B})^{\dagger}((Z^l - Y^u, Z^m - Y^m, Z^u - Y^l)) \\ &= \begin{pmatrix} (-\frac{2}{3}, -\frac{1}{12}, \frac{1}{2}) \\ (-\frac{1}{2}, -\frac{1}{4}, 0) \end{pmatrix}.\end{aligned}$$

Now, we compute the x_A^l, x_A^m, x_A^u as follows

$$x_A^l = \begin{pmatrix} -\frac{1}{12} \\ -\frac{3}{4} \end{pmatrix}, \quad x_A^m = \begin{pmatrix} -\frac{1}{12} \\ -\frac{1}{4} \end{pmatrix}, \quad x_A^u = \begin{pmatrix} \frac{11}{12} \\ \frac{1}{4} \end{pmatrix}.$$

Finally, we obtain the algebraic solution of the system as follows

$$\widetilde{X}_A = \begin{pmatrix} (-\frac{1}{12}, -\frac{1}{12}, \frac{11}{12}) \\ (-\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}) \end{pmatrix}.$$

Every element in $\widetilde{X}_A$ is a triangular fuzzy number, so the algebraic solution is acceptable.

5. Conclusions

In this paper, we obtained a simple method to solve the dual fuzzy matrix equations of the form $\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{\mathbf{X}} + \widetilde{\mathbf{D}}$ algebraically. In the system under consideration, $\mathbf{A}$ and $\mathbf{C}$ are $n \times n$ crisp matrices, $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{D}}$ are $n \times n$ fuzzy numbers matrices. A necessary and sufficient condition for the existence of unique algebraic solution of a dual fuzzy matrix equations is presented. More generally, We have also considered the dual fuzzy matrix equation $\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{\mathbf{X}} + \widetilde{\mathbf{D}}$, in which $\mathbf{A}$ and $\mathbf{C}$ are $m \times n$ matrices and $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{D}}$ are $m \times p$ fuzzy numbers matrices and the dual fuzzy linear systems $\mathbf{A}\widetilde{X} + \widetilde{Y} = \mathbf{B}\widetilde{X} + \widetilde{Z}$ whose coefficient matrices are $m \times n$ matrices and the left and right hand sides vectors are triangular fuzzy numbers matrices based on the generalized inverses of matrices. Finally, some examples are presented to illustrate our results. Our results will be useful in developing the theory of fuzzy matrix equations and fuzzy linear systems. Compared to existing methods, Not need to transform a dual fuzzy matrix equation into two crisp matrix equations is the main advantage of our method. In the future, we will further use method established in this article to explore some more complex forms of dual fuzzy matrix equations and dual fuzzy linear systems.

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Conflict of interest

The authors declare that they have no conflict of interest.

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