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Research article

Projection methods for quasi-nonexpansive multivalued mappings in Hilbert spaces

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Abstract: This paper proposes a modified D-iteration to approximate the solutions of three quasi-nonexpansive multivalued mappings in a real Hilbert space. Due to the incorporation of an inertial step in the iteration, the sequence generated by the modified method converges faster to the common fixed point of the mappings. Furthermore, the generated sequence strongly converges to the required solution using a shrinking technique. Numerical results obtained indicate that the proposed iteration is computationally efficient and outperforms the standard forward-backward with inertial step.

Keywords: quasi-nonexpansive multivalued mappings; weak and strong convergence; inertial technical term; D-iteration; projection methods
Mathematics Subject Classification: 47H09, 47H10

1. Introduction

Let $\mathbb{K}$ be a nonempty closed and convex subset of real Hilbert space $\mathbb{H}$. Define $S : \mathbb{K} \to \mathbb{K}$ to be a continuous mapping. A point $\bar{u} \in \mathbb{K}$ is said to be a fixed point of S if $S(\bar{u}) = \bar{u}$. Also, the $F(S)$ represents the set of all fixed points of S. Several authors have investigated the existence of fixed points for theorems of single-valued nonexpansive mappings (for example, [1–5]).

Mann [6] proposed the following method in 1953 for approximating the fixed point of a nonexpansive mapping S in a Hilbert space $\mathbb{H}$:

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n, \quad \forall n \geq 1, \tag{1.1}$$

where $\{a_n\}$ is a sequence in $[0, 1]$.

Ishikawa [7] generalized Mann's iterative algorithm (1.1) in 1974 by introducing the iteration:

$$\begin{cases} u_0 \in \mathbb{K} \text{ chosen arbitrary,} \\ v_n = (1-a_n)u_n + a_n S u_n, \\ u_{n+1} = (1-b_n)u_n + b_n S v_n, \quad n \geq 0, \end{cases} \tag{1.2}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $[0,1]$.

Noor [8] introduced and generalized Ishikawa's iterative algorithm (1.2) in 2000 by introducing the following iterative procedure for solving the fixed point problem of a single-valued nonlinear mapping:

$$\begin{cases} u_1 \in \mathbb{K} \text{ chosen arbitrary,} \\ v_n = (1-a_n)u_n + a_n S u_n, \\ \rho_n = (1-b_n)u_n + b_n S v_n, \\ u_{n+1} = (1-c_n)u_n + c_n S \rho_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$.

Yildirim and Özdemir [9] introduced a new iteration process in 2009 which is an n-step for finding the common fixed points. It is produced by the following processes:

$$\begin{cases} u_1 \in \mathbb{K} \text{ chosen arbitrary,} \\ v_n = P((1-a_{rn})u_n + a_{rn} S_r (PS_r)^{n-1} u_n), \\ v_{n+1} = P((1-a_{(r-1)n})v_n + a_{(r-1)n} S_{r-1} (PS_{r-1})^{n-1} v_n), \\ \qquad \vdots \\ v_{n+r-2} = P((1-a_{2n})v_{n+r-3} + a_{2n} S_2 (PS_2)^{n-1} v_{n+r-3}), \\ u_{n+1} = P((1-a_{1n})v_{n+r-2} + a_{1n} S_1 (PS_1)^{n-1} v_{n+r-2}), \quad n \geq 1 \text{ and } r \geq 2, \end{cases} \tag{1.4}$$

where $\{a_{jn}\}$ be a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, for each $j \in \{1, 2, \ldots, r\}$.

Sainuan [10] developed a new iteration called P-iteration in 2015. The P-iteration is defined as:

$$\begin{cases} u_1 \in \mathbb{K} \text{ chosen arbitrary,} \\ v_n = (1-a_n)u_n + a_n S u_n, \\ \rho_n = (1-b_n)v_n + b_n S v_n, \\ u_{n+1} = (1-c_n)S v_n + c_n S \rho_n, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$.

The D-iteration was introduced in 2018 by Daengsaen and Khemphet [11], who used the Sainuan's iteration concept. It is produced by the following processes:

$$\begin{cases} u_1 \in \mathbb{K} \text{ chosen arbitrary,} \\ v_n = (1-a_n)u_n + a_n S u_n, \\ \rho_n = (1-b_n)S u_n + b_n S v_n, \\ u_{n+1} = (1-c_n)S v_n + c_n S \rho_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$.

The heavy ball method, which was studied in [12, 13] for maximal monotone operators by the proximal point algorithm, was used by Alvarez and Attouch [19]. This algorithm is known as the inertial proximal point algorithm, and it is written as follows:

$$\begin{cases} u_0, u_1 \in \mathbb{K} \text{ chosen arbitrary}, \\ t_n = u_n + \lambda_n(u_n - u_{n-1}), \\ u_{n+1} = (I + \gamma_n B)^{-1} t_n, \quad n \geq 1, \end{cases} \tag{1.7}$$

where I is the identity mapping. It was proved that if $\{\gamma_n\}$ is non-decreasing and $\{\lambda_n\} \subset [0,1)$ with

$$\sum_{n=1}^{\infty} \lambda_n \|u_n - u_{n-1}\|^2 < \infty, \tag{1.8}$$

then algorithm (1.7) converges weakly to a zero of B.

Nakajo and Takahashi [18] proposed modifying Mann's iteration method (1.1) to obtain a strong convergence theorem in Hilbert spaces $\mathbb{H}$:

$$\begin{cases} u_0 \in \mathbb{K}, \text{ chosen arbitrary}, \\ v_n = (1 - a_n)u_n + a_n S u_n, \\ \mathbb{K}_n = \{x \in \mathbb{K} : \|v_n - x\| \leq \|u_n - x\|\}, \\ R_n = \{x \in \mathbb{K} : \langle u_0 - u_n, u_n - x \rangle\}, \\ u_{n+1} = P_{\mathbb{K}_n \cap R_n} u_0, \quad \forall\, n \geq 0, \end{cases} \tag{1.9}$$

where $\{a_n\} \subseteq [0, a]$ for some $a \in [0,1)$. They proved that the sequence $\{u_n\}$ converges strongly to $P_{F(S)} u_0$.

In 2021, Chaolamjiak et al. [14] proposed modifying SP iteration method (1.4) to obtain a strong convergence theorem in Hilbert spaces $\mathbb{H}$:

$$\begin{cases} u_0, u_1 \in \mathbb{K}, \ R_1 = \mathbb{K}, \\ t_n = u_n + \lambda_n(u_n - u_{n-1}), \\ v_n \in (1 - a_n)t_n + a_n S_1 t_n, \\ \rho_n \in (1 - b_n)v_n + b_n S_2 v_n, \\ w_n \in (1 - c_n)\rho_n + c_n S_3 \rho_n, \\ \mathbb{K}_n = \{x \in \mathbb{K} : \|w_n - x\|^2 \leq \|u_n - x\|^2 + 2\lambda_n^2 \|u_n - u_{n-1}\|^2 - 2\lambda_n \langle u_n - x, u_{n-1} - u_n \rangle\}, \\ R_n = \{x \in R_{n-1} : \langle u_1 - u_n, u_n - x \rangle \geq 0\}, \\ u_{n+1} = P_{\mathbb{K}_n \cap R_n} u_1, \end{cases} \tag{1.10}$$

for all $n \geq 1$, where $\{a_n\}$, $\{b_n\}$ and $\{c_n\} \subset (0,1)$. They proved that the sequence $\{u_n\}$ converges strongly to a common fixed point of S_1, S_2 and S_3.

The results [11,18,19,21] provide incentive. In order to locate a common fixed point of three quasi-nonexpansive multivalued mappings, we introduce the D-iterative approach with the inertial technical term. We can prove strong convergence theorems by combining shrinking projection methods with inertial D-iteration. Finally, we compare our inertial projection method to the traditional projection method and conduct numerical tests to support our major findings with different choices of the initial values x_0 and x_1 in 4 case.

2. Preliminaries

Let $CB(\mathbb{K})$ and $K(\mathbb{K})$ denote the families of nonempty closed bounded, and compact, respectively. The Hausdorff metric on $CB(\mathbb{K})$ is defined by:

$$H(C,Q) = \max\left\{\sup_{u\in C} d(u,Q), \sup_{v\in Q} d(v,C)\right\}, \quad \forall\, C,Q \in CB(\mathbb{K}),$$

where $d(u,Q) = \inf_{\alpha\in Q}\{\|u-\alpha\|\}$.
A single-valued mapping $S:\mathbb{K}\to\mathbb{K}$ is said to be nonexpansive if

$$\|Su - Sv\| \le \|u-v\|, \quad \forall\, u,v\in\mathbb{K}.$$

A multivalued mapping $S:\mathbb{K}\to CB(\mathbb{K})$ if $\bar{u}\in S\bar{u}$ and

$$H(Su, S\bar{u}) \le \|u-\bar{u}\|, \quad \forall\, u\in\mathbb{K} \text{ and } \bar{u}\in F(S).$$

Then S is said to be quasi-nonexpansive.

Condition (A). Let $\mathbb{H}$ be a Hilbert space and $\mathbb{K}$ be a subset of $\mathbb{H}$. A multivalued mapping $S:\mathbb{K}\to CB(\mathbb{K})$ is said to satisfy Condition (A) if $\|u-\bar{u}\| = d(u,S\bar{u})$ for all $u\in\mathbb{H}$ and $\bar{u}\in F(S)$.

We now give the example of quasi-nonexpansive multivalued mapping S which satisfies Condition (A) and the fixed point set $F(S)$ contains more than one element.

Example. In Euclidean space $\mathbb{R}$, let $\mathbb{K} = [0,2]$ and $S:\mathbb{K}\to CB(\mathbb{K})$ be defined by

$$Su = \begin{cases} \left[0, \dfrac{u}{2}\right], & \text{if } u\le 1,\\ \{2\}, & \text{if } u>1.\end{cases}$$

It is easy to see that $F(S) = \{0,2\}$.

Lemma 2.1. [14] *Let $\mathbb{H}$ be a real Hilbert space. Let $S:\mathbb{H}\to CB(\mathbb{H})$ be a quasi-nonexpansive mapping with $F(S)\neq\emptyset$. Then, $F(S)$ is closed, and if S satisfies Condition (A), then $F(S)$ is convex.*

A multivalued mapping $S:\mathbb{K}\to CB(\mathbb{K})$ is said to be hybrid if

$$3H(Su,Sv)^2 \le \|u-v\|^2 + d(v,Su)^2 + d(u,Sv)^2, \quad \forall\, u,v\in\mathbb{K}.$$

Lemma 2.2. [15] *Let $\mathbb{K}$ be a closed convex subset of a real Hilbert space $\mathbb{H}$. Let $S:\mathbb{K}\to K(\mathbb{K})$ be a hybrid multivalued mapping. Let $\{u_n\}$ be a sequence in $\mathbb{K}$ such that $u_n\to\bar{u}$ and* $\lim_{n\to\infty}\|u_n - x_n\| = 0$ *for some $x_n\in Su_n$. Then, $\bar{u}\in S\bar{u}$.*

Lemma 2.3. [16] *Let $\mathbb{X}$ be a Banach space satisfying Opial's condition and let $\{u_n\}$ be a sequence in $\mathbb{X}$. Let $x,y\in\mathbb{X}$ be such that* $\lim_{n\to\infty}\|u_n-x\|$ *and* $\lim_{n\to\infty}\|u_n-y\|$ *exist. If $\{u_{n_k}\}$ and $\{u_{m_k}\}$ are subsequences of $\{u_n\}$ which converge weakly to x and y, respectively, then $x=y$.*

Lemma 2.4. [17] *Let $\mathbb{K}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. For each $x,y\in\mathbb{H}$ and $v\in\mathbb{R}$, the set*

$$D = \{u\in\mathbb{K} : \|y-u\|^2 \le \|x-u\|^2 + \langle z,u\rangle + v\},$$

is closed and convex.

Lemma 2.5. [18] *Let $\mathbb{K}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$ and $P_{\mathbb{K}} : \mathbb{H} \to \mathbb{K}$ be the metric projection from $\mathbb{H}$ onto $\mathbb{K}$. Then*

$$\|v - P_{\mathbb{K}}u\|^2 + \|u - P_{\mathbb{K}}u\|^2 \le \|u - v\|^2,$$

for all $u \in \mathbb{H}$ and $v \in \mathbb{K}$.

Lemma 2.6. [19] *Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be the sequences in $[0, \infty)$ such that*

$$\alpha_{n+1} \le \alpha_n + \beta_n(\alpha_n - \alpha_{n-1}) + \gamma_n,$$

for all $n \ge 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, and there exists a real number β with $0 \le \beta_n \le \beta < 1$ for all $n \ge 1$. Then, the followings hold

(a) $\sum_{n\ge1}[\alpha_n - \alpha_{n-1}]_+ < \infty$, *where* $[t]_+ = \max\{t, 0\}$;
(b) *there exists* $\alpha^* \in [0, \infty)$ *such that* $\lim_{n\to\infty} \alpha_n = \alpha^*$.

Lemma 2.7. [20] *Let $\mathbb{H}$ be a real Hilbert space. Then, for each $u, v \in \mathbb{H}$ and $t \in [0, 1]$*

(a) $\|u - v\|^2 \le \|u\|^2 + \|v\|^2 - 2\langle u, v\rangle$;
(b) $\|tu - (1 - t)v\|^2 = t\|u\|^2 + (1 - t)\|v\|^2 - t(1 - t)\|u - v\|^2$;
(c) *If $\{u_n\}$ is a sequence in $\mathbb{H}$ such that $u_n \rightharpoonup u$, then*

$$\limsup_{n\to\infty} \|u_n - v\|^2 = \limsup_{n\to\infty} \left(\|u_n - u\|^2 + \|u - v\|^2\right).$$

3. Main results

Theorem 3.1. *Let $\mathbb{K}$ be a closed convex subset of a real Hilbert space $\mathbb{H}$ and $S_1, S_2, S_3 : \mathbb{H} \to CB(\mathbb{K})$ be quasi-nonexpansive multivalued mappings with $\Upsilon := F(S_1) \cap F(S_2) \cap F(S_3) \ne \emptyset$ and $I - S_i$ is demiclosed at 0 for all $i \in \{1, 2, 3\}$. Let $\{u_n\}$ be a sequence generated by*

$$\begin{cases} u_0, u_1 \in \mathbb{K} \text{ chosen arbitrary}, \\ t_n = u_n + \lambda_n(u_n - u_{n-1}), \\ v_n \in (1 - a_n)t_n + a_n S_1 t_n, \\ \rho_n \in (1 - b_n)S_1 t_n + b_n S_2 v_n, \\ u_{n+1} \in (1 - c_n)S_2 v_n + c_n S_3 \rho_n, \end{cases} \tag{3.1}$$

for all $n \ge 1$, where $\{a_n\}$, $\{b_n\}$ and $\{c_n\} \subset (0, 1)$. Assume that the following conditions hold

(a) $\sum_{n=1}^{\infty} \lambda_n\|u_n - u_{n-1}\| < \infty$;
(b) $0 < \liminf_{n\to\infty} a_n < \limsup_{n\to\infty} a_n < 1$;
(c) $0 < \liminf_{n\to\infty} b_n < \limsup_{n\to\infty} b_n < 1$;
(d) $0 < \liminf_{n\to\infty} c_n < \limsup_{n\to\infty} c_n < 1$.

If S_1, S_2 and S_3 satisfy Condition (A), then the sequence $\{u_n\}$ converges weakly to a common fixed point of S_1, S_2 and S_3.

Proof. Let $\bar{u} \in \Upsilon$. From S_1, S_2 and S_3 satisfy Condition (A), for $x_n \in S_1 t_n$, $y_n \in S_2 v_n, z_n \in S_3 \rho_n$ and using (3.1), we obtain

$$\begin{aligned} \|t_n - \bar{u}\| &= \|u_n + \lambda_n(u_n - u_{n-1}) - \bar{u}\| \\ &\leq \|u_n - \bar{u}\| + \lambda_n\|u_n - u_{n-1}\|, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \|v_n - \bar{u}\|| &= \|(1 - a_n)t_n + a_n x_n - \bar{u}\| \\ &= \|(1 - a_n)(t_n - \bar{u}) + a_n(x_n - \bar{u})\| \\ &\leq (1 - a_n)\|t_n - \bar{u}\| + a_n\|x_n - \bar{u}\| \\ &= (1 - a_n)\|t_n - \bar{u}\| + a_n d(x_n, S_1\bar{u})\| \\ &\leq (1 - a_n)\|t_n - \bar{u}\| + a_n H(S_1 t_n, S_1\bar{u}) \\ &\leq (1 - a_n)\|t_n - \bar{u}\| + a_n\|t_n - \bar{u})\| \\ &= \|t_n - \bar{u}\| - a_n\|t_n - \bar{u}\| + a_n\|t_n - \bar{u})\| \\ &= \|t_n - \bar{u}\|, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \|\rho_n - \bar{u}\| &= \|(1 - b_n)x_n + b_n y_n - \bar{u}\| \\ &= \|(1 - b_n)(x_n - \bar{u}) + b_n(y_n - \bar{u})\| \\ &\leq (1 - b_n)\|x_n - \bar{u}\| + b_n\|y_n - \bar{u}\| \\ &= (1 - b_n)d(x_n, S_1\bar{u}) + b_n d(y_n, S_2\bar{u}) \\ &\leq (1 - b_n)H(S_1 t_n, S_1\bar{u}) + b_n H(S_2 v_n, S_2\bar{u}) \\ &\leq (1 - b_n)\|t_n - \bar{u}\| + b_n\|v_n - \bar{u}\| \\ &\leq (1 - b_n)\|t_n - \bar{u}\| + b_n\|t_n - \bar{u}\| \\ &= \|t_n - \bar{u}\| - b_n\|t_n - \bar{u}\| + b_n\|t_n - \bar{u}\| \\ &= \|t_n - \bar{u}\| \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \|u_{n+1} - \bar{u}\| &= \|(1 - c_n)y_n + c_n z_n - \bar{u}\| \\ &= \|(1 - c_n)(y_n - \bar{u}) + c_n(z_n - \bar{u})\| \\ &\leq (1 - c_n)\|y_n - \bar{u}\| + c_n\|z_n - \bar{u}\| \\ &= (1 - c_n)d(y_n, S_2\bar{u}) + c_n d(z_n, S_3\bar{u}) \\ &\leq (1 - c_n)H(S_2 v_n, S_2\bar{u}) + c_n H(S_3\rho_n, S_3\bar{u}) \\ &\leq (1 - c_n)\|v_n - \bar{u}\| + c_n\|\rho_n - \bar{u}\| \\ &\leq (1 - c_n)\|t_n - \bar{u}\| + c_n\|t_n - \bar{u}\| \\ &= \|t_n - \bar{u}\| - c_n\|t_n - \bar{u}\| + c_n\|t_n - \bar{u}\| \\ &= \|t_n - \bar{u}\| \\ &\leq \|u_n - \bar{u}\| + \lambda_n\|u_n - u_{n-1}\|. \end{aligned} \tag{3.5}$$

Using Lemma 2.6, (3.5) and the assumption (a), we have $\lim_{n\to\infty} \|u_n - \bar{u}\|$ exists. Thus, $\{u_n\}$ is bounded and also $\{\rho_n\}$, $\{v_n\}$ and $\{t_n\}$. From Lemma 2.7(b), we get

$$\begin{aligned}
\|v_n - \bar{u}\|^2 &= \|(1 - a_n)t_n + a_n x_n - \bar{u}\|^2 \\
&= \|(1 - a_n)(t_n - \bar{u}) + a_n(x_n - \bar{u})\|^2 \\
&= (1 - a_n)\|t_n - \bar{u}\|^2 + a_n\|x_n - \bar{u}\|^2 - a_n(1 - a_n)\|t_n - x_n\|^2 \\
&= (1 - a_n)\|t_n - \bar{u}\|^2 + a_n d(x_n, S_1\bar{u})^2 - a_n(1 - a_n)\|t_n - x_n\|^2 \\
&\leq (1 - a_n)\|t_n - \bar{u}\|^2 + a_n H(S_1 t_n, S_1\bar{u})^2 - a_n(1 - a_n)\|t_n - x_n\|^2 \\
&\leq (1 - a_n)\|t_n - \bar{u}\|^2 + a_n\|t_n - \bar{u}\|^2 - a_n(1 - a_n)\|t_n - x_n\|^2 \\
&= \|t_n - \bar{u}\|^2 - a_n(1 - a_n)\|t_n - x_n\|^2,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\|\rho_n - \bar{u}\|^2 &= \|(1 - b_n)x_n + b_n y_n - \bar{u}\|^2 \\
&= \|(1 - b_n)(x_n - \bar{u}) + b_n(y_n - \bar{u})\|^2 \\
&= (1 - b_n)\|x_n - \bar{u}\|^2 + b_n\|y_n - \bar{u}\|^2 - b_n(1 - b_n)\|x_n - y_n\|^2 \\
&= (1 - b_n)d(x_n, S_1\bar{u})^2 + b_n d(y_n, S_2\bar{u})^2 - b_n(1 - b_n)\|x_n - y_n\|^2 \\
&\leq (1 - b_n)H(S_1 t_n, S_1\bar{u})^2 + b_n H(S_2 v_n, S_2\bar{u})^2 - b_n(1 - b_n)\|x_n - y_n\|^2 \\
&\leq (1 - b_n)\|t_n - \bar{u}\|^2 + b_n\|v_n - \bar{u}\|^2 - b_n(1 - b_n)\|x_n - y_n\|^2 \\
&\leq (1 - b_n)\|t_n - \bar{u}\|^2 + b_n\|t_n - \bar{u}\|^2 - a_n(1 - a_n)b_n\|t_n - x_n\|^2 - b_n(1 - b_n)\|x_n - y_n\|^2 \\
&= \|t_n - \bar{u}\|^2 - a_n(1 - a_n)b_n\|t_n - x_n\|^2 - b_n(1 - b_n)\|x_n - y_n\|^2
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\|u_{n+1} - \bar{u}\|^2 &= \|(1 - c_n)y_n + c_n z_n - \bar{u}\|^2 \\
&= \|(1 - c_n)(y_n - \bar{u}) + c_n(z_n - \bar{u})\|^2 \\
&= (1 - c_n)\|y_n - \bar{u}\|^2 + c_n\|z_n - \bar{u}\|^2 - c_n(1 - c_n)\|y_n - z_n\|^2 \\
&= (1 - c_n)d(y_n, S_2\bar{u})^2 + c_n d(z_n, S_3\bar{u})^2 - c_n(1 - c_n)\|y_n - z_n\|^2 \\
&\leq (1 - c_n)H(S_2 v_n, S_2\bar{u})^2 + c_n H(S_3\rho_n, S_3\bar{u})^2 - c_n(1 - c_n)\|y_n - z_n\|^2 \\
&\leq (1 - c_n)\|v_n - \bar{u}\|^2 + c_n\|\rho_n - \bar{u}\|^2 - c_n(1 - c_n)\|y_n - z_n\|^2.
\end{aligned} \tag{3.8}$$

Combination (3.6)–(3.8), we get

$$\begin{aligned}
\|u_{n+1} - \bar{u}\|^2 \leq\ & (1 - c_n)\|t_n - \bar{u}\|^2 - a_n(1 - a_n)(1 - c_n)\|t_n - x_n\|^2 - c_n(1 - c_n)\|y_n - z_n\|^2 \\
& + c_n\|t_n - \bar{u}\|^2 - a_n(1 - a_n)b_n c_n\|t_n - x_n\|^2 - b_n(1 - b_n)c_n\|x_n - y_n\|^2 \\
\leq\ & \|t_n - \bar{u}\|^2 - a_n(1 - a_n)b_n(1 - c_n)\|t_n - x_n\|^2 \\
& - c_n(1 - c_n)\|y_n - z_n\|^2 - a_n(1 - a_n)b_n c_n\|t_n - x_n\|^2 \\
& - b_n(1 - b_n)c_n\|x_n - y_n\|^2 \\
\leq\ & \|u_n - \bar{u}\|^2 + 2\lambda_n\langle u_n - u_{n-1}, t_n - \bar{u}\rangle - a_n(1 - a_n)b_n(1 - c_n)\|t_n - x_n\|^2 \\
& - c_n(1 - c_n)\|y_n - z_n\|^2 - a_n(1 - a_n)b_n c_n\|t_n - x_n\|^2 \\
& - b_n(1 - b_n)c_n\|x_n - y_n\|^2 \\
\leq\ & \|u_n - \bar{u}\|^2 + 2\lambda_n\langle u_n - u_{n-1}, t_n - \bar{u}\rangle - a_n(1 - a_n)b_n\|t_n - x_n\|^2 \\
& - c_n(1 - c_n)\|y_n - z_n\|^2 - b_n(1 - b_n)c_n\|x_n - y_n\|^2.
\end{aligned} \tag{3.9}$$

The inequality (3.9) implies that

$$\begin{aligned} &a_n(1-a_n)b_n\|t_n-x_n\|^2+c_n(1-c_n)\|y_n-z_n\|^2+b_n(1-b_n)c_n\|x_n-y_n\|^2 \\ &\leq \|u_n-\bar{u}\|^2-\|u_{n+1}-\bar{u}\|^2+2\lambda_n\langle u_n-u_{n-1},t_n-\bar{u}\rangle. \end{aligned} \tag{3.10}$$

Using conditions (a)–(d), $\lim_{n\to\infty}\|u_n-\bar{u}\|$ exists and (3.10), we obtain

$$\lim_{n\to\infty}\|t_n-x_n\|=\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|y_n-z_n\|=0. \tag{3.11}$$

This implies that

$$\lim_{n\to\infty}\|t_n-u_n\|=\lambda_n\lim_{n\to\infty}\|u_n-u_{n-1}\|=0. \tag{3.12}$$

$$\lim_{n\to\infty}\|v_n-t_n\|=a_n\lim_{n\to\infty}\|t_n-x_n\|=0. \tag{3.13}$$

$$\lim_{n\to\infty}\|\rho_n-x_n\|=b_n\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.14}$$

Because $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k}\rightharpoonup\bar{u}$ some $\bar{u}\in\mathbb{K}$. From (3.12), we have $t_{n_k}\rightharpoonup\bar{u}$. Because $I-S_1$ is demiclosed at 0 and (3.11), we obtain $\bar{u}\in S_1\bar{u}$. From (3.13), we have $v_{n_k}\rightharpoonup\bar{u}$. Because $I-S_2$ is demiclosed at 0 and (3.11), we obtain $\bar{u}\in S_2\bar{u}$. It follows from (3.14) that $\rho_{n_k}\rightharpoonup\bar{u}$. Again, because $I-S_3$ is demiclosed at 0 and (3.11), we have $\bar{u}\in S_3\bar{u}$. This implies that $\bar{u}\in\Upsilon$. Now, we show that $\{u_n\}$ converges weakly to $\bar{u}$. We take another subsequence $\{u_{m_k}\}$ of $\{u_n\}$ converging weakly to some $u^*\in\Upsilon$. Because $\lim_{n\to\infty}\|u_n-\bar{u}\|$ exists and Lemma 2.3.Thus, we have $\bar{u}=u^*$. □

Theorem 3.2. *Let $\mathbb{K}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$ and $S_1, S_2, S_3:\mathbb{K}\to CB(\mathbb{K})$ be quasi-nonexpansive multivalued mappings with $\Upsilon:=F(S_1)\cap F(S_2)\cap F(S_3)\neq\emptyset$ and $I-S_i$ is demiclosed at 0 for all $i\in\{1,2,3\}$. Let $\{u_n\}$ be a sequence generated by*

$$\begin{cases} u_0,u_1\in\mathbb{K},\ \ \mathbb{K}_1=\mathbb{K}, \\ t_n=u_n+\lambda_n(u_n-u_{n-1}), \\ v_n\in(1-a_n)t_n+a_nS_1t_n, \\ \rho_n\in(1-b_n)S_1t_n+b_nS_2v_n, \\ w_n\in(1-c_n)S_2v_n+c_nS_3\rho_n, \\ \mathbb{K}_{n+1}=\{x\in\mathbb{K}_n:\|w_n-x\|^2\leq\|u_n-x\|^2+2\lambda_n^2\|u_n-u_{n-1}\|^2-2\lambda_n\langle u_n-x,u_{n-1}-u_n\rangle\}, \\ u_{n+1}=P_{\mathbb{K}_{n+1}}u_1, \end{cases} \tag{3.15}$$

for all $n\geq 1$, where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}\subset(0,1)$. Assume that the following conditions hold

(a) $\sum_{n=1}^{\infty}\lambda_n\|u_n-u_{n-1}\|<\infty$;
(b) $0<\liminf_{n\to\infty}a_n<\limsup_{n\to\infty}a_n<1$;
(c) $0<\liminf_{n\to\infty}b_n<\limsup_{n\to\infty}b_n<1$;
(d) $0<\liminf_{n\to\infty}c_n<\limsup_{n\to\infty}c_n<1$.

If S_1, S_2 and S_3 satisfy Condition (A), then the sequence $\{u_n\}$ converges strongly to a common fixed point of S_1, S_2 and S_3.

Proof. Step I. Show that $\{u_n\}$ is well defined. Using S_1, S_2 and S_3 satisfy Condition (A), Lemma 2.1, Υ is closed and convex. Firstly, we show that $\mathbb{K}_n$ is closed and convex for all $n \geq 1$. Since induction on n that $\mathbb{K}_n$ is closed and convex. For $n = 1$, $\mathbb{K}_1 = \mathbb{K}$ is closed and convex. Suppose that $\mathbb{K}_n$ is closed and convex for some $n \geq 1$. Using the definition $\mathbb{K}_{n+1}$ and Lemma 2.4, we have that $\mathbb{K}_{n+1}$ is closed and convex. Thus, $\mathbb{K}_n$ is closed and convex for all $n \geq 1$. Next, we show that $\Upsilon \subseteq \mathbb{K}_n$ for each $n \geq 1$. From Lemma 2.6(b) and S_1, S_2 and S_3 satisfy Condition (A), let $\bar{u} \in \Upsilon$ for $x_n \in S_1 t_n$, $y_n \in S_2 v_n, z_n \in S_3 \rho_n$ and using (3.15), we obtain

$$\begin{aligned}
\|v_n - \bar{u}\|^2 &= \|(1-a_n)t_n + a_n x_n - \bar{u}\|^2 \\
&= \|(1-a_n)(t_n - \bar{u}) + a_n(x_n - \bar{u})\|^2 \\
&= (1-a_n)\|t_n - \bar{u}\|^2 + a_n\|x_n - \bar{u}\|^2 - a_n(1-a_n)\|t_n - x_n\|^2 \\
&\leq (1-a_n)\|t_n - \bar{u}\|^2 + a_n\|x_n - \bar{u}\|^2 \\
&= (1-a_n)\|t_n - \bar{u}\|^2 + a_n d(x_n, S_1\bar{u})^2 \\
&\leq (1-a_n)\|t_n - \bar{u}\|^2 + a_n H(S_1 t_n, S_1\bar{u})^2 \\
&\leq (1-a_n)\|t_n - \bar{u}\|^2 + a_n\|t_n - \bar{u}\|^2 \\
&= \|t_n - \bar{u}\|^2,
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\|\rho_n - \bar{u}\|^2 &= \|(1-b_n)x_n + b_n y_n - \bar{u}\|^2 \\
&= \|(1-b_n)(x_n - \bar{u}) + b_n(y_n - \bar{u})\|^2 \\
&= (1-b_n)\|x_n - \bar{u}\|^2 + b_n\|y_n - \bar{u}\|^2 - b_n(1-b_n)\|x_n - y_n\|^2 \\
&\leq (1-b_n)\|x_n - \bar{u}\|^2 + b_n\|y_n - \bar{u}\|^2 \\
&= (1-b_n)d(x_n, S_1\bar{u})^2 + b_n d(y_n, S_2\bar{u})^2 \\
&\leq (1-b_n)H(S_1 t_n, S_1\bar{u})^2 + b_n H(S_2 v_n, S_2\bar{u})^2 \\
&\leq (1-b_n)\|t_n - \bar{u}\|^2 + b_n\|v_n - \bar{u}\|^2 \\
&\leq (1-b_n)\|t_n - \bar{u}\|^2 + b_n\|t_n - \bar{u}\|^2 \\
&= \|t_n - \bar{u}\|^2
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
\|w_n - \bar{u}\|^2 &= \|(1-c_n)y_n + c_n z_n - \bar{u}\|^2 \\
&= \|(1-c_n)(y_n - \bar{u}) + c_n(z_n - \bar{u})\|^2 \\
&= (1-c_n)\|y_n - \bar{u}\|^2 + c_n\|z_n - \bar{u}\|^2 - c_n(1-c_n)\|y_n - z_n\|^2 \\
&\leq (1-c_n)\|y_n - \bar{u}\|^2 + c_n\|z_n - \bar{u}\|^2 \\
&= (1-c_n)d(y_n, S_2\bar{u})^2 + c_n d(z_n, S_3\bar{u})^2 \\
&\leq (1-c_n)H(S_2 v_n, S_2\bar{u})^2 + c_n H(S_3\rho_n, S_3\bar{u})^2 \\
&\leq (1-c_n)\|v_n - \bar{u}\|^2 + c_n\|\rho_n - \bar{u}\|^2 \\
&\leq (1-c_n)\|t_n - \bar{u}\|^2 + c_n\|t_n - \bar{u}\|^2 \\
&= \|t_n - \bar{u}\|^2 \\
&= \|u_n + \lambda_n(u_n - u_{n-1}) - \bar{u}\|^2 \\
&\leq \|u_n - \bar{u}\|^2 + 2\lambda_n^2\|u_n - u_{n-1}\|^2 - 2\lambda_n\langle u_n - \bar{u}, u_n - u_{n-1}\rangle.
\end{aligned} \tag{3.18}$$

Therefore, from (3.18), $\bar{u} \in \mathbb{K}_n$, $n \geq 1$. This implies that $\Upsilon \subseteq \mathbb{K}_n$ for each $n \geq 1$, and hence, $\mathbb{K}_n \neq \emptyset$. Thus, $\{u_n\}$ is well defined.

Step II. Show that $u_n \to u \in \mathbb{K}$ as $n \to \infty$. Since $u_n \in P_{\mathbb{K}_n} u_1$, $\mathbb{K}_{n+1} \subseteq \mathbb{K}_n$, and $u_{n+1} \in \mathbb{K}_n$, we obtain

$$\|u_n - u_1\| \leq \|u_{n+1} - u_1\|, \quad \forall\, n \geq 1. \tag{3.19}$$

Since $\Upsilon \subseteq \mathbb{K}_n$, we obtain

$$\|u_n - u_1\| \leq \|x - u_1\|, \quad \forall\, n \geq 1, \tag{3.20}$$

for all $x \in \Upsilon$. The inequalities (3.19) and (3.20) imply that the sequence $\{u_n - u_1\}$ is bounded and non-decreasing. Therefore, $\lim_{n\to\infty} \|u_n - u_1\|$ exists.

For $m > n$, from the definition of $\mathbb{K}_n$, we obtain $u_m \in P_{\mathbb{K}_m} u_1 \in \mathbb{K}_m \subseteq \mathbb{K}_n$. Using Lemma 2.5, we have

$$\|u_m - u_n\|^2 \leq \|u_m - u_1\|^2 - \|u_n - u_1\|^2. \tag{3.21}$$

From $\lim_{n\to\infty} \|u_n - u_1\|$ exists and follows (3.21), we have that $\lim_{n\to\infty} \|u_n - u_m\| = 0$. Therefore, $\{u_n\}$ is a Cauchy sequence in $\mathbb{K}$, and so $u_n \to u \in \mathbb{K}$ as $n \to \infty$.

Step III. Show that $\lim_{n\to\infty} \|t_n - x_n\| = \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|y_n - z_n\| = 0$, where $x_n \in S_1 t_n$, $y_n \in S_2 v_n$ and $z_n \in S_3 \rho_n$. From Step II, we obtain $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$. Because $u_{n+1} \in \mathbb{K}_n$, we have that

$$\begin{aligned} \|w_n - u_n\| &\leq \|w_n - u_{n+1}\| + \|u_{n+1} - u_n\| \\ &\leq \sqrt{\|u_n - u_{n+1}\|^2 + 2\lambda_n^2 \|u_n - u_{n-1}\|^2 - 2\lambda_n \langle u_n - u_{n+1}, u_{n-1} - u_n \rangle} + \|u_{n+1} - u_n\|. \end{aligned} \tag{3.22}$$

Using the assumption (a) and (3.22), we have

$$\lim_{n\to\infty} \|w_n - u_n\| = 0. \tag{3.23}$$

Because S_1 satisfies condition (A) and using Lemma 2.7, we obtain

$$\|w_n - \bar{u}\|^2 \leq (1 - c_n)\|v_n - \bar{u}\|^2 + c_n \|\rho_n - \bar{u}\|^2 - c_n(1 - c_n)\|y_n - z_n\|^2. \tag{3.24}$$

Using (3.6), (3.7) and (3.24), we have

$$\begin{aligned} \|w_n - \bar{u}\|^2 &\leq (1 - c_n)\|t_n - \bar{u}\|^2 - a_n(1 - a_n)(1 - c_n)\|t_n - x_n\|^2 \\ &\quad + c_n\|t_n - \bar{u}\|^2 - a_n(1 - a_n)b_n c_n \|t_n - x_n\|^2 - b_n(1 - b_n)c_n\|x_n - y_n\|^2 \\ &\quad - c_n(1 - c_n)\|y_n - z_n\|^2 \\ &\leq (1 - c_n)\|t_n - \bar{u}\|^2 - a_n(1 - a_n)b_n(1 - c_n)\|t_n - x_n\|^2 \\ &\quad + c_n\|t_n - \bar{u}\|^2 - a_n(1 - a_n)b_n c_n \|t_n - x_n\|^2 - b_n(1 - b_n)c_n\|x_n - y_n\|^2 \\ &\quad - c_n(1 - c_n)\|y_n - z_n\|^2 \\ &= \|t_n - \bar{u}\|^2 - a_n(1 - a_n)b_n\|t_n - x_n\|^2 - b_n(1 - b_n)c_n\|x_n - y_n\|^2 \\ &\quad - c_n(1 - c_n)\|y_n - z_n\|^2 \\ &\leq \|u_n - \bar{u}\|^2 + 2\lambda_n\langle u_n - u_{n-1}, t_n - \bar{u}\rangle - a_n(1 - a_n)b_n\|t_n - x_n\|^2 \\ &\quad - b_n(1 - b_n)c_n\|x_n - y_n\|^2 - c_n(1 - c_n)\|y_n - z_n\|^2. \end{aligned} \tag{3.25}$$

The inequality (3.25) implies that

$$\begin{aligned}&a_n(1-a_n)b_n\|t_n-x_n\|^2+b_n(1-b_n)c_n\|x_n-y_n\|^2+c_n(1-c_n)\|y_n-z_n\|^2\\&\leq\|u_n-\bar{u}\|^2-\|w_n-\bar{u}\|^2+2\lambda_n\langle u_n-u_{n-1},t_n-\bar{u}\rangle.\end{aligned}\tag{3.26}$$

From conditions (a)–(d), (3.23) and (3.25), we have (3.11). From (3.13), (3.14) and the same proof in Theorem 3.1, we have

$$\lim_{n\to\infty}\|t_n-u_n\|=\lim_{n\to\infty}\|v_n-t_n\|=\lim_{n\to\infty}\|\rho_n-x_n\|=0.\tag{3.27}$$

From Step II, we know that $u_n\to u\in\mathbb{K}$. It follows (3.27), we obtain that $t_n\to u$. Because $I-S_1$ is demiclosed at 0, we have $u\in F(S_1)$. In the same way, we have that $u\in F(S_2)$ and $u\in F(S_3)$. This implies that $u\in\Upsilon$.

Step IV. Show that $u=P_\Upsilon u_1$. From $u\in\Upsilon$ and (3.19), we obtain

$$\|u-u_1\|\leq\|x-u_1\|,\quad\forall\, x\in\Upsilon.$$

Using the definition of the projection operator, we can conclude that $u=P_\Upsilon u_1$. □

Theorem 3.3. *Let $\mathbb{K}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$ and $S_1, S_2, S_3:\mathbb{K}\to CB(\mathbb{K})$ be quasi-nonexpansive multivalued mappings with $\Upsilon:=F(S_1)\cap F(S_2)\cap F(S_3)\neq\emptyset$ and $I-S_i$ is demiclosed at 0 for all $i\in\{1,2,3\}$. Let $\{u_n\}$ be a sequence generated by*

$$\begin{cases}u_0,u_1\in\mathbb{K},\ R_1=\mathbb{K},\\t_n=u_n+\lambda_n(u_n-u_{n-1}),\\v_n\in(1-a_n)t_n+a_nS_1t_n,\\\rho_n\in(1-b_n)S_1t_n+b_nS_2v_n,\\w_n\in(1-c_n)S_2v_n+c_nS_3\rho_n,\\\mathbb{K}_n=\{x\in\mathbb{K}:\|w_n-x\|^2\leq\|u_n-x\|^2+2\lambda_n^2\|u_n-u_{n-1}\|^2-2\lambda_n\langle u_n-x,u_{n-1}-u_n\rangle\},\\R_n=\{x\in R_{n-1}:\langle u_1-u_n,u_n-x\rangle\geq0\},\\u_{n+1}=P_{\mathbb{K}_n\cap R_n}u_1,\end{cases}\tag{3.28}$$

for all $n\geq1$, where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}\subset(0,1)$. Assume that the following conditions hold

(a) $\sum_{n=1}^{\infty}\lambda_n\|u_n-u_{n-1}\|<\infty$;
(b) $0<\liminf_{n\to\infty}a_n<\limsup_{n\to\infty}a_n<1$;
(c) $0<\liminf_{n\to\infty}b_n<\limsup_{n\to\infty}b_n<1$;
(d) $0<\liminf_{n\to\infty}c_n<\limsup_{n\to\infty}c_n<1$.

If S_1, S_2 and S_3 satisfy Condition (A), then the sequence $\{u_n\}$ converges strongly to a common fixed point of S_1, S_2 and S_3.

Proof. From the same method of Theorem 3.2 step by step, we can conclude the proof by replacing $\mathbb{K}_{n+1}$ by $\mathbb{K}_n$, expect in Step 1. Showing that $\Upsilon\subseteq\mathbb{K}_n$ for each $n\geq1$. Next, we show that $\Upsilon\subseteq R_n$ for all $n\geq1$. Indeed, by mathematical induction, for $n=1$, we obtain $\Upsilon\subseteq\mathbb{K}=R_1$. Suppose that $\Upsilon\subseteq R_n$ for all $n\geq1$. Because u_{n+1} is the projection of u_1 onto $\mathbb{K}_n\cap R_n$, we obtain

$$\langle u_1-u_{n+1},u_{n+1}-x\rangle\geq0,\quad\forall\, x\in\mathbb{K}_n\cap R_n.$$

Therefore, $\Upsilon \subseteq \mathbb{K}_{n+1}$. Hence, $\Upsilon \subseteq \mathbb{K}_n \cap R_n$. This implies that $\{u_n\}$ is well defined.

Next, we show that $u_n \to q \in \mathbb{K}$ as $n \to \infty$. Using the definition of R_n, we obtain $u_n = P_{R_n} u_1$. Because $u_{n+1} \in R_n$, we have the inequality (3.19) and

$$\|u_n - u_1\| \leq \|q - u_1\|, \quad \forall\, q \in \Upsilon. \tag{3.29}$$

From (3.19) and (3.29) we have the sequence $\{u_n - u_1\}$ is bounded and non-decreasing, and so $\lim_{n\to\infty} \|u_n - u_1\|$ exists. For $m > n$, by definition of R_n, we have $u_m = P_{R_m} u_1 \in R_m \subseteq R_n$. Using Lemma 2.5, we have (3.21). Because $\lim_{n\to\infty} \|u_n - u_1\|$ exists, it follows (3.21), we obtain $\lim_{n\to\infty} \|u_m - u_n\| = 0$. Therefore, $\{u_n\}$ is a Cauchy sequence in $\mathbb{K}$, and hence $u_n \to q \in \mathbb{K}$ as $n \to \infty$. In fact, we have $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$. Using the same proof of Steps 3 and 4 in Theorem 3.2, we obtain $q = P_\Upsilon u_1$. □

4. Numerical results

It is commonly known that computing the projection of a point on an intersection is quite difficult. However, this can also be stated as the following optimization problem for computing purposes

$$P_{\mathbb{K}^*} := \min_{x \in \mathbb{K}^*} \|x - u\|^2, \tag{4.1}$$

where $\mathbb{K}^* = \mathbb{K}_n \cap R_n$. See [22] for a list of several more approaches to handle projection onto intersection of sets computationally.

The set $\mathbb{K}_{n+1}$ can be found by $\mathbb{K}_n \cap R_n$, where

$$R_n = \{x \in \mathbb{H} : \|w_n - x\|^2 \leq \|u_n - x\|^2 + 2\lambda_n^2 \|u_n - u_{n-1}\|^2 - 2\lambda_n \langle u_n - x, u_{n-1} - u_n \rangle\}. \tag{4.2}$$

The projection can be thought of as the following optimization problem by point (4.2):

$$P_{\mathbb{K}_{n+1}} := \min_{x \in \mathbb{K}_{n+1}} \|x - u\|^2, \tag{4.3}$$

where $\mathbb{K}_{n+1} = \mathbb{K}_n \cap R_n$.

Example. Let $\mathbb{H} = \mathbb{R}^3$ and $\mathbb{K} = [2, 5]^3$.
Let $\mathbb{K}_1 = \{u = (u_1, u_2, u_3) \in \mathbb{R}^3 : \sqrt{(u_1 - 5)^2 + (u_2 - 5)^2 + (u_3 - 5)^2} \leq 2\}$. We defined $S_1, S_2, S_3 : \mathbb{R}^3 \to CB(\mathbb{R}^3)$ as:

$$S_1 u = \begin{cases} \{(5, 5, 5)\} & \text{if } u \in \mathbb{K}_1, \\ \{v = (v_1, v_2, v_3) \in \mathbb{K} : \sqrt{(v_1 - 5)^2 + (v_2 - 5)^2 + (v_3 - 5)^2} \leq \dfrac{1}{\|u\|_1}\} & \text{otherwise,} \end{cases}$$

$$S_2 u = \begin{cases} \{(5, 5, 5)\} & \text{if } u \in \mathbb{K}_1, \\ \{v = (5, v_2, 5) \in \mathbb{K} : v_2 \in [(u_2 + 5)(\dfrac{\arctan(19u_2 - 65)}{2}) + u_2, 5]\} & \text{otherwise,} \end{cases}$$

and

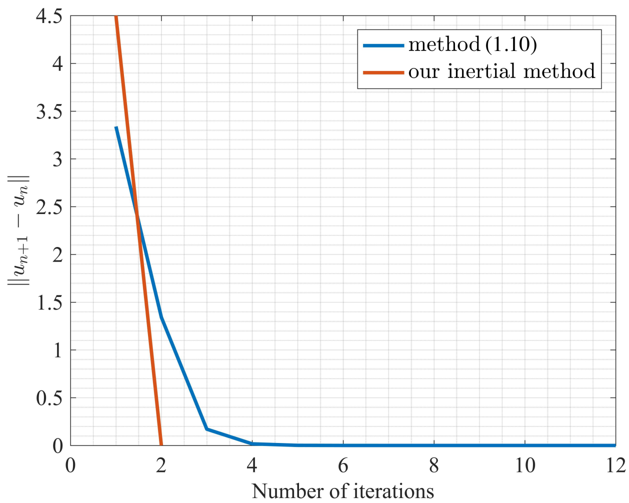
$$S_3 u = \begin{cases} \{(5, 5, 5)\} & \text{if } u \in \mathbb{K}_1, \\ \{v = (5, 5, v_3) \in \mathbb{K} : v_3 \in [(u_2 - 5)(\dfrac{\sin(19u_2 - 10)}{5}) + u_2, 5]\} & \text{otherwise.} \end{cases}$$

We see that S_1, S_2 and S_3 are quasi-nonexpansive and $F(S_1) \cap F(S_2) \cap F(S_3) = \{(5, 5, 5)\}$. Let $a_n = \frac{n+4}{5n+5}$, $b_n = \frac{n+2}{7n+2}$, $c_n = \frac{7n+7}{9n+9}$ and

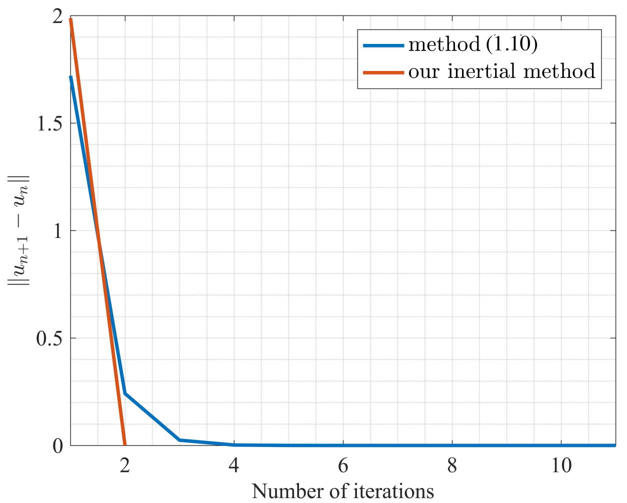
$$\lambda_n = \begin{cases} \min\left\{\dfrac{1}{(n+1)^2\|u_n - u_{n-1}\|}, 0.035\right\} & \text{if } u_n \neq u_{n-1}, \\ 0.035 & \text{otherwise.} \end{cases}$$

We compare a numerical test between our inertial method defined in Theorem 3.3 and method (1.10). The stopping criterion is defined by $\|u_{n+1} - u_n\| < 10^{-10}$. We make different choices of the initial values x_0 and x_1 as follows (see Figure 1 and Table 1)

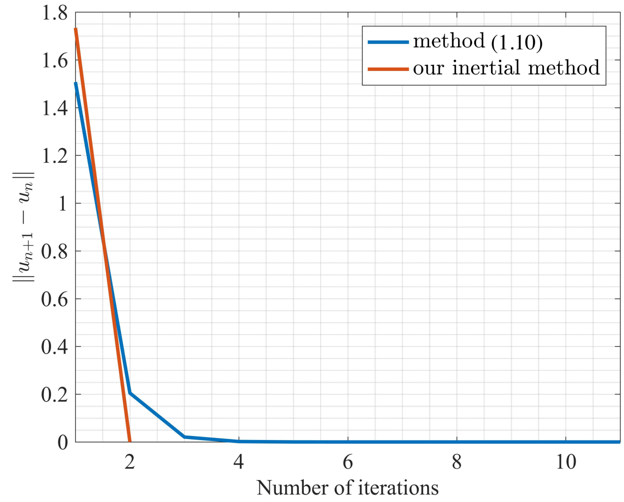
Case 1 : $x_0 = (2.6816, 2.4389, 2.8891)$ and $x_1 = (2.7733, 2.2146, 2.2555)$.
Case 2 : $x_0 = (3.9587, 4.6121, 2.9779)$ and $x_1 = (3.7123, 3.4894, 4.8867)$.
Case 3 : $x_0 = (4.9401, 3.9274, 2.8475)$ and $x_1 = (3.8675, 3.7185, 4.7133)$.
Case 4 : $x_0 = (3.9933, 4.9899, 4.2187)$ and $x_1 = (3.7722, 3.6190, 2.9152)$.



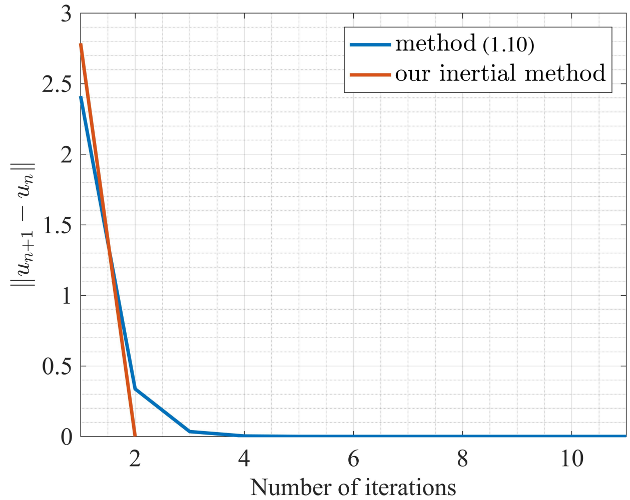
Case 1



Case 2



Case 3



Case 4

Figure 1. Valued of $\|u_{n+1} - u_n\|$ in Cases 1–4.

Table 1. Numerical results.

Case		our inertial method	method (1.10)
1	CPU time (sec)	0.01	0.03
	Number of Iterations	2	12
2	CPU time (sec)	0.04	0.12
	Number of Iterations	2	11
3	CPU time (sec)	0.03	0.09
	Number of Iterations	2	11
4	CPU time (sec)	0.01	0.08
	Number of Iterations	2	11

5. Conclusions

In this paper, we proved convergence theorems in Hilbert spaces using a modified D-iteration.We proved the weak and strong convergence of the iterative algorithms to the common fixed point under some suitable assumptions.

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Conflict of interest

The authors declare that they have no competing interests.

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