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Research article

Some novel estimates of Jensen and Hermite-Hadamard inequalities for h-Godunova-Levin stochastic processes

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Abstract: It is undeniable that convex and non-convex functions play an important role in optimization. As a result of its behavior, convexity also plays a significant role in discussing inequalities. It is clear that convexity and stochastic processes are intertwined. The stochastic process is a mathematical model that describes how systems or phenomena fluctuate randomly. Probability theory generally says that the convex function applied to the expected value of a random variable is bounded above by the expected value of the random variable's convex function. Furthermore, the deep connection between convex inequalities and stochastic processes offers a whole new perspective on the study of inequality. Although Godunova-Levin functions are well known in convex theory, their properties enable us to determine inequality terms with greater accuracy than those obtained from convex functions. In this paper, we established a more refined form of Hermite-Hadamard and Jensen type inequalities for generalized interval-valued h-Godunova-Levin stochastic processes. In addition, we provide some examples to demonstrate the validity of our main findings.

Keywords: Jensen inequality; Hermite-Hadamard inequality; Godunova-Levin stochastic process; interval valued functions
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1. Introduction

In mathematical analysis and general topology, interval analysis is an example of set-valued analysis. Modeling some deterministic real-world phenomena mathematically may have interval uncertainty that can be handled by using this technique. Historically, Archimede's circumference-calculation method is an example of an interval enclosure. As for interval analysis, it was Moore who first introduced these concepts in numerical analysis in his acclaimed book [1]. In the past fifty years, interval analysis has become highly valuable because of its many applications in different fields such as computer graphics [2], interval differential equation [3], automatic error analysis [4] and neural network output optimization [5], etc.

Among elementary mathematics, the Hermite-Hadamard inequality has attracted attention since it is the first geometrical interpretation of convex mappings. The function has been generalized, refined, and extended to cover various classes of convexity. The convexity of functions plays a significant role in mathematics and other scientific fields, such as economics, probability theory, and optimal control theory, which has revealed several inequalities over time, see e.g. [6, 7].

A classical version of $\mathbb{H}-\mathbb{H}$ inequality is given as:

$$\varphi\left(\frac{i+j}{2}\right) \leq \frac{1}{j-i}\int_i^j \varphi(\mu)d\mu \leq \frac{\varphi(i)+\varphi(j)}{2}, \tag{1.1}$$

where $\varphi : O \subseteq \mathcal{R} \to \mathcal{R}$ be a convex on interval O and $i, j \in \ O$ with $i < j$. This inequality can be extended or generalized in various ways, see e.g. [8,9]. Later on, many researchers have been fascinated by stochastic processes. A variety of applications are found in physics, engineering, economics, and mathematics, among others. Mathematical and probabilistic applications of stochastic convexity have a long history of relevance. Initially, Nikodem [10] introduced convex stochastic processes in 1980 and demonstrated a few of their basic properties. The Jensen convexity for stochastic processes was further investigated by Skowronski [11] in 1992. As a result, D Kotrys developed the $\mathbb{H}-\mathbb{H}$ inequality for convex stochastic processes, see e.g. [12, 13], and also calculated the bounds for integrals using the famous $\mathbb{H}-\mathbb{H}$ inequality. According to this inequality, if one considers stochastic processes $\varphi : I\times\omega \to \mathcal{R}$ that is mean-square continuous as well as Jensen-convex in the given provided interval I then, we have

$$\varphi\left(\frac{i+j}{2}, .\right) \leq \frac{1}{j-i}\int_i^j \varphi(\mu, .)d\mu \leq \frac{\varphi(i, .)+\varphi(j, .)}{2}. \tag{1.2}$$

For various results and applications related to stochastic convex processes, see e.g. [14–17]. Here are some recent results regarding convex stochastic processes, see e.g. [18–23]. In the beginning, h-convex was developed by Varoşanec [24] in 2007. Several researchers have developed $\mathbb{H}-\mathbb{H}$ inequalities containing h-convex functions, see e.g. [25–30]. Nelson Merentes and his co-authors developed $\mathbb{H}-\mathbb{H}$ and Jensen type inequalities for h-convex stochastic processes [31]. Furthermore, Zhao et al. [32] present new $\mathbb{H}-\mathbb{H}$ and Jensen type inequalities for interval-valued functions using h-convexity. In 2019, Ohud Almutairi and his co-author proved the following inequality using the h-Godunova-Levin function in [33]. It's interesting to study convex functions because of their interesting properties, namely that maximum values are reached at boundary points, and they also have global minimum values. As a result of their enormous applications in optimization theory, this topic of research attracted many researchers from different fields. Over the last few decades, a lot has been done on convex stochastic processes. Using strongly convex stochastic optimization, [34] examined gradient descent optimality. With a continuous-time financial portfolio selection model based on expected utility maximization, a convex stochastic optimization problem is essentially solved in terms of terminal wealth and budget constraints, see [35].

Taking motivation from Zhao et al. [32], Nelson Merentes [31], and Ohud Almutairi [33]. Using interval h-Godunova-Levin stochastic processes, we established some variants of H-H and Jensen type inequalities.

As a final note, the paper is structured as follows. Preliminaries and mathematical backgrounds are provided in Section 2. Section 3 describes the problem and our main findings. The conclusion is given in Section 4.

2. Mathematical backgrounds and preliminaries

It is important to begin by recalling a brief overview of the terms, notation, and properties used throughout this paper. Assume that interval I define the bundle of all intervals of $\mathcal{R}$, [i] $\in I$ is defined as:

$$[i] = [\underline{i}, \bar{i}] = \{x \in R | \underline{i} \leq x \leq \bar{i}\}, \underline{i}, \bar{i} \in R$$

where real interval $[i]$ is a compact subset of R. The interval $[i]$ shows degeneration when $\underline{i}$=$\bar{i}$. We state $[i]$ is positive when $\underline{i} > 0$ or negative when $\bar{i} < 0$. We denote the pack of all intervals by $\mathcal{R}_I$ of $\mathcal{R}$ and use $\mathcal{R}_I^-$ and $\mathcal{R}_I^+$ for negative and positive intervals respectively. The inclusion $\subseteq$ is denoted as:

$$[i] \subseteq [j] \Longleftrightarrow [\underline{i}, \bar{i}] \subseteq [\underline{j}, \bar{j}] \Longleftrightarrow \underline{j} \leq \underline{i}, \bar{i} \leq \bar{j}.$$

Pick any real number μ and $[i]$, the interval $\mu[i]$ is given as:

$$\mu.[\underline{i}, \bar{i}] = \begin{cases} [\mu\underline{i}, \mu\bar{i}], if \mu > 0 \\ \{0\}, if \mu = 0 \\ [\mu\bar{i}, \mu\underline{i}], if \mu < 0. \end{cases}$$

For $[i] = [\underline{i}, \bar{i}]$, and $[j] = [\underline{j}, \bar{j}]$, algebraic operations are defined as:

$$[i] + [j] = [\underline{i} + \underline{j}, \bar{i} + \bar{j}],$$

$$[i] - [j] = [\underline{i} - \underline{j}, \bar{i} - \bar{j}],$$

$$[i].[j] = [min\{\underline{i}\underline{j}, \underline{i}\bar{j}, \bar{i}\underline{j}, \bar{i}\bar{j}\}, max\{\underline{i}\underline{j}, \underline{i}\bar{j}, \bar{i}\bar{j}, \bar{i}\underline{j}\}],$$

$$[i]/[j] = [min\{\underline{i}/\underline{j}, \underline{i}/\bar{j}, \bar{i}/\underline{j}, \bar{i}/\bar{j}\}, max\{\underline{i}/\underline{j}, \underline{i}/\bar{j}, \bar{i}/\bar{j}, \bar{i}/\underline{j}\}],$$

where

$$0 \notin [\underline{i}, \bar{j}].$$

For intervals, the Hausdorff-Pompeiu distance is defined as:

$$d([\underline{i}, \bar{i}], [\underline{j}, \bar{j}]) = max\{|\underline{i} - \underline{j}|, |\bar{i} - \bar{j}|\}.$$

In general, the metric space (R_I, d) is complete.

Definition 2.1. *See [31]. Consider* $(\omega, \mathbb{A}, \mathbb{P})$ *as a probability space. If* $\varphi : \omega \to \mathcal{R}$ *is* $\mathbb{A}$*-measurable, it is referred to as a random variable. A function* $\varphi : I \times \omega \to \mathcal{R}$ *is known as a stochastic process if* $\forall$ $\alpha \in I$ *the function* $\varphi(\alpha, .)$ *is a random variable.*

2.1. Properties of stochastic process

- **Continuous**: A function $\varphi : I \times \omega \to \mathcal{R}$ is continuous on interval I, if $\forall\ n_o \in I$

$$P - \lim_{n \to n_o} \varphi(n, .) = \varphi(n_o, .)$$

where the probability limit is denoted by $P - lim$.

- **Mean square continuous**: A function $\varphi : I \times \omega \to \mathcal{R}$ is known as a mean square continuous on I, if $\forall\ n_o \in I$

$$\lim_{n \to n_o} \mathcal{F}(\varphi(n, .) - \varphi(n_o, .))^2 = 0$$

where expectation of random variable is denoted as $\mathcal{F}[\varphi(n, .)]$.

- **Mean square differentiable**: A function $\varphi : I \times \omega \to \mathcal{R}$ is called mean square differentiable at $k \in I$, if the random variable $\varphi'(n, .) : I \times \omega \to \mathcal{R}$ such that $n_o \in I$

$$\varphi'(n_o, .) = P - \lim_{n \to n_o} \left(\frac{\varphi(n, .) - \varphi(n_o, .)}{n - n_o} \right). \tag{2.1}$$

- **Mean square integral**: Consider $\varphi : I \times \omega \to \mathcal{R}$ be a stochastic process with $\mathcal{F}[\varphi(i, .)^2] < \infty$ then, we can state that a random variable $\beta : \omega \to \mathcal{R}$ is known as mean square- integrable on $[i, j]$, if for every partition of the normal sequence of the interval $[i, j]$, $i = m_o < m_1, .., m_r = j$ and $\forall$ $A_i \in [m_{i-1}, m_i]$ one has

$$\lim_{n \to \infty} \mathcal{F}\left[\left(\sum_{i=1}^{n} \varphi(A_i, .)(m_i - m_{i-1}) - \beta(.) \right)^2 \right] = 0$$

$$\beta(.) = \int_{c_1}^{c_2} \varphi(s, .) ds\, (a.e). \tag{2.2}$$

Definition 2.2. *See [31]. Consider $(\omega, \mathbb{A}, \mathbb{P})$ as a probability space. A stochastic process $\varphi : I \times \Omega \to \mathcal{R}$ is known as convex stochastic process, if $\forall\ i, j \in I$ and $\mu \in [0, 1]$, we have*

$$\varphi(\mu i + (1 - \mu) j, .) \leq \mu \varphi(i, .) + (1 - \mu) \varphi(j, .). \tag{2.3}$$

Definition 2.3. *Consider $(\omega, \mathbb{A}, \mathbb{P})$ as a probability space. A stochastic process $\varphi : I \times \Omega \to \mathcal{R}$ is known as Godunova-Levin-convex stochastic process, if $\forall\ i, j \in I$ and $\mu \in (0, 1)$, one has*

$$\varphi(\mu i + (1 - \mu) j, .) \leq \frac{\varphi(i, .)}{\mu} + \frac{\varphi(j, .)}{(1 - \mu)}. \tag{2.4}$$

Definition 2.4. *See [31]. Consider $h : (0, 1) \to \mathcal{R}$, $h \neq 0$. A stochastic process $\varphi : I \times \Omega \to \mathcal{R}$ is known as h-convex stochastic process, if $\forall\ i, j \in I$ and $\mu \in [0, 1]$, we have*

$$\varphi(\mu i + (1 - \mu) j, .) \leq h(\mu) \varphi(i, .) + h(1 - \mu) \varphi(j, .). \tag{2.5}$$

Definition 2.5. *Consider $h : (0, 1) \to \mathcal{R}$, $h \neq 0$. A stochastic process $\varphi : I \times \omega \to \mathcal{R}$ is known as h-Godunova-Levin (GL) convex stochastic process, if $\forall\ i, j \in I$ and $\mu \in (0, 1)$, we have*

$$\varphi(\mu i + (1 - \mu) j, .) \leq \frac{\varphi(i, .)}{h(\mu)} + \frac{\varphi(j, .)}{h(1 - \mu)}. \tag{2.6}$$

Remark 2.1. • *If $h(\mu) = 1$, then Definition (2.5) gives result for stochastic p-function.*
- *If $h(\mu) = \frac{1}{h(\mu)}$, then Definition (2.5) gives result for stochastic h-convex.*
- *If $h(\mu) = \mu$ then Definition (2.5) gives result for stochastic Godunova-Levin function.*
- *If $h(\mu) = \frac{1}{\mu^s}$ then Definition (2.5) gives result for stochastic s-convex function.*
- *If $h(\mu) = \mu^s$, then Definition (2.5) gives result for stochastic s-Godunova-Levin function.*

3. Main results

Now we are ready to define a newly introduced class of convexity called the interval-valued h-Godunova-Levin stochastic process. Throughout we can use I-V-F, and I-V-Fs for interval-valued function and functions respectively.

Definition 3.1. *Consider $(\omega, \mathcal{A}, \mathcal{P})$ as a probability space and $h : (0,1) \to \mathcal{R}$. A stochastic process $\varphi : I \times \omega \to \mathcal{R}_I{}^+$ where $I \subseteq \mathcal{R}$ is known as h-Godunova-Levin convex stochastic process for I-V-F, if for all $i, j \in I$ and $\mu \in (0,1)$, we have*

$$\frac{\varphi(i,.)}{h(\mu)} + \frac{\varphi(j,.)}{h(1-\mu)} \subseteq \varphi(\mu i + (1-\mu)j,.). \tag{3.1}$$

If the above inequality is inverted, it is referred to h-GL concave stochastic processes for I-V-F. The set of all h-GL convex and concave stochastic processes for I-V-Fs are denoted by $\varphi \in SGX(h, I, \mathcal{R}_I{}^+,.)$ and $\varphi \in SGV(h, I, \mathcal{R}_I{}^+,.)$, respectively.

Proposition 3.1. *Let $\varphi : I \times \omega \to \mathcal{R}_I{}^+$ be h-GL convex stochastic process for I-V-F defined as $\varphi(\mu,.) = [\underline{\varphi}(\mu,.), \overline{\varphi}(\mu,.)]$. Then if $\varphi \in SGX(h, I, \mathcal{R}_I{}^+)$ iff $\underline{\varphi} \in SGX(h, I, \mathcal{R}_I{}^+)$ and if $\overline{\varphi} \in SGV(h), I, \mathcal{R}_I{}^+)$.*

Proof. Let φ be h-GL convex stochastic process for I-V-F and conider $x, y \in [i, j]$, $\mu \in (0,1)$, then

$$\frac{\varphi(x,.)}{h(\mu)} + \frac{\varphi(y,.)}{h(1-\mu)} \subseteq \varphi\Big(\mu x + (1-\mu)y,.\Big),$$

that is,

$$\left[\frac{\underline{\varphi}(x,.)}{h(\mu)} + \frac{\underline{\varphi}(y,.)}{h(1-\mu)}, \frac{\overline{\varphi}(x,.)}{h(\mu)} + \frac{\overline{\varphi}(x,.)}{h(1-\mu)}\right]$$

$$\subseteq \left[\underline{\varphi}\Big(\mu x + (1-\mu)y,.\Big), \overline{\varphi}\Big(\mu x + (1-\mu)y,.\Big)\right]. \tag{3.2}$$

It follows that we have

$$\frac{\underline{\varphi}(x,.)}{h(\mu)} + \frac{\underline{\varphi}(y,.)}{h(1-\mu)} \geq \underline{\varphi}\Big(\mu x + (1-\mu)y,.\Big), \tag{3.3}$$

and

$$\frac{\overline{\varphi}(x,.)}{h(\mu)} + \frac{\overline{\varphi}(y,.)}{h(1-\mu)} \leq \overline{\varphi}\Big(\mu x + (1-\mu)y,.\Big) \tag{3.4}$$

This shows that $\underline{\varphi} \in SGX(h, [i, j], \mathcal{R}_I{}^+,.)$ and $\overline{\varphi} \in SGV(h, [i, j], \mathcal{R}_I{}^+,.)$. Conversely, suppose that if $\underline{\varphi} \in SGX(h, [i, j], \overline{\mathcal{R}_I}{}^+,.)$ and $\overline{\varphi} \in SGV(h, [i, j], \mathcal{R}_I{}^+,.)$. Then from above definition and set inclusion we have $\varphi \in SGX(h, [i, j], \mathcal{R}_I{}^+,.)$. This complete the proof. □

Example 3.1. *Let $h : (0,1) \to \mathcal{R}^+$, $h(\mu) = \frac{1}{\mu}$ for $\mu \in (0,1)$, $[x,y] = [-1,1]$, and $\varphi : [x,y] \subseteq I \times \Omega \to \mathcal{R}_I{}^+$ be defined as*

$$\varphi(\mu,.) = [\underline{\varphi}(\mu,.), \overline{\varphi}(\mu,.)] = [\mu^2, 4 - e^{\mu}].$$

Choose $\mu = \frac{1}{2}$, $x = \frac{1}{2}, y = 1$, we have

$$\frac{\underline{\varphi}(x,.)}{h(\mu)} + \frac{\underline{\varphi}(y,.)}{h(1-\mu)} \geq \underline{\varphi}\Big(\mu x + (1-\mu)y,.\Big) \Rightarrow \frac{5}{8} \geq \frac{9}{16}, \tag{3.5}$$

and

$$\frac{\overline{\varphi}(x,.)}{h(\mu)} + \frac{\overline{\varphi}(y,.)}{h(1-\mu)} \leq \overline{\varphi}\Big(\mu x + (1-\mu)y,.\Big) \Rightarrow \frac{-e - \sqrt{e} + 8}{2} \leq 4 - e^{\frac{3}{4}}. \tag{3.6}$$

Consequently, we have

$$\left[\frac{5}{8}, \frac{-e - \sqrt{e} + 8}{2}\right] \subseteq \left[\frac{9}{16}, 4 - e^{\frac{3}{4}}\right].$$

Hence, proved.

Proposition 3.2. *Let $\varphi : I \times \Omega \to \mathcal{R}_I{}^+$ be h-GL concave stochastic process for I-V-F defined as $\varphi(\mu,.) = [\underline{\varphi}(\mu,.), \overline{\varphi}(\mu,.)]$. Then if $\varphi \in SGV(h, I, \mathcal{R}_I{}^+, .)$ iff $\underline{\varphi} \in SGV(h, I, \mathcal{R}_I{}^+, .)$ and if $\overline{\varphi} \in SGX(h), I, \mathcal{R}_I{}^+)$.*

Proof. This can be similar to Proposition 3.1. □

3.1. Interval Hermite-Hadamard inequality

Theorem 3.1. *Let $h : (0,1) \to \mathcal{R}$ and $h\left(\frac{1}{2}\right) \neq 0$. A function $\varphi : I \times \omega \to \mathcal{R}_I{}^+$ is known as h-GL stochastic mean square integrable process for I-V-F. For every i,j $\in I$, $(i < j)$, if $\varphi \in SGX(h, I, \mathcal{R}_I{}^+, .)$ and $\varphi \in \mathcal{IR}_I$. Almost everywhere, the following inequality is satisfied*

$$\frac{h\left(\frac{1}{2}\right)}{2}\varphi\left(\frac{i+j}{2},.\right) \supseteq \frac{1}{j-i}\int_i^j \varphi(\mu,.)d\mu \supseteq [\varphi(i,.) + \varphi(j,.)]\int_0^1 \frac{dx}{h(x)}. \tag{3.7}$$

Proof. By supposition we have

$$\frac{\varphi\left(xi + (1-x)j,.\right)}{h\left(\frac{1}{2}\right)} + \frac{\varphi((1-x)i + xj,.)}{h\left(\frac{1}{2}\right)} \subseteq \varphi\left(\frac{i+j}{2},.\right).$$

It follows that

$$\int_0^1 \underline{\varphi}(xi + (1-x)j,.)dx + \int_0^1 \underline{\varphi}((1-x)i + xj,.)dx \geq h\left(\frac{1}{2}\right)\int_0^1 \underline{\varphi}\left(\frac{i+j}{2},.\right)dx, \tag{3.8}$$

$$\int_0^1 \overline{\varphi}(xi + (1-x)j,.)dx + \int_0^1 \overline{\varphi}((1-x)i + xj,.)dx \leq h\left(\frac{1}{2}\right)\int_0^1 \overline{\varphi}\left(\frac{i+j}{2},.\right)dx. \tag{3.9}$$

Consequently,

$$\frac{2}{j-i}\int_i^j \underline{\varphi}(\mu,.)d\mu \geq h\left(\frac{1}{2}\right)\int_0^1 \underline{\varphi}\left(\frac{i+j}{2},.\right)dx = h\left(\frac{1}{2}\right)\underline{\varphi}\left(\frac{i+j}{2},.\right).$$

Similarily

$$\frac{2}{j-i}\int_i^j \overline{\varphi}(\mu,.)d\mu \leq h\left(\frac{1}{2}\right)\int_0^1 \overline{\varphi}\left(\frac{i+j}{2},.\right)dx = h\left(\frac{1}{2}\right)\overline{\varphi}\left(\frac{i+j}{2},.\right).$$

This implies that

$$\frac{h\left(\frac{1}{2}\right)}{2}\left[\underline{\varphi}\left(\frac{i+j}{2},.\right),\overline{\varphi}\left(\frac{i+j}{2},.\right)\right] \supseteq \frac{1}{j-i}\int_i^j \varphi(\mu,.)d\mu, \tag{3.10}$$

From Definition 3.1, we have

$$\frac{\varphi(i,.)}{h(x)} + \frac{\varphi(j,.)}{h(1-x)} \subseteq \varphi(xi+(1-x)j,.).$$

With integration over $(0,1)$, we have

$$\varphi(i,.)\int_0^1 \frac{dx}{h(x)} + \varphi(j,.)\int_0^1 \frac{dx}{h(1-x)} \subseteq \int_0^1 \varphi(xj+(1-x)j,.)dx.$$

Accordingly,

$$[\varphi(i,.)+\varphi(j,.)]\int_0^1 \frac{dx}{h(x)} \subseteq \frac{1}{j-i}\int_i^j \varphi(\mu,.)d\mu. \tag{3.11}$$

Now combining Eqs (3.10) and (3.11) we get required result

$$\frac{h\left(\frac{1}{2}\right)}{2}\varphi\left(\frac{i+j}{2},.\right) \supseteq \frac{1}{j-i}\int_i^j \varphi(\mu,.)d\mu \supseteq [\varphi(i,.)+\varphi(j,.)]\int_0^1 \frac{dx}{h(x)}. \tag{3.12}$$

□

Remark 3.1.

- *If we set $h(x) = 1$, Theorem 3.1 gives the result for I-V-F P-convex stochastic process:*

$$\frac{1}{2}\varphi\left(\frac{i+j}{2},.\right) \supseteq \frac{1}{j-i}\int_i^j \varphi(\mu,.)d\kappa \supseteq \left[\varphi(i,.)+\varphi(j,.)\right].$$

- *If we set $h(x) = \frac{1}{x}$, Theorem 3.1 gives the result for I-V-F convex stochastic process :*

$$\varphi\left(\frac{i+j}{2},.\right) \supseteq \frac{1}{j-i}\int_i^j \varphi(\mu,.)d\kappa \supseteq \frac{[\varphi(i,.)+\varphi(j,.)]}{2}.$$

- *If we set $h(x) = \frac{1}{(x)^s}$, then Theorem 3.1 gives the result for I-V-F s-convex stochastic process :*

$$2^{s-1}\varphi\left(\frac{i+j}{2},.\right) \supseteq \frac{1}{j-i}\int_i^j \varphi(\mu,.)d\mu \supseteq \frac{[\varphi(i,.)+\varphi(j,.)]}{s+1}.$$

- *If $\underline{\varphi} = \overline{\varphi}$ then Theorem 3.1 gives the result of Ohud Almutairi [33], Theorem 1.*

Example 3.2. *Let $h:(0,1)\to\mathcal{R}^+$, $h(x)=\frac{1}{x}$ for $x\in(0,1)$, $[i,j]=[-2,2]$, and $\varphi:[i,j]\subseteq I\times\Omega\to\mathcal{R}_I{}^+$ be defined by $\varphi(\mu,.)=[\mu^2, 7-e^{\mu}]$. Then*

$$\frac{h\left(\frac{1}{2}\right)}{2}\varphi\left(\frac{i+j}{2},.\right) = \varphi(0,.) = [0,6],$$

$$\frac{1}{j-i}\int_i^j \varphi(\mu,.)\,d\mu = \left[\frac{4}{3}, -\frac{e^4-28e^2-1}{4e^2}\right],$$

$$[\varphi(i,.)+\varphi(j,.)]\int_0^1 \frac{dx}{h(x)} = \left[4, \frac{14e^2-1-e^4}{2e^2}\right].$$

As a result,

$$[0,6] \supseteq \left[\frac{4}{3}, -\frac{e^4-28e^2-1}{4e^2}\right] \supseteq \left[4, \frac{14e^2-1-e^4}{2e^2}\right].$$

Hence, proved.

Theorem 3.2. *Let $h:(0,1)\to\mathcal{R}$ and $h\left(\frac{1}{2}\right)\neq 0$. A function $\varphi: I\times\omega\to{\mathcal{R}_I}^+$ is known as h-GL stochastic mean square integrable process for I-V-F. For every i,j $\in I$, $(i<j)$, if $\varphi\in SGX(h,I,{\mathcal{R}_I}^+,.)$ and $\varphi\in\ \mathcal{IR}_I$. Almost everywhere, the following inequality is satisfied*

$$\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\varphi\left(\frac{i+j}{2},.\right) \supseteq \triangle_1 \supseteq \frac{1}{j-i}\int_i^j \varphi(\mu,.)d\mu \supseteq \triangle_2 \supseteq \left\{[\varphi(i,.)+\varphi(j,.)]\left[\frac{1}{2}+\frac{1}{h\left(\frac{1}{2}\right)}\right]\right\}\int_0^1 \frac{dx}{h(x)},$$

where

$$\triangle_1 = \frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left[\varphi\left(\frac{3i+j}{4},.\right)+\varphi\left(\frac{3j+i}{4},.\right)\right],$$

$$\triangle_2 = \left[\varphi\left(\frac{i+j}{2},.\right)+\frac{\varphi(i,.)+\varphi(j,.)}{2})\right]\int_0^1 \frac{dx}{h(x)}.$$

Proof. Consider $\left[i, \frac{i+j}{2}\right]$

$$\frac{\varphi\left(xi+(1-x)\frac{i+j}{2},.\right)}{h\left(\frac{1}{2}\right)} + \frac{\varphi\left((1-x)i+x\frac{i+j}{2},.\right)}{h\left(\frac{1}{2}\right)} \subseteq \varphi\left(\frac{3i+j}{2},.\right),$$

It follows that

$$\frac{1}{h\left(\frac{1}{2}\right)}\left[\int_0^1 \varphi(xi+(1-x)\frac{i+j}{2},.)dx + \int_0^1 \varphi(x\frac{i+j}{2}+(1-x)i,.)dx\right] \subseteq \varphi\left(\frac{3i+j}{2},.\right).$$

Then above inclusion become as

$$\frac{1}{j-i}\int_i^{\frac{i+j}{2}} \varphi(\mu,.)d\mu \subseteq \frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\varphi\left(\frac{3i+j}{2},.\right). \tag{3.13}$$

Similarly for interval $\left[\frac{i+j}{2}, j\right]$, we have

$$\frac{1}{j-i}\int_{\frac{i+j}{2}}^j \varphi(\mu,.)d\mu \subseteq \frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\varphi\left(\frac{3j+i}{2},.\right). \tag{3.14}$$

Adding Eqs (3.13) and (3.14) we get

$$\triangle_1 = \frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left[\varphi\left(\frac{3i+j}{4},.\right)+\varphi\left(\frac{3j+i}{4},.\right)\right] \supseteq \left[\frac{1}{j-i}\int_i^j \varphi(\mu,.)d\mu\right]$$

$$= \frac{1}{2}\left[\frac{2}{j-i}\int_i^{\frac{i+j}{2}} \varphi(\mu,.)d\mu + \frac{2}{j-i}\int_{\frac{i+j}{2}}^j \varphi(\mu,.)d\mu\right]$$

$$= \frac{1}{2}\left[\left\{\varphi(i,.)+\varphi(j,.)+2\varphi\left(\frac{i+j}{2},.\right)\right\}\int_0^1 \frac{dx}{h(x)}\right]$$

$$= \left[\frac{\varphi(i,.)+\varphi(j,.)}{2}+\varphi\left(\frac{i+j}{2},.\right)\right]\int_0^1 \frac{dx}{h(x)} = \triangle_2.$$

Now,

$$\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\varphi\left(\frac{i+j}{2},.\right) = \frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\varphi\left(\frac{1}{2}\left(\frac{3i+j}{4},.\right)+\frac{1}{2}\left(\frac{3j+i}{4},.\right)\right)$$

$$= \frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left[\varphi\left(\frac{3i+j}{4},.\right)+\varphi\left(\frac{3i+j}{4},.\right)\right] = \triangle_1$$

$$\supseteq \frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left\{\frac{1}{h\left(\frac{1}{2}\right)}\left[\varphi(i,.)+\varphi\left(\frac{i+j}{2},.\right)\right]+\frac{1}{h\left(\frac{1}{2}\right)}\left[\varphi(j,.)+\varphi\left(\frac{i+j}{2},.\right)\right]\right\}$$

$$\supseteq \frac{1}{2}\left[\frac{\varphi(i,.)+\varphi(j,.)}{2}+\varphi\left(\frac{i+j}{2},.\right)\right] \supseteq \left[\frac{\varphi(i,.)+\varphi(j,.)}{2}+\varphi\left(\frac{i+j}{2},.\right)\right]\int_0^1 \frac{dx}{h(x)} = \triangle_2$$

$$\supseteq \left[\frac{\varphi(i,.)+\varphi(j,.)}{2}+\frac{1}{h\left(\frac{1}{2}\right)}\left[\varphi(i,.)+\varphi(j,.)\right]\right]\int_0^1 \frac{dx}{h(x)}$$

$$= \left\{\left[\varphi(i,.)+\varphi(j,.)\right]\left[\frac{1}{2}+\frac{1}{h\left(\frac{1}{2}\right)}\right]\right\}\int_0^1 \frac{dx}{h(x)}.$$

This completes the proof. □

Example 3.3. *Recall to Example 3.2, we have*

$$\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\varphi\left(\frac{i+j}{2},.\right) = \varphi(0) = [0,6]$$

$$\triangle_1 = \frac{1}{2}\left[\varphi(-1,.)+\varphi(1,.)\right] = \left[1, \frac{14e-1-e^2}{2e}\right]$$

$$\frac{1}{j-i}\int_i^j \varphi(\mu,.)d\mu = \left[\frac{4}{3}, -\frac{e^4-28e^2-1}{4e^2}\right]$$

$$\triangle_2 = \left[\frac{\varphi(i,.)+\varphi(j,.)}{2}+\varphi\left(\frac{i+j}{2},.\right)\right]\int_0^1 \frac{dx}{h(x)} = \left[2, \frac{-e^4+26e^2-1}{4e^2}\right].$$

Thus, we obtain

$$[0,6] \supseteq \left[1, \frac{14e-1-e^2}{2e}\right] \supseteq \left[\frac{4}{3}, -\frac{e^4-28e^2-1}{4e^2}\right] \supseteq \left[2, \frac{-e^4+26e^2-1}{4e^2}\right] \subseteq \left[4, \frac{14e^2-1-e^4}{4e^2}\right],$$

which demonstrates the result described in Theorem 3.2.

Theorem 3.3. *Let $h : (0,1) \to \mathcal{R}$ and $h\left(\frac{1}{2}\right) \neq 0$. A functions $\varphi, \phi : I \times \omega \to \mathcal{R}_I{}^+$ is known as h-GL stochastic mean square integrable process for I-V-Fs. For every i,j $\in I$, $(i < j)$, if $\varphi \in SGX(h_1, I, \mathcal{R}_I{}^+, .)$, $\phi \in SGX(h_2, I, \mathcal{R}_I{}^+, .)$ and $\varphi, \phi \in \mathcal{IR}_I$. Almost everywhere, the following inequality is satisfied*

$$\frac{1}{j-i}\int_i^j \varphi(\mu,.)\phi(\mu),.d\mu \supseteq M(i,j)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx + N(i,j)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx,$$

where

$$M(i,j) = \varphi(i,.)\phi(j,.) + \varphi(j,.)\phi(j,.),$$

and

$$N(i,j) = \varphi(i,.)\phi(j,.) + \varphi(j,.)\phi(i,.).$$

Proof. We assume that $\varphi \in SGX(h_1), [i,j], \mathcal{R}_I{}^+, .)$, $\phi \in SGX(h_2), [i,j], \mathcal{R}_I{}^+, .)$, then

$$\frac{\varphi(i,.)}{h_1(x)} + \frac{\varphi(j,.)}{h_1(1-x)} \subseteq \varphi\left(ix+(1-x)j,.\right),$$

and

$$\frac{\phi(i,.)}{h_2(x)} + \frac{\phi(j,.)}{h_2(1-x)} \subseteq \phi\left(ix+(1-x)j,.\right).$$

Then

$$\begin{aligned}
&\varphi\left(ix+(1-x)j,.\right)\phi\left(ix+(1-x)j,.\right)\\
\supseteq&\frac{\varphi(i,.)\phi(i,.)}{h_1(x)h_2(x)} + \frac{\varphi(i,.)\phi(j,.)}{h_1(x)h_2(1-x)} + \frac{\varphi(j,.)\phi(i,.)}{h_1(1-x)h_2(x)} + \frac{\varphi(j,.)\phi(j,.)}{h_1(1-x)h_2(1-x)}.
\end{aligned}$$

With integration over $(0,1)$, we have

$$\begin{aligned}
&\int_0^1 \varphi\left(ix+(1-x)j,.\right)\phi\left(ix+(1-x)j,.\right)dx\\
=&\left[\int_0^1 \underline{\varphi}\left(ix+(1-x)j,.\right)\underline{\phi}\left(ix+(1-x)j,.\right)dx, \int_0^1 \overline{\varphi}\left(ix+(1-x)j,.\right)\overline{\phi}\left(ix+(1-x)j,.\right)dx\right]\\
=&\left[\frac{1}{j-i}\int_i^j \underline{\varphi}(\mu,.)\underline{\phi}(\mu,.)d\mu, \frac{1}{j-i}\int_i^j \overline{\varphi}(\mu,.)\overline{\phi}(\mu,.)d\mu\right]\\
=&\frac{1}{j-i}\int_i^j \varphi(\mu,.)\phi(\mu,.)d\mu \supseteq\\
&\int_0^1 \frac{[\varphi(i,.)\phi(i,.)+\varphi(j,.)\phi(j,.)]}{h_1(x)h_2(x)}dx + \int_0^1 \frac{[\varphi(i,.)\phi(j,.)+\varphi(j,.)\phi(i,.)]}{h_1(x)h_2(1-x)}dx.
\end{aligned}$$

It follows that

$$\frac{1}{j-i}\int_i^j \varphi(\mu,.)\phi(\mu,.)d\mu \supseteq M(i,j)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx + N(i,j)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx.$$

The proof is completed. □

Example 3.4. *Consider* $h_1(x) = \frac{1}{x}, h_2(x) = 1$ *where* $x \in (0,1)$, $[i, j] = [0,1]$, *and*

$$\varphi(\mu, .) = [\mu^2, 8 - e^{\mu}], \phi(\mu, .) = [\mu, 7 - \mu^2].$$

Then,

$$\frac{1}{j-i}\int_i^j \varphi(\mu, .)\phi(\mu, .)d\mu = \left[\frac{1}{4}, -\frac{18e - 175}{3}\right],$$
$$M(i, j)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx = \left[\frac{1}{2}, \frac{17}{2} - e\right],$$

and

$$N(i, j)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx = \left[0, \frac{9}{2} - \frac{3e}{4}\right].$$

It follows that

$$\left[\frac{1}{4}, -\frac{18e - 175}{3}\right] \supseteq \left[\frac{1}{2}, \frac{17}{2} - e\right] + \left[0, \frac{9}{2} - \frac{3e}{4}\right] = \left[\frac{1}{2}, 8 - \frac{7e}{4}\right],$$

which demonstrates the result described in Theorem 3.3.

Theorem 3.4. *Let* $h : (0,1) \to \mathcal{R}$ *and* $h\left(\frac{1}{2}\right) \neq 0$. *A functions* $\varphi, \phi : I \times \omega \to \mathcal{R}_I{}^+$ *is known as h-GL stochastic mean square integrable process for I-V-Fs. For every i,j* $\in I$, $(i < j)$, *if* $\varphi \in SGX(h_1, I, \mathcal{R}_I{}^+, .)$, $\phi \in SGX(h_2, I, \mathcal{R}_I{}^+, .)$ *and* $\varphi, \phi \in \mathcal{IR}_I$. *Almost everywhere, the following inequality is satisfied*

$$\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{2}\varphi\left(\frac{i+j}{2}, .\right)\phi\left(\frac{i+j}{2}, .\right)$$
$$\supseteq \frac{1}{j-i}\int_i^j \varphi(\mu, .)\phi(\mu, .)d\mu + M(i, j)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx + N(i, j)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx.$$

Example 3.5. *Consider* $h_1(x) = \frac{1}{x}, h_2(x) = 2$ *where* $x \in (0,1)$, $[i, j] = [0,1]$, *and*

$$\varphi(\mu, .) = [\mu^2, 8 - e^{\mu}], \phi(\mu, .) = [\mu, 7 - \mu^2].$$

Then

$$\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{2}\varphi\left(\frac{i+j}{2}, .\right)\phi\left(\frac{i+j}{2}, .\right) = \left[\frac{1}{4}, \frac{54\left(8 - \sqrt{e}\right)}{4}\right],$$
$$\frac{1}{j-i}\int_i^j \varphi(\mu, .)\phi(\mu, .)d\mu = \left[\frac{1}{4}, -\frac{18e - 175}{3}\right],$$
$$M(v, w)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx = \left[\frac{1}{4}, \frac{17}{4} - \frac{e}{2}\right],$$

and

$$N(v,w)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx = \left[0, \frac{9}{4} - \frac{3e}{8}\right]. \tag{3.15}$$

It follows that

$$\left[\frac{1}{4}, \frac{54\left(8-\sqrt{e}\right)}{4}\right] \supseteq \left[\frac{1}{4}, -\frac{18e-175}{3}\right] + \left[\frac{1}{4}, \frac{17}{4} - \frac{e}{2}\right] + \left[0, \frac{9}{4} - \frac{3e}{8}\right] = \left[\frac{1}{2}, \frac{13}{2} + \frac{-165e+1400}{24}\right].$$

This verifies the above theorem.

3.2. Jensen type inequality

Theorem 3.5. *Let $s_1, s_2, s_3...s_k \in \mathcal{R}^+$ with $k \geq 2$. If h is non-negative function and $\varphi : I \times \omega \to \mathcal{R}$ is non-negative interval-valued h-GL stochastic process almost everywhere or $\varphi \in SGX(h, P, {\mathcal{R}_I}^+, .)$ with $z_1, z_2, z_3, ..., z_k \in I$. Then the inequality become as :*

$$\varphi\left(\frac{1}{S_k}\sum_{i=1}^{k} s_i z_i, .\right) \supseteq \sum_{i=1}^{k}\left[\frac{\varphi(z_i, .)}{h\left(\frac{s_i}{S_k}\right)}\right], \tag{3.16}$$

where $S_k = \sum_{i=1}^{k} s_i$.

Proof. For $k = 2$, the inclusion (3.16) is trivially true. Now, we assume that it also true for $k-1$, then

$$\begin{aligned}
\varphi\left(\frac{1}{S_k}\sum_{i=1}^{k} s_i z_i, .\right) &= \varphi\left(\frac{s_k}{S_k} z_k + \sum_{i=1}^{k-1} \frac{s_i}{S_k} z_i, .\right)\\
&= \varphi\left(\frac{s_k}{S_k} z_k + \frac{S_{k-1}}{S_k}\sum_{i=1}^{k-1} \frac{s_i}{S_{k-1}} z_i, .\right)\\
&\supseteq \frac{\varphi(z_k, .)}{h\left(\frac{s_k}{S_k}\right)} + \frac{\varphi\left(\sum_{i=1}^{k-1} \frac{s_i}{S_{k-1}} z_i, .\right)}{h\left(\frac{S_{k-1}}{S_k}\right)}\\
&\supseteq \frac{\varphi(z_k, .)}{h\left(\frac{s_k}{S_k}\right)} + \sum_{i=1}^{k-1}\left[\frac{\varphi(z_i, .)}{h\left(\frac{s_i}{S_{k-1}}\right)}\right]\frac{1}{h\left(\frac{S_{k-1}}{S_k}\right)}\\
&\supseteq \frac{\varphi(z_k, .)}{h\left(\frac{s_k}{S_k}\right)} + \sum_{i=1}^{k-1}\left[\frac{\varphi(z_i, .)}{h(\frac{s_i}{S_k})}\right]\\
&= \sum_{i=1}^{k}\left[\frac{\varphi(z_i, .)}{h\left(\frac{s_i}{S_k}\right)}\right].
\end{aligned}$$

Therefore, the result is proven using mathematical induction. □

4. Conclusions

In this paper, we introduce the h-Godunova-Levin stochastic process for I-V-F. The purpose of the above concept was to study Jensen and $\mathbb{H}-\mathbb{H}$ inequalities using I-V-F. The inequalities previously established by Nelson Merentes [31], and Ohud Almutairi [33] are generalized in our study. To support our main conclusions, we also provide several useful examples. As part of this research direction, we will investigate Jensen and $\mathbb{H}-\mathbb{H}$ type inequalities for I-V-F and fuzzy-valued functions over time scales. Our research on interval integral operator-type integral inequalities will help to expand the range of practical problems that integrals can be used to solve. Integrals are widely used in engineering technology, including in various types of mathematical modelling. Using these concepts, a new approach to convex optimization can be developed. We hope that by embracing this concept, other authors will be able to secure their roles in various scientific fields.

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Conflict of interest

The authors declare no conflicts of interest.

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