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Research article

Dynamics of the positive almost periodic solution to a class of recruitment delayed model on time scales

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Abstract: By employing the operator theory, the Lyapunov function on time scales and the famous Gronwall's inequality, this paper addresses some dynamic properties of almost periodic solutions for a class of two species co-existence delayed model on time scales with almost periodic coefficients and Ricker, as well as the Beverton-Holt type function. First, we establish the existence and uniqueness of the almost periodic solution with a positive infimum by transforming the initial model into an equivalent integral equation. Second, we investigate the global exponential stability and uniformly asymptotic stability of the positive almost periodic solution. Finally, we give two examples to illustrate the main presented results.

Keywords: almost periodic solution; existence and uniqueness; stability; time scales
Mathematics Subject Classification: 34C27, 34D23

1. Introduction

In the research on population dynamics in biological applications, a recruitment-delayed model

$$x'(t) = B(x(t-\tau)) - D(x(t)) \tag{1.1}$$

is persistently used, where $x(t)$ is the population size of mature adults at time t, B is the birth function involving maturation delay τ and D represents the death rate [1]. In particular, the birth function has two common forms, where one is the Ricker-type function $Pxe^{-\alpha x}$ for $P \in \mathbb{R}^+$, and the other is the Beverton-Holt function $\frac{px}{q+x^m}$ for $m \in \mathbb{Z}^+$ and $\frac{P}{q} \in \mathbb{R}^+$. In order to model a laboratory fly population, Nicholson considered Eq (1.1) with $B(x) = Pxe^{-bx}$ and $D(x) = \delta x$, which is the well-known Nicholson's blowflies equation [2]; here, P is the per capita daily maximum egg production rate, $\frac{1}{b}$ is the size at which the population reproduces at its maximum rate, τ is the time generated from birth to maturity and $\delta \in \mathbb{R}^+$ is adult mortality rate per capita daily.

In recent years, there have been many researchers who have taken great interest in the investigation of dynamic behaviors based on Eq (1.1) and its analogous equations with Ricker's type or Beverton-Holt type functions, such as the existence and uniqueness of periodic solutions, oscillation, persistence, stability, etc.; see [3–8] for details. However, as far as we know, few authors have considered the problem on both Ricker's and Beverton-Holt type differential equations, let alone its positive almost periodic solutions and qualitative behavior.

In many ecological dynamical systems, on the one hand, the growth rate of a population for a natural mature adult species in the real world would not react instantaneously to changes in its own total amount or that of an interacting species, but it certainly does after a time lag. In addition, the different delays and variable coefficients in differential equations are much more suitable and realistic to depict the variations in a natural environment [9,10]. On the other hand, the frequent almost periodic varying environment has been the target of extensive analysis on evolutionary theory because of the superiority and actuality as compared with a periodic environment; see [11–14] for details. Therefore, investigating the dynamical behavior in delayed models with an almost periodic term is an interesting and worthy topic.

In spite of the fact that both continuous and discrete systems have a huge amount of research achievements, as a link between continuity and dispersion, the theory of time scales has occupied an irreplaceable position in fields such as the application of population models, quantum physics, etc. In recent years, there have been incremental investigations into a huge number of mathematical modelings with time scales, such as the permanence, existence and stability of periodic and almost periodic solutions; see [15–17] for details; this trend has become inevitable in the research on dynamical systems. Therefore, it is valuable in the exploration of dynamic equations on time scales.

Inspired by the above discussions, in this paper, we consider the two species co-existence delayed model with the almost periodic coefficients and Ricker- and Beverton-Holt-type functions on time scales, as follows:

$$\begin{cases} x^{\Delta}(t) = -\alpha_1(t)x(t) + \dfrac{\beta(t)x(t)}{\gamma(t)+x(t)} + p_1(t)x(t)e^{-bx(t)} \\ \qquad\quad + h_1(t)\displaystyle\int_{-\infty}^{0} K_1(t,s)y(t+s)e^{-by(t+s)}\Delta s, \\ y^{\Delta}(t) = -\alpha_2(t)y(t) + p_2(t)y(t)e^{-by(t)} \\ \qquad\quad + h_2(t)\displaystyle\int_{-\infty}^{0} K_2(t,s)x(t+s)e^{-bx(t+s)}\Delta s, \end{cases} \tag{1.2}$$

where $x(t)$ and $y(t)$ are the population densities of two species coexisting at time $t \in \mathbb{T}$, respectively, for the almost periodic time scale $\mathbb{T}$; $x^{\Delta}(t)$ and $y^{\Delta}(t)$ are the delta derivatives of the functions $x(t)$ and $y(t)$ respectively; α_i, β, γ, p_i and h_i are positive almost periodic functions for $i = 1,2$; $\frac{\beta(t)x(t)}{\gamma(t)+x(t)}$ is the Beverton-Holt type birth function; $p_1(t)x(t)e^{-bx(t)}$ and $p_2(t)y(t)e^{-by(t)}$ represent the Ricker-type birth function; $K_i(t,s)$ represents the transmission delay kernels for $i = 1,2$. Under the suitable assumptions, by transforming the model (1.2) into an equivalent integral equation and using the fixed point theorem in a normal and solid cone in Banach space, we establish some sufficient conditions for the existence and uniqueness of the positive almost periodic solution; further, we investigate the global exponential stability and uniformly asymptotic stability of this positive solution.

The remainder of this paper is organized as follows. In Section 2, we present some necessary preliminaries. In Section 3, we will give the main results on the existence, uniqueness and the stability of its positive almost periodic solutions of (1.2). In Section 4, we present two examples to illustrate the main results.

2. Preliminaries

Assume that $\mathbb{C} = \mathbb{C}((-\infty, 0]_{\mathbb{T}}, \mathbb{R})$ and $H^* \in \mathbb{R}^+$. Let $\mathbb{C}_{H^*} = \left\{\varphi : \varphi \in \mathbb{C}, \|\varphi\| = \sup_{\vartheta\in(-\infty,0]_{\mathbb{T}}} |\varphi(\vartheta)| < H^*\right\}$ and $\mathbb{S}_{H^*} = \{x : x \in \mathbb{R}, \|x\| < H^*\}$. For convenience, throughout this work, we denote the following nonnegative values:

$$\alpha^+ = \sup_{t\in\mathbb{R}} \alpha(t), \quad \alpha^- = \inf_{t\in\mathbb{R}} \alpha(t), \quad \beta^+ = \sup_{t\in\mathbb{R}} \beta(t),$$

$$\gamma^- = \inf_{t\in\mathbb{R}} \gamma(t), \quad p^+ = \sup_{t\in\mathbb{R}} p(t), \quad p^- = \inf_{t\in\mathbb{R}} p(t), \quad h^+ = \sup_{t\in\mathbb{R}} h(t).$$

In addition, for some definitions, lemmas and preliminary results on time scales and almost periodic functions, one can see [4, 11, 14, 17] for more details, which are valuable in proving the main results in Section 3.

Let the symbol $\mathbb{T}$ be a time scale, which is a closed nonempty subset of $\mathbb{R}$. In fact, $\mathbb{R}$ and $\cup_{k\in\mathbb{Z}}[2k, 2k+1]$ are some examples of $\mathbb{T}$. Assume that the forward and backward jump operators σ, ρ: $\mathbb{T} \to \mathbb{T}$ and the graininess μ: $\mathbb{T} \to \mathbb{R}^+$ are respectively defined by

$$\sigma(t) = \sup\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \inf\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. In addition, if $\mathbb{T}$ has a left-scattered maximum m, then define $\mathbb{T}^k = \mathbb{T} - m$; otherwise, let $\mathbb{T}^k = \mathbb{T}$; if $\mathbb{T}$ has a right-scattered minimum m, then define $\mathbb{T}_k = \mathbb{T} - m$; otherwise, let $\mathbb{T}_k = \mathbb{T}$.

Definition 2.1. A function f: $\mathbb{T} \to \mathbb{R}$ is right-dense/left-dense continuous provided that it is continuous at right-dense/left-dense points in $\mathbb{T}$ and its left-dense/right-dense limits exist (finite) at left-dense/right-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$.

Definition 2.2. Let the function f: $\mathbb{T} \to \mathbb{R}$, and define $f^\Delta(t)$ to be the number (if it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood U of t such that, for all $s \in U$,

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|.$$

Definition 2.3. We call $f^\Delta(t)$ the delta derivative of f at t. If $F^\Delta(t) = f(t)$, then the delta integral is defined by

$$\int_r^t f(s)\Delta s = F(t) - F(r) \text{ for } r, t \in \mathbb{T}.$$

Definition 2.4. A function p: $\mathbb{T} \to \mathbb{R}$ is called regressive provided that $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all such regressive and rd-continuous functions will be denoted by $\Re = \Re(\mathbb{T}, \mathbb{R})$. Let the set $\Re^+ = \Re^+(\mathbb{T}, \mathbb{R}) = \{p \in \Re : 1 + \mu(t)p(t) > 0,\ t \in \mathbb{T}\}$.

Definition 2.5. If p is a regressive function, then the generalized exponential function e_p is given as the unique solution of the initial value problem $y^{\Delta} = p(t)y$, $y(s) = 1$, where $s \in \mathbb{T}$. An explicit formula for $e_p(t, s)$ is defined as

$$e_p(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\} \text{ for all } s, t \in \mathbb{T}$$

with

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, \text{ for } h \neq 0, \\ z, \text{ for } h = 0. \end{cases}$$

Definition 2.6. Let Γ be a collection of sets which is constructed by subsets of $\mathbb{R}$. A time scale $\mathbb{T}$ is called an almost periodic time scale with respect to Γ if

$$\Gamma^* = \{\pm\tau \in \cap\Lambda : \Lambda \in \mathbb{T},\ t \pm \tau \in \mathbb{T},\ \text{for } t \in \mathbb{T}\},$$

and Γ^* is called the smallest almost periodic set of $\mathbb{T}$.

Definition 2.7. Let $\mathbb{T}$ be an almost periodic time scale with respect to Γ. A function $x : \mathbb{T} \to \mathbb{R}$ is called almost periodic if, for any $\varepsilon > 0$, the set $\mathbb{E}(\varepsilon, x) = \{\tau \in \Gamma^* : |x(t + \tau) - x(t)| < \varepsilon \text{ for } t \in \mathbb{T}\}$ is relatively dense in $\mathbb{T}$.

$\mathbb{E}(\varepsilon, x)$ and τ are called the ε-translation set and ε-translation number of x, respectively. Denote the space of all such almost periodic functions by $AP(\mathbb{T}, \mathbb{R})$.

Definition 2.8. Let $\mathbb{T}$ be an almost periodic time scale with respect to Γ. A function $f : \mathbb{T} \times \mathbb{B} \to \mathbb{R}$ is called almost periodic in $t \in \mathbb{T}$ uniformly for $x \in \mathbb{S}$ if the ε-translation set of f

$$\mathbb{E}(\varepsilon, f, \mathbb{S}) = \{\tau \in \Gamma^* : |f(t + \tau, x) - f(t, x)| < \varepsilon \text{ for } (t, x) \in (\mathbb{T} \times \mathbb{S})\}$$

is relatively dense in $\mathbb{T}$ for all $\varepsilon > 0$ and for each compact subset $\mathbb{S}$ of $\mathbb{B}$.

Definition 2.9. Let $Q(t)$ be an $n \times n$ continuous matrix defined on $\mathbb{T}$. The linear system

$$x^{\Delta}(t) = Q(t)x(t), \quad t \in \mathbb{T}, \tag{2.1}$$

is said to admit exponential dichotomy if there are positive constants $\bar{k}$ and $\bar{\alpha}$, a projection $\bar{P}$ and the fundamental solution matrix $X(t)$ of (2.1) satisfying

$$\|X(t)\bar{P}X^{-1}(\sigma(s))\| \leq \bar{k}e_{\ominus\bar{\alpha}}(t, \sigma(s)) \text{ for all } t \geq \sigma(s),\ s,\ t \in \mathbb{T},$$

$$\|X(t)(1 - \bar{P})X^{-1}(\sigma(s))\| \leq \bar{k}e_{\ominus\bar{\alpha}}(\sigma(s), t) \text{ for all } t \leq \sigma(s),\ s,\ t \in \mathbb{T}.$$

Lemma 2.1. If the linear system (2.1) admits an exponential dichotomy, then the almost periodic system

$$x^{\Delta}(t) = Q(t)x(t) + g(t)$$

has a unique almost periodic solution $x(t)$ and

$$x(t) = \int_{-\infty}^{t} X(t)\bar{P}X^{-1}(\sigma(s))g(s)\triangle s - \int_{t}^{+\infty} X(t)(1 - \bar{P})X^{-1}(\sigma(s))g(s)\triangle s.$$

Lemma 2.2. Let $\bar{c}_i(t)$ be an almost periodic function on $\mathbb{T}$, where $\bar{c}_i(t) > 0$, $-\bar{c}_i(t) \in \mathbb{R}^+$ and $\min_{1\le i\le n}\{\inf_{t\in\mathbb{T}} \bar{c}_i(t)\} > 0$. Then, the linear system

$$x^{\Delta}(t) = \mathrm{diag}(-\bar{c}_1(t), -\bar{c}_2(t), \cdots, -\bar{c}_n(t))x(t)$$

admits an exponential dichotomy.

Lemma 2.3. Let $-C = \mathrm{diag}(-\bar{c}_1(t), -\bar{c}_2(t), \cdots, -\bar{c}_n(t))$; then, $X(t) = e_{-C}(t, t_0)$ is a fundamental solution matrix of the linear system

$$x^{\Delta}(t) = \mathrm{diag}(-\bar{c}_1(t), -\bar{c}_2(t), \cdots, -\bar{c}_n(t))x(t).$$

Lemma 2.4. Assume that $a > 0$ and $-a \in \Re^+$; it follows that

(i) if $x^{\Delta}(t) \le b - ax(t)$, then $\limsup_{t\to+\infty} x(t) \le \frac{b}{a}$;

(ii) if $x^{\Delta}(t) \ge b - ax(t)$, then $\liminf_{t\to+\infty} x(t) \ge \frac{b}{a}$.

Definition 2.10. Let X be a Banach space and P be a closed nonempty subset of X; P is called a cone if (i) $x \in P$, $\lambda \ge 0$ implies $\lambda x \in P$ and (ii) $x \in P$, $-x \in P$ implies $x = \theta$.

Definition 2.11. A cone P of X is called a normal cone if there exists a positive constant ϵ such that $||x + y|| \ge \epsilon$ for any $x, y \in P$, $||x|| = ||y|| = 1$.

Lemma 2.5. Let C be a normal and solid cone in a real Banach space X and Φ: $C^0 \to C^0$ be a nondecreasing operator, where C^0 is the interior of C. Suppose further that there exists a function ϕ: $(0, 1) \times C^0 \to (0, +\infty)$ such that for each $\lambda \in (0, 1)$ and $x \in C^0$, $\phi(\lambda, x) > \lambda$, $\phi(\lambda, \cdot)$ is nondecreasing in C^0 and

$$\Phi(\lambda x) \ge \phi(\lambda, x)\Phi(x).$$

In addition, assume that there exists $z \in C^0$ such that $\Phi(z) \ge z$. Then, Φ has a unique fixed point x^* in C^0. Moreover, for any initial $x_0 \in C^0$, the iterative sequence

$$x_n = \Phi(x_{n-1}),\ n \in \mathbb{N},$$

satisfies

$$||x_n - x^*|| \to 0 \text{ as } n \to +\infty.$$

Consider the system

$$x^{\Delta} = f(t, x), \tag{2.2}$$

where $f(t, \phi)$ is continuous in $(\mathbb{R}, \mathbb{C})$, and almost periodic in t uniformly for $\phi \in \mathbb{C}_{H^*} \subseteq \mathbb{C}$; for any $\kappa > 0$, there exists $L(\kappa) > 0$ such that $|f(t, \phi)| \le L(\kappa)$. For the purpose of investigating the uniformly asymptotic stability of the almost periodic solution of System (1.2), the next conclusion is needed.

Lemma 2.6. Assume that there exists a Lyapunov function $V(t, x, y)$ defined on $\mathbb{T}^+\times\mathbb{S}_{H^*}\times\mathbb{S}_{H^*}$ satisfying the following conditions:

(i) $a_1(|x - y|) \le V(t, x, y) \le a_2(|x - y|)$, where a_i: $\mathbb{R}^+ \to \mathbb{R}^+$ are continuous and increasing and $a_i(0) = 0$ for $i = 1, 2$;

(ii) $|V(t, x_1, y_1) - V(t, x_2, y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|)$, where $L > 0$ is a constant;

(iii) $D^+V^{\Delta}(t, x, y) \le -cV(t, x, y)$, where $c > 0$ and $-c \in \Re^+$.

Moreover, if there exists a solution $x(t) \in \mathbb{S}$ of System (2.2) for $t \in T^+$, where $\mathbb{S} \subset \mathbb{S}_{H^*}$ is a compact set, then System (2.2) has a unique almost periodic solution in $\mathbb{S}$, which is uniformly asymptotically stable.

3. Existence, uniqueness and stability

Before establishing the main results, we list the following assumptions.

(H1) $p_i^- > \alpha_i^+$ and $-\alpha_i^- \in \Re^+$ for $i = 1, 2$.

(H2) $K_i(t,s) \le k_i e^{-m_i(t-s)}$ for $k_i, m_i \in \mathbb{R}^+$.

(H3) $\left(p_1^+ + \frac{h_1^+ k_1}{m_1}\right) e^{\frac{\beta^+}{\gamma^- \alpha_1^-} - 1} < \alpha_1^-$ and $\left(p_2^+ + \frac{h_2^+ k_2}{m_2}\right) < e\alpha_2^-$.

(H4) $\frac{\beta^+}{\gamma^-} + e^{-2} p_1^+ + \frac{2e^{-2} h_1^+ k_1}{m_1} < \alpha_1^-$ and $e^{-2} p_2^+ + \frac{2e^{-2} h_2^+ k_2}{m_2} < \alpha_2^-$.

From Lemma 2.4, it is not difficult to deduce the next result.

Lemma 3.1. Assume that (H1) and (H2) hold. Then, System (1.2) is permanent, that is, there exist constants $x^*, y^*, x_*, y_* \in \mathbb{R}^+$ that are independent of the solutions of System (1.2), satisfying

$$x_* \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le x^* \text{ and } y_* \le \liminf_{t\to+\infty} y(t) \le \limsup_{t\to+\infty} y(t) \le y^*$$

for any positive solution $(x(t), y(t))$ of (1.2).

In order to use the fixed-point theorem directly, we first transform System (1.2) into an equivalent equation because of its not nondecreasing nonlinear term.

Lemma 3.2. Assume that (H2) and (H3) hold. Then, System (1.2) is equivalent to the following integral equation in the sense of an almost periodic nonnegative solution:

$$\begin{cases} x(t) = \displaystyle\int_{-\infty}^{t} e_{-\alpha_1}(t, \sigma(s)) \Big[f_1(x(s)) + p_1(s) f_2(x(s)) \\ \qquad + h_1(s) \displaystyle\int_{-\infty}^{0} K_1(s,u) f_3(y(s+u)) \Delta u \Big] \Delta s, \\ y(t) = \displaystyle\int_{-\infty}^{t} e_{-\alpha_2}(t, \sigma(s)) \Big[p_2(s) g_1(y(s)) \\ \qquad + h_2(s) \displaystyle\int_{-\infty}^{0} K_2(s,u) g_2(x(s+u)) \Delta u \Big] \Delta s, \end{cases} \tag{3.1}$$

where

$$f_1(x) = \begin{cases} \dfrac{\beta(t) x}{\gamma(t) + x}, & 0 \le x \le \dfrac{1}{b}, \\ \dfrac{\beta(t)}{b\gamma(t) + 1}, & x > \dfrac{1}{b}, \end{cases} \qquad f_2(x) = g_2(x) = \begin{cases} x e^{-bx}, & 0 \le x \le \dfrac{1}{b}, \\ \dfrac{1}{be}, & x > \dfrac{1}{b}, \end{cases}$$

$$f_3(y) = g_1(y) = \begin{cases} y e^{-by}, & 0 \le y \le \dfrac{1}{b}, \\ \dfrac{1}{be}, & y > \dfrac{1}{b}. \end{cases}$$

Proof. Let $(\psi(t), \eta(t))$ be a nonnegative almost periodic solution of (1.2); then, from the almost periodicity of $\alpha_i^- > 0$ for $i = 1, 2$ and Lemmas 2.1 and 2.2, it is not difficult to obtain that

$$\begin{cases} \psi(t) = \displaystyle\int_{-\infty}^{t} e_{-\alpha_1}(t, \sigma(s)) \Bigg[\frac{\beta(s)\psi(s)}{\gamma(s) + \psi(s)} + p_1(s)\psi(s)e^{-b\psi(s)} \\ \qquad + h_1(s) \displaystyle\int_{-\infty}^{0} K_1(s, u)\eta(s + u)e^{-b\eta(s+u)}\Delta u \Bigg] \Delta s, \\ \eta(t) = \displaystyle\int_{-\infty}^{t} e_{-\alpha_2}(t, \sigma(s)) \Big[p_2(s)\eta(s)e^{-b\eta(s)} \\ \qquad + h_2(s) \displaystyle\int_{-\infty}^{0} K_2(s, u)\psi(s + u)e^{-b\psi(s+u)}\Delta u \Big] \Delta s. \end{cases}$$

By using the fact that

$$\sup_{t\in\mathbb{R}} f_1(t) = \frac{\beta(t)}{b\gamma(t) + 1}, \quad \sup_{t\in\mathbb{R}} f_2(t) = \sup_{t\in\mathbb{R}} f_3(t) = \sup_{t\in\mathbb{R}} g_1(t) = \sup_{t\in\mathbb{R}} g_2(t) = \frac{1}{be},$$

it follows from (H2) that

$$\begin{aligned} \psi(t) \leq & \int_{-\infty}^{t} e_{-\alpha_1^-}(t, \sigma(s)) \left[\frac{\beta^+\psi(s)}{\gamma^-} + \frac{p_1^+}{be} + \frac{h_1^+}{be} \int_{-\infty}^{0} K_1(s, u)\Delta u \right] \Delta s \\ \leq & \int_{-\infty}^{t} e_{-\alpha_1^-}(t, \sigma(s)) \left[\frac{\beta^+\psi(s)}{\gamma^-} + \frac{p_1^+}{be} + \frac{h_1^+ k_1}{be} \int_{-\infty}^{0} e^{-m_1(s-u)}\Delta u \right] \Delta s \\ \leq & \frac{\beta^+}{\gamma^-} \int_{-\infty}^{t} e_{-\alpha_1^-}(t, \sigma(s))\psi(s) ds + \frac{1}{be\alpha_1^-} \left(p_1^+ + \frac{h_1^+ k_1}{m_1} \right). \end{aligned}$$

Based on the well known Gronwall's inequality and (H3), it follows that

$$\psi(t) \leq \frac{1}{b\alpha_1^-} \left(p_1^+ + \frac{h_1^+ k_1}{m_1} \right) e^{\frac{\beta^+}{\gamma^-\alpha_1^-} - 1} < \frac{1}{b}.$$

Similarly, it deduces

$$\begin{aligned} \eta(t) \leq & \int_{-\infty}^{t} e_{-\alpha_2^-}(t, \sigma(s)) \left[\frac{p_2^+}{be} + \frac{h_2^+}{be} \int_{-\infty}^{0} K_2(s, u)\Delta u \right] \Delta s \\ \leq & \frac{1}{be\alpha_2^-} \left(p_2^+ + \frac{h_2^+ k_2}{m_2} \right) < \frac{1}{b}. \end{aligned}$$

Therefore, we have

$$f_1(\psi(s)) = \frac{\beta(s)\psi(s)}{\gamma(s) + \psi(s)}, \quad f_2(\psi(s)) = \psi(s)e^{-b\psi(s)}, \quad f_3(\eta(s + u)) = \eta(s + u)e^{-b\eta(s+u)},$$

$$g_1(\eta(s)) = \eta(s)e^{-b\eta(s)}, \quad g_2(\psi(s + u)) = \psi(s + u)e^{-b\psi(s+u)} \text{ for } t \in \mathbb{T}.$$

Further, it follows for $t \in \mathbb{T}$ that

$$\begin{cases} \psi(t) = \displaystyle\int_{-\infty}^{t} e_{-\alpha_1}(t,\sigma(s))\Big[f_1(\psi(s)) + p_1(s)f_2(\psi(s)) \\ \qquad +h_1(s)\displaystyle\int_{-\infty}^{0} K_1(s,u)f_3(\eta(s+u))du\Big]\Delta s, \\ \eta(t) = \displaystyle\int_{-\infty}^{t} e_{-\alpha_2}(t,\sigma(s))\Big[p_2(s)g_1(\eta(s-\tau(s))) \\ \qquad +h_2(s)\displaystyle\int_{-\infty}^{0} K_2(s,u)g_2(\psi(s+u))\Delta u\Big]\Delta s. \end{cases}$$

Therefore, it follows that (ψ, η) is an almost periodic solution of System (3.1). Analogously, for every nonnegative almost periodic solution (ψ, η) of System (3.1), it yields that (ψ, η) is an almost periodic solution of System (1.2).

Theorem 3.1. Assume that (H1)–(H3) hold. Then, System (1.2) exists as exactly one almost periodic solution (x^*, y^*) with a positive infimum. Moreover, for any almost periodic initial (x_0, y_0) with a positive infimum, the iterative sequence

$$\begin{cases} x_n(t) = \displaystyle\int_{-\infty}^{t} e_{-\alpha_1}(t,\sigma(s))\Big[\frac{\beta(s)x_{n-1}(s)}{\gamma(s)+x_{n-1}(s)} + p_1(s)x_{n-1}(s)e^{-bx_{n-1}(s)} \\ \qquad +h_1(s)\displaystyle\int_{-\infty}^{0} K_1(s,u)y_{n-1}(s+u)e^{-by_{n-1}(s+u)}\Delta u\Big]\Delta s, \\ y_n(t) = \displaystyle\int_{-\infty}^{t} e_{-\alpha_2}(t,\sigma(s))\Big[p_2(s)y_{n-1}(s)e^{-by_{n-1}(s)} \\ \qquad +h_2(s)\displaystyle\int_{-\infty}^{0} K_2(s,u)x_{n-1}(s+u)e^{-bx_{n-1}(s+u)}\Delta u\Big]\Delta s, \ n = 1,2,\cdots \end{cases}$$

satisfies $\|(x_n, y_n) - (x^*, y^*)\| \to 0$ as $n \to +\infty$.

Proof. Define

$$\begin{aligned} C &= \{(x,y) \in AP(\mathbb{T},\mathbb{R}) : x(t) \geq 0, y(t) \geq 0, \forall\, t \in \mathbb{T}\}, \\ C^0 &= \{(x,y) \in AP(\mathbb{T},\mathbb{R}) : \exists\, \varepsilon > 0 \text{ such that } x(t) > \varepsilon, y(t) > \varepsilon, \forall\, t \in \mathbb{T}\}, \end{aligned}$$

it is obvious that C is a normal and solid cone in Banach space $AP(\mathbb{T},\mathbb{R})$, and that C^0 is the interior of C. From Lemma 3.2, define the operator (Ψ_1, Ψ_2) on $C^0 \times C^0$ as follows:

$$\begin{cases} \Psi_1(x,y)(t) = \displaystyle\int_{-\infty}^{t} e_{-\alpha_1}(t,\sigma(s))\Big[f_1(x(s)) + p_1(s)f_2(x(s)) \\ \qquad +h_1(s)\displaystyle\int_{-\infty}^{0} K_1(s,u)f_3(y(s+u))\Delta u\Big]\Delta s, \\ \Psi_2(x,y)(t) = \displaystyle\int_{-\infty}^{t} e_{-\alpha_2}(t,\sigma(s))\Big[p_2(s)g_1(y(s)) \\ \qquad +h_2(s)\displaystyle\int_{-\infty}^{0} K_2(s,u)g_2(x(s+u))\Delta u\Big]\Delta s. \end{cases}$$

Next, we will complete the proof in three steps.

Step 1. Ψ_i: $C^0 \times C^0 \to C^0 \times C^0$ is set as a nondecreasing operator.

Due to the fact that f_i and g_j are both nondecreasing on $(0, +\infty)$ for $i = 1, 2, 3,\ j = 1, 2$, it follows that Ψ_i is also nondecreasing. In view of the bounded properties of f_i and g_j, it is not difficult to derive that f_i and g_j satisfies the Lipschitz condition. Based on the composition theorem and the invariance of convolution for the almost periodic functions, it is not difficult to deduce that Ψ_i is a self-map operator on $AP(\mathbb{T}, \mathbb{R})$. Moreover, we have

$$\begin{cases} \Psi_1(x, y)(t) \geq p_1^- \displaystyle\int_{-\infty}^{t} e_{-\alpha_1^+}(t, \sigma(s)) \min_{s\in\mathbb{R}} f_2(x(s))\Delta s > 0, \\ \Psi_2(x, y)(t) \geq p_2^- \displaystyle\int_{-\infty}^{t} e_{-\alpha_2^+}(t, \sigma(s)) \min_{s\in\mathbb{R}} g_1(y(s))\Delta s > 0, \end{cases}$$

which yields that there exists a suitable $\varepsilon_1 > 0$ such that $\Psi_i(x, y)(t) > \varepsilon_1$ for any $t \in \mathbb{R}$ and $i = 1, 2$.

Step 2. There exists a function ψ_i: $(0, 1) \times C^0 \times C^0 \to (0, +\infty)$ such that for each $\lambda \in (0, 1)$ and $x, y \in C^0 \times C^0$, $\psi_i(\lambda, x, y) > \lambda$, $\psi_i(\lambda, x, y)$ is nondecreasing in $C^0 \times C^0$ and

$$\Psi_i(\lambda x, \lambda y) \geq \psi_i(\lambda, x, y)\Psi_i(x, y) \text{ where } i = 1, 2.$$

Let

$$\phi_1(\lambda, x) = \begin{cases} \lambda, & 0 \leq x \leq \dfrac{1}{b\lambda}, \\ 1, & x > \dfrac{1}{b\lambda}, \end{cases}$$

$$\phi_2(\lambda, x) = \varphi_2(\lambda, x) = \begin{cases} \lambda e^{b(1-\lambda)x}, & 0 \leq x \leq \dfrac{1}{b}, \\ b\lambda x e^{1-b\lambda x}, & \dfrac{1}{b} < x \leq \dfrac{1}{b\lambda}, \\ 1, & x > \dfrac{1}{b\lambda}, \end{cases}$$

$$\phi_3(\lambda, y) = \varphi_1(\lambda, y) = \begin{cases} \lambda e^{b(1-\lambda)y}, & 0 \leq y \leq \dfrac{1}{b}, \\ b\lambda y e^{1-b\lambda y}, & \dfrac{1}{b} < y \leq \dfrac{1}{b\lambda}, \\ 1, & y > \dfrac{1}{b\lambda}; \end{cases}$$

therefore, for each $\lambda \in (0, 1)$ and $(x, y) \in (C^0 \times C^0)$, there exist functions ϕ_i, φ_j: $(0, 1) \times C^0 \to (0, +\infty)$ for $i = 1, 2, 3,\ j = 1, 2$ such that

(1) $\phi_i(\lambda, \cdot) > \lambda$ and $\varphi_j(\lambda, \cdot) > \lambda$.

(2) $\phi_i(\lambda, \cdot)$, $\varphi_j(\lambda, \cdot)$ is nondecreasing in C^0.

(3) $f_i(\lambda, \cdot) \geq \phi_i(\lambda, \cdot) f_i(\cdot)$ and $g_j(\lambda, \cdot) \geq \varphi_i(\lambda, \cdot) g_j(\cdot)$.

Let

$$\psi_1(\lambda, x, y) = \min\{\phi_1(\lambda, x), \phi_2(\lambda, x), \phi_3(\lambda, y)\}$$

and

$$\psi_2(\lambda, x, y) = \min\{\varphi_1(\lambda, y), \varphi_2(\lambda, x)\};$$

then, it follows that $\psi_i(\lambda, x, y) > \lambda$ and nondecreasing in $C^0 \times C^0$ for $i = 1, 2$. Moreover, it follows that

$$\begin{aligned}
\Psi_1(\lambda x, \lambda y)(t) = &\int_{-\infty}^{t} e_{-\alpha_1}(t, \sigma(s)) \left[f_1(\lambda x(s)) + p_1(s) f_2(\lambda x(s)) \right. \\
&\left. + h_1(s) \int_{-\tau}^{0} K_1(s, u) f_3(\lambda y(s + u)) \Delta u \right] \Delta s \\
\geq &\int_{-\infty}^{t} e_{-\alpha_1}(t, \sigma(s)) \left[\phi_1(\lambda, x) f_1(x(s)) + p_1(s) \phi_2(\lambda, x) f_2(x(s)) \right. \\
&\left. + h_1(s) \int_{-\tau}^{0} K_1(s, u) \phi_3(\lambda, y) f_3(y(s + u)) \Delta u \right] \Delta s \\
\geq &\psi_1(\lambda, x, y) \Psi_1(x, y)(t).
\end{aligned}$$

Similarly, one obtains that $\Psi_2(\lambda x, \lambda y)(t) \geq \psi_2(\lambda, x, y)\Psi_2(x, y)(t)$.

Step 3. There exists $(z, z) \in C^0 \times C^0$ such that $(\Psi_1(z, z), \Psi_2(z, z)) \geq (z, z)$.

Choose an appropriate $\varepsilon \in \left(0, \frac{1}{b}\right)$; from (H1), it deduces that

$$\begin{cases}
\Psi_1(\varepsilon, \varepsilon)(t) \geq p_1^- \displaystyle\int_{-\infty}^{t} e_{-\alpha_1^+}(t, \sigma(s)) f_2(\varepsilon(s)) \Delta s \geq \dfrac{p_1^- \varepsilon e^{-b\varepsilon}}{\alpha_1^+} \geq \varepsilon, \\
\Psi_2(\varepsilon, \varepsilon)(t) \geq p_2^- \displaystyle\int_{-\infty}^{t} e_{-\alpha_2^+}(t, \sigma(s)) g_1(\varepsilon(s)) \Delta s \geq \dfrac{p_2^- \varepsilon e^{-b\varepsilon}}{\alpha_2^+} \geq \varepsilon.
\end{cases}$$

Based on the above discussion and Lemma 2.5, it follows that (Ψ_1, Ψ_2) has a unique fixed point $(x^*, y^*) \in C^0 \times C^0$. Moreover, for any initial $(x_0, y_0) \in C^0 \times C^0$, the iterative sequence

$$(x_n, y_n) = \big(\Psi_1(x_{n-1}, y_{n-1}), \Psi_2(x_{n-1}, y_{n-1})\big), \ n \in \mathbb{N},$$

satisfies $\|(x_n, y_n) - (x^*, y^*)\| \to 0$ as $n \to +\infty$.

Remark 3.1. Based on the conditions (H1)–(H3), Theorem 3.1 implies the existence and uniqueness of the positive almost periodic solution (x^*, y^*) of System (1.2). If we attach another condition (H4), is the unique solution (x^*, y^*) globally stable? Even more generally, is it uniformly asymptotically stable? The next conclusions will give the answer.

Theorem 3.2. Assume that (H1)–(H4) hold. Then the solution $(x(t; t_0, \xi_1), y(t; t_0, \xi_2))$ of System (1.2) converges exponentially to the positive almost periodic solution (x^*, y^*) as $t \to +\infty$.

Proof. Let $x(t) = x(t; t_0, \xi_1)$, $y(t) = y(t; t_0, \xi_2)$ and $z_1(t) = x(t) - x^*(t)$, $z_2(t) = y(t) - y^*(t)$; then, one has

$$\begin{cases}
z_1^{\Delta}(t) = \beta(t) \left[\dfrac{x(t)}{\gamma(t) + x(t)} - \dfrac{x^*(t)}{\gamma(t) + x^*(t)} \right] + p_1(t) \left[x(t) e^{-bx(t)} - x^*(t) e^{-bx^*(t)} \right] \\
\qquad + h_1(t) \displaystyle\int_{-\infty}^{0} K_1(t, s) \left[y(t + s) e^{-by(t+s)} - y^*(t + s) e^{-by^*(t+s)} \right] \Delta s \\
\qquad - \alpha_1(t) z_1(t), \\
z_2^{\Delta}(t) = h_2(t) \displaystyle\int_{-\infty}^{0} K_2(t, s) \left[y(t + s) e^{-by(t+s)} - y^*(t + s) e^{-by^*(t+s)} \right] \Delta s \\
\qquad - \alpha_2(t) z_2(t) + p_2(t) \left[x(t) e^{-bx(t)} - x^*(t) e^{-bx^*(t)} \right].
\end{cases}$$

Let $\zeta_i \in [0,1]$ for $i = 1, 2$ and

$$\begin{cases} \Phi_1(\zeta_1) = -\alpha_1^- + \zeta_1(1+\mu^+\alpha_1^+) + \dfrac{\beta^+}{\gamma^-}(1+\zeta_1\mu^+) + (1+\zeta_1\mu^+)e^{-2}p_1^+ \\ \qquad\qquad + \dfrac{2(1+\zeta_1\mu^+)e^{\zeta_1-2}h_1^+k_1}{m_1}, \\ \Phi_2(\zeta_2) = -\alpha_2^- + \zeta_2(1+\mu^+\alpha_2^+) + (1+\zeta_2\mu^+)e^{-2}p_2^+ + \dfrac{2(1+\zeta_2\mu^+)e^{\zeta_2-2}h_2^+k_2}{m_2}; \end{cases}$$

in view of (H4), one obtains

$$\begin{cases} \Phi_1(0) = -\alpha_1^- + \dfrac{\beta^+}{\gamma^-} + e^{-2}p_1^+ + \dfrac{2e^{-2}h_1^+k_1}{m_1} < 0, \\ \Phi_2(0) = -\alpha_2^- + e^{-2}p_2^+ + \dfrac{2e^{-2}h_2^+k_2}{m_2} < 0, \end{cases}$$

which yields that there exist constants $\lambda_i \in (0,1]$, $i = 1, 2$, satisfying

$$\Phi_i(\lambda_i) < 0. \tag{3.2}$$

Choose the appropriate Lyapunov functional

$$V(t, z_1(t), z_2(t)) = |z_1(t)(t)|e^{\lambda_1 t} + |z_2(t)(t)|e^{\lambda_2 t}, \text{ where } \lambda_1, \lambda_2 \in (0,1], \tag{3.3}$$

and calculate its upper right derivative D^+V^Δ along the solution of Eq (3.3); it follows for $t > t_0$ that

$$\begin{aligned} & D^+V^\Delta(t, z_1(t), z_2(t)) \\ \le & \operatorname{sgn}(z_1(t))z_1^\Delta(t)e^{\lambda_1 t} + \lambda_1|z_1(t)|e^{\lambda_1 t} + \operatorname{sgn}(z_2(t))z_2^\Delta(t)e^{\lambda_2 t} + \lambda_2|z_2(t)|e^{\lambda_2 t} \\ \le & \sum_{i=1}^{7} V_i(t, z_1(t), z_2(t)), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} V_1 =& e^{\lambda_1 t}[-\alpha_1^- + \lambda_1(1+\mu^+\alpha_1^+)]|z_1(t)|, \\ V_2 =& e^{\lambda_2 t}[-\alpha_2^- + \lambda_2(1+\mu^+\alpha_2^+)]|z_2(t)|, \\ V_3 =& e^{\lambda_1 t}(1+\lambda_1\mu^+)\beta(t)\left|\frac{x(t)}{\gamma(t)+x(t)} - \frac{x^*(t)}{\gamma(t)+x^*(t)}\right|, \\ V_4 =& e^{\lambda_1 t}(1+\lambda_1\mu^+)p_1(t)\left|x(t)e^{-bx(t)} - x^*(t)e^{-bx^*(t)}\right|, \\ V_5 =& e^{\lambda_1 t}(1+\lambda_1\mu^+)h_1(t)\int_{-\infty}^{0} K_1(t,s)\left|y(t+s)e^{-by(t+s)} - y^*(t+s)e^{-by^*(t+s)}\right|\Delta s, \\ V_6 =& e^{\lambda_2 t}(1+\lambda_2\mu^+)p_2(t)\left|y(t)e^{-by(t)} - y^*(t)e^{-by^*(t)}\right|, \\ V_7 =& e^{\lambda_2 t}(1+\lambda_2\mu^+)h_2(t)\int_{-\infty}^{0} K_2(t,s)\left|x(t+s)e^{-bx(t+s)} - x^*(t+s)e^{-bx^*(t+s)}\right|\Delta s. \end{aligned}$$

It follows that there exists a constant M that satisfies $V(t, z_1(t), z_2(t)) < M$ for $t > t_0$; otherwise, there exists $t_* > t_0$ such that

$$V(t_*, z_1(t), z_2(t)) - M = 0 \text{ and } V(t, z_1(t), z_2(t)) - M < 0 \text{ for } -r \le t < t_*. \tag{3.5}$$

Consider (3.5) with the inequalities

$$\left|\vartheta e^{-a_1\vartheta} - \varrho e^{-a_1\varrho}\right| \le e^{-2}\left|\vartheta - \varrho\right| \text{ and } \left|\frac{\vartheta}{a_2+\vartheta} - \frac{\varrho}{a_2+\varrho}\right| \le \frac{1}{a_2}\left|\vartheta - \varrho\right|, \tag{3.6}$$

for $\vartheta, \varrho \in C^0$ and $a_1, a_2 \in \mathbb{R}^+$; it follows that

$$V_3(t^*, z_1(t^*), z_2(t^*)) \le \frac{\beta(t_*)}{\gamma(t_*)}(1+\lambda_1\mu^+)|z_1(t_*)|e^{\lambda t_*} < \frac{\beta^+}{\gamma^-}(1+\lambda_1\mu^+)M, \tag{3.7}$$

$$\begin{aligned} V_4(t^*, z_1(t^*), z_2(t^*)) \le &(1+\lambda_1\mu^+)p_1(t_*)e^{-2}|z_1(t_*)|e^{\lambda_1(t_*)} \\ < &\frac{(1+\lambda_1\mu^+)p_1^+M}{e^2}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} V_6(t^*, z_1(t^*), z_2(t^*)) \le &(1+\lambda_2\mu^+)p_2(t_*)e^{-2}|z_2(t_*)|e^{\lambda_2(t_*)} \\ < &\frac{(1+\lambda_2\mu^+)p_2^+M}{e^2}. \end{aligned} \tag{3.9}$$

In addition, consider

$$\begin{aligned} &\int_{-\infty}^{0} K_1(t,s)\left|y(t+s)e^{-by(t+s)} - y^*(t+s)e^{-by^*(t+s)}\right|\Delta s \\ \le &e^{-2}\int_{-\infty}^{0} K_1(t,s)\left|\int_t^{t+s}\left(y^\Delta(m) - y^{*\Delta}(m)\right)\Delta m + y(t) - y^*(t)\right|\Delta s \\ \le &\frac{2e^{-2}k_1}{m_1}|y(t) - y^*(t)|; \end{aligned}$$

one further deduces that

$$V_5(t^*, z_1(t^*), z_2(t^*)) \le \frac{2(1+\lambda_1\mu^+)e^{\lambda_1-2}h_1^+k_1M}{m_1}, \tag{3.10}$$

$$V_7(t^*, z_1(t^*), z_2(t^*)) \le \frac{2(1+\lambda_2\mu^+)e^{\lambda_2-2}h_2^+k_2M}{m_2}. \tag{3.11}$$

Substituting (3.7)–(3.11) into (3.4), then

$$\begin{aligned} 0 \le &D^+(V(t_*, z_1(t), z_2(t)) - M) \\ \le &\left\{[-\alpha_1^- + \lambda_1(1+\mu^+\alpha_1^+)] + [-\alpha_2^- + \lambda_2(1+\mu^+\alpha_2^+)] + \frac{\beta^+}{\gamma^-}(1+\lambda_1\mu^+)\right. \\ &\left. + \frac{(1+\lambda_1\mu^+)}{e^2}\left(p_1^+ + \frac{2e^{\lambda_1}h_1^+k_1}{m_1}\right) + \frac{(1+\lambda_2\mu^+)}{e^2}\left(p_2^+ + \frac{2e^{\lambda_2}h_2^+k_2}{m_2}\right)\right\}M, \end{aligned}$$

which contradicts (3.2). Therefore, $V(t, z_1(t), z_2(t)) < M$ for $t > t_0$; choose $\lambda = \min_{t\in\mathbb{T}}\{\lambda_1, \lambda_2\}$, that is

$$|(z_1(t), z_2(t))| < Me^{-\lambda t} \text{ for } t > t_0.$$

Theorem 3.3. Assume that (H1)–(H3) hold. Then there exists a unique uniformly asymptotically stable positive almost periodic solution of System (1.2) provided that $\Theta = \min\{\Theta_1, \Theta_2\} > 0$ and $-\Theta \in \Re^+$, where

$$\begin{aligned}
\Theta_1 =&\alpha_1^-\left[(2-\mu^-\alpha_1^-) + \frac{(\mu^+ + \mu^-)\beta^+}{\gamma^-} + \frac{(\mu^+ + \mu^-)}{e^2}\left(p_1^+ + \frac{k_1 h_1^+}{m_1}\right)\right] \\
&- \left(2 + \frac{\mu^+\beta^+}{\gamma^-} + \frac{\mu^+ p_1^+}{e^2}\right)\left(\frac{\beta^+}{\gamma^-} + \frac{p_1^+}{e^2} + \frac{k_1 h_1^+}{m_1 e^2}\right) - \frac{k_1\mu^+ h_1^+}{m_1 e^2}\left(\frac{\beta^+}{\gamma^-} + \frac{p_1^+}{e^2}\right), \\
&- \left[2 + \frac{\mu^+}{e^2}\left(2p_2^+ + \frac{4k_2 h_2^+}{m_2}\right) - (\mu^- + \mu^+)\alpha_2^-\right]\frac{k_2 h_2^+}{m_2 e^2}, \\
\Theta_2 =&\alpha_2^-\left[(2-\mu^-\alpha_2^-) + \frac{(\mu^+ + \mu^-)}{e^2}\left(p_2^+ + \frac{k_2 h_2^+}{m_2}\right)\right] \\
&- \frac{1}{e^2}\left(2 + \frac{\mu^+ p_2^+}{e^2}\right)\left(p_2^+ + \frac{k_2 h_2^+}{m_2}\right) - \frac{k_2\mu^+ h_2^+ p_2^+}{m_2 e^4}, \\
&- \left[2 + \mu^+\left(\frac{2\beta^+}{\gamma^-} + \frac{2p_1^+}{e^2} + \frac{4k_1 h_1^+}{m_1 e^2}\right) - (\mu^- + \mu^+)\alpha_1^-\right]\frac{k_1 h_1^+}{m_1 e^2}.
\end{aligned}$$

Proof. Consider the Lyapunov function defined on $\mathbb{T}\times\mathbb{C}\times\mathbb{C}$ by

$$V(t, X(t), Y(t)) = [x(t) - x_1(t)]^2 + [y(t) - y_1(t)]^2, \tag{3.12}$$

where $X(t) = (x(t), y(t))^T$ and $Y(t) = (x_1(t), y_1(t))^T$ are the almost periodic solutions of (1.2). Based on Theorem 4.2 in [15], it follows that the condition (i) in Lemma 2.6 holds. By using the fact that

$$(s_1 - s_2)^2 - (s_3 - s_4)^2 \le 4\max\{|s_1|, |s_2|, |s_3|, |s_4|\}\left(|s_1 - s_3| + |s_2 - s_4|\right) \text{ for } s_i \in \mathbb{R},$$

it follows for $X'(t) = (x'(t), y'(t))^T$ and $Y'(t) = (x_1'(t), y_1'(t))^T$ that

$$|V(t, X(t), Y(t)) - V(t, X'(t), Y'(t))| \le M^*\left(|X(t) - X'(t)| + |Y(t) - Y'(t)|\right),$$

with $M^* = 4\max\{|x^*|, |x_*|, |y^*|, |y_*|\}$. Further, the condition (ii) in Lemma 2.6 holds.

Calculating the right derivative D^+V^Δ of V along the solution of (3.12) yields

$$\begin{aligned}
&D^+V^\Delta(t, X(t), Y(t)) \\
=&\left[2(x(t) - x_1(t)) + \mu(t)(x(t) - x_1(t))^\Delta\right](x(t) - x_1(t))^\Delta \\
&+\left[2(y(t) - y_1(t)) + \mu(t)(y(t) - y_1(t))^\Delta\right](y(t) - y_1(t))^\Delta \\
=&:\Pi_1(t) + \Pi_2(t),
\end{aligned} \tag{3.13}$$

where

$$\begin{cases}
(x - x_1)^\Delta(t) =\beta(t)\left[\dfrac{x(t)}{\gamma(t) + x(t)} - \dfrac{x_1(t)}{\gamma(t) + x_1(t)}\right] \\
\qquad + p_1(t)\left[x(t)e^{-bx(t)} - x_1(t)e^{-bx_1(t)}\right] - \alpha_1(t)[x(t) - x_1(t)] \\
\qquad + h_1(t)\displaystyle\int_{-\infty}^0 K_1(t,s)\left[y(t+s)e^{-by(t+s)} - y_1(t+s)e^{-by_1(t+s)}\right]\Delta s, \\
(y - y_1)^\Delta(t) = -\alpha_2(t)[y(t) - y_1(t)] + p_2(t)\left[y(t)e^{-by(t)} - y_1(t)e^{-by_1(t)}\right] \\
\qquad + h_2(t)\displaystyle\int_{-\infty}^0 K_2(t,s)\left[x(t+s)e^{-bx(t+s)} - x_1(t+s)e^{-bx_1(t+s)}\right]\Delta s.
\end{cases}$$

Based on the inequalities of (3.6), and for any $z \in \mathbb{C}$ satisfying

$$\int_{-\infty}^{0} K_1(t,s)\left[z(t+s)e^{-bz(t+s)} - z_1(t+s)e^{-bz_1(t+s)}\right]\Delta s \le \frac{2k_1}{m_1e^2}[z(t)-z_1(t)],$$

one can deduce

$$\begin{aligned}
\Pi_1(t) \le& \left\{\left(2-\mu^-\alpha_1^- + \frac{\mu^+\beta^+}{\gamma^-} + \frac{\mu^+p_1^+}{e^2}\right)[x(t)-x_1(t)] + \frac{2k_1\mu^+h_1^+}{m_1e^2}[y(t)-y_1(t)]\right\}\cdot \\
&\left\{\left(-\alpha_1^- + \frac{\beta^+}{\gamma^-} + \frac{p_1^+}{e^2}\right)[x(t)-x_1(t)] + \frac{2k_1h_1^+}{m_1e^2}[y(t)-y_1(t)]\right\} \\
\le&\left(2-\mu^-\alpha_1^- + \frac{\mu^+\beta^+}{\gamma^-} + \frac{\mu^+p_1^+}{e^2}\right)\left(-\alpha_1^- + \frac{\beta^+}{\gamma^-} + \frac{p_1^+}{e^2}\right)[x(t)-x_1(t)]^2 \\
&+\left[2-(\mu^-+\mu^+)\alpha_1^- + \frac{2\mu^+\beta^+}{\gamma^-} + \frac{2\mu^+p_1^+}{e^2}\right]\frac{2k_1h_1^+}{m_1e^2}[x(t)-x_1(t)][y(t)-y_1(t)] \\
&+\mu^+\left(\frac{2k_1h_1^+}{m_1e^2}\right)^2[y(t)-y_1(t)]^2 \\
\le&(A_1-B_1)[x(t)-x_1(t)]^2 + (A_2-B_2)[y(t)-y_1(t)]^2,
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
A_1 =&\left(2+\frac{\mu^+\beta^+}{\gamma^-} + \frac{\mu^+p_1^+}{e^2}\right)\left(\frac{\beta^+}{\gamma^-} + \frac{p_1^+}{e^2} + \frac{k_1h_1^+}{m_1e^2}\right) \\
&+\frac{k_1\mu^+h_1^+}{m_1e^2}\left(\frac{\beta^+}{\gamma^-} + \frac{p_1^+}{e^2}\right) + \mu^-\left(\alpha_1^-\right)^2, \\
B_1 =&\alpha_1^-\left[2+\frac{\mu^+\beta^+}{\gamma^-} + \frac{\mu^+}{e^2}\left(p_1^+ + \frac{k_1h_1^+}{m_1}\right) + \mu^-\left(\frac{\beta^+}{\gamma^-} + \frac{p_1^+}{e^2} + \frac{k_1h_1^+}{m_1e^2}\right)\right], \\
A_2 =&\left[2+\mu^+\left(\frac{2\beta^+}{\gamma^-} + \frac{2p_1^+}{e^2} + \frac{4k_1h_1^+}{m_1e^2}\right)\right]\frac{k_1h_1^+}{m_1e^2}, \\
B_2 =&\frac{(\mu^-+\mu^+)\alpha_1^-k_1h_1^+}{m_1e^2},
\end{aligned}$$

and

$$\begin{aligned}
\Pi_2(t) \le& \left\{\left(2-\mu^-\alpha_2^- + \frac{\mu^+p_2^+}{e^2}\right)[y(t)-y_1(t)] + \frac{2k_2\mu^+h_2^+}{m_2e^2}[x(t)-x_1(t)]\right\}\cdot \\
&\left\{\left(-\alpha_2^- + \frac{p_2^+}{e^2}\right)[y(t)-y_1(t)] + \frac{2k_2h_2^+}{m_2e^2}[x(t)-x_1(t)]\right\} \\
\le&(A_3-B_3)[x(t)-x_1(t)]^2 + (A_4-B_4)[y(t)-y_1(t)]^2,
\end{aligned} \tag{3.15}$$

where

$$A_3 = \left[2+\frac{\mu^+}{e^2}\left(2p_2^+ + \frac{4k_2h_2^+}{m_2}\right)\right]\frac{k_2h_2^+}{m_2e^2},$$

$$\begin{aligned}
B_3 =& \frac{(\mu^- + \mu^+)\alpha_2^- k_2 h_2^+}{m_2 e^2},\\
A_4 =& \frac{1}{e^2}\left(2 + \frac{\mu^+ p_2^+}{e^2}\right)\left(p_2^+ + \frac{k_2 h_2^+}{m_2}\right) + \frac{k_2 \mu^+ h_2^+ p_2^+}{m_2 e^4} + \mu^-(\alpha_2^-)^2,\\
B_4 =& \alpha_2^-\left[2 + \frac{(\mu^+ + \mu^-)}{e^2}\left(p_2^+ + \frac{k_2 h_2^+}{m_2}\right)\right].
\end{aligned}$$

Substituting (3.14) and (3.15) into (3.13), it follows that

$$D^+ V^\Delta(t, X(t), Y(t)) \le -\Theta V(t, X(t), Y(t)),$$

where

$$\Theta = \min\{(B_1 + B_3) - (A_1 + A_3), (B_2 + B_4) - (A_2 + A_4)\} > 0.$$

Combine Lemma 2.6 with Theorem 3.1; it follows that the unique positive almost periodic solution of System (1.2) is uniformly asymptotically stable.

4. Examples

In this section, we introduce some suitable examples to support the main results.

Example 4.1. Let us illustrate that System (1.2) exists as exactly one almost periodic solution with a positive infimum. Assume that

$$\begin{aligned}
\alpha_1(t) &= 0.79 + 0.001\sin(\sqrt{5}t), & \alpha_2(t) &= 0.84 + 0.002\sin(\sqrt{3}t),\\
\beta(t) &= 0.592 - \sin^2(\pi t) - \sin^2 t, & \gamma(t) &= 1.001 - 0.001\sin(\sqrt{3}t),\\
p_1(t) &= 0.8 + 0.005\cos(\sqrt{3}t), & p_2(t) &= 0.85 + 0.005\cos(\sqrt{2}t),\\
h_1(t) &= 0.1 + 0.04\sin(\sqrt{2}t), & h_2(t) &= 0.2 + 0.06\sin(\sqrt{6}t),\\
K_1(t,s) &= K_2(t,s) = e^{-2(t-s)};
\end{aligned}$$

then,

$$\begin{aligned}
&\alpha_1^+ = 0.791, && \alpha_1^- = 0.789, && \alpha_2^+ = 0.842, && \alpha_2^- = 0.838,\\
&\beta^+ = 0.592, && \gamma^- = 1, && p_1^+ = 0.805, && p_1^- = 0.795,\\
&p_2^+ = 0.855, && p_2^- = 0.845, && h_1^+ = 0.14, && h_2^+ = 0.26,\\
&k_1 = k_2 = 1, && m_1 = m_2 = 2.
\end{aligned}$$

Obviously, $p_1^- > \alpha_1^+$ and $p_2^- > \alpha_2^+$; further, one chooses $e = 2.718$ and calculates that

$$\left(p_1^+ + \frac{h_1^+ k_1}{m_1}\right)e^{\frac{\beta^+}{\gamma^- \alpha_1^-} - 1} \approx 0.681 < 0.789 = \alpha_1^-, \qquad \left(p_2^+ + \frac{h_2^+ k_2}{m_2}\right) = 0.985 < 2.278 = e\alpha_2^-,$$

which indicates that $(H1)$–$(H3)$ hold. Therefore, according to Theorem 3.1, it follows that System (1.2) exists with exactly one almost periodic solution (x^*, y^*) with a positive infimum.

Example 4.2. Let us illustrate the stability of the positive almost periodic solution of System (1.2). Assume that the conditions in Example 4.1 hold; then,

$$\frac{\beta^+}{\gamma^-}+e^{-2}p_1^+ + \frac{2e^{-2}h_1^+k_1}{m_1}\approx 0.72<0.789=\alpha_1^-,\qquad e^{-2}p_2^+ + \frac{2e^{-2}h_2^+k_2}{m_2}\approx 0.151<0.838=\alpha_2^-,$$

that is, $(H4)$ is satisfied. Therefore, from Theorem 3.2, it obtains that the unique almost periodic solution is exponentially stable.

Let $\mu^+=\mu^-=1$; then,

$$\begin{aligned}\Theta_1 =&\alpha_1^-\left[(2-\mu^-\alpha_1^-)+\frac{(\mu^++\mu^-)\beta^+}{\gamma^-}+\frac{(\mu^++\mu^-)}{e^2}\left(p_1^+ + \frac{k_1h_1^+}{m_1}\right)\right]-\frac{k_1\mu^+h_1^+}{m_1e^2}\left(\frac{\beta^+}{\gamma^-}+\frac{p_1^+}{e^2}\right)\\ &-\left(2+\frac{\mu^+\beta^+}{\gamma^-}+\frac{\mu^+p_1^+}{e^2}\right)\left(\frac{\beta^+}{\gamma^-}+\frac{p_1^+}{e^2}+\frac{k_1h_1^+}{m_1e^2}\right)-\left[2+\frac{\mu^+}{e^2}\left(2p_2^+ + \frac{4k_2h_2^+}{m_2}\right)-(\mu^-+\mu^+)\alpha_2^-\right]\frac{k_2h_2^+}{m_2e^2}\\ \approx&0.141>0,\end{aligned}$$

and

$$\begin{aligned}\Theta_2 =&\alpha_2^-\left[(2-\mu^-\alpha_2^-)+\frac{(\mu^++\mu^-)}{e^2}\left(p_2^+ + \frac{k_2h_2^+}{m_2}\right)\right]-\frac{1}{e^2}\left(2+\frac{\mu^+p_2^+}{e^2}\right)\left(p_2^+ + \frac{k_2h_2^+}{m_2}\right)-\frac{k_2\mu^+h_2^+p_2^+}{m_2e^4}\\ &-\left[2+\mu^+\left(\frac{2\beta^+}{\gamma^-}+\frac{2p_1^+}{e^2}+\frac{4k_1h_1^+}{m_1e^2}\right)-(\mu^-+\mu^+)\alpha_1^-\right]\frac{k_1h_1^+}{m_1e^2}\\ \approx&0.896>0.\end{aligned}$$

Moreover, $\Theta=\min\{0.141,0.896\}=0.141>0$ and $-\Theta=-0.141\in\Re^+$. From Example 4.1, it follows that $(H1)$–$(H3)$ hold; hence, Theorem 3.3 implies that System (1.2) has a unique uniformly asymptotically stable positive almost periodic solution.

5. Conclusions

In this paper, we introduced a class of two species co-existence delayed model with the almost periodic coefficients on time scales defined as System (1.2). Based on the operator theory, Lyapunov function and Gronwall's inequality, by choosing an appropriate Lyapunov function, this paper addresses some dynamic properties of almost periodic solutions of this model. First, we presented System (1.2) is permanent and further established the existence and uniqueness of the almost periodic solution with a positive infimum by transforming the initial model into an equivalent integral equation. Second, we investigated the global exponential stability and uniformly asymptotic stability of the positive almost periodic solution. In some existing works, for example, the work in [4,8], for a class of continuous system, which is a particular case of systems on time, the authors only studied the existence and uniqueness of the positive almost periodic solution, but did not further verify whether the obtained solution is stable or not. In addition, in [17], although the authors explored a class of high-order neural networks model with variable delays on time scales and showed some sufficient conditions to prove the existence and uniqueness of the almost periodic solution, the results were obtained based on the Lipschitz condition. Compared to these existing works on almost periodic solutions, our conclusions are valuable in the exploration of dynamic equations on time scales.

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Conflict of interest

The author declares no conflict of interest regarding the publication of this paper.

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