AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 7142–7162.
DOI: 10.3934/math.2023360
Received: 12 August 2022
Revised: 24 November 2022
Accepted: 25 November 2022
Published: 11 January 2023

Research article

Comparison of two modified analytical approaches for the systems of time fractional partial differential equations

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Abstract: The aim of this article is to present a comparison of two analytical approaches toward obtaining the solution of the time-fractional system of partial differential equations. The newly proposed approaches are the new approximate analytical approach (NAAA) and Mohand variational iteration transform approach (MVITA). The NAAA is based on the Caputo-Riemann operator and its basic properties with the decomposition procedure. The NAAA provides step wise series form solutions with fractional order, which quickly converge to the exact solution for integer order. The MVITA is based on a variational iteration procedure and uses the Mohand integral transform. The MVITA also provides a series solution without a stepwise solution. Both approaches provide a series form of solutions to the proposed problems. The analytical procedures and obtained results are compared for the proposed problems. The obtained results were also compared with exact solutions for the problems. The obtained result and plots have shown the validity and applicability of the proposed algorithms. Both approaches can be extended for the analytical solution of other physical phenomena in science and technology.

Keywords: fractional order systems of PDEs; Mohand variational iteration transform; new approximate analytical approach; analytical solutions; Caputo-Riemann operator
Mathematics Subject Classification: 33B15, 34A34, 35A20, 35A22, 44A10

1. Introduction

Fractional calculus is considered to be the main research area nowadays due to its numerous applications in applied sciences. These applications have attracted the attention of researchers. The advanced applications in the field of applied science and technology include the nonlinear oscillation fractional order earth quack model, fractional order airfoil model [1], fractional-order fluid dynamic traffic model [2], financial model with fractional order [3], financial models chaos theory [4], fractional order Zener model [5], fractional Zener model of signal processing phenomena [6], Poisson-NerstPlanck diffusion model of fractional order [7], electrodynamics [8], fractional model of cancer chemotherapy [9], optics [10], fractional model for the dynamics of hepatitis B virus [11], fractional model for tuberculosis [12], fractional-order pine wilt disease model [13] and various others [14–17].

Whenever the applications attract attention, on the other side, the solution of fractional order physical models also gains the attention of researchers. The solution has become an important issue for researchers [18–20]. Various analytical and numerical techniques have been introduced by researchers, such as the homotopy perturbation technique [21], Laplace Adomian decomposition method [22], Laplace homotopy perturbation transform technique [23], Crank-Nicholson finite difference method [24], p-homotopy analysis method [25], novel operational matrix-based method [26], Harr wavelet collection approach [27], natural transform decomposition method [28], dynamical system method [29], new approximate analytical method [30], new modified decomposition method [31], new analytical technique [32], approximate analytical method [33], kernal Method [34], fully Petrov-Galerkin spectral method [35] and various others [36–38].

In the connection of these numerical and analytical methods, we have also used two different analytical approaches and tested for the solution of the non-linear time-fractional system of partial differential equations (PDEs). These approaches are the new approximate analytical approach (NAAA) and Mohand variation iteration transform approach (MVITA). The NAAA is a decomposition procedure that uses Caputo-Riemann operators of derivatives and integration, respectively. Using the combined property of both operators provides a series form solution with fractional order. The obtained fractional-order series-form solution rapidly converges to the exact solution by approaching integer order. The MVITA is a variational iteration strategy that use a new integral transform (Mohand transform). It provides the series form solution without using any decomposition, He's polynomial or discretization. The series form solution provides an exact form solution for the problems by using a specific value of summation, i.e., an integer value instead of fractional order.

We have tested both of the approaches by solving the following systems of time-fractional differential equations [39]

$$\begin{aligned} D_\tau^{\eth}\zeta - \zeta^2\xi + \frac{1}{4}\left(\zeta_{\vartheta\vartheta} - \zeta_{\gamma\gamma}\right) + 2\zeta = 0, \\ D_\tau^{\eth}\xi + \zeta^2\xi - \frac{1}{4}\left(\xi_{\vartheta\vartheta} - \xi_{\gamma\gamma}\right) - \zeta = 0, \end{aligned} \tag{1.1}$$

where $\vartheta, \tau \in \mathbb{R}, \eth \in (0, 1]$ and

$$\begin{aligned} D_\tau^{\eth}\zeta + \zeta_\vartheta\xi_\vartheta + \zeta_\gamma\xi_\gamma + \zeta = 0, \\ D_\tau^{\eth}\xi + \xi_\vartheta\Im_\vartheta - \xi_\gamma\Im_\gamma - \xi = 0, \\ D_\tau^{\eth}\Im + \Im_\vartheta\zeta_\vartheta + \Im_\gamma\zeta_\gamma + \Im = 0, \end{aligned} \tag{1.2}$$

where $\vartheta, \tau \in \mathbb{R}$ and $\eth \in (0, 1]$; we can verify the obtained results with the exact solution and for different fractional orders. We also compared the results of the NAAA and MVITA numerically and by plotting.

The rest of the article is organized as follows. Some related basic definitions and results are elaborated in Section 2. In Section 3, we explained the generalized procedure of the NAAA; in Section 4, we presented the generalized procedure of the MVITA. In Section 5, we tested some systems of PDEs based on the NAAA and MVITA. Finally, we concluded the present research results in Section 6.

2. Preliminaries and basic concepts

In this section, we have defined some relative elementary concepts of the research work.

Definition 2.1. [40] The Riemann-Liouville fractional partial integral $\wr_\tau^\eth$, where $\eth \in \mathbb{N}$ and $\eth \geq 0$, is defined as

$$\wr_\tau^\eth \zeta(\vartheta, \tau) = \begin{cases} \dfrac{1}{\Gamma(\eth)} \displaystyle\int_0^\tau \zeta(\vartheta, \tau) d\tau, & \text{if } \eth, \tau > 0; \\ \zeta(\vartheta, \tau), & \text{if } \eth = 0, \tau > 0, \end{cases} \tag{2.1}$$

where $\eth$ is denotes the gamma function.

Let $\eth, \beta \in \mathbb{R} \setminus \mathbb{N}$ and $\eth, \beta > 0, \alpha > -1$ then, for the function $\zeta(\vartheta, \tau)$, the operator $\wr_\tau^\eth$ has the following properties [40]

(i) $\wr_\tau^\eth \zeta(\vartheta, \tau) \wr_\tau^\beta \zeta(\vartheta, \tau) = \wr_\tau^{\eth+\beta} \zeta(\vartheta, \tau)$;
(ii) $\wr_\tau^\eth \zeta(\vartheta, \tau) \wr_\tau^\beta \zeta(\vartheta, \tau) = \wr_\tau^\beta \zeta(\vartheta, \tau) \wr_\tau^\eth \zeta(\vartheta, \tau)$;
(iii) $\wr_\tau^\eth \tau^\alpha = \dfrac{\Gamma(\alpha + 1)}{\Gamma(\eth + \alpha + 1)} t^{\eth+\alpha}$.

Definition 2.2. [40] The Caputo operator of the fractional partial derivative is defined by

$$D_\tau^\eth \zeta(\vartheta, \tau) = \frac{\partial^\eth \zeta(\vartheta, \tau)}{\partial \tau^\eth} = \begin{cases} \wr^{n-\eth} \left[\dfrac{\partial^\eth \zeta(\vartheta, \tau)}{\partial \tau^\eth} \right], & \text{if } n - 1 < \eth \leq n, n \in \mathbb{N}; \\ \dfrac{\partial^\eth \zeta(\vartheta, \tau)}{\partial \tau^\eth}, & \text{if } n = \eth. \end{cases} \tag{2.2}$$

Theorem 2.1. [41] *Let* $\eth, \tau \in \mathbb{R}, \tau > 0$ *and* $m - 1 < \vartheta < m \in \mathbb{N}$; *then,*

$$\wr_\tau^\eth D_\tau^\eth \zeta(\vartheta, \tau) = \zeta(\vartheta, \tau) - \sum_{k=0}^{m-1} \frac{\tau^k}{k!} \frac{\partial^k \upsilon(\vartheta, 0^+)}{\partial \tau^k}, \tag{2.3}$$

and

$$D_\tau^\eth \wr_\tau^\eth \zeta(\vartheta, \tau) = \zeta(\vartheta, \tau). \tag{2.4}$$

The Mohand transform, which is represented by $M(\cdot)$ for a function $\zeta(\tau)$, is defined as

$$M\{\zeta(\tau)\} = R(\upsilon) = \upsilon^2 \int_0^\infty \zeta(\tau) e^{-\upsilon\tau} d\tau, \quad k_1 \leq \upsilon \leq k_2. \tag{2.5}$$

The Mohand transform of a function $\zeta(\tau)$ is $R(\upsilon)$; then, $\zeta(\tau)$ is called the inverse of $R(\upsilon)$, which is expressed as

$$M^{-1}\{R(\upsilon)\} = \zeta(\tau), \tag{2.6}$$

where M^{-1} is the inverse Mohand operator.

The Mohand transform for nth derivatives is defined by [41]

$$M\{\zeta^n(\tau)\} = \upsilon^n R(\upsilon) - \upsilon^{n+1}\zeta_0 - \upsilon^n\zeta_0' - \cdots - \upsilon^2\zeta_0^{n-1}. \tag{2.7}$$

3. Generalized procedure for implementation of NAAA

In this section, we extend the NAAA implementation for the analytical treatment of the general equation of non-linear fractional order PDEs defined as [30]:

$$D_\tau^{\eth}\zeta(\vartheta,\gamma,\tau) = \pounds\zeta(\vartheta,\gamma,\tau) + \aleph\zeta(\vartheta,\gamma,\tau) + g(\vartheta,\gamma,\tau), \quad \eth \in (0,2) \tag{3.1}$$

with the initial sources

$$\zeta(\vartheta,\gamma,0) = \zeta(\vartheta,\gamma), \quad D_\tau^{\eth}\zeta(\vartheta,\gamma,0) = \zeta(\vartheta,\gamma),$$

where £ is the linear operator, $\aleph$ is the non-linear operator and $\delta(\vartheta,\gamma,\tau)$ is a source function.

The computational work will use the following basic concepts.

Lemma 3.1. [30] *For $\zeta(\vartheta,\gamma,\tau) = \sum_{k=0}^{\infty}\lambda^k\zeta_k(\vartheta,\gamma,\tau)$, the linearity of $\pounds\zeta(\vartheta,\gamma,\tau)$ is given the following result:*

$$\pounds\zeta(\vartheta,\gamma,\tau) = \pounds\left(\sum_{k=0}^{\infty}\lambda^k\zeta_k(\vartheta,\gamma,\tau)\right) = \sum_{k=0}^{\infty}\pounds\left(\zeta_k(\vartheta,\gamma,\tau)\right). \tag{3.2}$$

Theorem 3.1. [30] *If the non-linear term with the η parameter $\zeta_k(\vartheta,\gamma,\tau) = \sum_0^{\infty}\lambda^k\zeta_k(\vartheta,\gamma,\tau)$, then the nonlinear operator $\aleph\zeta(\vartheta,\gamma,\tau)$ satisfies the following property:*

$$\aleph\left(\zeta_k(\vartheta,\gamma,\tau)\right) = \aleph\left(\sum_0^{\infty}\eta^k\zeta_k(\vartheta,\gamma,\tau)\right) = \sum_0^{\infty}\left[\frac{1}{n!}\frac{d^n}{d\eta^n}\left[\aleph\left(\sum_0^{\infty}\eta^k\zeta_k(\vartheta,\gamma,\tau)\right)\right]_{\eta=0}\right]\eta^n. \tag{3.3}$$

Definition 3.1. [30] The non-linear term $\aleph\left(\zeta_k(\vartheta,\gamma,\tau)\right)$ is uniquely expressed in summation form as follows:

$$\aleph\left(\zeta_k(\vartheta,\gamma,\tau)\right) = \sum_{k=0}^{\infty}\lambda^k\aleph_k. \tag{3.4}$$

The following results verify the existence and uniqueness of the NAAM solution.

Theorem 3.2. *Let $\delta(\vartheta,\gamma,\tau)$ and $\zeta(\vartheta,\gamma,\tau)$ be defined for $n-1 < \eth < n$ in* (3.1)*; the wave model with variable coefficients* (3.1) *yields the unique solution as*

$$\zeta(\vartheta,\gamma,\tau) = \delta_\tau^{-\eth}(\vartheta,\gamma,\tau) + \zeta(\vartheta,\gamma,0) + D_\tau^{\eth}\zeta(\vartheta,\gamma,0) + \sum_{k=1}^{\infty}\left[\pounds_\tau^{-\eth}(\zeta_{(k-1)}) + \aleph_{(k-1)\tau}^{-\eth}\right], \tag{3.5}$$

where $\pounds_\tau^{-\eth}(\zeta_{(k-1)})$ and $\Im_{(k-1)\tau}^{-\eth}$ represent the Riemmen fractional order ($\eth$) integration with the parameter τ.

Proof. The solution $\zeta(\vartheta,\gamma,\tau)$ of the wave model is achieved by using the expansion as

$$\zeta(\vartheta,\gamma,\tau)=\sum_{k=0}^{\infty}\zeta_k(\vartheta,\gamma,\tau). \tag{3.6}$$

Similarly, the solution is further summarized by the following procedure

$$\zeta_k(\vartheta,\gamma,\tau)=\sum_{0}^{\infty}\lambda^k\zeta_k(\vartheta,\gamma,\tau). \tag{3.7}$$

To evaluate the solution of the wave model (3.1), we compute as

$$D_\tau^{\eth}\zeta(\vartheta,\gamma,\tau)=\lambda[\pounds\zeta(\vartheta,\gamma,\tau)+\aleph\zeta(\vartheta,\gamma,\tau)+\delta(\vartheta,\gamma,\tau)],\quad \eth\in(0,2) \tag{3.8}$$

with the initial conditions

$$\zeta(\vartheta,\gamma,0)=\zeta(\vartheta,\gamma),\quad D_\tau^{\eth}\zeta(\vartheta,\gamma,0)=\varphi(\vartheta,\gamma); \tag{3.9}$$

by using the Riemann integral operator (2.1), with respect to τ, and by applying the basic property, we have

$$\zeta_\lambda\zeta(\vartheta,\gamma)=\zeta(\vartheta,\gamma,0)+D_\tau^{\eth}\zeta(\vartheta,\gamma,0)+\lambda\,\wr_\tau^{\eth}\left[\pounds\zeta(\vartheta,\gamma,\tau)+\aleph\zeta(\vartheta,\gamma,\tau)+\delta(\vartheta,\gamma,\tau)\right]. \tag{3.10}$$

Now, by substituting the initial conditions and (3.5) in (3.8), we get

$$\begin{aligned}\sum_{k=0}^{\infty}\lambda^k\zeta_\lambda(\vartheta,\gamma,\tau)&=\zeta(\vartheta,\gamma)+\varphi(\vartheta,\gamma)+\lambda\,\wr_\tau^{\eth}\left[\delta(\vartheta,\gamma,\tau)\right]\\&\quad+\lambda\,\wr_\tau^{\eth}\left[\pounds\left(\sum_{k=0}^{\infty}\lambda^k\zeta(\vartheta,\gamma,\tau)\right)+\aleph\left(\sum_{k=0}^{\infty}\lambda^k\zeta(\vartheta,\gamma,\tau)\right)\right],\end{aligned} \tag{3.11}$$

with using Lemma 3.1 and Definition 3.1, (3.11) becomes

$$\sum_{k=0}^{\infty}\lambda^k\zeta_k(\vartheta,\gamma,\tau)=\zeta(\vartheta,\gamma)+\varphi(\vartheta,\gamma)+\lambda\wr_\tau^{\eth}[\delta(\vartheta,\gamma,\tau)]+\lambda\wr_\tau^{\eth}\left[\pounds\left(\sum_{k=0}^{\infty}\lambda^k\zeta_k(\vartheta,\gamma,\tau)\right)\right]+\lambda\wr_\tau^{\eth}\left[\left(\sum_{0}^{\infty}\lambda^k\aleph_k\right)\right]. \tag{3.12}$$

By equating the identical power of (λ), the iterative scheme becomes

$$\begin{aligned}\zeta_0(\vartheta,\gamma,\tau)&=\zeta(\vartheta,\gamma)+\varphi(\vartheta,\gamma),\\ \zeta_1(\vartheta,\gamma,\tau)&=\delta(\vartheta,\gamma,\tau)+\pounds_\tau^{-\eth}\zeta_0+\aleph_{\tau 0}^{-\eth},\end{aligned}$$

and

$$\zeta_k(\vartheta,\gamma,\tau)=\pounds_\tau^{-\eth}\zeta_{(k-1)}+\aleph_{\tau(k-1)}^{-\eth}$$

for $k=2,3,\ldots$. □

4. Generalized procedure for implementation of MVITA

Consider the general equation defined by (3.1) with the initial condition

$$\zeta(\vartheta,\gamma,0)=\zeta(\vartheta,\gamma).$$

Now, by applying the Mohand transform to (3.1), we get

$$M\left\{D_{\tau}^{\eth}\zeta(\vartheta,\gamma,\tau)\right\}=M\left\{\pounds\zeta(\vartheta,\gamma,\tau)+\aleph\zeta(\vartheta,\gamma,\tau)+g(\vartheta,\gamma,\tau)\right\}. \tag{4.1}$$

Using the iteration property of transformation, we obtain

$$M\left\{\zeta(\vartheta,\gamma,\tau)\right\}-\sum_{k=0}^{m-1}\upsilon^{\vartheta-k-1}\frac{\partial^{k}\psi(\chi,y,\tau)}{\partial\tau^{k}}\bigg|_{\tau=0}=M\left\{\pounds\zeta(\vartheta,\gamma,\tau)+\aleph\zeta(\vartheta,\gamma,\tau)+g(\vartheta,\gamma,\tau)\right\}. \tag{4.2}$$

And, by using the iterative technique with the Lagrange multiplier $(-\lambda(\upsilon))$, we have

$$M\left\{\zeta_{n+1}(\vartheta,\gamma,\tau)]\right\}=M\left\{\zeta_{n}(\vartheta,\gamma,\tau)\right\}-\lambda(\upsilon)\left[M\{\zeta_{n}(\vartheta,\gamma,\tau)\}-\sum_{k=0}^{m-1}\upsilon^{\vartheta-k-1}\frac{\partial^{k}\zeta(\vartheta,\gamma,0)}{\partial\tau^{k}}\right], \tag{4.3}$$

where $\lambda(\upsilon)=-\frac{1}{\upsilon^{\eth}}$. Putting (4.3) in (4.2), we get

$$\begin{aligned}M\left\{\zeta_{n+1}(\vartheta,\gamma,\tau)\right\}&=M\left\{\zeta_{n}(\vartheta,\gamma,\tau)\right\}\\&-\lambda(\upsilon)\left[M\{\zeta_{n}(\vartheta,\gamma,\tau)\}-\sum_{k=0}^{m-1}\upsilon^{\vartheta-k-1}\frac{\partial^{k}\zeta(\vartheta,\gamma,0)}{\partial\tau^{k}}+M\left\{\pounds\zeta(\vartheta,\gamma,\tau)+\aleph\zeta(\vartheta,\gamma,\tau)+g(\vartheta,\gamma,\tau)\right\}\right].\end{aligned} \tag{4.4}$$

By applying the inverse Mohand transform to (4.4), we obtain

$$\zeta_{n+1}(\vartheta,\gamma,\tau)=\zeta_{n}(\vartheta,\gamma,\tau)+M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^{k}\zeta}{\partial\tau^{k}}\bigg|_{\tau=0}+M\left\{\pounds\zeta(\vartheta,\gamma,\tau)+\aleph\zeta(\vartheta,\gamma,\tau)+g(\vartheta,\gamma,\tau)\right\}\right]\right\}.$$

The initial condition becomes

$$\zeta_{0}(\vartheta,\gamma,\tau)=M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^{k}\zeta(\vartheta,\gamma,0)}{\partial\tau^{k}}\right\}.$$

The recursive scheme becomes

$$\begin{aligned}\zeta_{n+1}(\vartheta,\gamma,\tau)&=\zeta_{n}(\vartheta,\gamma,\tau)\\&+M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^{k}\zeta(\vartheta,\gamma,\tau)}{\partial\tau^{k}}\bigg|_{\tau=0}M\left\{\pounds\zeta(\vartheta,\gamma,\tau)+\aleph\zeta(\vartheta,\gamma,\tau)+g(\vartheta,\gamma,\tau)\right\}\right]\right\}.\end{aligned} \tag{4.5}$$

The approximate solution is obtained through the recursive scheme defined by (4.5).

5. NAAA and MVITA testing analysis

In this section, we test the NAAA approach for the solution of the time fractional time system of PDEs [29].

Example 5.1. Consider a non-linear time fractional time system of PDEs in the form of

$$\begin{aligned} D_\tau^{\eth}\zeta &= \zeta^2\xi + \frac{1}{4}\left(\zeta_{\vartheta\vartheta} + \zeta_{\gamma\gamma}\right) - 2\zeta, \\ D_\tau^{\eth}\xi &= -\zeta^2\xi + \frac{1}{4}\left(\xi_{\vartheta\vartheta} + \xi_{\gamma\gamma}\right) + \zeta, \end{aligned} \tag{5.1}$$

where $\vartheta, \tau \in \mathbb{R}$ and $\eth \in (0, 1]$ and we have the following conditions:

$$\begin{aligned} \zeta(\vartheta, \gamma, 0) &= e^{-\vartheta-\gamma}, \\ \xi(\vartheta, \gamma, 0) &= e^{\vartheta+\gamma}; \end{aligned}$$

the exact solution for integer order is

$$\begin{aligned} \zeta(\vartheta, \gamma, \tau) &= e^{-\vartheta-\gamma-\tau}, \\ \xi(\vartheta, \gamma, \tau) &= e^{\vartheta+\gamma+\tau}. \end{aligned}$$

5.1. Testing Problem 5.1 by using the NAAA

For the solution of (5.1), we assume the final procedure described by (3.10) and the final recursive scheme becomes

$$\begin{aligned} \zeta_0(\vartheta, \gamma, \tau) &= \zeta(\vartheta, \gamma), \\ \xi_0(\vartheta, \gamma, \tau) &= \xi(\vartheta, \gamma), \end{aligned} \tag{5.2}$$

$$\begin{aligned} \zeta_1(\vartheta, \gamma, \tau) &= \wr_\tau^{-\eth}\left(\frac{1}{4}\left(\zeta_{0\vartheta\vartheta} + \zeta_{0\gamma\gamma}\right) - 2\zeta_0\right) + \aleph_{0\tau}^{-\eth}, \\ \xi_1(\vartheta, \gamma, \tau) &= \wr_\tau^{-\eth}\left(\frac{1}{4}\left(\xi_{0\vartheta\vartheta} + \xi_{0\gamma\gamma}\right) + \zeta_0\right) + \aleph_{0\tau}^{-\eth} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \zeta_k(\vartheta, \gamma, \tau) &= \wr_\tau^{-\eth}\left(\frac{1}{4}\left(\zeta_{(k-1)\vartheta\vartheta} + \zeta_{(k-1)\gamma\gamma}\right) - 2\zeta_{(k-1)}\right) + \aleph_{(k-1)\tau}^{-\eth}, \\ \xi_k(\vartheta, \gamma, \tau) &= \wr_\tau^{-\eth}\left(\frac{1}{4}\left(\xi_{(k-1)\vartheta\vartheta} + \xi_{(k-1)\gamma\gamma}\right) + \zeta_{(k-1)}\right) + \aleph_{(k-1)\tau}^{-\eth} \end{aligned} \tag{5.4}$$

for $k = 2, 3, \ldots$. Consequently, we obtain

$$\begin{aligned} \zeta_0(\vartheta, \gamma, \tau) &= e^{-\vartheta-\gamma}, \\ \xi_0(\vartheta, \gamma, \tau) &= e^{\vartheta+\gamma} \end{aligned} \tag{5.5}$$

and by using

$$\begin{aligned}\zeta_1(\vartheta,\gamma,\tau) &= \wr_\tau^{-\eth}\left(\frac{1}{4}\left(\zeta_{0\vartheta\vartheta}+\zeta_{0\gamma\gamma}\right)-2\zeta_0\right)+\aleph_{0\tau}^{-\eth},\\ \xi_1(\vartheta,\gamma,\tau) &= \wr_\tau^{-\eth}\left(\frac{1}{4}\left(\xi_{0\vartheta\vartheta}+\xi_{0\gamma\gamma}\right)+\zeta_0\right)+\aleph_{0\tau}^{-\eth},\end{aligned} \tag{5.6}$$

we get the second approximation as

$$\begin{aligned}\zeta_1(\vartheta,\gamma,\tau) &= -e^{-\vartheta-\gamma}\frac{\tau^{\eth}}{\eth!},\\ \xi_1(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma}\frac{\tau^{\eth}}{\eth!}.\end{aligned} \tag{5.7}$$

Now, we apply the general recursive scheme as follows:

$$\begin{aligned}\zeta_k(\vartheta,\gamma,\tau) &= \wr_\tau^{-\eth}\left(\frac{1}{4}\left(\zeta_{(k-1)\vartheta\vartheta}+\zeta_{(k-1)\gamma\gamma}\right)-2\zeta_{(k-1)}\right)+\aleph_{(k-1)\tau}^{-\eth},\\ \xi_k(\vartheta,\gamma,\tau) &= \wr_\tau^{-\eth}\left(\frac{1}{4}\left(\xi_{(k-1)\vartheta\vartheta}+\xi_{(k-1)\gamma\gamma}\right)+\zeta_{(k-1)}\right)+\aleph_{(k-1)\tau}^{-\eth}.\end{aligned} \tag{5.8}$$

$$\begin{aligned}\zeta_2(\vartheta,\gamma,\tau) &= e^{-\vartheta-\gamma}\frac{\tau^{2\eth}}{(2\eth)!},\\ \xi_2(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma}\frac{\tau^{2\eth}}{(2\eth)!},\\ \zeta_3(\vartheta,\gamma,\tau) &= -e^{-\vartheta-\gamma}\frac{\tau^{3\eth}}{(3\eth)!},\\ \xi_3(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma}\frac{\tau^{3\eth}}{(3\eth)!},\\ &\vdots\end{aligned} \tag{5.9}$$

The NAAM becomes

$$\begin{aligned}\zeta(\vartheta,\gamma,\tau) &= \zeta_0+\zeta_1+\zeta_2+\zeta_3+\cdots,\\ \xi(\vartheta,\gamma,\tau) &= \xi_0+\xi_1+\xi_2+\xi_3+\cdots;\end{aligned} \tag{5.10}$$

by substituting the values of (5.5), (5.7) and (5.9) in (5.10), we obtain

$$\begin{aligned}\zeta(\vartheta,\gamma,\tau) &= e^{-\vartheta-\gamma}-e^{-\vartheta-\gamma}\frac{\tau^{\eth}}{\eth!}+e^{-\vartheta-\gamma}\frac{\tau^{2\eth}}{(2\eth)!}-e^{-\vartheta-\gamma}\frac{\tau^{3\eth}}{(3\eth)!}+\cdots,\\ \xi(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma}+e^{\vartheta+\gamma}\frac{\tau^{\eth}}{\eth!}+ee^{\vartheta+\gamma}\frac{\tau^{2\eth}}{(2\eth)!}+e^{\vartheta+\gamma}\frac{\tau^{3\eth}}{(3\eth)!}+\cdots.\end{aligned} \tag{5.11}$$

For the special case $\gamma = 1$, the NAAA solution in series form becomes

$$\begin{aligned}\zeta(\vartheta,\gamma,\tau) &= e^{-\vartheta-\gamma}\left(1-\frac{\tau}{1!}+\frac{\tau^2}{(2)!}-\frac{\tau^3}{(3)!}+\cdots\right),\\ \xi(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma}\left(1+\frac{\tau}{1!}+\frac{\tau^2}{(2)!}+\frac{\tau^3}{(3)!}+\cdots\right);\end{aligned} \tag{5.12}$$

this series form solution directly converges to the exact solution of the problem:

$$\begin{aligned}\zeta(\vartheta,\gamma,\tau) &= e^{-\vartheta-\gamma-\tau},\\ \xi(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma+\tau}.\end{aligned} \tag{5.13}$$

5.2. Testing Problem 5.1 by using the MVITA

Consider the same non-linear system of time-fractional PDEs defined by (5.1) with the initial conditions

$$\begin{aligned}\zeta(\vartheta,\gamma,0) &= e^{-\vartheta-\gamma},\\ \xi(\vartheta,\gamma,0) &= e^{\vartheta+\gamma}.\end{aligned}$$

By using the general recursive scheme (4.5), we get

$$\zeta_{n+1}(\vartheta,\gamma,\tau) = \zeta_n(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\zeta(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{\zeta_n^2\xi_n + \frac{1}{4}\left(\zeta_{n\vartheta\vartheta}+\zeta_{n\gamma\gamma}\right) - 2\zeta_n\right\}\right]\right\},$$

$$\xi_{n+1}(\vartheta,\gamma,\tau) = \xi_n(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\xi(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{-\zeta_n^2\xi_n + \frac{1}{4}\left(\xi_{n\vartheta\vartheta}+\xi_{n\gamma\gamma}\right) + \zeta_n\right\}\right]\right\}.$$

So, the initial condition becomes

$$\begin{aligned}\zeta_0(\vartheta,\gamma,\tau) &= e^{-\vartheta-\gamma},\\ \xi_0(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma}.\end{aligned}$$

Now, for different values of n ($n = 0, 1, 2, \ldots$), we have

$$\zeta_1(\vartheta,\gamma,\tau) = \zeta_0(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\zeta(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{\zeta_0^2\xi_0 + \frac{1}{4}\left(\zeta_{0\vartheta\vartheta}+\zeta_{0\gamma\gamma}\right) - 2\zeta_0\right\}\right]\right\},$$

$$\xi_1(\vartheta,\gamma,\tau) = \xi_0(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\xi(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{-\zeta_0^2\xi_0 + \frac{1}{4}\left(\xi_{0\vartheta\vartheta}+\xi_{0\gamma\gamma}\right) + \zeta_0\right\}\right]\right\}.$$

By simplifying with the initial condition, we get

$$\begin{aligned}\zeta_1(\vartheta,\gamma,\tau) &= e^{-\vartheta-\gamma} + e^{-\vartheta-\gamma}\frac{\tau^{\eth}}{\eth!},\\ \xi_1(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma} + e^{\vartheta+\gamma}\frac{\tau^{\eth}}{\eth!}.\end{aligned} \tag{5.14}$$

Consequently, we get

$$\zeta_2(\vartheta,\gamma,\tau) = \zeta_1(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\zeta(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{\zeta_1^2\xi_0 + \frac{1}{4}(\zeta_{1\vartheta\vartheta}+\zeta_{1\gamma\gamma}) - 2\zeta_1\right\}\right]\right\},$$

$$\xi_2(\vartheta,\gamma,\tau) = \xi_1(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\xi(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{-\zeta_1^2\xi_1 + \frac{1}{4}(\xi_{1\vartheta\vartheta}+\xi_{1\gamma\gamma}) + \zeta_1\right\}\right]\right\}.$$

The series form solution, we get

$$\begin{aligned}
\zeta_2(\vartheta,\gamma,\tau) &= e^{-\vartheta-\gamma} + e^{-\vartheta-\gamma}\frac{\tau^{\eth}}{\eth!} + e^{-\vartheta-\gamma}\frac{\tau^{2\eth}}{(2\eth)!},\\
\xi_2(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma} + e^{\vartheta+\gamma}\frac{\tau^{\eth}}{\eth!} + e^{\vartheta+\gamma}\frac{\tau^{2\eth}}{(2\eth)!},\\
\zeta_3(\vartheta,\gamma,\tau) &= e^{-\vartheta-\gamma} + e^{-\vartheta-\gamma}\frac{\tau^{\eth}}{\eth!} + e^{-\vartheta-\gamma}\frac{\tau^{2\eth}}{(2\eth)!} + e^{-\vartheta-\gamma}\frac{\tau^{3\eth}}{(3\eth)!},\\
\xi_3(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma} + e^{\vartheta+\gamma}\frac{\tau^{\eth}}{\eth!} + e^{\vartheta+\gamma}\frac{\tau^{2\eth}}{(2\eth)!} + e^{\vartheta+\gamma}\frac{\tau^{3\eth}}{(3\eth)!},\\
&\vdots
\end{aligned}$$

Thus, the solution becomes

$$\begin{aligned}
\zeta(\vartheta,\gamma,\tau) &= \sum_{n=0}^{\infty}\zeta_n(\vartheta,\gamma,\tau) = e^{-\vartheta-\gamma}\left(\sum_{n=0}^{\infty}\frac{(-1)^n\tau^{n\eth}}{(n\eth)!}\right),\\
\xi(\vartheta,\gamma,\tau) &= \sum_{n=0}^{\infty}\xi_n(\vartheta,\gamma,\tau) = e^{\vartheta+\gamma}\left(\sum_{n=0}^{\infty}\frac{\tau^{n\eth}}{(n\eth)!}\right).
\end{aligned} \tag{5.15}$$

For a specific integer value $\eth = 1$, the MVITA solution becomes

$$\begin{aligned}
\zeta(\vartheta,\gamma,\tau) &= e^{-\vartheta-\gamma-\tau},\\
\xi(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma-\tau},
\end{aligned}$$

which is the exact solution of the problem.

In Figure 1, the approximate solution of $\zeta(\vartheta,\gamma,\tau)$ is obtained by using the NAAA for Example 5.1. Figure 2, the approximate solution of $\zeta(\vartheta,\gamma,\tau)$ is obtained by using the MVITA for Example 5.1. Figure 3, the approximate solution of $\xi(\vartheta,\gamma,\tau)$ is obtained by using the NAAA for Example 5.1. And Figure 4, the approximate solution of $\xi(\vartheta,\gamma,\tau)$ is obtained by using the MVITA for Example 5.1.

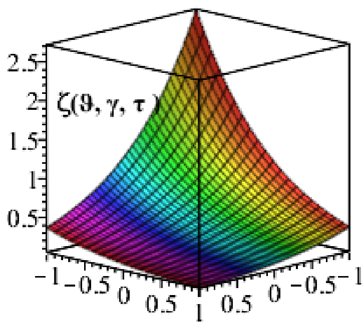

Figure 1. The approximate solution of $\zeta(\vartheta,\gamma,\tau)$ as obtained by using the NAAA for Example 5.1.

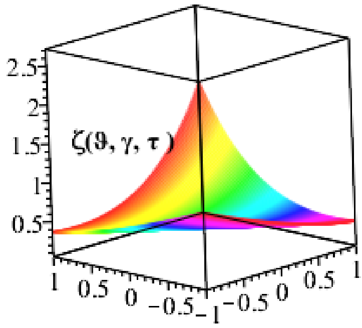

Figure 2. The approximate solution of $\zeta(\vartheta, \gamma, \tau)$ as obtained by using the MVITA for Example 5.1.

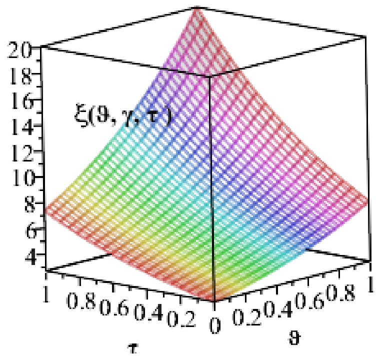

Figure 3. The approximate solution of $\xi(\vartheta, \gamma, \tau)$ as obtained by using the NAAA for Example 5.1.

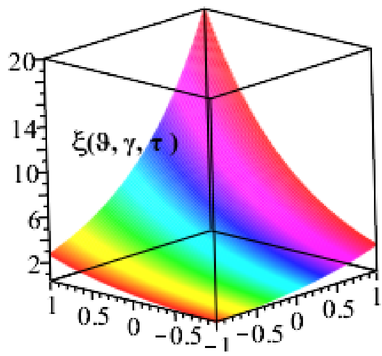

Figure 4. The approximate solution of $\xi(\vartheta, \gamma, \tau)$ as obtained by using the MVITA for Example 5.1.

Example 5.2. Consider a non-linear time-fractional system of PDEs in the form of

$$
\begin{aligned}
D_\tau^{\eth}\zeta &= -\zeta_\vartheta \xi_\vartheta - \zeta_\gamma \xi_\gamma - \zeta, \\
D_\tau^{\eth}\xi &= -\xi_\vartheta \Im_\vartheta + \xi_\gamma \Im_\gamma + \xi, \\
D_\tau^{\eth}\Im &= -\Im_\vartheta \zeta_\vartheta - \Im_\gamma \zeta_\gamma - \Im,
\end{aligned}
\tag{5.16}
$$

where $\vartheta, \tau \in \mathbb{R}$ and $\eth \in (0, 1]$ with the conditions

$$\zeta(\vartheta, \gamma, 0) = e^{\vartheta+\gamma},$$
$$\xi(\vartheta, \gamma, 0) = e^{\vartheta-\gamma},$$
$$\Im(\vartheta, \gamma, 0) = e^{-\vartheta+\gamma};$$

the exact solution for integer order is

$$\zeta(\vartheta, \gamma, \tau) = e^{\vartheta+\gamma-\tau},$$
$$\xi(\vartheta, \gamma, \tau) = e^{\vartheta-\gamma+\tau},$$
$$\Im(\vartheta, \gamma, \tau) = e^{-\vartheta+\gamma-\tau}.$$

5.3. *Testing Problem 5.2 by using the NAAA*

For the solution of (5.16), we assume the final procedure described by (3.10), and the final recursive scheme becomes

$$\begin{aligned} \zeta_0(\vartheta, \gamma, \tau) &= \zeta(\vartheta, \gamma), \\ \xi_0(\vartheta, \gamma, \tau) &= \xi(\vartheta, \gamma), \\ \Im_0(\vartheta, \gamma, \tau) &= \Im(\vartheta, \gamma), \end{aligned} \tag{5.17}$$

$$\begin{aligned} \zeta_1(\vartheta, \gamma, \tau) &= -\wr_\tau^{-\eth}(\zeta_0) - \aleph_{10\tau}^{-\eth} - \aleph_{20\tau}^{-\eth}, \\ \xi_1(\vartheta, \gamma, \tau) &= \wr_\tau^{-\eth}(\xi_0) - \aleph_{30\tau}^{-\eth} + \aleph_{40\tau}^{-\eth}, \\ \Im_1(\vartheta, \gamma, \tau) &= -\wr_\tau^{-\eth}(\Im_0) - \aleph_{50\tau}^{-\eth} + \aleph_{60\tau}^{-\eth} \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} \zeta_k(\vartheta, \gamma, \tau) &= -\wr_\tau^{-\eth}\left(\zeta_{(k-1)}\right) - \aleph_{1(k-1)\tau}^{-\eth} - \aleph_{2(k-1)\tau}^{-\eth}, \\ \xi_k(\vartheta, \gamma, \tau) &= \wr_\tau^{-\eth}\left(\xi_{(k-1)}\right) - \aleph_{3(k-1)\tau}^{-\eth} + \aleph_{4(k-1)\tau}^{-\eth}, \\ \Im_k(\vartheta, \gamma, \tau) &= -\wr_\tau^{-\eth}\left(\Im_{(k-1)}\right) - \aleph_{5(k-1)\tau}^{-\eth} + \aleph_{6(k-1)\tau}^{-\eth}, \end{aligned} \tag{5.19}$$

where $\aleph_1 = -\zeta_\vartheta\xi_\vartheta, \aleph_2 = -\zeta_\gamma\xi_\gamma, \aleph_3 = -\xi_\vartheta\Im_\vartheta, \aleph_4 = \xi_\gamma\Im_\gamma, \aleph_5 = -\Im_\vartheta\zeta_\vartheta$ *and* $\aleph_6 = -\Im_\gamma\zeta_\gamma$ are the non-linear terms of the given problem.

Consequently, we can obtain the approximated terms as

$$\begin{aligned} \zeta_0(\vartheta, \gamma, \tau) &= e^{\vartheta+\gamma}, \\ \xi_0(\vartheta, \gamma, \tau) &= e^{\vartheta-\gamma}, \\ \Im_0(\vartheta, \gamma, \tau) &= e^{-\vartheta+\gamma}. \end{aligned} \tag{5.20}$$

Now, by using

$$\begin{aligned} \zeta_1(\vartheta, \gamma, \tau) &= -\wr_\tau^{-\eth}(\zeta_0) - \aleph_{10\tau}^{-\eth} - \aleph_{20\tau}^{-\eth}, \\ \xi_1(\vartheta, \gamma, \tau) &= \wr_\tau^{-\eth}(\xi_0) - \aleph_{30\tau}^{-\eth} + \aleph_{40\tau}^{-\eth}, \\ \Im_1(\vartheta, \gamma, \tau) &= -\wr_\tau^{-\eth}(\Im_0) - \aleph_{50\tau}^{-\eth} + \aleph_{60\tau}^{-\eth}, \end{aligned} \tag{5.21}$$

we obtain the second approximation as

$$\begin{aligned}\zeta_1(\vartheta,\gamma,\tau) &= -e^{\vartheta+\gamma}\frac{\tau^{\eth}}{\eth!},\\ \xi_1(\vartheta,\gamma,\tau) &= e^{\vartheta-\gamma}\frac{\tau^{\eth}}{\eth!},\\ \Im_1(\vartheta,\gamma,\tau) &= -e^{-\vartheta+\gamma}\frac{\tau^{\eth}}{\eth!}.\end{aligned} \tag{5.22}$$

The other approximated terms by obtained by using

$$\begin{aligned}\zeta_k(\vartheta,\gamma,\tau) &= -\wr_{\tau}^{-\eth}\left(\zeta_{(k-1)}\right) - \aleph_{1(k-1)\tau}^{-\eth} - \aleph_{2(k-1)\tau}^{-\eth},\\ \xi_k(\vartheta,\gamma,\tau) &= \wr_{\tau}^{-\eth}\left(\xi_{(k-1)}\right) - \aleph_{3(k-1)\tau}^{-\eth} + \aleph_{4(k-1)\tau}^{-\eth},\\ \Im_k(\vartheta,\gamma,\tau) &= -\wr_{\tau}^{-\eth}\left(\Im_{(k-1)}\right) - \aleph_{5(k-1)\tau}^{-\eth} + \aleph_{6(k-1)\tau}^{-\eth}.\end{aligned} \tag{5.23}$$

For different values of k, we get

$$\begin{aligned}\zeta_2(\vartheta,\gamma,\tau) &= e^{-\vartheta-\gamma}\frac{\tau^{2\eth}}{(2\eth)!},\\ \xi_2(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma}\frac{\tau^{2\eth}}{(2\eth)!},\\ \Im_2(\vartheta,\gamma,\tau) &= e^{-\vartheta+\gamma}\frac{\tau^{2\eth}}{(2\eth)!},\\ \zeta_3(\vartheta,\gamma,\tau) &= -e^{-\vartheta-\gamma}\frac{\tau^{3\eth}}{(3\eth)!},\\ \xi_3(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma}\frac{\tau^{3\eth}}{(3\eth)!},\\ \Im_3(\vartheta,\gamma,\tau) &= -e^{-\vartheta+\gamma}\frac{\tau^{3\eth}}{(3\eth)!},\\ &\vdots\end{aligned} \tag{5.24}$$

The NAAM becomes

$$\begin{aligned}\zeta(\vartheta,\gamma,\tau) &= \zeta_0+\zeta_1+\zeta_2+\zeta_3+\cdots,\\ \xi(\vartheta,\gamma,\tau) &= \xi_0+\xi_1+\xi_2+\xi_3+\cdots,\\ \Im(\vartheta,\gamma,\tau) &= \Im_0+\Im_1+\Im_2+\Im_3+\cdots.\end{aligned} \tag{5.25}$$

By substituting the values of (5.20), (5.22) and (5.24) in (5.25), we obtain

$$\begin{aligned}\zeta(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma} - e^{\vartheta+\gamma}\frac{\tau^{\eth}}{\eth!} + e^{\vartheta+\gamma}\frac{\tau^{2\eth}}{(2\eth)!} - e^{\vartheta+\gamma}\frac{\tau^{3\eth}}{(3\eth)!}+\cdots,\\ \xi(\vartheta,\gamma,\tau) &= e^{\vartheta-\gamma} + e^{\vartheta-\gamma}\frac{\tau^{\eth}}{\eth!} + e^{\vartheta-\gamma}\frac{\tau^{2\eth}}{(2\eth)!} + e^{\vartheta-\gamma}\frac{\tau^{3\eth}}{(3\eth)!}+\cdots,\\ \Im(\vartheta,\gamma,\tau) &= e^{-\vartheta+\gamma} - e^{-\vartheta+\gamma}\frac{\tau^{\eth}}{\eth!} + e^{-\vartheta+\gamma}\frac{\tau^{2\eth}}{(2\eth)!} - e^{-\vartheta+\gamma}\frac{\tau^{3\eth}}{(3\eth)!}+\cdots.\end{aligned} \tag{5.26}$$

For the special case $\gamma = 1$, the NAAA solution in series form becomes

$$\zeta(\vartheta,\gamma,\tau) = e^{\vartheta+\gamma}\left(1 - \frac{\tau}{1!} + \frac{\tau^2}{(2)!} - \frac{\tau^3}{(3)!} + \cdots\right),$$
$$\xi(\vartheta,\gamma,\tau) = e^{\vartheta+\gamma}\left(1 + \frac{\tau}{1!} + \frac{\tau^2}{(2)!} + \frac{\tau^3}{(3)!} + \cdots\right),$$
$$\Im(\vartheta,\gamma,\tau) = e^{-\vartheta+\gamma}\left(1 - \frac{\tau}{1!} + \frac{\tau^2}{(2)!} - \frac{\tau^3}{(3)!} + \cdots\right),$$

this series form solution directly converges to the exact solution of the problem:

$$\begin{aligned}\zeta(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma-\tau},\\ \xi(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma+\tau},\\ \Im(\vartheta,\gamma,\tau) &= e^{-\vartheta+\gamma-\tau}.\end{aligned} \tag{5.27}$$

5.4. Testing Problem 5.2 by using the MVITA

Consider the same non-linear system of time-fractional PDEs defined by (5.16) with the initial conditions

$$\begin{aligned}\zeta(\vartheta,\gamma,0) &= e^{\vartheta+\gamma},\\ \xi(\vartheta,\gamma,0) &= e^{\vartheta-\gamma},\\ \Im(\vartheta,\gamma,0) &= e^{-\vartheta+\gamma}.\end{aligned}$$

By using the general recursive scheme (4.5), we get

$$\zeta_{n+1}(\vartheta,\gamma,\tau) = \zeta_n(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\zeta(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{-\zeta_{n\vartheta}\xi_{n\vartheta} - \zeta_{n\gamma}\xi_{n\gamma} - \zeta_n\right\}\right]\right\},$$
$$\xi_{n+1}(\vartheta,\gamma,\tau) = \xi_n(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\xi(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{-\xi_{n\vartheta}\Im_{n\vartheta} + \xi_{n\gamma}\Im_{n\gamma} + \xi_n\right\}\right]\right\},$$
$$\Im_{n+1}(\vartheta,\gamma,\tau) = \Im_n(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\Im(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{-\Im_{n\vartheta}\zeta_{n\vartheta} - \Im_{n\gamma}\zeta_{n\gamma} - \Im_n\right\}\right]\right\}.$$

So, the initial guess becomes

$$\begin{aligned}\zeta_0(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma},\\ \xi_0(\vartheta,\gamma,\tau) &= e^{\vartheta-\gamma},\\ \Im_0(\vartheta,\gamma,\tau) &= e^{-\vartheta+\gamma}.\end{aligned}$$

Now, for different values of n ($n = 0, 1, 2, \ldots$), we have

$$\zeta_1(\vartheta,\gamma,\tau) = \zeta_0(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\zeta(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{-\zeta_{0\vartheta}\xi_{0\vartheta} - \zeta_{0\gamma}\xi_{0\gamma} - \zeta_0\right\}\right]\right\},$$
$$\xi_1(\vartheta,\gamma,\tau) = \xi_0(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\xi(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{-\xi_{0\vartheta}\Im_{0\vartheta} + \xi_{0\gamma}\Im_{0\gamma} + \xi_0\right\}\right]\right\},$$
$$\Im_1(\vartheta,\gamma,\tau) = \Im_0(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\Im(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{-\Im_{0\vartheta}\zeta_{0\vartheta} - \Im_{0\gamma}\zeta_{0\gamma} - \Im_0\right\}\right]\right\}.$$

By simplifying with the initial conditions, we get

$$\zeta_1(\vartheta,\gamma,\tau) = e^{\vartheta+\gamma} - e^{\vartheta+\gamma}\frac{\tau^{\eth}}{(\eth)!},$$
$$\xi_1(\vartheta,\gamma,\tau) = e^{\vartheta-\gamma} + e^{\vartheta-\gamma}\frac{\tau^{\eth}}{(\eth)!},$$
$$\Im_1(\vartheta,\gamma,\tau) = e^{-\vartheta+\gamma} - e^{-\vartheta+\gamma}\frac{\tau^{\eth}}{(\eth)!}.$$

Consequently, we get

$$\zeta_2(\vartheta,\gamma,\tau) = \zeta_1(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\zeta(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{-\zeta_{1\vartheta}\xi_{0\vartheta} - \zeta_{0\gamma}\xi_{1\gamma} - \zeta_1\right\}\right]\right\},$$
$$\xi_2(\vartheta,\gamma,\tau) = \xi_1(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\xi(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{-\xi_{1\vartheta}\Im_{1\vartheta} + \xi_{1\gamma}\Im_{1\gamma} + \xi_1\right\}\right]\right\},$$
$$\Im_2(\vartheta,\gamma,\tau) = \Im_1(\vartheta,\gamma,\tau) + M^{-1}\left\{\frac{1}{\upsilon^{\eth}}\left[\sum_{k=0}^{m-1}\upsilon^{\eth-k-1}\frac{\partial^k\Im(\vartheta,\gamma,\tau)}{\partial\tau^k}\bigg|_{\tau=0} + M\left\{-\Im_{1\vartheta}\zeta_{1\vartheta} - \Im_{1\gamma}\zeta_{1\gamma} - \Im_1\right\}\right]\right\}.$$

$$\zeta_2(\vartheta,\gamma,\tau) = e^{\vartheta+\gamma} - e^{\vartheta+\gamma}\frac{\tau^{\eth}}{(\eth)!} + e^{\vartheta+\gamma}\frac{\tau^{2\eth}}{(2\eth)!},$$
$$\xi_2(\vartheta,\gamma,\tau) = e^{\vartheta-\gamma} + e^{\vartheta-\gamma}\frac{\tau^{\eth}}{(\eth)!} + e^{\vartheta-\gamma}\frac{\tau^{2\eth}}{(2\eth)!},$$
$$\Im_2(\vartheta,\gamma,\tau) = e^{-\vartheta+\gamma} - e^{-\vartheta+\gamma}\frac{\tau^{\eth}}{(\eth)!} + e^{-\vartheta+\gamma}\frac{\tau^{2\eth}}{(2\eth)!},$$

similarly, we get

$$\zeta_3(\vartheta,\gamma,\tau) = e^{\vartheta+\gamma} - e^{\vartheta+\gamma}\frac{\tau^{\eth}}{(\eth)!} + e^{\vartheta+\gamma}\frac{\tau^{2\eth}}{(2\eth)!} - e^{\vartheta+\gamma}\frac{\tau^{3\eth}}{(3\eth)!},$$
$$\xi_3(\vartheta,\gamma,\tau) = e^{\vartheta-\gamma} + e^{\vartheta-\gamma}\frac{\tau^{\eth}}{(\eth)!} + e^{\vartheta-\gamma}\frac{\tau^{2\eth}}{(2\eth)!} + e^{\vartheta-\gamma}\frac{\tau^{3\eth}}{(3\eth)!},$$
$$\Im_3(\vartheta,\gamma,\tau) = e^{-\vartheta+\gamma} - e^{-\vartheta+\gamma}\frac{\tau^{\eth}}{(\eth)!} + e^{-\vartheta+\gamma}\frac{\tau^{2\eth}}{(2\eth)!} - e^{-\vartheta+\gamma}\frac{\tau^{3\eth}}{(3\eth)!},$$
$$\vdots$$

Thus the solution becomes

$$\begin{aligned}
\zeta(\vartheta,\gamma,\tau) &= \sum_{n=0}^{\infty}\zeta_n(\vartheta,\gamma,\tau) = e^{\vartheta+\gamma}\left(\sum_{n=0}^{\infty}\frac{(-1)^n\tau^{n\eth}}{(n\eth)!}\right),\\
\xi(\vartheta,\gamma,\tau) &= \sum_{n=0}^{\infty}\xi_n(\vartheta,\gamma,\tau) = e^{\vartheta-\gamma}\left(\sum_{n=0}^{\infty}\frac{\tau^{n\eth}}{(n\eth)!}\right),\\
\Im(\vartheta,\gamma,\tau) &= \sum_{n=0}^{\infty}\Im_n(\vartheta,\gamma,\tau) = e^{-\vartheta+\gamma}\left(\sum_{n=0}^{\infty}\frac{(-1)^n\tau^{n\eth}}{(n\eth)!}\right).
\end{aligned} \tag{5.28}$$

For a specific integer value $\eth = 1$, the MVITA solution becomes

$$\begin{aligned}
\zeta(\vartheta,\gamma,\tau) &= e^{\vartheta+\gamma-\tau},\\
\xi(\vartheta,\gamma,\tau) &= e^{\vartheta-\gamma+\tau},\\
\Im(\vartheta,\gamma,\tau) &= e^{-\vartheta+\gamma-\tau},
\end{aligned}$$

which is the exact solution of Example 5.2.

In Figure 5, the approximate solution of $\zeta(\vartheta,\gamma,\tau)$ is obtained by using the NAAA for Example 5.2. Figure 6, the approximate solution of $\xi(\vartheta,\gamma,\tau)$ is obtained by using the NAAA for Example 5.2. Figure 7, the approximate solution of $\Im(\vartheta,\gamma,\tau)$ is obtained by using the NAAA for Example 5.2. Figure 8, the approximate solution of $\zeta(\vartheta,\gamma,\tau)$ is obtained by using the MVITA for Example 5.2. Figure 9, the approximate solution of $\xi(\vartheta,\gamma,\tau)$ is obtained by using the MVITA for Example 5.2. Figure 10, the approximate solution of $\Im(\vartheta,\gamma,\tau)$ is obtained by using the MVITA for Example 5.2.

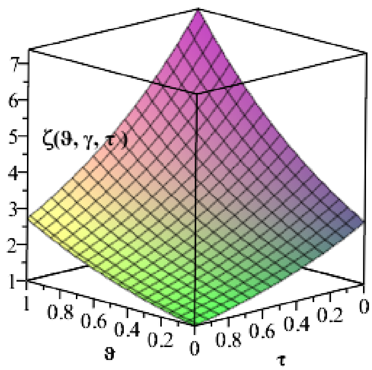

Figure 5. The approximate solution of $\zeta(\vartheta,\gamma,\tau)$ as obtained by using the NAAA for Example 5.2.

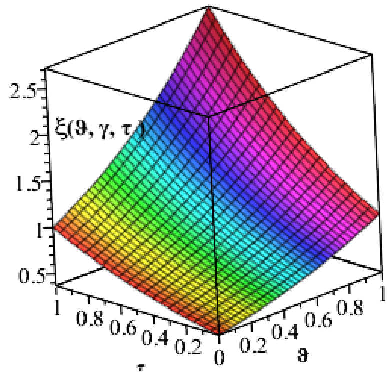

Figure 6. The approximate solution of $\xi(\vartheta, \gamma, \tau)$ as obtained by using the NAAA for Example 5.2.

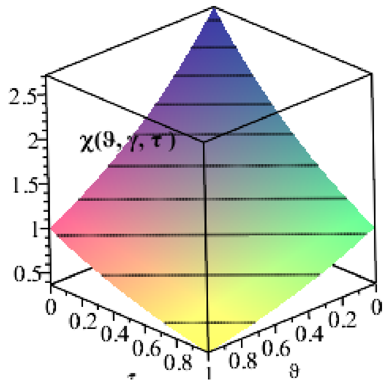

Figure 7. The approximate solution of $\Im(\vartheta, \gamma, \tau)$ as obtained by using the NAAA for Example 5.2.

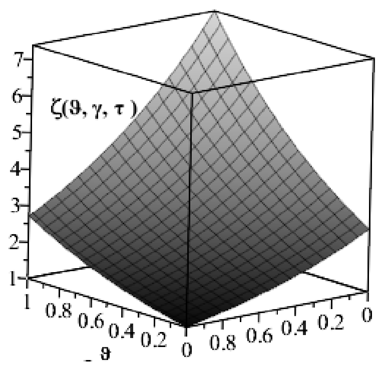

Figure 8. The approximate solution of $\zeta(\vartheta, \gamma, \tau)$ as obtained by using the MVITA for Example 5.2.

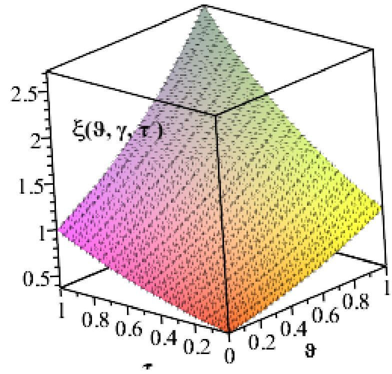

Figure 9. The approximate solution of $\xi(\vartheta, \gamma, \tau)$ as obtained by using the MVITA for Example 5.2.

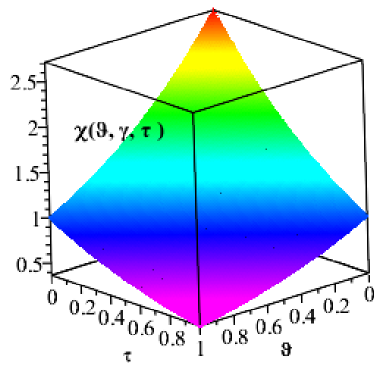

Figure 10. The approximate solution of $\Im(\vartheta, \gamma, \tau)$ as obtained by using the MVITA for Example 5.2.

6. Conclusions

The current article has introduced two analytical approaches, the NAAA and MVITA, to solve PDEs. Their validity and applicability were checked by solving time-fractional systems of PDEs. For Example 5.1, the approximations defined by (5.11) and (5.15) have shown that the MVITA has rapid convergence to the exact solution for the problem for integer order; this was also demonstrated by plotting. Similarly, for Example 5.2, the approximated results represented by (5.26) and (5.28) have shown that the MVITA has a series-form solution which quickly converges to the exact solution for the problem; this was checked by plotting. Overall, it has been demonstrated that the MVITA has less computational work and rapid convergence without decomposition, He's polynomial or discretization, while, in the case of the NAAA, the non-linearity is decomposed by an Adomian decomposition procedure, and it has a slow rate of convergence for non-linear problems. The MVITA can easily be extended to high order non-linear physical models.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Large Groups (R.G.P2./41/43).

Conflict of interest

The authors declare no conflicts of interest.

References

1. J. H. He, Homotopy perturbation technique, *Comput. Methods Appl. Mech. Eng.*, **178** (1999), 257–262. https://doi.org/10.1016/S0045-7825(99)00018-3
2. X. J. Wu, D. R. Lai, H. T. Lu, Generalized synchronization of the fractional-order chaos in weighted complex dynamical networks with nonidentical nodes, *Nonlinear Dynam.*, **69** (2012), 667–683. https://doi.org/10.1007/s11071-011-0295-9
3. H. Sheng, Y. Q. Chen, T. S. Qiu, *Fractional Processes and Fractional-Order Signal Processing*, London: Springer, 2011. https://doi.org/10.1007/978-1-4471-2233-3
4. H. Nasrolahpour, A note on fractional electrodynamics, *Commun. Nonlinear Sci.*, **18** (2013), 2589–2593. https://doi.org/10.1016/j.cnsns.2013.01.005
5. P. Veeresha, D. G. Prakasha, H. M. Baskonus, New numerical surfaces to the mathematical model of cancer chemotherapy effect in Caputo fractional derivatives, *Chaos An Interdiscip. J. Nonlinear Sci.*, **29** (2019), 013119. https://doi.org/10.1063/1.5074099
6. S. Longhi, Fractional Schrodinger equation in optics, *Opt. lett.*, *40* (2015), 1117–1120. https://doi.org/10.1364/OL.40.001117
7. S. Ullah, M. A. Khan, M. Farooq, A new fractional model for the dynamics of the hepatitis B virus using the Caputo-Fabrizio derivative, *Eur. Phys. J. Plus*, **133** (2018), 237. https://doi.org/10.1140/epjp/i2018-12072-4
8. M. A. Khan, S. Ullah, M. Farooq, A new fractional model for tuberculosis with relapse uia Atangana-Baleanu deriuatiue, *Chaos Soliton Fract.*, **116** (2018), 227–238. https://doi.org/10.1016/j.chaos.2018.09.039
9. M. A. Khan, S. Ullah, K. O. Okosun, K. Shah, A fractional order pine wilt disease model with Caputo-Fabrizio deriuatiue, *Adv. Differ. Equ.*, **2018** (2018), 410. https://doi.org/10.1186/s13662-018-1868-4
10. J. Singh, D. Kumar, D. Baleanu, On the analysis of fractional diabetes model with exponential law, *Adv. Differ. Equ.*, **2018** (2018), 231. https://doi.org/10.1186/s13662-018-1680-1
11. R. Hilfer, *Applications of Fractional Calculus in Physics*, 2000. https://doi.org/10.1142/3779
12. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Amsterdam, Boston: Elsevier, 2006.
13. S. Das, A note on fractional diffusion equations, *Chaos Soliton Fract.*, **42** (2009), 2074–2079. https://doi.org/10.1016/j.chaos.2009.03.163

14. R. Ye, P. Liu, K. B. Shi, B. Yan, State damping control: A novel simple method of rotor UAV with high performance, *IEEE Access*, **8** (2020), 214346–214357. https://doi.org/10.1109/ACCESS.2020.3040779

15. K. Liu, Z. X. Yang, W. F. Wei, B. Gao, D. L. Xin, C. M. Sun, et al., Novel detection approach for thermal defects: Study on its feasibility and application to vehicle cables, *High Volt.*, 2022, 1–10. https://doi.org/10.1049/hve2.12258

16. P. Liu, J. P. Shi, Z. A. Wang, Pattern formation of the attraction-repulsion Keller-Segel system, *Discrete Continuous Dyn. Syst. B*, **18** (2013), 2597–2625. https://doi.org/10.3934/dcdsb.2013.18.2597

17. J. G. Liu, X. J. Yang, J. J. Wang, A new perspective to discuss Korteweg-de Vries-like equation, *Phys. Lett. A*, **451** (2022), 128429. https://doi.org/10.1016/j.physleta.2022.128429

18. Z. H. Xie, X. A. Feng, X. J. Chen, Partial least trimmed squares regression, *Chemometr. Intell. Lab. Syst.*, **221** (2022), 104486. https://doi.org/10.1016/j.chemolab.2021.104486

19. Z. T. Shao, Q. Z. Zhai, Z. H. Han, X. H. Guan, A linear AC unit commitment formulation: An application of data-driven linear power flow model, *Int. J. Elec. Power*, **145** (2023), 108673. https://doi.org/10.1016/j.ijepes.2022.108673

20. V. N. Kovalnogov, R.V. Fedorov, T. V. Karpukhina, T. E. Simos, C. Tsitouras, Runge-Kutta pairs of orders 5(4) trained to best address keplerian type orbits, *Mathematics*, **9** (2021), 2400. https://doi.org/10.3390/math9192400

21. N. A. Shah, H. A. Alyousef, S. A. El-Tantawy, J. D. Chung, Analytical investigation of fractional-order Korteweg-De-Vries-type equations under Atangana-Baleanu-Caputo operator: Modeling nonlinear waves in a plasma and fluid, *Symmetry*, **14** (2022), 739. https://doi.org/10.3390/sym14040739

22. R. Shah, H. Khan, D. Baleanu, P. Kumam, M. Arif, The analytical investigation of time-fractional multi-dimensional Navier-Stokes equation, *Alex. Eng. J.*, **59** (2020), 2941–2956. https://doi.org/10.1016/j.aej.2020.03.029

23. K. Nonlaopon, M. Naeem, A. M. Zidan, R. Shah, A. Alsanad, A. Gumaei, Numerical investigation of the time-fractional Whitham-Broer-Kaup equation involving without singular kernel operators, *Complexity*, **2021** (2021), 7979365. https://doi.org/10.1155/2021/7979365

24. S. Alyobi, A. Khan, N. A. Shah, K. Nonlaopon, Fractional analysis of nonlinear boussinesq equation under Atangana-Baleanu-Caputo operator, *Symmetry*, **14** (2022), 2417. https://doi.org/10.3390/sym14112417

25. S. Mukhtar, S. Noor, The numerical investigation of a fractional-order multi-dimensional model of Navier-Stokes equation via novel techniques, *Symmetry*, **14** (2022), 1102. https://doi.org/10.3390/sym14061102

26. V. B. L. Chaurasia, D. Kumar, Solution of the time-fractional Navier-Stokes equation, *Gen. Math. Notes*, **4** (2011), 49–59.

27. A. Prakash, P. Veeresha, D. G. Prakasha, M. Goyal, A new efficient technique for solving fractional coupled Navier–Stokes equations using q-homotopy analysis transform method, *Pramana*, **93** (2019), 6. https://doi.org/10.1007/s12043-019-1763-x

28. A. A. Alderremy, N. A. Shah, S. Aly, K. Nonlaopon, Evaluation of fractional-order pantograph delay differential equation via modified laguerre wavelet method, *Symmetry*, **14** (2022), 2356. https://doi.org/10.3390/sym14112356
29. H. Thabet, S. Kendre, J. Peters, Travelling wave solutions for fractional Korteweg-de Vries equations via an approximate-analytical method, *AIMS Mathematics*, **4** (2019), 1203–1222. https://doi.org/10.3934/math.2019.4.1203
30. B. R. Sontakke, A. S. Shaikh, Properties of Caputo operator and its applications to linear fractional differential equations, *Int. J. Eng. Res. Appl.*, **5** (2015), 22–27.
31. S. Momani, Z. Odibat, Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method, *Appl. Math. Comput.*, **177** (2006), 488–494. https://doi.org/10.1016/j.amc.2005.11.025
32. N. A. Shah, Y. S. Hamed, K. M. Abualnaja, J. D. Chung, A. Khan, A comparative analysis of fractional-order kaup-kupershmidt equation within different operators, *Symmetry*, **14** (2022), 986. https://doi.org/10.3390/sym14050986
33. M. M. Al-Sawalha, O. Y. Ababneh, K. Nonlaopon, Numerical analysis of fractional-order Whitham-Broer-Kaup equations with non-singular kernel operators, *AIMS Mathematics*, **8** (2023), 2308–2336. https://doi.org/10.3934/math.2023120
34. T. Botmart, R. P. Agarwal, M. Naeem, A. Khan, R. Shah, On the solution of fractional modified Boussinesq and approximate long wave equations with non-singular kernel operators, *AIMS Mathematics*, **7** (2022), 12483–12513. https://doi.org/10.3934/math.2022693
35. D. Kumar, J. Singh, A. Prakash, R. Swroop, Numerical simulation for system of time-fractional linear and nonlinear differential equations, *Prog. Fract. Differ. Appl.*, **5** (2019), 65–77. https://doi.org/10.18576/pfda/050107
36. M. Goyal, H. M. Baskonus, A. Prakash, An efficient technique for a time fractional model of lassa hemorrhagic fever spreading in pregnant women, *Eur. Phys. J. Plus*, **134** (2019), 482. https://doi.org/10.1140/epjp/i2019-12854-0
37. H. M. He, J. G. Peng, H. Y. Li, Iterative approximation of fixed point problems and variational inequality problems on Hadamard manifolds, *UPB Sci. Bull. Series A*, **84** (2022), 25–36.
38. L. Liu, L. Zhang, G. Pan, S. Zhang, Robust yaw control of autonomous underwater vehicle based on fractional-order PID controller, *Ocean Eng.*, **257** (2022), 111493. https://doi.org/10.1016/j.oceaneng.2022.111493
39. A. Prakash, H. Kaur, Analysis and numerical simulation of fractional Biswas-Milovic model, *Math. Comput. Simulat.*, **181** (2021), 298–315. https://doi.org/10.1016/j.matcom.2020.09.016
40. A. Prakash, M. Goyal, S. Gupta, A reliable algorithm for fractional Bloch model arising in magnetic resonance imaging, *Pramana*, **92** (2019), 18. https://doi.org/10.1007/s12043-018-1683-1
41. M. Goyal, H. M. Baskonus, A. Prakash, Regarding new positive, bounded and convergent numerical solution of nonlinear time fractional HIV/AIDS transmission model, *Chaos Soliton Fract.*, **139** (2020), 110096. https://doi.org/10.1016/j.chaos.2020.110096

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