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Research article

Common fixed point results via $\mathcal{A}_\vartheta$-α-contractions with a pair and two pairs of self-mappings in the frame of an extended quasi b-metric space

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Abstract: In this paper, we take advantage of implicit relationships to come up with a new concept called "$\mathcal{A}_\vartheta$-α-contraction mapping". We utilized our new notion to formulate and prove some common fixed point theorems for two and four self-mappings over complete extended quasi b-metric spaces under a set of conditions. Our main results widen and improve many existing results in the literature. To support our research, we present some examples as applications to our main findings.

Keywords: common fixed point; extended quasi b-metric space; $\mathcal{A}_\vartheta$-α-contractions; compatible mapping
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1. Introduction

Banach's fixed point theory, also referred to as "the contraction mapping theorem", is one of the most significant sources of existence and uniqueness theorems in many areas of analysis, it ensures the existence of a unique fixed point for self-mappings under appropriate contraction conditions over complete metric spaces.

In recent years some author have succeeded in obtaining many fixed and common fixed point findings for different classes of mappings by weakening their hypothesis or changing the Lipschitz constant to real valued functions such that their values are less than unity or in some other way by extending the fixed and common fixed point results from metric spaces to another spaces such as quasi metric spaces, cone metric spaces, b-metric space etc.

Some authors used A-contraction to introduce some new results, see [3–6, 12]. Nurwahyu et al. [15] studied some fixed points for mapping of cyclic form. Recently, Ali et al. [7, 9, 10] applied the dynamic iteration to generate some new findings. Also, Shatanawi et al. [21,26] have linked some known results to cone metric spaces. Shatanawi [20] also studied some fixed point results in orbitally metric spaces. Some researchers have used ω-distance to obtain new results, see [1, 2, 16, 24]. Also, others have obtained results on b-metric spaces [11, 14, 18, 22, 25], extended b-metric spaces [17, 27] and quasi metric spaces, see [19,23,28,30]. Very recently, Ali et al. [8] have obtained new results on generalized θ_b-contractions. Song et al. [29] utilized fuzzy sets for having their own results on fuzzy metric spaces.

In the current paper, we introduce the concept of $\mathcal{A}_\vartheta$-α-contractions. We then take advantage of our new concept to formulate and prove some common fixed point results for self-mappings in the frame of an extended quasi b-metric space.

2. Preliminaries

The purpose of this section is to collect the basic concepts from literature about extended quasi b-metric spaces, which we will need in our current work.

Definition 2.1. *[30] A quasi metric space $(\mathcal{Y}, d)$ consists of a non-empty set $\mathcal{Y}$ and a function d: $\mathcal{Y} \times \mathcal{Y} \longrightarrow [0, \infty)$ such that*

(1) $d(\mu, \nu) = 0$ if $\mu = \nu$, $\forall \mu, \nu \in \mathcal{Y}$.

(2) $d(\mu, \omega) \leq d(\mu, \nu) + d(\nu, \omega)$, $\forall \mu, \nu, \omega \in \mathcal{Y}$.

A function d that satisfies the above conditions is called a quasi-metric.

Definition 2.2. *[15] On a nonempty set $\mathcal{Y} \times \mathcal{Y}$, let $\theta : \mathcal{Y} \times \mathcal{Y} \longrightarrow [1, \infty)$ be a function. A function d_θ: $\mathcal{Y} \times \mathcal{Y} \longrightarrow [0, \infty)$ is called an extended quasi b-metric if for all $\mu, \nu, \omega \in \mathcal{Y}$, we have*

(1) $d_\theta(\mu, \nu) = 0$ if $\mu = \nu$.

(2) $d_\theta(\mu, \omega) \leq \theta(\mu, \omega)[d_\theta(\mu, \nu) + d_\theta(\nu, \omega)]$.

The pair $(\mathcal{Y}, d_\theta)$ is referred to as an extended quasi b-metric space.

If d_θ satisfying (1) *and* (2) *with $\theta = s \geq 1$, then $(\mathcal{Y}, d_s)$ is referred to as a quasi b-metric space with parameter s.*

Example 2.1. *[15] Consider $\mathcal{Y} = [0, 1]$, define θ: $\mathcal{Y} \times \mathcal{Y} \to [1, \infty)$ by $\theta(\mu, \nu) = 2^{1-(\mu+\nu)/2}$ and d_θ: $\mathcal{Y} \times \mathcal{Y} \to [0, \infty)$ by $d_\theta(\mu, \nu) = |2^{\mu-\nu} - 1|$. Then $(\mathcal{Y}, d_\theta)$ is an extended quasi b-metric space.*

Definition 2.3. *[27] Let (μ_n) be a sequence in the extended quasi b-metric space $(\mathcal{Y}, d_\theta)$. Then, we say that (μ_n) converges to $\mu \in \mathcal{Y}$ if*

$$\lim_{n\to\infty} d_\theta(\mu_n, \mu) = \lim_{n\to\infty} d_\theta(\mu, \mu_n) = 0.$$

Definition 2.4. *[27] Let (μ_n) be a sequence in the extended quasi b-metric space $(\mathcal{Y}, d_\theta)$. Then, we say that*

(1) (μ_n) is left-Cauchy if and only if for every $\zeta > 0$, $\exists N \in \mathbb{N}$ such that $d_\theta(\mu_n, \mu_\iota) < \zeta$ for all $n \geq \iota > N$.

(2) (μ_n) is right-Cauchy if and only if for every $\zeta > 0$, $\exists N \in \mathbb{N}$ such that $d_\theta(\mu_\iota, \mu_n) < \zeta$ for all $n \geq \iota > N$.

(3) (μ_n) is Cauchy if and only if it is left-Cauchy and right-Cauchy.

Definition 2.5. *[27] Let $(\mathcal{Y}, d_\theta)$ be an extended quasi b-metric space. Then, we say that*

(1) $(\mathcal{Y}, d_\theta)$ is left-complete if and only if each left-Cauchy sequence in $\mathcal{Y}$ converges.

(2) $(\mathcal{Y}, d_\theta)$ is right-complete if and only if each right-Cauchy sequence in $\mathcal{Y}$ converges.

(3) $(\mathcal{Y}, d_\theta)$ is complete if and only if each Cauchy sequence in $\mathcal{Y}$ converges.

We adopt [13, 15] to generate the following definition:

Definition 2.6. *Let $(\mathcal{Y}, d_\theta)$ be an extended quasi b-metric space. Two self-mappings f and ℓ on $\mathcal{Y}$ are said to be compatible if $\lim\limits_{n\to\infty} d_\theta(f\ell\mu_n, \ell f\mu_n) = 0$ and $\lim\limits_{n\to\infty} d_\theta(\ell f\mu_n, f\ell\mu_n) = 0$ when (μ_n) is a sequence in $\mathcal{Y}$ such that $\lim\limits_{n\to\infty} f\mu_n = \lim\limits_{n\to\infty} \ell\mu_n = \nu$ for some $\nu \in \mathcal{Y}$.*

Akram and Siddiqui [4] introduced a new class of functions, denoted by A, as follows: $\tau \in A$ if τ: $\mathbb{R}_+^3 \to \mathbb{R}_+$ satisfies the following assertions:

(i) τ is continuous on $\mathbb{R}_+^3$.

(ii) $\kappa_1 \leq \lambda\eta_1$ for some $\lambda \in [0, 1)$, when $\kappa_1 \leq \tau(\kappa_1, \eta_1, \eta_1)$ or $\kappa_1 \leq \tau(\eta_1, \kappa_1, \eta_1)$ or $\kappa_1 \leq \tau(\eta_1, \eta_1, \kappa_1)$ for $\kappa_1, \eta_1 \in \mathbb{R}_+$.

Akram and Siddiqui [4] took advantage of class A to introduce a new concept of contractions called A-contraction as follows:

Definition 2.7. *[4] On a metric space $(\mathcal{Y}, d)$, a self-mapping ℓ is called A-contraction if there exists $\tau \in A$ such that*

$$d(\ell\mu, \ell\nu) \leq \tau(d(\mu, \nu), d(\mu, \ell\mu), d(\nu, \ell\nu))$$

holds for all for all $\mu, \nu \in \mathcal{Y}$.

Based on the above definitions, we extend the class of contraction into a new class known as $\mathcal{A}_\vartheta$, from which we derive some common fixed point theorems, as described in the next section.

3. Main results

In this section, we introduce a new concept of contractions called $\mathcal{A}_\vartheta$-α-contraction. We then take advantage of our concept to prove the existence and uniqueness common fixed point for self-mappings in complete extended quasi b-metric spaces.

To begin our work, we introduce a new class of functions, denoted by $\mathcal{A}_\vartheta$, as follows: ϑ: $\mathbb{R}_+^4 \to \mathbb{R}_+ \in \mathcal{A}_\vartheta$ if ϑ satisfies the following conditions:

(i) ϑ is continuous.

(ii) ϑ is non-decreasing in all of its variables.

(iii) If $\kappa \leq \vartheta(\eta, \kappa, \eta, \alpha(\kappa + \eta))$, $\kappa \leq \vartheta(\kappa, \eta, \eta, \alpha(\kappa + \eta))$, or $\kappa \leq \vartheta(\eta, \eta, \kappa, \alpha(\kappa + \eta))$ for $\kappa, \eta \in \mathbb{R}_+$ and $\alpha \in (0, 1)$, then $\kappa \leq \lambda\eta$ for some $\lambda \in [0, 1)$.

Example 3.1. *Define the function ϑ: $\mathbb{R}_+^4 \to \mathbb{R}_+$ by $\vartheta(\kappa, \eta, \grave{\kappa}, \grave{\eta}) = \frac{1}{6}(\kappa + \eta + \grave{\kappa} + \grave{\eta})$. Then $\vartheta \in \mathcal{A}_\vartheta$.*

Proof. Note that the function ϑ is well-defined. Moreover, one can easily see (i) and (ii) are satisfied. To prove (iii), we assume that $\kappa \leq \vartheta(\eta, \kappa, \eta, \alpha(\kappa+\eta))$, $\kappa \leq \vartheta(\kappa, \eta, \eta, \alpha(\kappa+\eta))$, or $\kappa \leq \vartheta(\eta, \eta, \kappa, \alpha(\kappa+\eta))$ for some $\alpha \in (0,1)$. Take $\lambda = \frac{2+\alpha}{5-\alpha}$, then $\lambda < \frac{3}{4}$. Moreover, with few calculations, one can prove that $\kappa \leq \lambda\eta$.

We have provided the background needed to initiate a new contraction, called $\mathcal{A}_\vartheta$-contraction:

Definition 3.1. *Let $\alpha \in (0,1)$, $(\mathcal{Y}, d_\theta)$ be an extended quasi b-metric space and f, ℓ be two self-mappings on $\mathcal{Y}$. Then the pair (f, ℓ) is said to be $\mathcal{A}_\vartheta$-α-contraction if there exists $\vartheta \in \mathcal{A}_\vartheta$ such that*

$$d_\theta(f\mu, \ell\nu) \leq \vartheta\Big(\alpha\theta(\mu,\nu)d_\theta(\mu,\nu), \alpha\theta(f\mu,\mu)d_\theta(f\mu,\mu), \alpha\theta(\ell\nu,\nu)d_\theta(\ell\nu,\nu), \alpha^3\theta(f\mu,\nu)d_\theta(f\mu,\nu)\Big) \tag{3.1}$$

and

$$d_\theta(\ell\nu, f\mu) \leq \vartheta\Big(\alpha\theta(\nu,\mu)d_\theta(\nu,\mu), \alpha\theta(\mu,f\mu)d_\theta(\mu,f\mu), \alpha\theta(\nu,\ell\nu)d_\theta(\nu,\ell\nu), \alpha^3\theta(\nu,f\mu)d_\theta(\nu,f\mu)\Big) \tag{3.2}$$

hold for all $\mu, \nu \in \mathcal{Y}$

Now, we will present and prove our main result.

Theorem 3.1. *Let $\mathcal{Y}$ be a nonempty set, $\alpha \in (0,1)$, $(\mathcal{Y}, d_\theta)$ be a complete extended quasi b-metric space and f, ℓ be two self-mappings on $\mathcal{Y}$. Assume the followings:*

(1) ℓ is continuous.

(2) $\theta(\kappa,\eta) \leq \frac{1}{\alpha}$ for all $\kappa, \eta \in \mathcal{Y}$.

(3) d_θ is continuous in its variables.

If (f,ℓ) is $\mathcal{A}_\vartheta$-α-contraction, then f and ℓ have a unique common fixed point provided that $\lambda < \alpha$, where λ is the constant satisfies condition (iii) of the definition $\mathcal{A}_\vartheta$.

Proof. Choose $\mu_0 \in \mathcal{Y}$. Take $\mu_1 = f\mu_0$ and $\mu_2 = \ell\mu_1$. Then, we construct a sequence (μ_n) as follows:

$$\mu_{2n+1} = f\mu_{2n} \text{ and } \mu_{2n+2} = \ell\mu_{2n+1}, \forall n \in \mathbb{N} \cup \{0\}.$$

Now, we verify that (μ_n) is left Cauchy. Look at

$$d_\theta(\mu_{2n+1}, \mu_{2n}) = d_\theta(f\mu_{2n}, \ell\mu_{2n-1}).$$

Using $\mathcal{A}_\vartheta$-α-contraction condition, we get

$$\begin{aligned} d_\theta(f\mu_{2n}, \ell\mu_{2n-1}) \leq\ & \vartheta\Big(\alpha\theta(\mu_{2n},\mu_{2n-1})d_\theta(\mu_{2n},\mu_{2n-1}), \alpha\theta(f\mu_{2n},\mu_{2n})d_\theta(f\mu_{2n},\mu_{2n}), \\ & \alpha\theta(\ell\mu_{2n-1},\mu_{2n-1})d_\theta(\ell\mu_{2n-1},\mu_{2n-1}), \alpha^3\theta(f\mu_{2n},\mu_{2n-1})d_\theta(f\mu_{2n},\mu_{2n-1})\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} d_\theta(\mu_{2n+1}, \mu_{2n}) \leq\ & \vartheta\Big(\alpha\theta(\mu_{2n},\mu_{2n-1})d_\theta(\mu_{2n},\mu_{2n-1}), \alpha\theta(\mu_{2n+1},\mu_{2n})d_\theta(\mu_{2n+1},\mu_{2n}), \\ & \alpha\theta(\mu_{2n},\mu_{2n-1})d_\theta(\mu_{2n},\mu_{2n-1}), \alpha^3\theta(\mu_{2n+1},\mu_{2n-1})d_\theta(\mu_{2n+1},\mu_{2n-1})\Big). \end{aligned}$$

Since θ is bounded by $\frac{1}{\alpha}$, we obtain

$$d_\theta(\mu_{2n+1},\mu_{2n}) \leq \vartheta\Big(d_\theta(\mu_{2n},\mu_{2n-1}), d_\theta(\mu_{2n+1},\mu_{2n}), d_\theta(\mu_{2n},\mu_{2n-1}), \alpha^2 d_\theta(\mu_{2n+1},\mu_{2n-1})\Big).$$

Due to the triangular inequality, we obtain

$$\begin{aligned} d_\theta(\mu_{2n+1},\mu_{2n}) &\leq \vartheta\Big(d_\theta(\mu_{2n},\mu_{2n-1}), d_\theta(\mu_{2n+1},\mu_{2n}), d_\theta(\mu_{2n},\mu_{2n-1}),\\ &\quad \alpha^2\theta(\mu_{2n+1},\mu_{2n-1})(d_\theta(\mu_{2n+1},\mu_{2n}) + d_\theta(\mu_{2n},\mu_{2n-1})\Big)\\ &\leq \vartheta\Big(d_\theta(\mu_{2n},\mu_{2n-1}), d_\theta(\mu_{2n+1},\mu_{2n}), d_\theta(\mu_{2n},\mu_{2n-1}), \alpha(d_\theta(\mu_{2n+1},\mu_{2n}) + d_\theta(\mu_{2n},\mu_{2n-1}))\Big). \end{aligned}$$

By putting $\kappa = d_\theta(\mu_{2n+1},\mu_{2n})$ and $\eta = d_\theta(\mu_{2n},\mu_{2n-1})$, we obtain $\kappa \leq \vartheta(\eta,\kappa,\eta,\alpha(\kappa+\eta))$. Thus we have $\kappa \leq \lambda\eta$ for some $\lambda \in [0,1)$. Hence

$$d_\theta(\mu_{2n+1},\mu_{2n}) \leq \lambda d_\theta(\mu_{2n},\mu_{2n-1}).$$

By induction, we get

$$d_\theta(\mu_{2n+1},\mu_{2n}) \leq \lambda d_\theta(\mu_{2n},\mu_{2n-1}) \leq \lambda^2 d_\theta(\mu_{2n-1},\mu_{2n-2}) \ldots \leq \lambda^{2n} d_\theta(\mu_1,\mu_0).$$

Thus

$$d_\theta(\mu_{2n+1},\mu_{2n}) \leq \lambda^{2n} d_\theta(\mu_1,\mu_0).$$

From here one can show that (μ_n) is left Cauchy. Similarly, we can show that (μ_n) is right Cauchy. As a result, (μ_n) is Cauchy. So (μ_n) converges to μ, for some $\mu \in \mathcal{Y}$; that is,

$$\lim_{n\to\infty} d_\theta(\mu_{2n},\mu) = \lim_{n\to\infty} d_\theta(\mu,\mu_{2n}) = 0 = d_\theta(\mu,\mu),$$

and

$$\lim_{n\to\infty} d_\theta(\mu_{2n-1},\mu) = \lim_{n\to\infty} d_\theta(\mu,\mu_{2n-1}) = 0 = d_\theta(\mu,\mu).$$

Claim. μ is a common fixed point of f and ℓ. Look at

$$d_\theta(f\mu,\mu_{2n}) = d_\theta(f\mu,\ell\mu_{2n-1}).$$

Since (f,ℓ) is $\mathcal{A}_\vartheta$-α-contraction, then

$$\begin{aligned} d_\theta(f\mu,\ell\mu_{2n-1}) &\leq \vartheta\Big(\alpha\theta(\mu,\mu_{2n-1})d_\theta(\mu,\mu_{2n-1}), \alpha\theta(f\mu,\mu)d_\theta(f\mu,\mu), \alpha\theta(\ell\mu_{2n-1},\mu_{2n-1})d_\theta(\ell\mu_{2n-1},\mu_{2n-1}),\\ &\quad \alpha^3\theta(f\mu,\mu_{2n-1})d_\theta(f\mu,\mu_{2n-1})\Big)\\ &= \vartheta\Big(\alpha\theta(\mu,\mu_{2n-1})d_\theta(\mu,\mu_{2n-1}), \alpha\theta(f\mu,\mu)d_\theta(f\mu,\mu), \alpha\theta(\mu_{2n},\mu_{2n-1})d_\theta(\mu_{2n},\mu_{2n-1}),\\ &\quad \alpha^3\theta(f\mu,\mu_{2n-1})d_\theta(f\mu,\mu_{2n-1})\Big). \end{aligned}$$

Since θ is bounded by $\frac{1}{\alpha}$ and by the triangular inequality, we get

$$d_\theta(f\mu,\mu_{2n}) \leq \vartheta\Big(d_\theta(\mu,\mu_{2n-1}), d_\theta(f\mu,\mu), d_\theta(\mu_{2n},\mu_{2n-1}), \alpha(d_\theta(f\mu,\mu) + d_\theta(\mu,\mu_{2n-1}))\Big). \tag{3.3}$$

By allowing $n \longrightarrow \infty$ in Inequality (3.3), the continuity of ϑ and d_θ in their variables implies that

$$d_\theta(f\mu,\mu) \quad \leq \quad \vartheta\Big(0, d_\theta(f\mu,\mu), 0, \alpha(d_\theta(f\mu,\mu)+0)\Big).$$

By taking $\kappa = d_\theta(f\mu,\mu)$ and $\eta = 0$, then $d_\theta(f\mu,\mu) \leq \lambda 0 = 0$ for some $\lambda \in [0,1)$. As a result, μ is a fixed point of f.

Due to the continuity of ℓ and the continuity of d_θ in its variables, we have

$$d_\theta(\ell\mu,\mu) = \lim_{n\to\infty} d_\theta(\ell\mu,\mu_{2n}) = \lim_{n\to\infty} d_\theta(\ell\mu,\ell\mu_{2n-1}) = d_\theta(\ell\mu,\ell\mu) = 0,$$

which implies that $\ell\mu = \mu$. So μ is a fixed point of ℓ.

Finally, to demonstrate the uniqueness, suppose μ^* is another common fixed point of f and ℓ such that $\mu^* \neq \mu$. So, we have

$$d_\theta(\mu,\mu^*) \quad = \quad d_\theta(f\mu,\ell\mu^*).$$

Since (f,ℓ) is $\mathcal{A}_\vartheta$-α-contraction, then

$$\begin{aligned} d_\theta(f\mu,\ell\mu^*) &\leq \vartheta\Big(\alpha\theta(\mu,\mu^*)d_\theta(\mu,\mu^*), \alpha\theta(f\mu,\mu)d_\theta(f\mu,\mu), \alpha\theta(\ell\mu^*,\mu^*)d_\theta(\ell\mu^*,\mu^*), \alpha^3\theta(f\mu,\mu^*)d_\theta(f\mu,\mu^*)\Big) \\ &\leq \vartheta\Big(d_\theta(\mu,\mu^*), d_\theta(\mu,\mu), d_\theta(\mu^*,\mu^*), \alpha(d_\theta(f\mu,\mu)+d_\theta(\mu,\mu^*))\Big) \\ &= \vartheta\Big(d_\theta(\mu,\mu^*), 0, 0, \alpha(d_\theta(\mu,\mu^*)+0)\Big). \end{aligned}$$

By taking $\kappa = d_\theta(\mu,\mu^*)$ and $\eta = 0$, then $\kappa \leq \lambda\eta$ for some $\lambda \in [0,1)$. Therefore

$$d_\theta(\mu,\mu^*) \leq \lambda 0 = 0.$$

Hence $\mu = \mu^*$, a contradiction. Thus μ is a unique common fixed point of f and ℓ.

We support our result with the following example, e denotes the Euler's number and π denotes the ratio of a circle's circumference to its diameter.

Example 3.2. *On $\mathcal{Y} = [0,1]$, define the mapping θ: $\mathcal{Y}\times\mathcal{Y} \to [1,\infty)$ via $\theta = 1 + |\mu - \nu|$ and define d_θ: $\mathcal{Y}\times\mathcal{Y} \to [0,\infty)$ via $d_\theta = |\mu - \nu|$. Then $(\mathcal{Y}, d_\theta)$ is a complete extended quasi b-metric space.*

Define the mappings f,ℓ: $\mathcal{Y} \to \mathcal{Y}$ by $f(\mu) = \frac{\mu^2}{20\mu^2+17\pi}$ and $\ell(\nu) = \frac{1}{50}\sin(\nu)$. Also, define ϑ: $\mathbb{R}_+^4 \to \mathbb{R}_+$ by

$$\vartheta(\kappa,\eta,\grave{\kappa},\grave{\eta}) = \frac{\sqrt{12}}{6(\sqrt{12}+\sqrt{e})}(\kappa+\eta+\grave{\kappa}+\grave{\eta}), \forall \kappa,\eta,\grave{\kappa},\grave{\eta} \in \mathbb{R}_+.$$

Note that ϑ is continuous and non-decreasing in all of its variables. Now, assume that

$$\kappa \leq \vartheta(\kappa,\eta,\eta,\frac{\sqrt{e}}{\sqrt{12}}(\kappa+\eta)), \kappa \leq \vartheta(\eta,\kappa,\eta,\frac{\sqrt{e}}{\sqrt{12}}(\kappa+\eta))$$

or

$$\kappa \leq \vartheta(\eta,\eta,\kappa,\frac{\sqrt{e}}{\sqrt{12}}(\kappa+\eta)) \, for \, \kappa,\eta \in \mathcal{Y}.$$

Take

$$\lambda = \frac{(2\sqrt{12}+\sqrt{e})}{5(\sqrt{12}+\sqrt{e})},$$

with few calculations, we find $\kappa \leq \lambda\eta$ for $\kappa, \eta \in \mathcal{Y}$. Indeed,

$$\begin{aligned}
\kappa &\leq \frac{\sqrt{12}}{6(\sqrt{12}+\sqrt{e})}(\kappa + 2\eta + \frac{\sqrt{e}}{\sqrt{12}}(\kappa+\eta)) \\
&= \frac{\sqrt{12}}{6(\sqrt{12}+\sqrt{e})}\Big(\frac{\kappa(\sqrt{12}+\sqrt{e})}{\sqrt{12}} + \frac{\eta(2\sqrt{12}+\sqrt{e})}{\sqrt{12}}\Big) \\
&= \frac{\kappa}{6} + \frac{\eta(2\sqrt{12}+\sqrt{e})}{6(\sqrt{12}+\sqrt{e})}.
\end{aligned}$$

Thus,

$$\kappa \leq \frac{(2\sqrt{12}+\sqrt{e})}{5(\sqrt{12}+\sqrt{e})}\eta = \lambda\eta.$$

Note that we can easily figure out:

(1) ℓ is continuous.

(2) $\theta(\mu, \nu) \leq \frac{1}{\alpha} = \frac{\sqrt{12}}{\sqrt{e}}$ for all $\mu, \nu \in \mathcal{Y}$.

(3) d_θ is continuous on its variables.

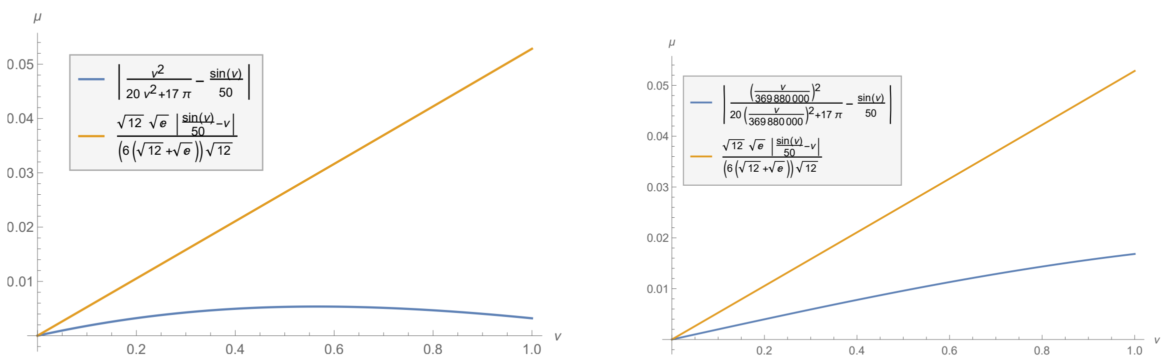
Given $\mu, \nu \in \mathcal{Y}$, with $\nu > \mu$, let $n \in [1, +\infty)$ such that $\mu = \frac{\nu}{n}$. Then

$$\begin{aligned}
d_\theta(f\mu, \ell\nu) &= \Big|\frac{\mu^2}{20\mu^2 + 17\pi} - \frac{1}{50}\sin(\nu)\Big| \\
&= \Big|\frac{(\frac{\nu}{n})^2}{20(\frac{\nu}{n})^2 + 17\pi} - \frac{1}{50}\sin(\nu)\Big|.
\end{aligned}$$

From Figure 1 (a and b), we deduce the following inequality:

$$\Big|\frac{(\frac{\nu}{n})^2}{20(\frac{\nu}{n})^2 + 17\pi} - \frac{1}{50}\sin(\nu)\Big| \leq \Big(\frac{\sqrt{12}}{6(\sqrt{12}+\sqrt{e})}\Big)\Big(\frac{\sqrt{e}}{\sqrt{12}}\Big)\Big|\frac{1}{50}\sin(\nu) - \nu\Big|.$$

For $n = 1$ For $n = 369880000$



(a) Comparison between $\Big|\frac{\nu^2}{20\nu^2+17\pi} - \frac{1}{50}\sin(\frac{\nu}{2})\Big|$ and $\Big(\frac{\sqrt{12}}{6(\sqrt{12}+\sqrt{e})}\Big)\Big(\frac{\sqrt{e}}{\sqrt{12}}\Big)\Big|\frac{1}{50}\sin(\nu) - \nu\Big|$.

(b) Comparison between $\Big|\frac{(\frac{\nu}{369880000})^2}{20(\frac{\nu}{369880000})^2+17\pi} - \frac{1}{50}\sin(\nu)\Big|$ and $\Big(\frac{\sqrt{12}}{6(\sqrt{12}+\sqrt{e})}\Big)\Big(\frac{\sqrt{e}}{\sqrt{12}}\Big)\Big|\frac{1}{50}\sin(\nu) - \nu\Big|$

Figure 1. Comparison between two functions.

Therefore

$$
\begin{aligned}
d_\theta(f\mu,\ell\nu) &\leq \Big(\frac{\sqrt{12}}{6(\sqrt{12}+\sqrt{e})}\Big)\Big(\frac{\sqrt{e}}{\sqrt{12}}\Big)\Big|\frac{1}{50}\sin(\nu)-\nu\Big| \\
&= \Big(\frac{\sqrt{12}}{6(\sqrt{12}+\sqrt{e})}\Big)\alpha d_\theta(\ell\nu,\nu) \\
&\leq \Big(\frac{\sqrt{12}}{6(\sqrt{12}+\sqrt{e})}\Big)\alpha\theta(\ell\nu,\nu)d_\theta(\ell\nu,\nu).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d_\theta(f\mu,\ell\nu) &\leq \frac{\sqrt{12}}{6(\sqrt{12}+\sqrt{e})}\Big(\alpha\theta(\mu,\nu)d_\theta(\mu,\nu)+\alpha\theta(f\mu,\mu)d_\theta(f\mu,\mu)+\alpha\theta(\ell\nu,\nu)d_\theta(\ell\nu,\nu) \\
&\quad +\alpha^3\theta(f\mu,\nu)d_\theta(f\mu,\nu)\Big) \\
&= \vartheta\Big(\alpha\theta(\mu,\nu)d_\theta(\mu,\nu),\alpha\theta(f\mu,\mu)d_\theta(f\mu,\mu),\alpha\theta(\ell\nu,\nu)d_\theta(\ell\nu,\nu),\alpha^3\theta(f\mu,\nu)d_\theta(f\mu,\nu)\Big).
\end{aligned}
$$

On a similar manner, we can get

$$
d_\theta(\ell\nu,f\mu) \leq \vartheta\Big(\alpha\theta(\nu,\mu)d_\theta(\nu,\mu),\alpha\theta(\mu,f\mu)d_\theta(\mu,f\mu),\alpha\theta(\nu,\ell\nu)d_\theta(\nu,\ell\nu),\alpha^3\theta(\nu,f\mu)d_\theta(\nu,f\mu)\Big).
$$

Thus (f,l) is $\mathcal{A}_\vartheta$-α-contraction. So all conditions of Theorem 3.1 are satisfied. Hence f and l have a common fixed point. Here, 0 is the unique common fixed of f and 0.

Corollary 3.1. *On the complete quasi b-metric space $(\mathcal{Y},d_s)$, let f and ℓ be two self-mappings on $\mathcal{Y}$. Suppose there exist $\alpha\in(0,1)$ and $\vartheta\in\mathcal{A}_\vartheta$ with*

$$
d_s(f\mu,\ell\nu)\leq\vartheta\Big(\alpha s d_s(\mu,\nu),\alpha s d_s(f\mu,\mu),\alpha s d_s(\ell\nu,\nu),\alpha^3 s d_s(f\mu,\nu)\Big)
$$

and

$$
d_s(\ell\nu,f\mu)\leq\vartheta\Big(\alpha s d_s(\nu,\mu),\alpha s d_s(\mu,f\mu),\alpha s d_s(\nu,\ell\nu),\alpha^3 s d_s(\nu,f\mu)\Big)
$$

hold for all $\mu,\nu\in\mathcal{Y}$. Then f and ℓ have a unique common fixed point in $\mathcal{Y}$ provided that ℓ is continuous, $s\leq\frac{1}{\alpha}$ and $\lambda<\alpha$, where λ is the constant satisfies condition (iii) of the definition $\mathcal{A}_\vartheta$.

Proof. The desired result will be obtained from Theorem (3.1) by defining $\theta:\mathcal{Y}\times\mathcal{Y}\to[1,+\infty)$ via $\theta(\kappa,\eta)=s$, $s\geq1$.

Corollary 3.2. *On the complete extended quasi b-metric space $(\mathcal{Y},d_\theta)$, let f be a continuous self-mapping on $\mathcal{Y}$. Assume there exist $\alpha\in(0,1)$ and $\vartheta\in\mathcal{A}_\vartheta$ such that*

$$
d_\theta(f\mu,f\nu)\leq\vartheta\Big(\alpha\theta(\mu,\nu)d_\theta(\mu,\nu),\alpha\theta(f\mu,\mu)d_\theta(f\mu,\mu),\alpha\theta(f\nu,\nu)d_\theta(f\nu,\nu),\alpha^3\theta(f\mu,\nu)d_\theta(f\mu,\nu)\Big)
$$

and

$$
d_\theta(f\nu,f\mu)\leq\vartheta\Big(\alpha\theta(\nu,\mu)d_\theta(\nu,\mu),\alpha\theta(\mu,f\mu)d_\theta(\mu,f\mu),\alpha\theta(\nu,f\nu)d_\theta(\nu,f\nu),\alpha^2\theta(\nu,f\mu)d_\theta(\nu,f\mu)\Big)
$$

hold for all $\mu,\nu\in\mathcal{Y}$. Then f has a unique fixed point in $\mathcal{Y}$ provided that θ is bounded by $\frac{1}{\alpha}$ and $\lambda<\alpha$, where λ is the constant satisfies condition (iii) of the definition $\mathcal{A}_\vartheta$.

Proof. The desired result will be obtained from Theorem (3.1) by taking $\ell = f$.

Corollary 3.3. *On the complete quasi b-metric space* $(\mathcal{Y}, d_s)$*, let* f *be a continuous mapping on* $\mathcal{Y}$*. Suppose there exist* $\alpha \in (0,1)$ *and* $\vartheta \in \mathcal{A}_\vartheta$ *such that*

$$d_s(f\mu, f\nu) \leq \vartheta\Big(\alpha s d_s(\mu,\nu), \alpha s d_s(f\mu,\mu), \alpha s d_s(f\nu,\nu), \alpha^3 s d_s(f\mu,\nu)\Big)$$

and

$$d_s(f\nu, f\mu) \leq \vartheta\Big(\alpha s d_s(\nu,\mu), \alpha s d_s(\mu,f\mu), \alpha s d_s(\nu,f\nu), \alpha^3 s d_s(\nu,f\mu)\Big)$$

hold for all $\mu, \nu \in \mathcal{Y}$*.*

If $s \leq \frac{1}{\alpha}$*, then the mapping* f *has a unique fixed point in* $\mathcal{Y}$ *provided that* $\lambda < \alpha$*, where* λ *is the constant satisfies condition (iii) of the definition* $\mathcal{A}_\vartheta$*.*

Proof. The desired result will be obtained from Corollary (3.1) by taking $f = \ell$. Our second main result for four self-mappings is as follows:

Theorem 3.2. *Let* $\mathcal{Y}$ *be a nonempty set,* $\alpha \in (0,1)$*,* $(\mathcal{Y}, d_\theta)$ *be a complete extended quasi b-metric space, and* f, l, g *and* h *be four self mappings on* $\mathcal{Y}$*. Assume the following conditions:*

(1) $f(\mathcal{Y}) \subseteq h(\mathcal{Y})$ *and* $g(\mathcal{Y}) \subseteq l(\mathcal{Y})$*.*

(2) f *or* l *is continuous.*

(3) $\theta(\kappa,\eta) \leq \frac{1}{\alpha}$ *for all* $\kappa, \eta \in \mathcal{Y}$*.*

(4) d_θ *is continuous in its variables.*

(5) (f,l) *and* (g,h) *are compatible.*

(6) There exists $\vartheta \in \mathcal{A}_\theta$ *such that*

$$\begin{aligned} d_\theta(f\mu, g\nu) \quad \leq \quad & \vartheta\Big(\alpha\theta(l\mu,h\nu)d_\theta(l\mu,h\nu), \alpha\theta(l\mu,f\mu)d_\theta(l\mu,f\mu), \alpha\theta(h\nu,g\nu)d_\theta(h\nu,g\nu), \\ & \alpha^3\theta(l\mu,g\nu)d_\theta(l\mu,g\nu)\Big), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} d_\theta(g\nu, f\mu) \quad \leq \quad & \vartheta\Big(\alpha\theta(h\nu,l\mu)d_\theta(h\nu,l\mu), \alpha\theta(f\mu,l\mu)d_\theta(f\mu,l\mu), \alpha\theta(g\nu,h\nu)d_\theta(g\nu,h\nu), \\ & \alpha^3\theta(g\nu,l\mu)d_\theta(g\nu,l\mu)\Big), \end{aligned} \tag{3.5}$$

hold for all $\mu, \nu \in \mathcal{Y}$*.*

Then f, l, g *and* h *have a unique common fixed point in* $\mathcal{Y}$ *provided that* $\lambda < \alpha$*, where* λ *is the constant satisfies condition (iii) of the definition* $\mathcal{A}_\vartheta$*.*

Proof. Choose μ_0 in $\mathcal{Y}$. Since $f(\mathcal{Y}) \subseteq h(\mathcal{Y})$ and $g(\mathcal{Y}) \subseteq l(\mathcal{Y})$, then $\exists \mu_1, \mu_2$ in $\mathcal{Y}$ such that $f\mu_0 = h\mu_1$, $g\mu_1 = l\mu_2$. By continuing this process, we construct a sequence (ν_n) in $\mathcal{Y}$ as follows:

$$\nu_{2n} = h\mu_{2n+1} = f\mu_{2n} \text{ and } \nu_{2n+1} = l\mu_{2n+2} = g\mu_{2n+1}.$$

By Condition (3.4), we get

$$\begin{aligned}
d_\theta(\nu_{2n},\nu_{2n+1}) &= d_\theta(f\mu_{2n},g\mu_{2n+1})\\
&\le \vartheta\Big(\alpha\theta(l\mu_{2n},h\mu_{2n+1})d_\theta(l\mu_{2n},h\mu_{2n+1}),\alpha\theta(l\mu_{2n},f\mu_{2n})d_\theta(l\mu_{2n},f\mu_{2n}),\\
&\quad \alpha\theta(h\mu_{2n+1},g\mu_{2n+1})d_\theta(h\mu_{2n+1},g\mu_{2n+1}),\alpha^3\theta(l\mu_{2n},g\mu_{2n+1})d_\theta(l\mu_{2n},g\mu_{2n+1})\Big)\\
&= \vartheta\Big(\alpha\theta(\nu_{2n-1},\nu_{2n})d_\theta(\nu_{2n-1},\nu_{2n}),\alpha\theta(\nu_{2n-1},\nu_{2n})d_\theta(\nu_{2n-1},\nu_{2n}),\\
&\quad \alpha\theta(\nu_{2n},\nu_{2n+1})d_\theta(\nu_{2n},\nu_{2n+1}),\alpha^3\theta(\nu_{2n-1},\nu_{2n+1})d_\theta(\nu_{2n-1},\nu_{2n+1})\Big).
\end{aligned}$$

Since θ is bounded by $\frac{1}{\alpha}$ and due to the triangular inequality of d_θ, we get

$$d_\theta(\nu_{2n},\nu_{2n+1}) \le \vartheta\Big(d_\theta(\nu_{2n-1},\nu_{2n}),d_\theta(\nu_{2n-1},\nu_{2n}),d_\theta(\nu_{2n},\nu_{2n+1}),\alpha\big(d_\theta(\nu_{2n-1},\nu_{2n})+d_\theta(\nu_{2n},\nu_{2n+1})\big)\Big).$$

By putting $\kappa = d_\theta(\nu_{2n},\nu_{2n+1})$ and $\eta = d_\theta(\nu_{2n-1},\nu_{2n})$, we obtain $\kappa \le \vartheta(\eta,\eta,\kappa,\alpha(\kappa+\eta))$. Hence $\kappa \le \lambda\eta$ for some $\lambda \in [0,1)$; that is

$$d_\theta(\nu_{2n},\nu_{2n+1}) \le \lambda d_\theta(\nu_{2n-1},\nu_{2n}).$$

Hence, we have

$$d_\theta(\nu_{2n},\nu_{2n+1}) \le \lambda d_\theta(\nu_{2n-1},\nu_{2n}) \le \lambda^2 d_\theta(\nu_{2n-2},\nu_{2n-1})\ldots \le \lambda^{2n} d_\theta(\nu_0,\nu_1).$$

Thus

$$d_\theta(\nu_{2n},\nu_{2n+1}) \le \lambda^{2n} d_\theta(\nu_0,\nu_1).$$

From here one can show that (ν_n) is right-Cauchy. On the same way, we can prove that (ν_n) is left-Cauchy. As a result, (ν_n) is a Cauchy sequence. So $\exists\varsigma \in \mathcal{Y}$ such that

$$\lim_{n\to\infty} h\mu_{2n+1} = \lim_{n\to\infty} f\mu_{2n} = \lim_{n\to\infty} l\mu_{2n+2} = \lim_{n\to\infty} g\mu_{2n+1} = \varsigma.$$

Claim. ς is a common fixed point for f,g,h and l. If l is continuous, then

$$\lim_{n\to\infty} lf\mu_{2n} = l\varsigma.$$

Since (f,l) is compatible, then $\lim_{n\to\infty} d_\theta(lf\mu_{2n},fl\mu_{2n}) = 0$. The triangular inequality of d_θ implies

$$\begin{aligned}
d_\theta(fl\mu_{2n},l\varsigma) &\le \theta(fl\mu_{2n},l\varsigma)\Big(d_\theta(fl\mu_{2n},lf\mu_{2n})+d_\theta(lf\mu_{2n},l\varsigma)\Big)\\
&\le \frac{1}{\alpha}\Big(d_\theta(fl\mu_{2n},lf\mu_{2n})+d_\theta(lf\mu_{2n},l\varsigma)\Big).
\end{aligned}$$

Letting $n\to\infty$ and recalling the continuity of d_θ, we get

$$\lim_{n\to\infty} d_\theta(fl\mu_{2n},l\varsigma) = 0.$$

Thus

$$\lim_{n\to\infty} fl\mu_{2n} = l\varsigma.$$

Now

$$\begin{aligned} d_\theta(l\varsigma, f\varsigma) &\le \theta(l\varsigma, f\varsigma)(d_\theta(l\varsigma, lf\mu_{2n}) + d_\theta(lf\mu_{2n}, f\varsigma)) \\ &\le \theta(l\varsigma, f\varsigma)d_\theta(l\varsigma, lf\mu_{2n}) + \theta(l\varsigma, f\varsigma)\theta(lf\mu_{2n}, f\varsigma)(d_\theta(lf\mu_{2n}, fl\mu_{2n}) + d_\theta(fl\mu_{2n}, f\varsigma)). \end{aligned}$$

By letting $n \to \infty$ in above inequalities, we arrive at $d_\theta(l\varsigma, f\varsigma) = 0$. Thus, ς is a coincidence point for f and l in $\mathcal{Y}$. Let $\mu = \varsigma$ and $\nu = \mu_{2n+1}$ in Inequality (3.4), we obtain

$$\begin{aligned} d_\theta(f\varsigma, g\mu_{2n+1}) \le\ & \vartheta\Big(\alpha\theta(l\varsigma, h\mu_{2n+1})d_\theta(l\varsigma, h\mu_{2n+1}), \alpha\theta(l\varsigma, f\varsigma)d_\theta(l\varsigma, f\varsigma), \\ & \alpha\theta(h\mu_{2n+1}, g\mu_{2n+1})d_\theta(h\mu_{2n+1}, g\mu_{2n+1}), \alpha^3\theta(l\varsigma, g\mu_{2n+1})d_\theta(l\varsigma, g\mu_{2n+1})\Big). \end{aligned}$$

By using triangle inequality and keeping in our account that θ is bounded by $\frac{1}{\alpha}$, we find

$$d_\theta(f\varsigma, g\mu_{2n+1}) \le \vartheta\Big(d_\theta(l\varsigma, h\mu_{2n+1}), d_\theta(l\varsigma, f\varsigma), d_\theta(h\mu_{2n+1}, g\mu_{2n+1}), \alpha\Big(d_\theta(l\varsigma, \varsigma) + d_\theta(\varsigma, g\mu_{2n+1})\Big)\Big).$$

By allowing $n \to \infty$ in above inequality, we obtain

$$\begin{aligned} d_\theta(l\varsigma, \varsigma) &\le \vartheta\Big(d_\theta(l\varsigma, \varsigma), d_\theta(l\varsigma, f\varsigma), d_\theta(\varsigma, \varsigma), \alpha\Big(d_\theta(l\varsigma, \varsigma) + d_\theta(\varsigma, \varsigma)\Big)\Big) \\ &\le \vartheta\Big(d_\theta(l\varsigma, \varsigma), 0, 0, \alpha\Big(d_\theta(l\varsigma, \varsigma) + 0\Big)\Big). \end{aligned}$$

By putting $\kappa = d_\theta(l\varsigma, \varsigma)$ and $\eta = 0$, we obtain that $\kappa \le \vartheta(\kappa, 0, 0, \alpha(\kappa + 0))$. Hence $\kappa \le \lambda\eta$ for some $\lambda \in [0, 1)$; that is

$$d_\theta(l\varsigma, \varsigma) \le \lambda 0.$$

Thus $l\varsigma = \varsigma$ and hence $f\varsigma = \varsigma$. Since $f(\mathcal{Y}) \subseteq h(\mathcal{Y})$, there exists $p \in \mathcal{Y}$ such that $\varsigma = f\varsigma = hp$.
By putting $\mu = \mu_{2n}$ and $\nu = p$ in Inequality (3.4), we obtain

$$\begin{aligned} d_\theta(f\mu_{2n}, gp) \le\ & \vartheta\Big(\alpha\theta(l\mu_{2n}, hp)d_\theta(l\mu_{2n}, hp), \alpha\theta(l\mu_{2n}, f\mu_{2n})d_\theta(l\mu_{2n}, f\mu_{2n}), \\ & \alpha\theta(hp, gp)d_\theta(hp, gp), \alpha^3\theta(l\mu_{2n}, gp)d_\theta(l\mu_{2n}, gp)\Big). \end{aligned}$$

Through the triangle inequality, given that θ is bounded by $\frac{1}{\alpha}$, we get

$$d_\theta(f\mu_{2n}, gp) \le \vartheta\Big(d_\theta(l\mu_{2n}, hp), d_\theta(l\mu_{2n}, f\mu_{2n}), d_\theta(hp, gp), \alpha(d_\theta(l\mu_{2n}, \varsigma) + d_\theta(\varsigma, gp))\Big).$$

By allowing $n \to \infty$ and since $hp = \varsigma$, we obtain

$$\begin{aligned} d_\theta(\varsigma, gp) &\le \vartheta\Big(d_\theta(\varsigma, \varsigma), d_\theta(\varsigma, \varsigma), d_\theta(\varsigma, gp), \alpha(d_\theta(\varsigma, \varsigma) + d_\theta(\varsigma, gp))\Big) \\ &= \vartheta\Big(0, 0, d_\theta(\varsigma, gp), \alpha(0 + d_\theta(\varsigma, gp))\Big). \end{aligned}$$

By putting $\kappa = d_\theta(\varsigma, gp)$ and $\eta = 0$, we get $\kappa \le \vartheta(0, 0, \kappa, \alpha(\kappa + 0))$. So $\kappa \le \lambda\eta$ for some $\lambda \in [0, 1)$; that is

$$d_\theta(\varsigma, gp) \le \lambda 0.$$

Therefore $gp = \varsigma$ and hence $gp = hp = \varsigma$.

Now,

$$\begin{aligned} d_\theta(h\varsigma, g\varsigma) &\leq \theta(h\varsigma, g\varsigma)(d_\theta(h\varsigma, hgp) + d_\theta(hgp, g\varsigma)) \\ &\leq \theta(h\varsigma, g\varsigma)d_\theta(h\varsigma, hgp) + \theta(h\varsigma, g\varsigma)\theta(hgp, g\varsigma)(d_\theta(hgp, ghp) + d_\theta(ghp, g\varsigma)). \end{aligned}$$

Since θ is bounded by $\frac{1}{\alpha}$ and (g, h) is compatible, we have $d_\theta(h\varsigma, g\varsigma) = 0$. Thus, $h\varsigma = g\varsigma$.

By putting $\mu = \mu_{2n}$ and $\nu = \varsigma$ in Inequality (3.4), we obtain

$$\begin{aligned} d_\theta(f\mu_{2n}, g\varsigma) \leq\ & \vartheta\Big(\alpha\theta(l\mu_{2n}, h\varsigma)d_\theta(l\mu_{2n}, h\varsigma), \alpha\theta(l\mu_{2n}, f\mu_{2n})d_\theta(l\mu_{2n}, f\mu_{2n}), \\ & \alpha\theta(h\varsigma, g\varsigma)d_\theta(h\varsigma, g\varsigma), \alpha^3\theta(l\mu_{2n}, g\varsigma)d_\theta(l\mu_{2n}, g\varsigma)\Big). \end{aligned}$$

Through the triangle inequality, given that θ bounded by $\frac{1}{\alpha}$, we get

$$d_\theta(f\mu_{2n}, g\varsigma) \leq \vartheta\Big(d_\theta(l\mu_{2n}, h\varsigma), d_\theta(l\mu_{2n}, f\mu_{2n}), d_\theta(h\varsigma, g\varsigma), \alpha\Big(d_\theta(l\mu_{2n}, \varsigma) + d_\theta(\varsigma, g\varsigma)\Big)\Big).$$

By allowing $n \to \infty$ in above inequality, we get

$$\begin{aligned} d_\theta(\varsigma, g\varsigma) &\leq \vartheta\Big(d_\theta(\varsigma, g\varsigma), d_\theta(\varsigma, \varsigma), d_\theta(g\varsigma, g\varsigma), \alpha\Big(d_\theta(\varsigma, \varsigma) + d_\theta(\varsigma, g\varsigma)\Big)\Big) \\ &\leq \vartheta\Big(d_\theta(\varsigma, g\varsigma), 0, 0, \alpha\Big(0 + d_\theta(\varsigma, g\varsigma)\Big)\Big). \end{aligned}$$

Setting $\kappa = d_\theta(\varsigma, g\varsigma)$ and $\eta = 0$, we obtain that $\kappa \leq \vartheta(\kappa, 0, 0, \alpha(\kappa + 0))$. Thus $\kappa \leq \lambda\eta$ for some $\lambda \in [0, 1)$; that is

$$d_\theta(\varsigma, g\varsigma) \leq \lambda 0.$$

So $g\varsigma = \varsigma$ and hence $g\varsigma = h\varsigma = \varsigma$. Therefore $l\varsigma = f\varsigma = g\varsigma = h\varsigma = \varsigma$.

For the sake of uniqueness, assume that $\varsigma*$ is another common fixed point for f, g, h and l such that $\varsigma* \neq \varsigma$. Then

$$\begin{aligned} d_\theta(\varsigma, \varsigma^*) &= d_\theta(f\varsigma, g\varsigma^*) \\ &\leq \vartheta\Big(\alpha\theta(l\varsigma, h\varsigma^*)d_\theta(l\varsigma, h\varsigma^*), \alpha\theta(l\varsigma, f\varsigma)d_\theta(l\varsigma, f\varsigma), \\ &\qquad \alpha\theta(h\varsigma^*, g\varsigma^*)d_\theta(h\varsigma^*, g\varsigma^*), \alpha^3\theta(l\varsigma, g\varsigma^*)d_\theta(l\varsigma, g\varsigma^*)\Big) \\ &\leq \vartheta\Big(d_\theta(\varsigma, \varsigma^*), d_\theta(\varsigma, \varsigma), d_\theta(\varsigma^*, \varsigma^*), \alpha(d_\theta(\varsigma, \varsigma) + d_\theta(\varsigma, \varsigma^*))\Big) \\ &\leq \vartheta\Big(d_\theta(\varsigma, \varsigma^*), 0, 0, \alpha(0 + d_\theta(\varsigma, \varsigma^*))\Big). \end{aligned}$$

Thus, we conclude that

$$d_\theta(\varsigma, \varsigma^*) \leq 0.$$

So $\varsigma = \varsigma^*$, a contradiction. Thus f, g, h and l have a unique common fixed point.

Now, we support Theorem 3.2 with the following example:

Example 3.3. *On the same complete space of Example* (3.2). *Define the mappings* $f, g, l, h : \mathcal{Y} \to \mathcal{Y}$ *by*

$$f(\mu) = \frac{1}{12}\sin\Big(\frac{\mu}{4}\Big),\ l(\mu) = \frac{6}{7}\mu,\ g(\mu) = \frac{\mu}{9\mu + 110} \text{ and } h(\mu) = \frac{3}{7}\mu.$$

Define ϑ: $\mathbb{R}_+^4 \to \mathbb{R}_+$ *by*

$$\vartheta(\kappa, \eta, \grave{\kappa}, \grave{\eta}) = \frac{1}{11}(\kappa + \eta + \grave{\kappa} + \grave{\eta}) \, for \; all \; \kappa, \eta, \grave{\kappa}, \grave{\eta} \in R_+.$$

It is clear that ϑ *is continuous and non-decreasing in all of its variables. Now, suppose that*

$$\kappa \leq \vartheta(\kappa, \eta, \eta, \frac{2}{5}(\kappa + \eta)) \, for \; \kappa, \eta \in \mathcal{Y}, \kappa \leq \vartheta(\eta, \kappa, \eta, \frac{2}{5}(\kappa + \eta))$$

or

$$\kappa \leq \vartheta(\eta, \eta, \kappa, \frac{2}{5}(\kappa + \eta)) \, for \; \kappa, \eta \in \mathcal{Y} \; for \; \kappa, \eta \in \mathcal{Y}.$$

Then, with a few calculations, we get $\kappa \leq \frac{1}{4}\eta$ *for* $\kappa, \eta \in \mathcal{Y}$.
Note that we can easily figure out:

(1) $f(\mathcal{Y}) \subseteq h(\mathcal{Y})$ *and* $g(\mathcal{Y}) \subseteq l(\mathcal{Y})$.

(2) f, g, h *and* l *are continuous.*

(3) $\theta(\mu, \nu) \leq \frac{1}{\alpha} = \frac{5}{2}$ *for all* $\mu, \nu \in \mathcal{Y}$.

(4) d_θ *is continuous in its variables.*

To show that (f, l) *is compatible, let* (μ_n) *be a sequence in* $\mathcal{Y}$ *such that*

$$\lim_{n\to\infty} f(\mu_n) = \lim_{n\to\infty} l(\mu_n) = \nu$$

for some $\nu \in \mathcal{Y}$. *So*

$$l\mu_n = \frac{6}{7}\mu_n \to \nu \; and \; f\mu_n = \frac{1}{12}\sin(\frac{\mu_n}{4}) \to \nu.$$

Therefore $\mu_n \to \frac{7}{6}\nu$ *and* $\sin(\frac{\mu_n}{4}) \to 12\nu$. *Hence* $\sin(\frac{\mu_n}{4}) \to \sin(\frac{7}{24}\nu)$ *and* $\sin(\frac{\mu_n}{4}) \to 12\nu$.
By the uniqueness of limit in Real numbers, we conclude that $\sin(\frac{7}{24}\nu) = 12\nu$. *Thus* $\nu = 0$ *and so* $\mu_n \to 0$.

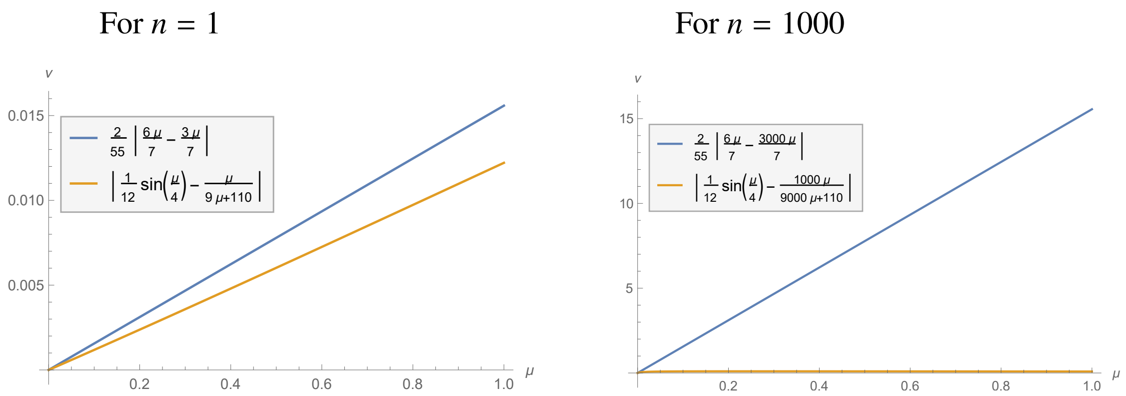
$$\lim_{n\to\infty} d_\theta(fl\mu_n, lf\mu_n) \;\; = \;\; \lim_{n\to\infty} |0 - 0| = 0.$$

So the pair (f, l) *is compatible. Similarly, one can show that the pair* (g, h) *is compatible.*
Given $\mu, \nu \in \mathcal{Y}$, *with* $\nu \geq \mu$, *let* $n \in [1, +\infty)$ *such that* $\nu = n\mu$

$$\begin{aligned} d_\theta(f\mu, g\nu) &= \Big|\frac{1}{12}\sin\Big(\frac{\mu}{4}\Big) - \frac{\nu}{9\nu + 110}\Big| \\ &= \Big|\frac{1}{12}\sin\Big(\frac{\mu}{4}\Big) - \frac{n\mu}{9n\mu + 110}\Big|. \end{aligned}$$

From Figure 2 (a and b), we deduce that

$$\Big|\frac{1}{12}\sin\Big(\frac{\mu}{4}\Big) - \frac{n\mu}{9n\mu + 110}\Big| \leq \frac{2}{55}\Big|\frac{6\mu}{7} - \frac{3n\mu}{7}\Big|.$$



(**a**) Comparison between $\frac{2}{55}\left|\frac{6\mu}{7} - \frac{3\mu}{7}\right|$ and $\left|\frac{1}{12}\sin\left(\frac{\mu}{4}\right) - -\frac{\mu}{9\mu+110}\right|$.

(**b**) Comparison between $\frac{2}{55}\left|\frac{6\mu}{7} - \frac{3000\mu}{7}\right|$ and $\left|\frac{1}{12}\sin\left(\frac{\mu}{4}\right) - -\frac{1000\mu}{9000\mu+110}\right|$.

Figure 2. Comparison between two functions.

Therefore

$$\begin{aligned} d_\theta(f\mu, g\nu) &\leq \frac{2}{55}\left|\frac{6\mu}{7} - \frac{3n\mu}{7}\right| \\ &= \Big(\frac{1}{11}\Big)\Big(\frac{2}{5}\Big)\left|\frac{6\mu}{7} - \frac{3n\mu}{7}\right| \\ &\leq \Big(\frac{1}{11}\Big)\alpha\theta(l\mu, h\nu)d_\theta(l\mu, h\nu). \end{aligned}$$

Thus,

$$\begin{aligned} d_\theta(f\mu, g\nu) &\leq \frac{1}{11}\Big(\alpha\theta(l\mu, h\nu)d_\theta(l\mu, h\nu) + \alpha\theta(l\mu, f\mu)d_\theta(l\mu, f\mu) + \alpha\theta(h\nu, g\nu)d_\theta(h\nu, g\nu) \\ &\quad + \alpha^3\theta(l\mu, g\nu)d_\theta(l\mu, g\nu)\Big) \\ &= \vartheta\Big(\alpha\theta(l\mu, h\nu)d_\theta(l\mu, h\nu), \alpha\theta(l\mu, f\mu)d_\theta(l\mu, f\mu), \alpha\theta(h\nu, g\nu)d_\theta(h\nu, g\nu), \alpha^3\theta(l\mu, g\nu)d_\theta(l\mu, g\nu)\Big). \end{aligned}$$

On a similar manner, we can get

$$d_\theta(g\nu, f\mu) \leq \vartheta\Big(\alpha\theta(h\nu, l\mu)d_\theta(h\nu, l\mu), \alpha\theta(f\mu, l\mu)d_\theta(f\mu, l\mu), \alpha\theta(g\nu, h\nu)d_\theta(g\nu, h\nu), \alpha^3\theta(g\nu, l\mu)d_\theta(g\nu, l\mu)\Big).$$

Therefore, all conditions of Theorem (3.2) *have been fulfilled. So the desired result is obtained.*

Corollary 3.4. *Let $\mathcal{Y}$ be a non-empty set, $(\mathcal{Y}, d_s)$ be a quasi b-metric space, $\alpha \in (0, 1)$, and f, l, h, g be four self-mappings on $\mathcal{Y}$. Suppose the following conditions:*

(1) $f(\mathcal{Y}) \subseteq h(\mathcal{Y})$ and $g(\mathcal{Y}) \subseteq l(\mathcal{Y})$.

(2) f or l is continuous.

(3) $s \leq \frac{1}{\alpha}$.

(4) d_θ is continuous in its variables.

(5) The pairs (f, l) and (g, h) are compatible.

(6) There exists $\vartheta \in \mathcal{A}_\vartheta$ such that

$$d_s(f\mu, g\nu) \leq \vartheta\Big(\alpha s d_s(l\mu, h\nu), \alpha s d_s(l\mu, f\mu), \alpha s d_s(h\nu, g\nu), \alpha^3 s d_s(l\mu, g\nu)\Big)$$

and

$$d_s(g\nu, f\mu) \leq \vartheta\Big(\alpha s d_s(h\nu, l\mu), \alpha s d_s(f\mu, l\mu), \alpha s d_\theta(g\nu, h\nu), \alpha^3 s d_s(g\nu, l\mu)\Big)$$

hold for all $\mu, \nu \in \mathcal{Y}$.

Then f, l, h and g have a unique common fixed point in $\mathcal{Y}$ provided that $\lambda < \alpha$, where λ is the constant satisfies condition (iii) of the definition $\mathcal{A}_\vartheta$..

Proof. The desired result will be obtained from Theorem (3.2) by defining θ: $\mathcal{Y} \times \mathcal{Y} \to [1, +\infty)$ via $\theta(\kappa, \eta) = s$.

4. Conclusions

In the current paper, we introduced a new concept called $\mathcal{A}_\vartheta$-α-contraction. We used our new concept to introduce and prove some common fixed point results for several self-mappings under a set of conditions over an extended quasi b-metric space. Also, we have provided some examples to show the novelty of our results.

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Conflict of interest

The authors declare no conflicts of interest.

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